

# A geometric spectral sequence in Khovanov homology

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**Abstract** The aim of this paper is to introduce and study a geometric spectral sequence on  $\mathbb{Z}_2$  Khovanov homology.

**AMS Classification**

**Keywords**

## 1 Introduction

The construction in the present paper was motivated by a joint work with Peter Ozsváth on the Heegaard Floer homology of double branched covers. In [12] a spectral sequence is constructed from the reduced Khovanov homology of a link  $L$  to the hat version of Heegaard Floer homology  $\widehat{HF}(-\Sigma_L)$  with  $\mathbb{Z}_2$  coefficients, where  $\Sigma_L$  denotes the double cover of  $S^3$  branched along  $L$ . The same construction also gives a spectral sequence from the mod 2 Khovanov homology  $Kh(L)$  to  $\widehat{HF}(-\Sigma_L \# (S^1 \times S^2))$ . (Note that the latter is just two copies of  $\widehat{HF}(-\Sigma_L)$ .)

Gauge-theoretic spectral sequences starting from Khovanov homology were constructed by Bloom [4] for Monopole Floer homology [7], and by Kronheimer and Mrowka [8] in the context of instanton Floer homology.

It was proved by Baldwin in [1] that the Heegaard Floer homology spectral sequence is a link invariant. The corresponding result for the Monopole Floer homology sequence was given in [4].

Given a diagram  $\mathcal{D}$  of the link  $L$  with  $n$  double points the constructions in [4], [8], and [12] assign higher differentials for the mod 2 Khovanov complex that correspond to  $k$ -dimensional faces of  $\{0, 1\}^n$ , with  $k \geq 2$ . Note that the differentials count solutions to the Seiberg-Witten equations for certain  $k - 1$  dimensional family of metrics in [4], instantons in [8], and certain pseudo holomorphic  $k + 2$ -gons in [12]. In particular the maps for a given hypercube

depend on some additional data, such as choices of metric and perturbations, or Heegaard diagrams and complex structures.

The construction in the present paper uses the same idea. However the extra data is more combinatorial: At each double point we will fix an orientation of the arc that connects the two segments of the 0-resolution, see Figure 1. This overall choice  $\mathbf{t}$  is called a decoration of the diagram. Given a decoration, each  $k$  dimensional face determines a collection of circles in the two-dimensional plane together with  $k$  oriented arcs that connect the circles. These configurations are discussed in Section 2. In Sections 3 and 4 we spell out a geometric rule that assigns non-trivial contributions to certain special configurations, see for example Figure 2 and 3. In Section 5 we prove that  $\mathbf{d}(\mathbf{t}) \cdot \mathbf{d}(\mathbf{t}) = 0$ , and so we get a chain complex  $\widehat{C}(\mathcal{D}, \mathbf{t}) = (C_{\mathcal{D}}, \mathbf{d}(\mathbf{t}))$  for a decorated diagram  $(\mathcal{D}, \mathbf{t})$ .

Another feature of the construction is that the higher differentials preserve the  $\delta$  grading in Khovanov homology, and that induces a  $\delta$  grading on the homology  $\widehat{H}(\mathcal{D}, \mathbf{t})$ . See also [5], [10] and [19] for discussions on extra gradings in Heegaard Floer homology. There is also a filtration on  $\widehat{C}(\mathcal{D}, \mathbf{t})$ , given by the homological grading, and that gives a spectral sequence from  $Kh(L)$  to  $\widehat{H}(\mathcal{D}, \mathbf{t})$ .

In Section 6 we prove that the homology theory  $\widehat{H}$  and the spectral sequence are well-defined invariants of  $L$ :

**Theorem 1.1** *Given an oriented link  $L$ , let  $\mathcal{D}_1, \mathcal{D}_2$  be two diagrams representing  $L$ , and let  $\mathbf{t}_1, \mathbf{t}_2$  be decorations for  $\mathcal{D}_1$  and  $\mathcal{D}_2$  respectively. Then there is a grading preserving isomorphism between  $\widehat{H}(\mathcal{D}_1, \mathbf{t}_1)$  and  $\widehat{H}(\mathcal{D}_2, \mathbf{t}_2)$ . Furthermore the spectral sequences  $Kh(L) \rightarrow \widehat{H}(\mathcal{D}_1, \mathbf{t}_1)$  and  $Kh(L) \rightarrow \widehat{H}(\mathcal{D}_2, \mathbf{t}_2)$  are also isomorphic.*

While the definitions of  $\widehat{H}(L) = \widehat{H}(\mathcal{D}, \mathbf{t})$  and  $\widehat{HF}(-\Sigma_L \# (S^1 \times S^2))$  use different tools, the two homology theories are rather similar. In fact it is natural to conjecture that the two homology theories are isomorphic as mod 2 graded vector spaces over  $\mathbb{Z}_2$ . The similarities are underscored by a few additional constructions, presented in Section 7. These include a reduced version of  $\widehat{H}(L)$  that could be the natural counterpart of  $\widehat{HF}(-\Sigma(L))$ , and an adaptation of the transverse invariant of [13] to  $\widehat{H}(L)$ , see also [14], [1]. Computations for the spectral sequence are given in a recent work of Seed, see [17].

While in this paper we work over mod 2 coefficients, an integer lift of the spectral sequence for odd Khovanov homology is given in [18], see also [3]. In a different direction it would be interesting to compare  $\widehat{H}(L)$  with the recent work of Lipshitz, Ozsváth and Thurston, see [9]

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## 2 Preliminary constructions

We will start by recalling some constructions from Khovanov homology, see [6], [2]. Let  $L$  be an oriented link, and  $\mathcal{D}$  be a diagram of  $L$  in the plane with  $n$  double points. At each crossing we have two resolutions 0 and 1, see Figure 1. Sometimes it will be helpful to use the one-point compactification of the plane and view  $\mathcal{D}$  and the resolutions in the 2-dimensional sphere.

By ordering the double-points we get an identification between the set of resolutions  $\mathcal{R}$  and  $\{0,1\}^n$ . Each resolution  $I \in \mathcal{R}$  gives a set of disjoint circles  $x_1, \dots, x_t$  in the sphere. The resolution  $I$  comes equipped with a  $2^t$  dimensional vector space  $V(I)$  over  $\mathbb{Z}_2$ . It will be useful for us to identify the basis of  $V(I)$  with monomials in  $x_i$ . This is done by associating a two dimensional vector space  $V(x_i)$  for each circle with generators 1 and  $x_i$ , and defining  $V(I)$  as the tensor product of  $V(x_i)$  for  $i = 1, \dots, t$ . Note that the original construction of Khovanov [2], [6] uses different notations, where  $v_-$  plays the role of  $x_i$ , and  $v_+$  the role of 1. Finally we define

$$C_{\mathcal{D}} = \bigoplus_{I \in \mathcal{R}} V(I)$$

A  $k$  dimensional face of  $C_{\mathcal{D}}$  corresponds  $(I, J) \in \mathcal{R} \times \mathcal{R}$  with  $I < J$  so that  $I$  and  $J$  differs at exactly  $k$  coordinates. In Khovanov homology the boundary map is defined by associating maps

$$D_{I,J} : V(I) \longrightarrow V(J)$$

to all the 1-dimensional faces (edges) of  $\mathcal{R}$ . Our goal is to define some higher differentials on  $C_{\mathcal{D}}$  by extending the definition of  $D_{I,J}$  to all  $k$  dimensional faces. To this end we will need to fix some extra data (decoration) at each crossing. First note that at each crossing there is an arc connecting the two segments of the 0 resolution. Making surgery along this arc produces the 1 resolution. The extra data is an assignment of orientation to all of these arcs. The oriented arc at the  $i$ -th crossing is denoted by  $\gamma_i$ . At each crossing there are two choices. Let  $\mathbf{t}$  denote an overall decoration for  $\mathcal{D}$ .

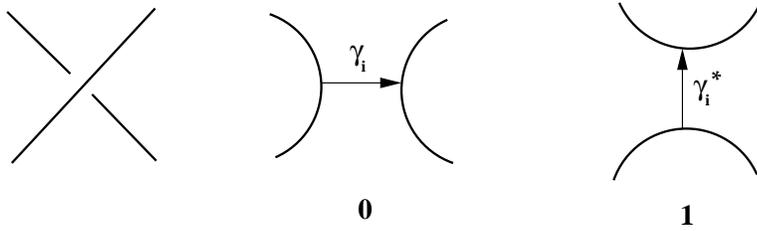


Figure 1: Resolutions and oriented arcs

A decorated  $k$ -dimensional face  $\mathcal{F} = (I, J, \mathbf{t})$  of  $(\mathcal{D}, \mathbf{t})$  determines a configuration in the sphere, that consist of the circles of  $I$  together with  $k$  oriented arcs  $\gamma_{j_1}, \dots, \gamma_{j_k}$ , where  $j_1, \dots, j_k$  are the coordinates where  $I$  and  $J$  differ.

Sometimes it will be useful to have a more abstract notion of this configuration, that makes no reference to the knot projection  $\mathcal{D}$ .

**Definition 2.1** A  $k$  dimensional *configuration* is a set of disjoint circles  $x_1, \dots, x_t$  in  $S^2$  together with  $k$  embedded oriented arcs  $\gamma_1, \dots, \gamma_k$ , with the properties that

- The arcs are disjoint from each other.
- The endpoints of the arcs lie on the circles.
- The inside of the arcs are disjoint from the circles.

**Definition 2.2** Given a decorated  $k$ -dimensional face  $(I, J, \mathbf{t})$  of  $\mathcal{D}$ , the corresponding  $k$  dimensional configuration is denoted by  $\mathcal{C}(I, J, \mathbf{t})$ .

**Definition 2.3** Given a configuration  $\mathcal{C} = (x_1, \dots, x_t, \gamma_1, \dots, \gamma_k)$ , the

- *undecorated configuration*  $\bar{\mathcal{C}}$  is given by forgetting the orientation on the  $\gamma$  arcs.
- *dual configuration*  $\mathcal{C}^* = (y_1, \dots, y_s, \gamma_1^*, \dots, \gamma_k^*)$  is given by the rule that transforms the 0-resolution of Figure 1 into the 1-resolution. In particular the dual circles  $y_i$  are constructed from the  $x$  circles by making surgeries along the  $\gamma$  arcs, and the dual arcs  $\gamma_i^*$  are given by rotating  $\gamma_i$  by 90 degrees counter-clockwise.
- *reverse*  $r(\mathcal{C})$  is defined by reversing the orientation for all the  $\gamma_i$  arcs. Note that  $(\mathcal{C}^*)^*$  is equivalent to  $r(\mathcal{C})$ .
- *mirror*  $m(\mathcal{C})$  is defined by reversing the orientation of the two-dimensional sphere.

Let's call  $x_i$  the *starting circles* of  $\mathcal{C}$ , and  $y_i$  the *ending circles* of  $\mathcal{C}$ . Let

$$V_0(\mathcal{C}) = \bigotimes_{i=1}^t V(x_i), \quad V_1(\mathcal{C}) = \bigotimes_{j=1}^s V(y_j)$$

Our goal is to define a map

$$F_{\mathcal{C}} : V_0(\mathcal{C}) \longrightarrow V_1(\mathcal{C})$$

for each configuration.

**Definition 2.4** Given a configuration  $\mathcal{C} = (x_1, \dots, x_t, \gamma_1, \dots, \gamma_k)$ , those  $x_i$  circles that are disjoint from all the  $\gamma$  arcs are called *passive* circles. Clearly the same circles are also passive for the dual configuration  $\mathcal{C}^*$ . A configuration is called (purely) *active*, if it has no passive circles. By deleting all the passive circles we get the *active part*  $\mathcal{C}_0$  of  $\mathcal{C}$ . The starting circles of  $\mathcal{C}_0$  are called the *active starting circles* of  $\mathcal{C}$ , and similarly the ending circles of  $\mathcal{C}_0$  are the *active ending circles* of  $\mathcal{C}$ . We also have a decomposition

$$V_0(\mathcal{C}) = V_0(\mathcal{C}_0) \otimes P(\mathcal{C}), \quad V_1(\mathcal{C}) = V_1(\mathcal{C}_0) \otimes P(\mathcal{C}),$$

where  $P(\mathcal{C})$  is the tensor product of  $V(w_i)$  of all the passive  $w_i$  circles of  $\mathcal{C}$ .

The map  $F_{\mathcal{C}}$  satisfies various properties. We will list these below.

**Definition 2.5** *Extension Formula.* For a configuration  $\mathcal{C}$  the map  $F_{\mathcal{C}}$  depends only on the active part  $\mathcal{C}_0$  and the number of passive circles by the following formula:

$$F_{\mathcal{C}}(a \cdot v) = F_{\mathcal{C}_0}(a) \cdot v,$$

where  $v \in P(\mathcal{C})$  and  $a \in V_0(\mathcal{C})$ .

Recall from [2], [6] that the Khovanov differential satisfies the same extension property. For one-dimensional faces (edges) there are two kinds of active configurations: splitting and joining.

- A splitting edge  $\mathcal{C}$ , has one active starting circle  $x_1$ , and two active ending circles  $y_1$  and  $y_2$ . The map  $F_{\mathcal{C}_0}$  is given by

$$F_{\mathcal{C}_0}(1) = y_1 + y_2, \quad F_{\mathcal{C}_0}(x_1) = y_1 y_2.$$

- A joining edge has two active starting circles,  $x_1$  and  $x_2$ , one active ending circle  $y_1$  and we have

$$F_{\mathcal{C}_0}(1) = 1, \quad F_{\mathcal{C}_0}(x_1) = y_1, \quad F_{\mathcal{C}_0}(x_2) = y_1, \quad F_{\mathcal{C}_0}(x_1 x_2) = 0.$$

**Definition 2.6** A configuration  $\mathcal{C}$  is called disconnected, if the active starting circles of  $\mathcal{C}_0$  can be partitioned into two non-empty sets,  $c_1, \dots, c_s, d_1, \dots, d_t$  so that none of the  $\gamma$  arcs connect  $c_i$  to  $d_j$ . Otherwise we call the configuration connected. Note that every 1-dimensional configuration is connected.

**Definition 2.7** *Disconnected Rule.* If  $\mathcal{C}$  is a disconnected configuration then

$$F_{\mathcal{C}} \equiv 0.$$

**Definition 2.8** *Conjugation Rule.* For each configuration  $\mathcal{C}$  we have

$$F_{\mathcal{C}} = F_{r(\mathcal{C})}$$

**Definition 2.9** *Naturality Rule.* Let  $\mathcal{C}$  and  $\mathcal{C}'$  be two  $k$  dimensional configurations, with the property that there is an orientation preserving diffeomorphism of the sphere that maps  $\mathcal{C}$  to  $\mathcal{C}'$ . Then the diffeomorphism induces natural identifications  $V_0(\mathcal{C}) = V_0(\mathcal{C}')$ , and  $V_1(\mathcal{C}) = V_1(\mathcal{C}')$ . Under these identifications we have

$$F_{\mathcal{C}} = F_{\mathcal{C}'}$$

Now we discuss the duality rule. First note that the monomials in  $V_0(\mathcal{C})$  and  $V_1(\mathcal{C})$  give natural basis for these vector spaces. In particular if  $a$  is a monomial in  $V_0(\mathcal{C})$  then we can write

$$F_{\mathcal{C}}(a) = \sum_b \alpha(a, b) \cdot b$$

where  $\alpha(a, b) \in \mathbb{Z}_2$  and the sum is over all the monomials  $b \in V_1(\mathcal{C})$ . We will call  $\alpha(a, b)$  the coefficient of  $F_{\mathcal{C}}(a)$  at  $b$ . Given a circle  $z$  we define the duality map on the 2 dimensional vector space  $V(z)$  by  $1^* = z, z^* = 1$ . This induces the duality maps on  $V_0(\mathcal{C})$  and  $V_1(\mathcal{C})$ .

**Definition 2.10** *Duality Rule.* Let  $\mathcal{C}$  be a configuration, and  $m(\mathcal{C}^*)$  the mirror of the dual configuration. Then for all pairs of monomials  $(a, b)$ ,  $a \in V_0(\mathcal{C})$ ,  $b \in V_1(\mathcal{C})$  the coefficient of  $F_{\mathcal{C}}(a)$  at  $b$  is equal to the coefficient of  $F_{m(\mathcal{C}^*)}(b^*)$  at  $a^*$ .

**Definition 2.11** *Filtration rule.* Let  $\mathcal{C}$  be a configuration,  $a \in V_0(\mathcal{C})$ ,  $b \in V_1(\mathcal{C})$  monomials. For a point  $P$  in the union of the starting circles, let  $x(P)$  and  $y(P)$  denote the starting and ending circles that go through  $P$ . If  $a$  is divisible by  $x(P)$  and the coefficient of  $F_{\mathcal{C}}(a)$  at  $b$  is non-zero, then  $b$  is divisible by  $y(P)$ .

The next property involves the grading shift of  $F$ . For  $a \in V(x)$  define

$$gr(1) = 1, \quad gr(x) = -1$$

For monomials  $a \in V_0(\mathcal{C}) = \otimes_{i=1}^t V(x_i)$ ,  $b \in V_1(\mathcal{C}) = \otimes_{j=1}^s V(y_j)$  define the grading to be the sum of the gradings in each factor.

**Definition 2.12** *Grading rule.* Let  $\mathcal{C}$  be a  $k$ -dimensional configuration,  $a \in V_0(\mathcal{C})$ ,  $b \in V_1(\mathcal{C})$  monomials. If the coefficient of  $F_{\mathcal{C}}(a)$  at  $b$  is non-trivial, then

$$gr(b) - gr(a) = k - 2.$$

Note that for 1-dimensional configurations  $F_{\mathcal{C}}$  satisfies the rules in Definition 2.7- 2.12

### 3 2-dimensional configurations

In this section we will define the  $F$  map for all 2-dimensional configurations. According to Definition 2.5 it is enough to spell out the rule for the active part of 2-dimensional configurations. Furthermore according to Definition 2.7, 2.8 and 2.9 it is enough to consider connected configurations modulo orientation preserving diffeomorphisms in the sphere, and reversals  $\mathcal{C} \rightarrow r(\mathcal{C})$ . The resulting equivalence classes are listed in Figure 2. The rules for the active map

$$F = F_{\mathcal{C}_0} : V_0(\mathcal{C}_0) \longrightarrow V_1(\mathcal{C}_0)$$

are given in Definition 3.1, where we use monomials in the active starting circles  $x_i$  as a basis of  $V_0(\mathcal{C}_0)$  and list only the non-zero terms of  $F$ .

**Definition 3.1**     • For a Type 1 configuration

$$F(1) = 1.$$

- For a Type 2 or Type 3 configuration there are three starting circles and one ending circle. If  $x_1$  denotes the starting circle that meets both  $\gamma$  arcs, then

$$F(x_2x_3) = y_1.$$

- For a Type 4 or Type 5 configuration,

$$F(1) = y_1,$$

where among the three ending circles,  $y_1$  denotes the unique circle that meets both of the dual  $\gamma^*$  arcs.

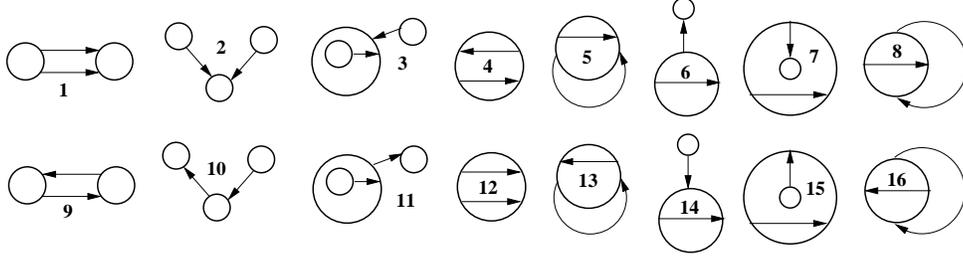


Figure 2: The classification of active, connected 2-dimensional configurations in the sphere, modulo the additional relation that  $\mathcal{C}$  is equivalent to  $r(\mathcal{C})$ .

- For a Type 6 or Type 7 configuration there are two starting and two ending circles. Let  $x_1$  denote the starting circle that meets both of the  $\gamma$  arcs, and  $y_1$  denotes the ending circle that meets both of the dual arcs. Then

$$F(x_2) = y_1$$

- For a Type 8 configuration

$$F(1) = 1, \quad F(x_1) = y_1.$$

- For a Type 9, configuration

$$F(x_1x_2) = y_1y_2.$$

- Finally for a Type 10, 11, 12, 13, 14, 15, 16 or for a disconnected configuration

$$F \equiv 0.$$

**Lemma 3.2** For 2-dimensional  $\mathcal{C}$  configurations the map  $F_{\mathcal{C}}$  satisfies the rules in Definition 2.7-2.12.

**Proof** The rules in Definition 2.7, 2.8, 2.9, 2.11, 2.12 follow immediately from the definition of  $F_{\mathcal{C}}$ .

According to Definition 2.5, it is enough to check the duality rule for the active 2-dimensional configurations. If the configuration  $\mathcal{C}$  is disconnected, then the dual configuration is also disconnected, so  $F_{\mathcal{C}} = 0$ ,  $F_{m(\mathcal{C}^*)} = 0$ .

After dividing with the relation that  $\mathcal{C}$  is equivalent to its reverse, the connected types of Figure 2 are related in the following way,  $1^* = 9$ ,  $2^* = 4$ ,  $3^* = 5$ ,  $6^* = 14$ ,  $7^* = 15$ ,  $8^* = 16$ ,  $10^* = 12$ ,  $11^* = 13$ . Furthermore  $m(i) = i$  for  $1 \leq i \leq 5$ , or  $9 \leq i \leq 13$ , and  $m(6) = 14, m(7) = 15, m(8) = 16$ . It

follows that  $m(1^*) = 9$ ,  $m(2^*) = 4$ ,  $m(3^*) = 5$ ,  $m(6^*) = 6$ ,  $m(7^*) = 7$ ,  $m(8^*) = 8$ , and  $m(i^*)$  is of type  $j$  with  $10 \leq j \leq 16$  if and only if  $10 \leq i \leq 16$ . Checking the duality formula is now rather straightforward. For example if  $\mathcal{C}$  is of Type 6 then  $F(x_2) = y_1$ ,  $x_2^* = x_1$ ,  $y_1^* = y_2$ , and indeed for the  $m(\mathcal{C}^*)$  configuration (which is also of Type 6) the circle  $y_2$  is mapped to  $x_1$ . The other 2-dimensional configurations are left for the interested reader to check.  $\square$

## 4 Contributions of $k$ -dimensional configurations.

Given  $k > 2$  we will distinguish 5 kinds of  $k$  dimensional configurations for which  $F_{\mathcal{C}} \neq 0$ .

**Definition 4.1** The configuration  $\mathcal{C} = (x_1, \dots, x_s, \gamma_1, \dots, \gamma_k)$  is of Type  $A_k$  if for each pair  $(i, j)$  with  $1 \leq i, j \leq k$  the two dimensional configuration  $(x_1, \dots, x_s, \gamma_i, \gamma_j)$  is of type 1, see Figure 3. It follows that  $\mathcal{C}$  has two active starting circles and  $k$  active ending circles. For a Type  $A_k$  configuration we define

$$F_{\mathcal{C}_0}(1) = 1.$$

**Definition 4.2** A  $k$ -dimensional configuration  $\mathcal{C}$  is of Type  $B_k$  if  $m(\mathcal{C}^*)$  is of Type  $A_k$ . It follows that  $\mathcal{C}$  has  $k$  active starting circles and two active ending circles. For a Type  $B_k$  configuration define

$$F_{\mathcal{C}_0}\left(\prod_{i=1}^k x_i\right) = y_1 y_2$$

**Definition 4.3** Let  $\mathcal{C}$  be a  $k$ -dimensional configuration, with the property that it has only one active starting circle  $x_1$ . This circle separates the sphere into two regions, and the set of arcs decomposes as

$$\{\gamma_1, \dots, \gamma_k\} = \{e_1, \dots, e_p\} \cup \{f_1, \dots, f_q\}$$

where  $e_i$  lie in one side of  $x_1$  and  $f_j$  lie on the other side. Then  $\mathcal{C}$  is of Type  $C_{p,q}$  if  $p \geq 1$ ,  $q \geq 1$  and for each  $(i, j)$  pair with  $1 \leq i \leq p$ ,  $1 \leq j \leq q$  the two dimensional configuration  $(x_1, e_i, f_j)$  is of Type 8. For a Type  $C_{p,q}$  configuration we define

$$F_{\mathcal{C}_0}(1) = 1.$$

Note that  $p + q = k$  and  $\mathcal{C}$  has  $k - 1$  active ending circles. Furthermore if  $\mathcal{C}$  is Type  $C_{p,q}$  then  $r(\mathcal{C})$  is also Type  $C_{p,q}$ .

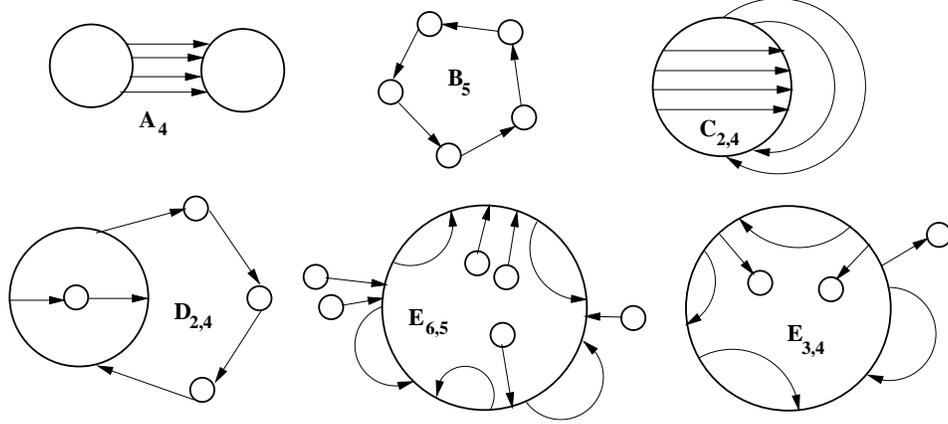


Figure 3: Examples for  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  types of configurations

**Definition 4.4** A  $k$ -dimensional configuration  $\mathcal{C}$  is of Type  $D_{p,q}$  if and only if  $m(\mathcal{C}^*)$  is of type  $C_{p,q}$ . In this case we define

$$F_{\mathcal{C}_0}(\prod_{i=1}^{k-1} x_i) = y_1.$$

**Definition 4.5** A  $k$ -dimensional configuration  $\mathcal{C} = (x_1, \dots, x_s, \gamma_1, \dots, \gamma_k)$  with  $p + 1$  active starting circles, and  $q + 1$  active ending circles, is called of type  $E_{p,q}$ , if for each pair  $(i, j)$  with  $1 \leq i, j \leq k$  the 2-dimensional configuration  $(x_1, \dots, x_s, \gamma_i, \gamma_j)$  is of type, 2, 3, 4, 5, 6, or 7. See Figure 3 for examples of  $E_{6,5}$  and  $E_{3,4}$ . Note that in a type  $E_{p,q}$  configuration there is a unique starting circle  $x_1$ , called the *central starting circle*, with the property that it meets all the  $\gamma$  arcs. The other active  $x_i$  are called degree 1 starting circles. Similarly there is a unique ending circle  $y_1$  with the property that  $y_1$  meets all the  $\gamma^*$  arcs. This circle is called *central ending circle*, and the other active  $y_i$  are called the dual degree 1 circles. Using this labeling, we define

$$F_{\mathcal{C}_0}(\prod_{i=2}^{p+1} x_i) = y_1,$$

when  $p \geq 1$  and

$$F_{\mathcal{C}_0}(1) = y_1,$$

when  $p = 0$ .

Note that if a  $k$  dimensional configuration  $\mathcal{C}$  is Type  $E_{p,q}$ , then  $k = p + q$ ,  $r(\mathcal{C})$  is Type  $E_{p,q}$  and  $m(\mathcal{C}^*)$  is of Type  $E_{q,p}$ .

**Lemma 4.6** For any  $k$ -dimensional  $\mathcal{C}$  configurations with  $k \geq 1$  the map  $F_{\mathcal{C}}$  satisfies the rules in Definition 2.7-2.12.

**Proof** This follows immediately from the above definitions and remarks and Lemma 3.2 □

It is also helpful to revisit now the contribution of the 2-dimensional configurations. In later calculations we will think of them as special cases of the  $A, B, C, D, E$  types. Note that for a two-dimensional configuration, Type 2 and 3 of Figure 2 are examples for  $E_{2,0}$ , Type 4 and 5 are  $E_{0,2}$ , and Type 6 and 7 are  $E_{1,1}$ . Similarly Type 1 is  $A_2$  and Type 9 is  $B_2$ .

There is also a special case, when  $\mathcal{C}$  is of Type 8, since then  $\mathcal{C}$  is in a sense both  $C_{1,1}$  and  $D_{1,1}$ . In fact, recall that  $F_{\mathcal{C}_0}$  has two nontrivial terms:

$$F_{\mathcal{C}_0}(1) = 1 \quad , \quad F_{\mathcal{C}_0}(x_1) = y_1$$

and the first term corresponds to the  $C_{p,q}$  rule, see Definition 4.3, and the second to the  $D_{p,q}$  rule of Definition 4.4.

## 5 The definition of $\mathbf{d}$ .

In this section we define  $\mathbf{d}$  and study the role of decorations.

Given a diagram  $\mathcal{D}$  with decoration  $\mathbf{t}$  and a  $k$ -dimensional face  $(I, J)$  the corresponding configuration is denoted by  $\mathcal{C}(I, J, \mathbf{t})$ . Clearly  $V(I)$  is naturally identified with  $V_0(\mathcal{C}(I, J, \mathbf{t}))$  and  $V(J) = V_1(\mathcal{C}(I, J, \mathbf{t}))$ .

Using these identifications, we define

$$D_{I,J,\mathbf{t}} : V(I) \longrightarrow V(J)$$

by the formula

$$D_{I,J,\mathbf{t}} = F_{\mathcal{C}(I,J,\mathbf{t})}.$$

**Definition 5.1** Let  $n$  denote the number of double-points in  $\mathcal{D}$ . For  $1 \leq k \leq n$  we define

$$\mathbf{d}_k(\mathbf{t}) : C_{\mathcal{D}} \longrightarrow C_{\mathcal{D}}$$

as the sum of  $D_{I,J,\mathbf{t}}$  for all  $k$ -dimensional faces  $(I, J, \mathbf{t})$ . Now the boundary map

$$\mathbf{d}(\mathbf{t}) : C_{\mathcal{D}} \longrightarrow C_{\mathcal{D}}$$

is defined by

$$\mathbf{d}(\mathbf{t}) = \sum_{k=1}^n \mathbf{d}_k(\mathbf{t}).$$

Note that  $\mathbf{d}_1(\mathbf{t})$  agrees with the Khovanov differential, and in particular  $\mathbf{d}_1(\mathbf{t})$  doesn't depend on  $\mathbf{t}$ .

### 5.1 The $H_m$ maps.

We define an “edge-homotopy” for one dimensional configurations. Similarly to the  $F$  maps,  $H$  is defined on the active part  $\mathcal{C}_0$  and then extended to

$$H_{\mathcal{C}} : V_0(\mathcal{C}) \longrightarrow V_1(\mathcal{C})$$

by the extension formula of Definition 2.5.

**Definition 5.2** If  $\mathcal{C}$  is a one-dimensional configuration, then

$$H_{\mathcal{C}_0} : V_0(\mathcal{C}_0) \longrightarrow V_1(\mathcal{C}_0)$$

is defined:

- For a splitting edge

$$H_{\mathcal{C}_0}(1) = 1.$$

- For a joining edge

$$H_{\mathcal{C}_0}(x_1 x_2) = y_1.$$

Furthermore

$$H_{\mathcal{C}}(a \cdot v) = H_{\mathcal{C}_0}(a) \cdot v,$$

for  $a \in V_0(\mathcal{C}_0)$ ,  $v \in P(\mathcal{C})$ .

Note that  $H$  doesn't depend on the orientation of the  $\gamma$  arcs.

**Definition 5.3** For a 1-dimensional face  $(I, J)$  we define

$$H_{I,J} : V(I) \longrightarrow V(J)$$

by the formula

$$H_{I,J} = H_{\mathcal{C}(I,J)}.$$

For the  $m$ -th double point in the diagram  $\mathcal{D}$  we define

$$H_m : C_{\mathcal{D}} \longrightarrow C_{\mathcal{D}}$$

by summing  $H_{I,J}$  over those 1-dimensional faces, where  $I$  and  $J$  differ only in the  $m$ -th coordinate.

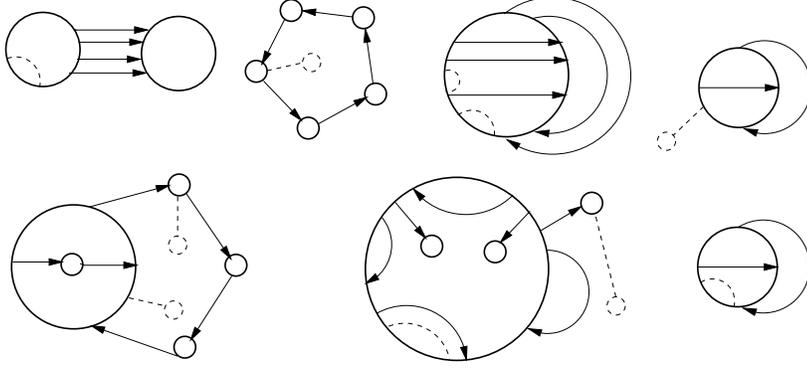


Figure 4:  $\delta$  arcs and  $w$  circles.

## 5.2 Dependence on the perturbations.

**Theorem 5.4** *Suppose that  $\mathbf{t}$  and  $\mathbf{t}'$  are decorations of the diagram  $\mathcal{D}$  that differ only at the  $m$ -th crossing. Then  $\mathbf{d}(\mathbf{t})$  and  $\mathbf{d}(\mathbf{t}')$  are related by the following formula:*

$$\mathbf{d}(\mathbf{t}') = \mathbf{d}(\mathbf{t}) + H_m \cdot \mathbf{d}(\mathbf{t}) + \mathbf{d}(\mathbf{t}) \cdot H_m.$$

**Proof** Let  $\delta$  denote the (unoriented) arc that correspond to the  $m$ -th crossing. For a  $k$ -dimensional face  $(I, J)$  with  $I(m) = 0$ ,  $J(m) = 1$ , let's define  $I'$  and  $J'$  as  $I'(m) = 1$ ,  $J'(m) = 0$ ,  $I'(i) = I(i)$ ,  $J'(i) = J(i)$  for  $i \neq m$ . Then Theorem 5.4 is equivalent to the following equation for all these  $k$  dimensional  $(I, J)$  faces:

$$(1) \quad D_{I,J,\mathbf{t}} - D_{I,J,\mathbf{t}'} = H_{I,I'} \cdot D_{I',J} + D_{I,J'} \cdot H_{J,J'}.$$

Of course, both sides depend only on the equivalence class of the configuration  $\mathcal{C} = \mathcal{C}(I, J, \mathbf{t})$  and the position of the  $\delta$  arc among the  $\gamma$  arcs in  $\mathcal{C}$ . Note also that  $\mathbf{t}$  and  $\mathbf{t}'$  agree on the  $k - 1$  dimensional faces  $(I, J')$  and  $(I', J)$  so we can safely omit that from the notation.

The rest of this section is devoted to the proof of Equation (1). Clearly it is enough to check the equation for active configurations. Another observation is that both the  $D$  and the  $H$  maps satisfy the duality rule, in particular, if Equation (1) holds for the pair  $(\mathcal{C}, \delta)$  then it holds for  $(m(\mathcal{C}^*), \delta^*)$ .

**$\mathcal{C}$  is a disconnected configuration.** By the disconnected rule we have  $D_{I,J,\mathbf{t}} = D_{I,J,\mathbf{t}'} = 0$ . Furthermore by extension formula for  $\mathbf{d}$  and  $H$  we

have

$$H_{I,I'} \cdot D_{I',J} = D_{I,J'} \cdot H_{J',J}.$$

**$\mathcal{C}$  is 2-dimensional.** In this case checking Equation (1) is an easy exercise. We illustrate this when  $\mathcal{C}$  is of Type 1. Let  $x_1, x_2$  denote the circles of  $I$ ,  $y_1, y_2$  denote the circles of  $J$ , and  $w_1$  the circle of  $I'$ . Since  $(I, I')$  is a join cobordism,

$$H_{I,I'}(x_1x_2) = w_1, \quad D_{I',J}(w_1) = y_1y_2.$$

For the other decomposition of  $(I, J)$  we have a non-trivial composition by

$$D_{I,J'}(1) = 1, \quad H_{J',J}(1) = 1.$$

It follows that the right hand side of Equation (1) maps 1 to 1, and  $x_1x_2$  to  $y_1y_2$ . Since  $\mathcal{C}(I, J, \mathbf{t})$  is of Type 1 and  $\mathcal{C}(I, J, \mathbf{t}')$  is of Type 9 the nontrivial terms of  $D$  are

$$D_{I,J,\mathbf{t}}(1) = 1, \quad D_{I,J,\mathbf{t}'}(x_1x_2) = y_1y_2,$$

and Equation (1) holds. We leave the rest of the  $k = 2$  cases to the reader.

For the connected  $k \geq 3$  cases, we start with:

**$D_{I,J,\mathbf{t}}$  or  $D_{I,J,\mathbf{t}'}$  is non-trivial.** We can then assume  $D_{I,J,\mathbf{t}} \neq 0$ , so  $\mathcal{C}(I, J, \mathbf{t})$  is of Type  $A_k, B_k, C_{p,q}, D_{p,q}$  or  $E_{p,q}$  with  $p + q = k$ . Since  $k \geq 3$  it follows that changing the orientation of one arc gives a configuration with trivial contribution. In particular  $D_{I,J,\mathbf{t}'} = 0$ . We claim that in each of these case either

$$(i) \quad H_{I,I'} \cdot D_{I',J} = 0, \quad D_{I,J'} \cdot H_{J',J} = D_{I,J,\mathbf{t}}$$

or

$$(ii) \quad H_{I,I'} \cdot D_{I',J} = D_{I,J,\mathbf{t}}, \quad D_{I,J'} \cdot H_{J',J} = 0$$

holds.

If  $\mathcal{C}(I, J, \mathbf{t})$  is of Type:

- $A_k$ , then  $\mathcal{C}(I, J')$  is  $A_{k-1}$  and (i) holds.
- $C_{p,q}$  with  $p \geq 2, q \geq 2$ , then  $\mathcal{C}(I, J')$  is  $C_{p,q-1}$  (or  $C_{p-1,q}$ ) and (i) holds.
- $C_{p,1}$ , with  $p \geq 3$ , and  $\delta$  is one of  $e_1, \dots, e_p$  then  $\mathcal{C}(I, J')$  is  $C_{p-1,1}$  and (i) holds.
- $C_{p,1}$  with  $p \geq 2$  and  $\delta$  equals to  $f_1$ , then  $(I', J)$  is of Type  $A_p$  and (ii) holds.
- $C_{2,1}$  and  $\delta$  is one of  $e_1, e_2$ , then  $\mathcal{C}(I, J')$  is of Type 8 and (i) holds.

- Type  $E_{p,q}$ , then  $\delta$  either connects the central starting circle  $x_1$  to  $x_t$  for  $2 \leq t \leq p+1$ , or connects  $x_1$  to itself. In the first case  $\mathcal{C}(I, J')$  and  $\mathcal{C}(I', J)$  are  $E_{p-1,q}$  and (i) holds. In the second case  $\mathcal{C}(I, J')$  and  $\mathcal{C}(I', J)$  are  $E_{p,q-1}$  and (ii) holds.

When  $\mathcal{C}$  is of Type  $B_k$  or Type  $D_{p,q}$ , then Equation (1) follows from the earlier duality argument.

It remains to check the case when:

**Both  $D_{I,J,\mathbf{t}}$  and  $D_{I,J,\mathbf{t}'}$  are trivial.** Using the duality property of  $D$  and  $H$ , it is enough to consider the case

$$D_{I,J'} \cdot H_{J',J} \neq 0.$$

Now we have to show that

$$(iii) \quad H_{I,I'} \cdot D_{I',J} = D_{I,J'} \cdot H_{J',J}$$

holds. We list all the cases and configurations, using the Type of  $\mathcal{C}(I, J')$  to determine all the possibilities: If  $\mathcal{C}(I, J')$  is of Type

- $A_{k-1}$ ,  $k \geq 3$  then  $D_{I,J'}(1) = 1$  and  $(J', J)$  needs to be a splitting edge. Let  $x_1$  and  $x_2$  denote the active circles in  $\mathcal{C}(I, J)$ . If the  $\delta$  connects  $x_1$  and  $x_2$  then either  $(I, J, p)$  or  $(I, J, p')$  is of type  $A_k$  and that contradicts our assumption. It follows that  $\delta$  connects  $x_1$  to itself or  $x_2$  to itself, see Figure 4. In these cases  $\mathcal{C}(I', J)$  and  $\mathcal{C}(I, J')$  are both type  $A_{k-1}$  and (iii) holds.
- $B_{k-1}$ ,  $k \geq 3$  then  $D_{I,J'} \cdot H_{J',J} \neq 0$  implies that  $(J', J)$  is a joining edge. If  $\delta$  connects  $y_1$  and  $y_2$ , then either  $\mathcal{C}(I, J, \mathbf{t})$  or  $\mathcal{C}(I, J, \mathbf{t}')$  is of type  $C_{k-1,1}$  (a case covered earlier). It follows that  $\delta$  connects another circle  $w$  to  $y_1$  or  $y_2$ , see Figure 4, and  $\mathcal{C}(I', J)$  is of Type  $B_{k-1}$ .
- $C_{p,q}$  with  $p+q \geq 3$ , then  $D_{I,J'} \cdot H_{J',J} \neq 0$  implies that  $(J', J)$  is a splitting edge, see Figure 4 for the possible choices for  $\delta$ . In all cases  $\mathcal{C}(I', J)$  is of Type  $C_{p,q}$ .
- $D_{p,q}$  with  $p+q \geq 3$  then  $(J', J)$  has to be a joining edge. Since  $\mathcal{C}(I, J')$  has only one active ending circle  $y_1$ , it follows that  $\mathcal{C}(I, J)$  has an additional active starting circle  $w$ . See Figure 4 for the possible choices for  $\delta$  and  $w$ .
- $E_{p,q}$  and  $(J', J)$  is splitting edge, then  $\delta$  has to split one of  $y_2, \dots, y_{q+1}$ . It follows that  $\mathcal{C}(I', J)$  is also of Type  $E_{p,q}$ .
- $E_{p,q}$  and  $(J', J)$  is a join edge, then  $\delta$  has to connect a new circle  $w$  to  $y_1$ . If  $\delta$  attaches  $w$  to portion of  $y_1$  that lies in the central starting circle

$x_1$ , then either  $\mathcal{C}(I, J, \mathbf{t})$  or  $\mathcal{C}(I, J, \mathbf{t}')$  is of Type  $E_{p+1,q}$ . It follows that  $\delta$  attaches  $w$  to the portion of  $y_1$  that lies in  $x_i$  for some  $2 \leq i \leq p+1$ , see Figure 4. It follows that  $\mathcal{C}(I', J)$  is of Type  $E_{p,q}$ .

- 8 and  $(J', J)$  is a join edge, then  $\delta$  connects a new circle  $w$  to  $y_1$ , see Figure 4, and  $\mathcal{C}(I', J)$  is also of type 8.
- 8 and  $(J', J)$  is a split edge then in  $\mathcal{C}(I, J, \mathbf{t})$  there is a dual circle that is only attached to  $\delta^*$  and none of the other  $\gamma$  arcs, see Figure 4, (otherwise either  $\mathcal{C}(I, J, \mathbf{t})$  or  $\mathcal{C}(I, J, \mathbf{t}')$  would be of type  $C(2, 1)$ ). It follows that  $\mathcal{C}(I', J)$  is also of type 8.

Checking (iii) is straightforward in all the cases. □

## 6 Proof of $\mathbf{d}(\mathbf{t}) \cdot \mathbf{d}(\mathbf{t}) = 0$ .

In this section (decorated) configurations are denoted as  $\mathcal{C}$  or  $(\overline{\mathcal{C}}, \mathbf{t})$  and undecorated configurations are denoted by  $\overline{\mathcal{C}}$ .

**Theorem 6.1** *For every  $k$ -dimensional configuration  $(\overline{\mathcal{C}}, \mathbf{t})$  we have*

$$\sum_{i=1}^{k-1} \mathbf{d}_i(\mathbf{t}) \cdot \mathbf{d}_{k-i}(\mathbf{t}) = 0$$

**Proof** We will use induction on  $k$ . The  $k = 2$  case is trivial, since  $\mathbf{d}_1(\mathbf{t})$  is the Khovanov differential.

**Lemma 6.2** *Let  $\mathbf{t}$  and  $\mathbf{t}'$  be two decorations on the  $k$  dimensional undecorated configuration  $\overline{\mathcal{C}}$ . If Equation (2) holds for all  $k - 1$  dimensional configurations, then*

$$\sum_{i=1}^{k-1} \mathbf{d}_{k-i}(\mathbf{t}) \left( \mathbf{d}_i(\mathbf{t})(a) \right) = \sum_{i=1}^{k-1} \mathbf{d}_{k-i}(\mathbf{t}') \left( \mathbf{d}_i(\mathbf{t}')(a) \right).$$

for all  $a \in V_0(\mathcal{C})$ .

**Proof** It is enough to consider the case when  $\mathbf{t}$  and  $\mathbf{t}'$  differ at a single crossing indexed by  $m$ . According to Theorem 5.4, we have

$$\mathbf{d}_j(\mathbf{t}') = \mathbf{d}_j(\mathbf{t}) + \mathbf{d}_{j-1}(\mathbf{t}) \cdot H_m + H_m \cdot \mathbf{d}_{j-1}(\mathbf{t}).$$

This formula together with the trivial observations:

$$H_m \cdot H_m = 0, \quad H_m \cdot \mathbf{d}_j \cdot H_m = 0$$

finishes the argument.  $\square$

The strategy to prove Theorem 6.1 is to consider undecorated configurations  $\bar{\mathcal{C}}$  and monomials  $a \in V_0(\bar{\mathcal{C}})$  and  $b \in V_1(\bar{\mathcal{C}})$ . We will need the following:

**Theorem 6.3** *Let  $\bar{\mathcal{C}}$  be an undecorated  $k$  dimensional configuration with  $k \geq 3$ , let  $a$  and  $b$  denote monomials  $a \in V_0(\bar{\mathcal{C}})$ ,  $b \in V_1(\bar{\mathcal{C}})$ . For every triple,  $(\bar{\mathcal{C}}, a, b)$  there exists a decoration  $\mathbf{t}$  on  $\bar{\mathcal{C}}$  so that the coefficient of*

$$\sum_{i=1}^{k-1} \mathbf{d}_{k-i}(\mathbf{t}) \left( \mathbf{d}_i(\mathbf{t})(a) \right)$$

at  $b$  equals to 0.

Theorem 6.1 follows immediately from Theorem 6.3 and Lemma 6.2 by induction on  $k$ .  $\square$

The rest of this section is devoted to the proof of Theorem 6.3. Let's start with a few notations and remarks. If for a given  $\bar{\mathcal{C}}$  the statement in Theorem 6.3 holds for all  $a \in V_0(\bar{\mathcal{C}})$  and  $b \in V_1(\bar{\mathcal{C}})$ , we say that Theorem 6.3 holds for  $\bar{\mathcal{C}}$ . Similarly if for a given  $(\bar{\mathcal{C}}, a)$  the statement holds for all  $b \in V_1(\bar{\mathcal{C}})$ , we say that Theorem 6.3 holds for  $(\bar{\mathcal{C}}, a)$ .

Note that  $\mathbf{d}_{k-i}(\mathbf{t})(\mathbf{d}_i(\mathbf{t})(a))$  can be written by summing

$$F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)),$$

over all decompositions of  $\mathcal{C}$  as  $\mathcal{C} = \mathcal{C}(1) * \mathcal{C}(2)$ , with  $\mathcal{C}(1)$  having dimension  $i$ , and  $\mathcal{C}(2)$  dimension  $k - i$ . Clearly  $\mathcal{C}(1)$  and  $\mathcal{C}(2)$  is determined by the decomposition of the  $\gamma$  index set  $\{1, 2, \dots, k\}$  as the union of sets  $U$  and  $V$ , where  $|U| = i$  and  $|V| = k - i$ , where  $U$  corresponds to  $\mathcal{C}(1)$ ,  $V$  corresponds to  $\mathcal{C}(2)$ .

In proving Theorem 6.3 note that according the extension property it is enough to consider active configurations. Next we consider the case when  $\bar{\mathcal{C}}$  is disconnected. If the active part of  $\bar{\mathcal{C}}$  has more than 2 connected components then either  $F_{\mathcal{C}(1)}$  or  $F_{\mathcal{C}(2)}$  is 0 by the disconnected rule. If there are 2 connected components then there are still two decompositions of the  $k$  dimensional cube to consider. However their contributions agree according to the extension formula.

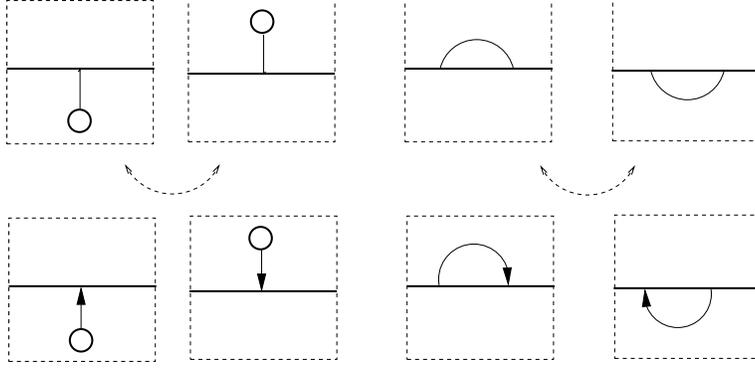


Figure 5: Rotation

From now on we will assume that  $\overline{\mathcal{C}}$  is both active and connected. (Note that  $\mathcal{C}(1)$  or  $\mathcal{C}(2)$  could be disconnected, or could have passive circles.)

Now we discuss a few moves on  $\overline{\mathcal{C}}$  in order to cut down the number of cases to consider:

**Lemma 6.4** *The statement in Theorem 6.3 holds for the triple  $(\overline{\mathcal{C}}, a, b)$  if and only if it holds for  $(m(\overline{\mathcal{C}}^*), b^*, a^*)$ .*

**Proof** This follows immediately from the duality rule.  $\square$

**Lemma 6.5** *Let  $\overline{\mathcal{C}}$  and  $\overline{\mathcal{C}}'$  be configurations that differ by a rotation move of Figure 5. Then Theorem 6.3 holds for  $(\overline{\mathcal{C}}, a, b)$  if and only if it holds for  $(\overline{\mathcal{C}}', a, b)$ .*

**Proof** The second row of Figure 5 indicates how to modify the decorations. Using these decorations all the maps in  $(\overline{\mathcal{C}}, \mathbf{t})$  agree with the maps in  $(\overline{\mathcal{C}}', \mathbf{t}')$ .  $\square$

**Lemma 6.6** *Suppose that  $\overline{\mathcal{C}}$  contains an active starting circle  $x_1$  that meets only one of  $\gamma$  arcs, say  $\gamma_1$ . We will call  $x_1$  a degree 1 starting circle. Let  $x_2$  denote the other circle that meets  $\gamma_1$ , and let  $a \in V_0(\overline{\mathcal{C}})$  be a monomial.*

- *If  $a$  is not divisible by  $x_1$  then Theorem 6.3 holds for  $(\overline{\mathcal{C}}, a)$ .*
- *If  $a$  is divisible by  $x_2$ , then Theorem 6.3 holds for  $(\overline{\mathcal{C}}, a)$ .*

**Proof** We will use the notation of  $\mathcal{C}(1)$ ,  $\mathcal{C}(2)$ ,  $U$  and  $V$  as above. If  $a$  is not divisible by  $x_1$  then there are only 2 possible decompositions with  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) \neq 0$ , corresponding to  $U = \{1\}$  and  $U' = \{2, \dots, k\}$ . However these terms cancel each other according to the extension formula.

For the second part, it is enough to consider the case when  $a$  is divisible by  $x_1 x_2$ . It follows that  $F_{\mathcal{C}(1)}(a) = 0$  if  $1 \in U$ . If  $1 \notin U$ , let  $w$  denote the ending circle of  $\mathcal{C}(1)$  that contains the intersection point between  $\gamma_1$  and  $x_2$ . According to the filtration rule  $F_{\mathcal{C}(1)}(a)$  is divisible by  $x_1 \cdot w$ . This implies that its image under  $F_{\mathcal{C}(2)}$  is trivial.  $\square$

**Lemma 6.7** *Supposes that  $\bar{\mathcal{C}}$  contains an active starting circle  $x_1$  that is connected to the other circles by two of the  $\gamma$  arcs  $\gamma_1$  and  $\gamma_2$ , see Figure 6. Let  $a \in V_0(\bar{\mathcal{C}})$  denote a monomial, and  $x, x'$  denote the other circles that meet  $\gamma_1$  and  $\gamma_2$ . If  $x = x'$  define  $p = x$ . Otherwise define  $p = x \cdot x'$*

- *If  $a$  is not divisible by  $x_1$ , then Theorem 6.3 holds for  $(\bar{\mathcal{C}}, a)$ .*
- *If  $a$  is divisible with  $p \cdot x_1$  then Theorem 6.3 holds for  $(\bar{\mathcal{C}}, a)$ .*

**Proof** For the first statement use the decoration as in the center of Figure 6. Consider decompositions with  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) \neq 0$ . If  $a$  is not divisible by  $x_1$  then there are only two possible non-trivial terms, corresponding to decompositions with  $U = \{1\}$  and  $U' = \{2\}$ . However according to the chosen decoration  $\mathcal{C}(2)$  and  $\mathcal{C}(2)'$  are equivalent configurations. Since  $a$  is not divisible by  $x_1$  we also have  $F_{\mathcal{C}(1)}(a) = F_{\mathcal{C}(1)'}(a)$ . It follows that the contributions cancel each other.

For the second part use the decoration as in the right of Figure 6. Suppose that  $F_{\mathcal{C}(1)}(a) \neq 0$ . If  $\{1, 2\} \subset U$ , then  $a$  being divisible with  $p \cdot x_1$  implies that  $\mathcal{C}(1)$  is Type  $B$  or  $D$ , but that contradicts the choice of decoration. If  $1 \in U, 2 \in V$  then  $\mathcal{C}(1)$  has to be a join edge or Type  $E$ , but that contradicts  $F_{\mathcal{C}}(a) \neq 0$  and the choice of  $a$ . The case  $1 \in V, 2 \in U$  is ruled out the same way. It remains to consider  $\{1, 2\} \subset V$ . In this case the filtration rule implies that  $F_{\mathcal{C}(1)}(a)$  is divisible by  $x_1$  and the other (one or two) ending circles of  $\mathcal{C}(1)$  that meets  $\gamma_1$  and  $\gamma_2$ . So  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) \neq 0$ , implies that  $\mathcal{C}(2)$  is Type  $B$  or  $D$ . However that again contradicts the chosen decoration.  $\square$

**Lemma 6.8** *Let  $\bar{\mathcal{C}}$  and  $\bar{\mathcal{C}}'$  be configurations that differ by a trading move of Figure 7. Then Theorem 6.3 holds for  $\bar{\mathcal{C}}$  if and only if it holds for  $\bar{\mathcal{C}}'$ .*

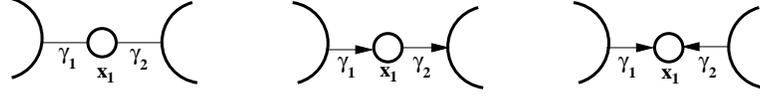


Figure 6:

**Proof** Let  $\bar{\mathcal{C}}$  denote the local configuration in the far left of Figure 7, and  $y_1$  denote the ending circle of  $\bar{\mathcal{C}}$  that is given by merging  $x_1$  and  $x_2$ . Also let  $y'_1$  denote the ending circle of  $\bar{\mathcal{C}}'$  that meets only  $(\gamma'_1)^*$ , and  $y_2$  be the other ending circle of  $\bar{\mathcal{C}}'$  that meets  $(\gamma'_1)^*$ . Let's write  $V_0(\bar{\mathcal{C}}) = V(x_1) \otimes V(x_2) \otimes W$  and  $V_1(\bar{\mathcal{C}}) = V(y_1) \otimes P$ . Then  $V_0(\bar{\mathcal{C}}') = V(x'_2) \otimes W$  and  $V_1(\bar{\mathcal{C}}') = V(y'_1) \otimes V(y'_2) \otimes P$ . According to Lemma 6.6 and the filtration rule, it is enough to consider  $(\bar{\mathcal{C}}, a, b)$  where  $a = x_1 \cdot w$  and  $b = y_1 \cdot p$ ,  $w \in W$ ,  $p \in P$ . Similarly using Lemma 6.4 and Lemma 6.6 it is enough to consider  $(\bar{\mathcal{C}}, a', b')$  where  $a' = w$  and  $b' = x'_2 \cdot p$ . On the other hand, for a fixed pair  $(w, p)$  and fixed  $U, V$  decomposition the coefficient of  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a))$  at  $b$  is equal to the coefficient of  $F_{\mathcal{C}(2)'}(F_{\mathcal{C}(1)'}(a'))$  at  $b'$ .

□

**Lemma 6.9** *Suppose that  $a \in V_0(\bar{\mathcal{C}})$  is the product of all the active starting circles. If the number of active starting circles of  $\bar{\mathcal{C}}$  is greater than 1, then Theorem 6.3 holds for  $(\bar{\mathcal{C}}, a)$*

**Proof** It is enough to consider the active part of  $\bar{\mathcal{C}}$ . Let  $W \subset \{1, \dots, n\}$  denote the index set of the  $\gamma$  arcs that connect  $x_1$  to  $x_i$  with  $i \geq 2$ . Since  $\bar{\mathcal{C}}$  is connected it follows that  $W$  is nonempty. Orient all these arcs away from  $x_1$ . Suppose  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) \neq 0$ , where  $a$  is the product of all the starting circles. Since  $F_{\mathcal{C}(1)}(a) \neq 0$ , it follows that either  $\mathcal{C}(1)$  is a split edge or it is Type  $B$  or  $D$ . It follows from the chosen decoration that  $W$  is disjoint from the index set  $U$  of  $\mathcal{C}(1)$ . Now according to the duality rule  $F_{\mathcal{C}(1)}(a)$  is the product of the starting circles of  $\mathcal{C}(2)$ . However the chosen decoration of the  $\gamma_j$  arcs with  $j \in W$  imply that  $\mathcal{C}(2)$  is not a split edge, neither Type  $B$  or Type  $D$ , so in fact  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) = 0$ .

□

**Lemma 6.10** *Suppose that the pair  $\bar{\mathcal{C}}$  contains at least one of the local configurations  $M_1, M_2, \dots, M_9$  in Figure 8. Then  $\bar{\mathcal{C}}$  satisfies Theorem 6.3.*

**Proof** By the trading and rotation operations we can reduce the  $M_2$  and  $M_3$  cases to  $M_1$ . Similarly  $M_5$  can be traded to  $M_4$ . Furthermore the symmetry

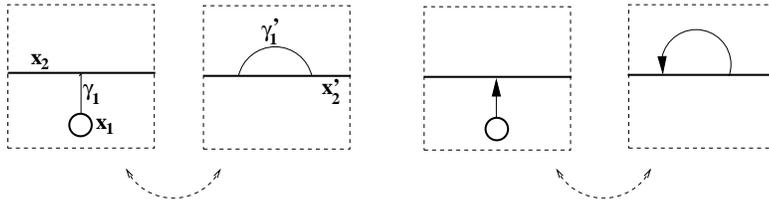


Figure 7: Trading

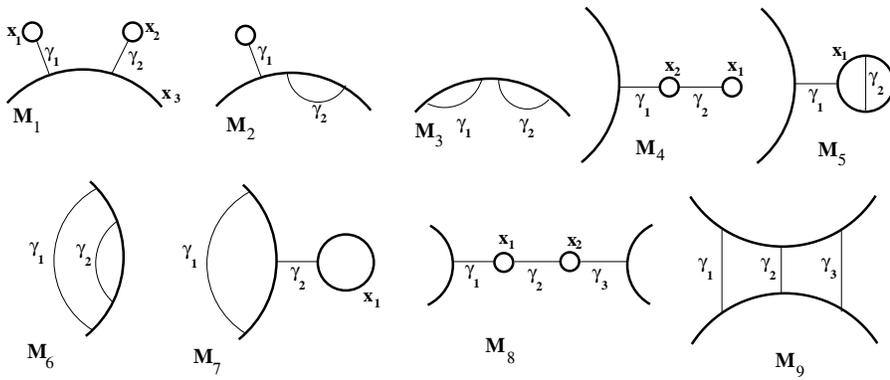


Figure 8:

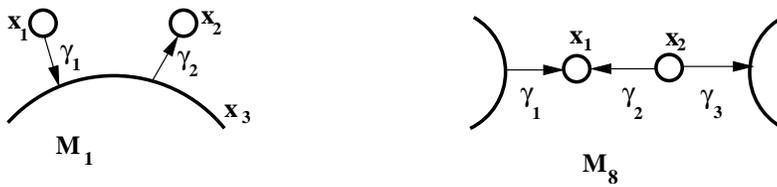


Figure 9:

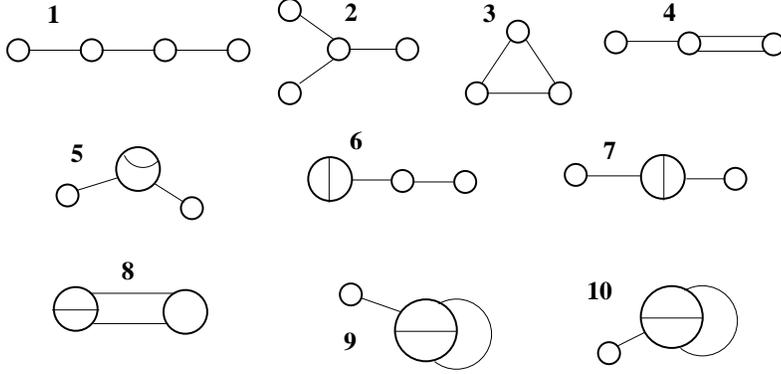


Figure 10: 3-dimensional undecorated configurations modulo rotation, and the additional symmetry  $\bar{\mathcal{C}} \rightarrow m(\bar{\mathcal{C}}^*)$

$\bar{\mathcal{C}} \rightarrow m(\bar{\mathcal{C}}^*)$  maps  $M_6$  to  $M_4$ ,  $M_7$  to  $M_5$ , and  $M_9$  to  $M_8$ . So, by Lemma 6.4, 6.8 it is enough to consider the local configurations  $M_1$ ,  $M_4$  and  $M_8$ .

For  $M_1$  let's write  $V_0(\bar{\mathcal{C}}) = V(x_1) \otimes V(x_2) \otimes V(x_3) \otimes W$ . It follows from Lemma 6.6 that it is enough to consider the case  $a = x_1 x_2 \cdot w$ , where  $w \in W$ . We use the decoration as in Figure 9, and fix a point  $P$  in the  $x_1$  circle. We look at decompositions  $U, V$  so that  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) \neq 0$ . The decoration rules out  $\{1, 2\} \subset U$ , and  $\{1, 2\} \subset V$ . In case  $1 \in U$  and  $2 \in V$ , the filtration rule implies that  $F_{\mathcal{C}(1)}(a)$  is divisible by  $x_2 \cdot y(P)$ , and so  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) = 0$ . The case of  $1 \in V$ ,  $2 \in U$  is ruled out similarly.

For  $M_4$  there are two cases to consider: If  $a$  is divisible by  $x_2$  then the second part of Lemma 6.6 applies. If  $a$  is not divisible by  $x_2$  then the first part of Lemma 6.7 finishes the argument.

For  $M_8$  use the decoration in Figure 9. By Lemma 6.7 it is enough to consider the case when  $a$  is divisible by  $x_1 x_2$ . In the cases  $\{1, 2\} \subset U$ ,  $\{1, 3\} \subset U$  or  $\{2, 3\} \subset U$ , the chosen decoration implies  $F_{\mathcal{C}(1)}(a) = 0$ . If  $2 \in U$ ,  $\{1, 3\} \subset V$  then  $F_{\mathcal{C}(1)}(a) = 0$  since  $a$  is divisible by  $x_1 x_2$ . If  $1 \in U$ ,  $\{2, 3\} \subset V$ , then  $F_{\mathcal{C}(1)}(a)$  is divisible by  $x_2$  and so  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) = 0$  because of the decoration. The case of  $3 \in U$ ,  $\{1, 2\} \subset V$  is ruled out similarly. Finally if  $\{1, 2, 3\} \subset V$  then  $F_{\mathcal{C}(2)} = 0$  because of the decoration.

□

**Lemma 6.11** *If  $\bar{\mathcal{C}}$  is 3-dimensional, then Theorem 6.3 holds for  $\bar{\mathcal{C}}$ .*

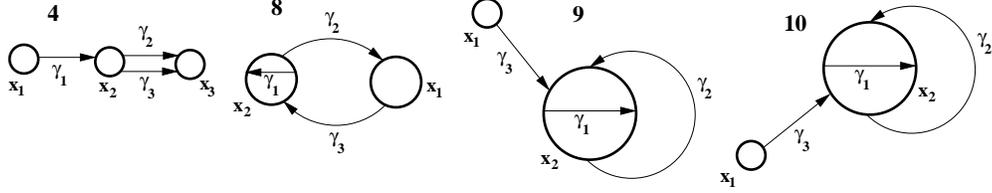


Figure 11:

**Proof** Figure 10 lists all the connected active 3 dimensional configurations, modulo rotation moves, and the additional move  $\bar{\mathcal{C}} \rightarrow m(\bar{\mathcal{C}}^*)$ . According to Lemma 6.5 and 6.4 it is enough to consider these cases. Among these, Lemma 6.10 settles Cases 1, 2, 3, 5 and 6 and 7. The rest of the configurations are given in Figure 11 with a choice of decoration.

For Case 4 it is enough to consider  $a = x_1 x_3$  by Lemma 6.6 and 6.7. Using the decoration in Figure 11 we see that for all six decompositions  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) = 0$ .

For Case 8 it is enough to consider  $a = x_1$  by Lemma 6.7. Again using the decoration in Figure 11 we see that for all decompositions  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) = 0$ .

For Case 9 it is enough to check  $a = x_1$  by Lemma 6.6. Using the given decoration we have nontrivial terms from  $U = \{1\}$  where  $\mathcal{C}(1)$  is split,  $\mathcal{C}(2)$  is type  $E_{2,0}$ , and  $U' = \{2, 3\}$  where  $\mathcal{C}(1)'$  is type  $E_{1,1}$ ,  $\mathcal{C}(2)'$  is join. We have

$$F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(x_1)) = F_{\mathcal{C}(2)'}(F_{\mathcal{C}(1)'}(x_1)) = y_1$$

so the terms cancel.

For Case 10 it is again enough to check  $a = x_1$  by Lemma 6.6. Using the given decoration we have nontrivial terms from  $U = \{1, 3\}$  where  $\mathcal{C}(1)$  is type  $E_{1,1}$ ,  $\mathcal{C}(2)$  is join, and  $U' = \{2, 3\}$  where  $\mathcal{C}(1)'$  is type  $E_{1,1}$ ,  $\mathcal{C}(2)'$  is join. We again have

$$F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(x_1)) = F_{\mathcal{C}(2)'}(F_{\mathcal{C}(1)'}(x_1)) = y_1$$

so the terms cancel. □

**Lemma 6.12** For a fixed  $(\bar{\mathcal{C}}, a, b)$  suppose that there exists a decoration  $\mathbf{t}$  on  $\bar{\mathcal{C}}$  so that the coefficient of

$$\mathbf{d}_1(\mathbf{t})(\mathbf{d}_{k-1}(\mathbf{t})(a))$$

at  $b$  is non-trivial. Then Theorem 6.3 holds for  $(\bar{\mathcal{C}}, a, b)$ .

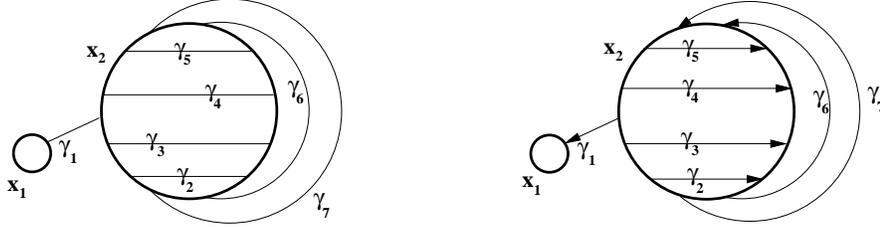


Figure 12:

**Proof** First we use  $\mathbf{d}_1(\mathbf{t})(\mathbf{d}_{k-1}(\mathbf{t})(a)) \neq 0$  to get some kind of structure result for  $\bar{\mathcal{C}}$ . In this family it is enough to consider  $\bar{\mathcal{C}}$  that doesn't contain  $M_i$  in Figure 9, according to Lemma 6.10. Using Lemma 6.4-6.9 further reduces the problem to a finite list of undecorated  $\bar{\mathcal{C}}$  configurations. Then for the remaining cases it will be helpful to use different  $\mathbf{t}'$  decorations on  $\bar{\mathcal{C}}$  to simplify the calculations. These steps are spelled out below.

According to Lemma 6.11 it is enough to consider the case when  $k \geq 4$ . We can also assume that  $\bar{\mathcal{C}}$  is active and connected. Let  $\mathcal{C} = (\bar{\mathcal{C}}, \mathbf{t})$ . Clearly there is a decomposition  $\mathcal{C} = \mathcal{C}(1) * \mathcal{C}(2)$  and a monomial  $z \in V_1(\mathcal{C}(1)) = V_0(\mathcal{C}(2))$  so that  $\mathcal{C}(1)$  is  $k - 1$  dimensional, and the coefficients of  $F_{\mathcal{C}(1)}(a)$  at  $z$ ,  $F_{\mathcal{C}(2)}(z)$  at  $b$  are both nontrivial. Now the active part of  $\mathcal{C}(1)$  is a  $k - 1$  dimensional configuration of Type  $A_{k-1}$ ,  $B_{k-1}$ ,  $C_{p,q}$ ,  $D_{p,q}$  or  $E_{p,q}$ , where  $p+q = k-1$ . The one dimensional configuration  $\mathcal{C}(2)$  is determined by an additional arc called  $\delta$ . For  $\delta$  there are 3 cases to consider

- (i)  $\delta$  joins two active ending circles of  $\mathcal{C}(1)$ .
- (ii)  $\delta$  joins an active ending circle of  $\mathcal{C}(1)$  to a new  $w$  circle.
- (iii)  $\delta$  splits one of the active ending circles of  $\mathcal{C}(1)$ .

If  $\mathcal{C}(1)$  is of Type  $A_{k-1}$ :

For case (i)  $\mathcal{C}(1)$  and  $\mathcal{C}$  have the same active starting circles. It follows that  $a = 1$ ,  $z = 1$ ,  $b = 1$ . Since  $b^*$  is the product of the active circles we can use Lemma 6.4 and 6.9. In case of (ii) or (iii),  $\delta$  meets only one of the  $k - 1$  ending circles of  $\mathcal{C}(1)$ , so  $\bar{\mathcal{C}}$  contains the  $M_9$  configuration.

If  $\mathcal{C}(1)$  is of Type  $B_{k-1}$ :

Case (ii) is covered by the second part of Lemma 6.6. Cases (i) and (iii) follow from Lemma 6.9.

If  $\mathcal{C}(1)$  is of Type  $C_{p,q}$ :

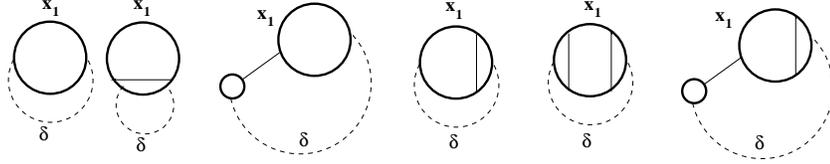


Figure 13: The core of the  $E \cup \delta$  configurations.

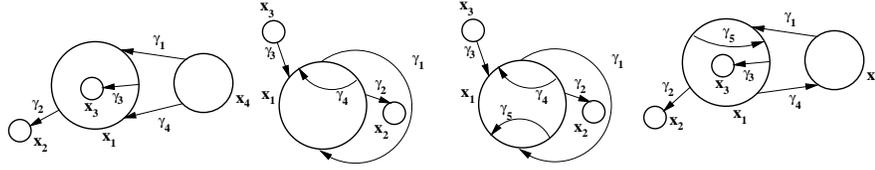


Figure 14: Possible  $E$  configurations.

For case (i) we only have to check for  $a = 1$ ,  $b = 1$ . Since  $b^*$  is the product of the ending circles of  $\bar{\mathcal{C}}$ , and  $\bar{\mathcal{C}}$  has  $p + q - 2 = k - 1 \geq 2$  ending circles, Lemma 6.4 together with 6.9 implies that  $(\bar{\mathcal{C}}, a, b)$  satisfies Theorem 6.3.

For case (ii) and  $p \geq 5$  or  $p, q \geq 3$ , the  $\bar{\mathcal{C}}$  configuration contains  $M_9$ . A maximal example that we still has to consider is given by the left of Figure 12. Modulo rotation and trading all the configurations without  $M_9$  part can be obtained from this picture by deleting some of the  $\gamma$  curves. In particular these examples correspond to the index set  $W \subset \{1, 2, 3, 4, 5, 6, 7\}$  with  $|W| \geq 4$ ,  $1 \in W$ ,  $W \cap \{2, 3, 4, 5\} \neq \emptyset$ ,  $W \cap \{6, 7\} \neq \emptyset$ . For such a  $\bar{\mathcal{C}}(W)$  configuration use the  $\mathbf{t}'$  decoration inherited from the right side of Figure 12. According to Lemma 6.6 it is enough to consider  $a = x_1$ . Now  $F_{\mathcal{C}(1)}(x_1) \neq 0$  implies that  $|U| = 1$  or  $U = \{1, 2\}$  or  $U = \{1, 3\}$ , and in each of these cases  $|W| \geq 4$  implies that  $F_{\mathcal{C}(2)} = 0$ . It follows that  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(x_1)) = 0$  for all decompositions of  $(\mathcal{C}(W), \mathbf{t}')$ .

In Case (iii) if  $\delta$  is parallel to the other  $\gamma$  circles then  $\bar{\mathcal{C}}$  is an undecorated  $C_{p+1, q}$  and contains  $M_9$ . Finally if  $\delta$  is not parallel we can use the trading operation of Lemma 6.8 to reduce to case (ii).

If  $\mathcal{C}(1)$  is of Type  $D_{p, q}$ :

Case (ii) follows from the second part of Lemma 6.6. Cases (i) and (iii) follow from Lemma 6.9.

If  $\mathcal{C}(1)$  is of Type  $E_{p, q}$ :

The central starting circle of  $\mathcal{C}(1)$  gives a starting circle for  $\bar{\mathcal{C}}$  that we denote

by  $x_1$ . If  $\delta$  connects  $x_i$  to  $w$ ,  $x_i$  to  $x_i$  or  $x_i$  to  $x_j$  with  $i, j \geq 2$  then up to rotation we get  $M_4$ ,  $M_5$  or  $M_8$  respectively.

In the remaining cases we argue as follows: Since  $\mathcal{C}(1)$  is of Type  $E$ , it contains  $k-1$  local configurations that could be traded as in Figure 7. Of course  $\delta$  could intersect some of them. In fact the endpoints of  $\delta$  could intersect in a degree one  $x_i$  circle, or a degree one dual  $y_i$  circle. By deleting all the local configurations that are disjoint from  $\delta$  we get the core of  $\mathcal{C}$  see Figure 13. Each picture contains at most two segments of  $x_1$  where the deleted configurations might have been attached. However two such (disjoint from  $\delta$ ) configurations on the same segment would form a local configuration of type  $M_1$ ,  $M_2$  or  $M_3$  (up to rotations). The first 2 cores give at most 3-dimensional examples. Using rotation and trading to simplify the list, we represent the examples from the other 4 cores in Figure 14. By forgetting the decoration for a moment, each picture on the right side of Figure 14 shows three  $\overline{\mathcal{C}}$  configurations: the maximal 5-dimensional, and the two 4-dimensionals obtained by deleting  $(\gamma_2, x_2)$  or  $(\gamma_3, x_3)$ . Using Lemma 6.6 it is enough to consider the case where  $a$  is the dual of  $x_1$ . Now each of these eight configurations have a preferred  $\mathbf{t}'$  decoration induced from Figure 14. It is easy to check that for the given decorations  $F_{\mathcal{C}(2)}(F_{\mathcal{C}(1)}(a)) = 0$  for all decompositions.  $\square$

**Lemma 6.13** *For a fixed  $(\overline{\mathcal{C}}, a, b)$  suppose that there exists a decoration  $\mathbf{t}$  on  $\overline{\mathcal{C}}$  so that the coefficient of*

$$\mathbf{d}_{k-1}(\mathbf{t})(\mathbf{d}_1(\mathbf{t})(a))$$

*at  $b$  is nontrivial. Then Theorem 6.3 holds for  $(\overline{\mathcal{C}}, a, b)$ .*

**Proof** This follows from Lemma 6.12 and Lemma 6.4.  $\square$

**Lemma 6.14** *Suppose that there exists a decoration  $\mathbf{t}$  on  $\overline{\mathcal{C}}$  so that the coefficient of*

$$\mathbf{d}_i(\mathbf{t})(\mathbf{d}_j(\mathbf{t})(a))$$

*at  $b$  is non-trivial, where  $i + j = k$ ,  $i \geq 2$ , and  $j \geq 2$ . Then Theorem 6.3 holds for  $(\overline{\mathcal{C}}, a, b)$ .*

**Proof** If  $\mathbf{d}_i(\mathbf{t})(\mathbf{d}_j(\mathbf{t})(a))$  at  $b$  is non-trivial, then there exists a decomposition of  $\mathcal{C} = (\overline{\mathcal{C}}, \mathbf{t})$  as  $\mathcal{C} = \mathcal{C}(1) * \mathcal{C}(2)$ , and a monomial  $z \in V_1(\mathcal{C}(1)) = V_0(\mathcal{C}(2))$ , so that the coefficient of  $F_{\mathcal{C}(1)}(a)$  at  $z$  is non-trivial, and  $F_{\mathcal{C}(2)}(z)$  at  $b$  is

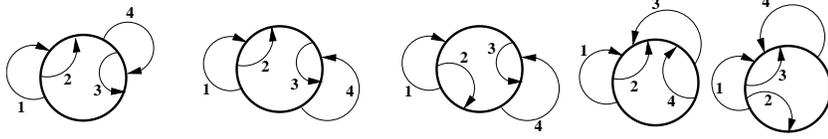


Figure 15:

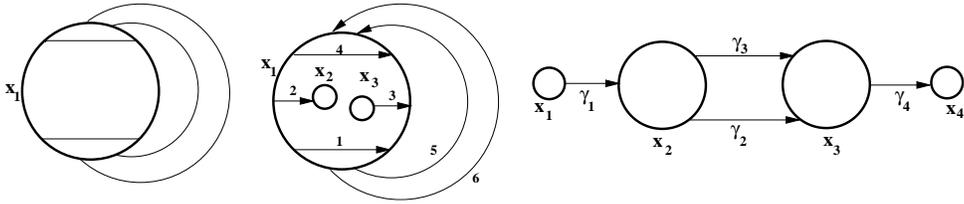


Figure 16:

non-trivial. By recording the types of  $\mathcal{C}(1)$  and  $\mathcal{C}(2)$  we get an element  $Q$  in  $\{A, B, C, D, E\}^2$ :

$$Q = (\text{Type of } \mathcal{C}(1), \text{Type of } \mathcal{C}(2)).$$

In the special case when  $\mathcal{C}(1)$  or  $\mathcal{C}(2)$  is the 2-dimensional configuration Type 8, we use the extra information on  $p$  and the the remark at the end of Section 4, to assign the value  $C$  or  $D$ .

It follows from the definition of the  $F$  map that  $Q \in \{A, C\} \times \{B, D\}$  and  $Q \in \{B, D\} \times \{A, C\}$  is not possible unless  $\mathcal{C}$  is disconnected (and disconnected configurations are already discussed.)

In case  $Q \in \{B, D\} \times \{B, D\}$  the monomial  $a$  is the product of the starting circles. According to Lemma 6.9 it is enough to consider the cases where  $\bar{\mathcal{C}}$  has only one starting circle. This still leaves the possibility where both  $\mathcal{C}(1)$  and  $\mathcal{C}(2)$  is of Type  $D_{1,1}$ . This gives 5 undecorated,  $\bar{\mathcal{C}}$  configurations. By assigning convenient  $\mathbf{t}'$  decorations, we get the list of  $\mathcal{C}' = (\bar{\mathcal{C}}, \mathbf{t}')$  in Figure 15. In all the five configurations  $F_{\mathcal{C}'(1)} \neq 0$  implies  $|U| = 1$ . Furthermore for these choices of  $U$  we have  $F_{\mathcal{C}'(2)} = 0$ .

The case  $Q \in \{A, C\} \times \{A, C\}$  follows from the previous  $Q \in \{B, D\} \times \{B, D\}$  case and Lemma 6.4.

The remaining cases all contain an  $E$  factor. Similarly to the proof of case  $E$  in Lemma 6.12 we determine the core of  $\bar{\mathcal{C}}$ , and use the trading and rotation operations to shorten the list of configurations to check. (It follows from the proof of Lemma 6.8 that rotation or trading doesn't change  $Q$ .)

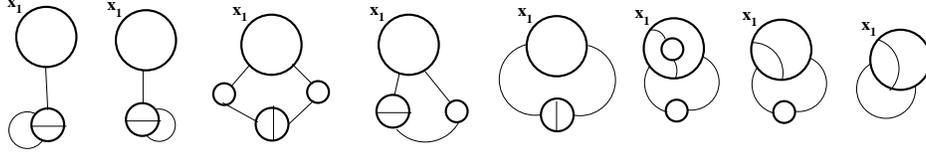


Figure 17:

**The case  $\mathbf{Q}=(\mathbf{E},\mathbf{A})$  :** Let  $y_1, \dots, y_s$  denote the active ending circles of  $\mathcal{C}(1)$ . Since  $\mathcal{C}(1)$  is Type  $E$ ,  $F_{\mathcal{C}(1)}(a)$  is divisible by the central ending circle  $y_1$ . It follows that  $y_1$  is a passive starting circle for  $\mathcal{C}(2)$ . In particular the  $\gamma$  arcs of  $\mathcal{C}(2)$  are parallel and connect either  $y_i$  to a new circle  $w$  or  $y_i$  to  $y_j$ , where  $i, j \geq 2$ . If the dimension of  $\mathcal{C}(2)$  is greater than 2 we get the local configuration  $M_8$ .

It remains to consider the case where  $\mathcal{C}(2)$  has dimension 2. If  $w$  is an active starting circle in  $\mathcal{C}(2)$  then  $w$  is a degree 2 circle in  $\bar{\mathcal{C}}$  and by Lemma 6.7 it is enough to consider the case where  $a$  is divisible by  $w$ . However then  $w$  divides  $z$  as well, and since  $\mathcal{C}(2)$  is type  $A$ , we have  $F_{\mathcal{C}(2)}(z) = 0$ . If  $y_i$  and  $y_j$  are the active starting circles of  $\mathcal{C}(2)$  then the core of  $\bar{\mathcal{C}}$  is given in the left of Figure 16. The corresponding configurations (modulo rotation and trading) are represented in the center of Figure 16. The picture represents four configurations, the maximal 6-dimensional and the subconfigurations where either  $(\gamma_2, x_2)$  or  $(\gamma_3, x_3)$  or both are deleted. By Lemma 6.6 is enough to consider  $a = x_1^*$ . For the given decoration  $F_{\mathcal{C}(1)}(a) \neq 0$  implies that  $|U| = 1$  or  $U = \{3, 4\}$  or  $U = \{1, 2\}$ . With these choices of  $U$  we have  $F_{\mathcal{C}(2)} = 0$ .

**The case  $\mathbf{Q}=(\mathbf{A},\mathbf{E})$  :** In this case  $\mathcal{C}(1)$  has  $s$  active ending circles  $y_1, \dots, y_s$  where  $s$  is the dimension of  $\mathcal{C}(1)$ . Furthermore none of these  $y_i$  divide  $z$ . It follows that exactly one of these circles, say  $y_1$  is the central starting circle of  $\mathcal{C}(2)$  and the others are passive starting circles for  $\mathcal{C}(2)$ . So if  $\bar{\mathcal{C}}$  doesn't contain  $M_8$ , then  $\mathcal{C}(1)$  is 2 dimensional. Using again the core idea, rotation and trading, we get one remaining  $\bar{\mathcal{C}}$  configuration to check, see  $(\bar{\mathcal{C}}, \mathbf{t}')$  in the right side of Figure 16. We need to consider  $a = x_1 x_4$ . This gives  $|U| = 1$  or  $U = \{2, 3\}$ . For these  $U$  we again have  $F_{\mathcal{C}(2)} = 0$ .

**The case  $\mathbf{Q}=(\mathbf{E},\mathbf{C})$ :** In this case the active starting circle of  $\mathcal{C}(2)$  agrees with one of the degree 1 ending circles of  $\mathcal{C}(1)$ . It follows that if the dimension of  $\mathcal{C}(1)$  is greater than 2, then  $\bar{\mathcal{C}}$  contains an  $M_1$ ,  $M_2$  or  $M_3$  configuration after rotations. Furthermore if  $\mathcal{C}(1)$  is two-dimensional, then  $\bar{\mathcal{C}}$  contains an  $M_6$  or  $M_7$  after rotations.

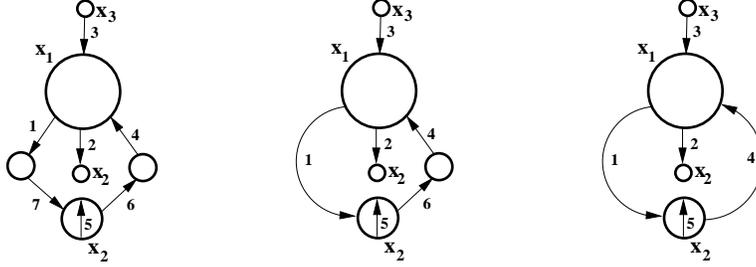


Figure 18:

**The case  $Q=(E,D)$  :** Clearly  $a$  is divisible by all the starting circles, except for  $x_1$ , the central starting circle of  $\mathcal{C}(1)$ . Since  $\mathcal{C}(2)$  is Type  $D_{p,q}$ , one of its active starting circle is the central ending circle of  $\mathcal{C}(1)$ , the other  $p + q - 2$  active starting circles of  $\mathcal{C}(2)$  have to be passive ending circles for  $\mathcal{C}(1)$ . Clearly if  $p$  or  $q$  is large, we get an  $M_8$  configuration in  $\bar{\mathcal{C}}$ . Further simplifying with the second part of Lemma 6.7 we get the cores as in Figure 17. The first two cores give configurations that contain  $M_4$  or  $M_5$  up to rotation. The next three cores gives the examples presented in Figure 18. The last three cores give Figure 19.

Note that each of the six pictures represent several  $(\bar{\mathcal{C}}, \mathbf{t}')$  configurations. These are obtained from the maximal configurations by erasing a subset of the degree 1 local configurations. (The only constraint is that the remaining index set  $W$  satisfies  $|W| \geq 4$ ). By Lemma 6.6 it is enough to consider  $a = x_1^*$ .

When checking  $C'(W)$  configurations in Figure 18,  $F_{C'(1)}(a) \neq 0$  implies that either  $|U| = 1$ ,  $U = \{1, 2\}$  or  $U = \{3, 4\}$ . In all these cases  $F_{C'(2)} = 0$ .

When checking  $C'(W)$  configurations in Figure 19: For the left family we get  $|U| = 1$ ,  $U \subset \{1, 3, 4, 6\}$  or  $U \subset \{2, 5, 7, 8\}$ . The middle family gives  $|U| = 1$ ,  $U \subset \{3, 4, 6\}$  or  $U \subset \{2, 5, 7\}$ . For the family on the right side we get  $|U| = 1$ ,  $U = \{5, 6\}$  or  $U = \{3, 6\}$ . For all but four of these  $(W, U)$  choices  $F_{C'(2)} = 0$ . The remaining cases correspond to the family on the right side of Figure 19, with  $W = \{1, 2, 5, 6\}$  and  $W = \{1, 3, 4, 6\}$ . For  $W = \{1, 2, 5, 6\}$  we get nontrivial terms with  $U = \{1\}$ ,  $U' = \{5, 6\}$ , both mapping  $x_1^*$  to  $y_1$ . For  $W = \{1, 3, 4, 6\}$  we get  $U = 1$ ,  $U' = \{3, 4\}$ , and again both decompositions map  $x_1^*$  to  $y_1$ .

The cases  $Q = (B, E)$ ,  $(E, B)$ ,  $(E, C)$  and  $(C, E)$  follow from the previous four cases by duality.

Finally if  $Q = (E, E)$  let  $r$  denote the number of circles that are both active ending circles of  $\mathcal{C}(1)$  and active starting circles of  $\mathcal{C}(2)$ . Since we are dealing

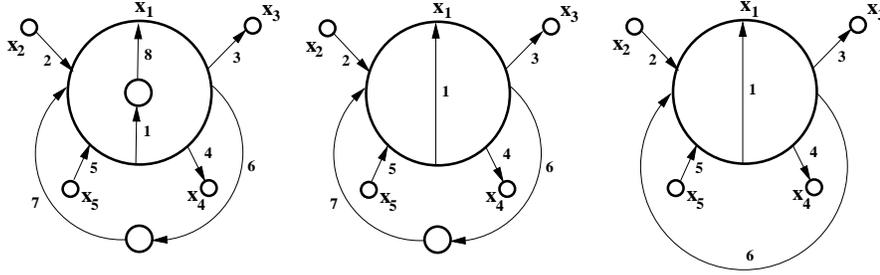


Figure 19:

with connected configurations we have  $r \geq 1$ . Let  $w$  be such a circle. If  $w$  divides the monomial  $z$  then  $w$  has to be the central ending circle of  $\mathcal{C}(1)$  and a degree 1 starting circle of  $\mathcal{C}(2)$ . If  $w$  doesn't divide  $z$  then  $w$  has to be a degree 1 ending circle of  $\mathcal{C}(1)$  and the central starting circle of  $\mathcal{C}(2)$ . It follows that  $r = 1$  or  $r = 2$ . When  $r = 1$   $\bar{\mathcal{C}}$  contains at least one of the  $M_1, M_2, M_3, M_4$  or  $M_5$  configurations after rotations. When  $r = 2$  after trading we get the configurations in the middle and in the right of Figure 19, that were dealt earlier.  $\square$

Now Theorem 6.3 follows immediately from Lemma 6.12, 6.13 and 6.14.

## 7 The spectral sequence.

Let  $L$  be an oriented link. Given a projection  $\mathcal{D}$  of  $L$ , let  $n^+(\mathcal{D})$  and  $n^-(\mathcal{D})$  denote the number of positive and negative crossings respectively.

Let's recall the bigrading from Khovanov homology, see [6], [15]. Given  $I \in \{0, 1\}^n$ , and a monomial  $z \in V(I)$ , the homological grading is

$$h(z) = |I| - n^-(\mathcal{D})$$

the  $q$  grading

$$q(z) = gr(z) + |I| + n^+(\mathcal{D}) - 2n^-(\mathcal{D}).$$

There is also a  $\delta$  grading, see [16], given by

$$\delta(z) = q(z) - 2h(z) = gr(z) - |I| + n^+(\mathcal{D}).$$

Note that in Khovanov boundary map  $\mathbf{d}_1$  shifts the  $(q, h)$  grading by  $(0, 1)$ , and gives a bigraded homology theory.

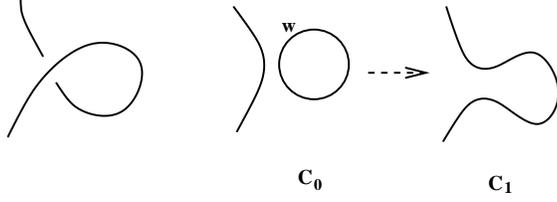


Figure 20:

According to the grading rule, see Lemma 2.12 the map  $\mathbf{d}_k$  shifts the  $(q, h)$  gradings by  $(2k - 2, k)$ , and decreases the  $\delta$  grading by 2.

**Definition 7.1** Given a diagram  $\mathcal{D}$  and a decoration  $\mathbf{t}$ , let  $\widehat{C}(\mathcal{D}, \mathbf{t})$  denote the chain complex  $(C_{\mathcal{D}}, \mathbf{d}(\mathbf{t}))$ , and let  $\widehat{H}(\mathcal{D}, \mathbf{t})$  denote its homology.

According to the grading shifts  $\delta$  gives a well-defined grading on  $\widehat{H}(\mathcal{D}, \mathbf{t})$  and we get a decomposition

$$\widehat{H}(\mathcal{D}, \mathbf{t}) = \oplus_i \widehat{H}_i(\mathcal{D}, \mathbf{t})$$

Given  $i \in \mathbf{Z}$ , let  $C_i \subset \widehat{C}(\mathcal{D}, \mathbf{t})$  be generated by pairs  $(I, z)$ , where  $z$  is a monomial in  $V(I)$  with  $h(z) \geq i$ .

Then  $C_i$  is a subcomplex of  $\widehat{C}(\mathcal{D}, \mathbf{t})$ . The corresponding filtration by  $C_i$ ,  $i \in \mathbf{Z}$  induces a spectral sequence. Since the first non-trivial differential  $\mathbf{d}(1)$  is the Khovanov differential we get a spectral sequence starting at the  $\mathbb{Z}_2$  Khovanov homology of  $L$  and ending at  $\widehat{H}(\mathcal{D}, \mathbf{t})$

**Theorem 7.2** Given an oriented link  $L$ , let  $\mathcal{D}_1, \mathcal{D}_2$  be two diagrams representing  $L$ , and let  $\mathbf{t}_1, \mathbf{t}_2$  be decorations for  $\mathcal{D}_1$  and  $\mathcal{D}_2$  respectively. Then there is a grading preserving isomorphism between  $\widehat{H}(\mathcal{D}_1, \mathbf{t}_1)$  and  $\widehat{H}(\mathcal{D}_2, \mathbf{t}_2)$ . Furthermore the two spectral sequences from Khovanov homology to  $\widehat{H}$  are also isomorphic.

**Proof** We start with a given diagram and two different decorations. Clearly it is enough to consider the case where  $\mathbf{t}$  and  $\mathbf{t}'$  differ only at one of the crossing, say the  $m$ -th crossing. Then we claim that

$$G : \widehat{C}(\mathcal{D}, \mathbf{t}) \longrightarrow \widehat{C}(\mathcal{D}, \mathbf{t}')$$

given by  $G(x) = x + H_m(x)$  is a filtered isomorphism between the chain complexes. We have

$$G(\mathbf{d}(\mathbf{t})(x)) = \mathbf{d}(\mathbf{t}')(x) + H_m(\mathbf{d}(\mathbf{t})(x))$$

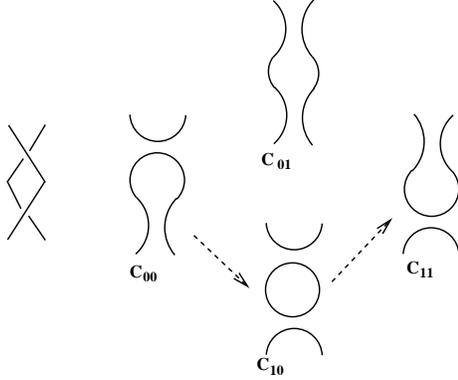


Figure 21:

and

$$\mathbf{d}(\mathbf{t}')(x + H_m(x)) = (\mathbf{d}(\mathbf{t}) + H_m \cdot \mathbf{d}(\mathbf{t}) + \mathbf{d}(\mathbf{t}) \cdot H_m)(x + H_m(x))$$

according to Theorem 5.4. Since  $H_m \cdot \mathbf{d}(\mathbf{t}) \cdot H_m = 0$  and  $H_m \cdot H_m = 0$  we get that  $G$  is chain map. Since  $G$  is defined as the identity plus lower order, it follows that both the kernel and cokernel of  $G$  is trivial.

The next step is to look at the Reidemeister moves. This works the same way as in Khovanov homology. However for the third Reidemeister move we have to pay attention to the decorations as well.

For the first Reidemeister move with positive stabilization we get the picture in Figure 20. Then  $C_0$  is a direct sum of  $C_0^+$  and  $C_0^-$ . Here  $C_0^-$  is generated by  $z$  that are divisible with  $w$ , and  $C_0^+$  is generated by  $z$  that are not divisible with  $w$ . Note that the higher differentials between  $C_0^+$  and  $C_1$  are trivial: Any configuration  $\bar{\mathcal{C}}$  that contains the  $\gamma_1$  arc will have  $w$  as a degree 1 starting circle. However then  $\bar{\mathcal{C}}$  needs to be a Type  $E$  configuration. Since  $z$  is not divisible by  $w$  it follows that  $F_{\bar{\mathcal{C}}}(z) = 0$ . However the Khovanov differential  $\mathcal{D}_1$  gives an isomorphism between  $C_0^+$  and  $C_1$ . Cancelling this subcomplex with trivial homology, we get the  $C_0^-$  is a quotient complex, which is isomorphic to the chain complex before stabilization. Since the isomorphism preserves the filtration we also get the same spectral sequence.

For the second Reidemeister move we again have the usual argument in Khovanov homology, see Figure 21. Here we first cancel with the subcomplex spanned by  $C_{10}^+ \oplus C_{11}$ . In the simplified complex we cancel with the quotient complex  $C_{00} \oplus C_{10}^-$ . The remaining  $C_{01}$  is isomorphic to the original complex.

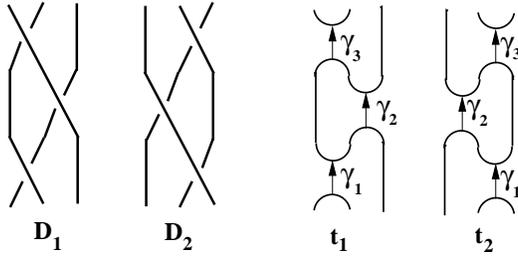


Figure 22:

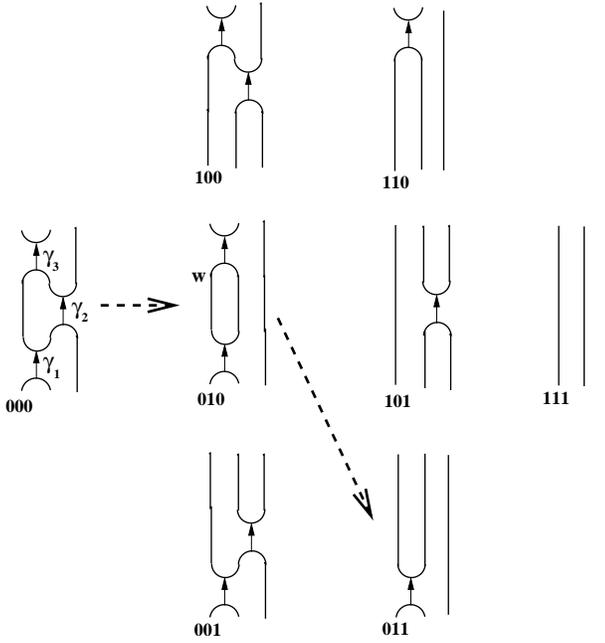


Figure 23:

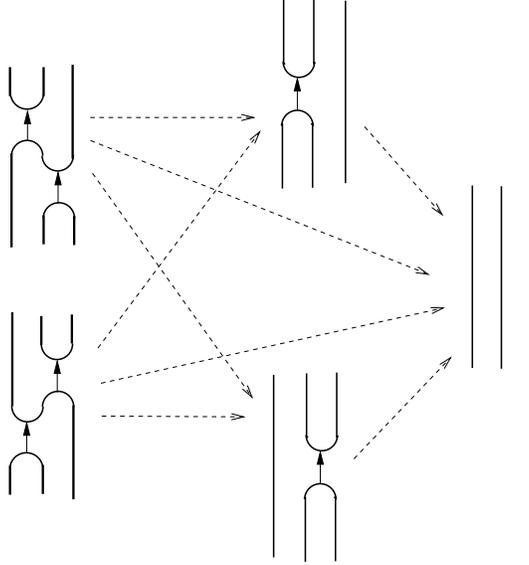


Figure 24:

For the third Reidemeister move we use the decorations as in Figure 22. Using the usual argument, see Figure 23 we first reduce the chain complex by cancelling the quotient complex  $C_{000} \oplus C_{010}^-$ . Then we reduce further by contracting with the edge map that connect  $C_{010}^+$  and  $C_{011}$ .

We have to determine the resulting complex. The first observation is that the complexes  $C_{011}$  and  $C_{110}$  are isomorphic.

When considering the restriction of the boundary map from  $C_{010}^+$  to  $C_{110}$  and  $C_{010}^+$  to  $C_{001}$  the one-dimensional configurations induce an isomorphism. The higher dimensional configurations contribute trivially: such a  $\mathcal{C}$  configuration would contain  $w$  as a degree one circle, so it has to be Type  $E$ , but  $a \in C_{010}^+$  is not divisible with  $w$ .

Finally we claim that the map from  $C_{010}^+$  to  $C_{111}$  is trivial: A configuration  $\mathcal{C}$  that connects them would contain  $w$  as an active starting circle, and both  $\gamma_2$  and  $\gamma_3$ . Because of the decoration it is enough to consider the case where  $\mathcal{C}$  is Type  $B$  or  $D$ . However  $a$  not divisible by  $w$  implies that  $F_{\mathcal{C}}(a) = 0$ .

It follows that the simplified chain complex is identical to the chain complex in Figure 24. Note that here the maps are still given by the usual decorated configurations, where we use small upwards or downwards isotopies supported in the region to identify the circles. A similar argument shows that  $\hat{C}(\mathcal{D}_2, \mathbf{t}_2)$

can be simplified to the same complex.

□

## 8 Further construction

First note that there is an alternating construction  $\mathbf{d}'$  for the boundary map, where  $F'_C$  is defined by

$$F'_C = F_{m(C)}.$$

So for example in this alternate world the 2 dimensional Type 8 configuration would contribute trivially, and for a Type 16 configuration  $C$  we would have  $F'_C(1) = 1$  and  $F'_C(x_1x_2) = y_1y_2$ . Clearly the construction carries through with  $\mathbf{d}'$  in place of  $\mathbf{d}$  and gives a twin version  $\widehat{H}'(L)$  of  $\widehat{H}(L)$ .

For braids one could simplify the choice of decorations as follows. First draw the braid diagram so that the strands are moving in the vertical direction. Then at each 0 resolution the  $\gamma$  arc is either vertical or horizontal. Now orient the arcs so that vertical arcs are oriented upward, and the horizontal arcs are oriented to the right.

For the transverse element in  $\widehat{H}(L)$  we follow the construction in [13], see also [1]: Given an  $n$ -braid we choose the unique resolution that gives back the  $n$ -parallel circles. For this resolution we choose the monomial  $z$  that is the product of all the  $n$  circles. Now using the braid decoration as above we see that  $z$  is a closed element in  $\widehat{C}(\mathcal{D}, \mathbf{t})$ : The choice of  $z$  implies that we only have to check Type  $B$  or Type  $D$  configurations. The geometry of the resolution shows that the only possibility is Type  $B_2$ . However all the oriented arcs are pointed to the right, so there are no Type  $B_2$  configurations that emanate from this resolution. Now we have to study braid moves as in [13], and see that the distinguished element maps to each other. However that is straightforward from the discussion of Section 6, where we view the second and the third moves as supported inside a braid. (Note that the braid decoration guarantees that both  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are decorated as in Figure 22.)

For a link with a distinguished component, we can define a reduced version  $\widehat{H}_{red}(L)$ . This is given by the usual point filtration in Khovanov homology. Take a point  $P$  in the diagram that lies in the distinguished component. Then for each resolution we get a special circle  $x(P)$  that contains  $P$ . Now define let  $\widehat{C}(\mathcal{D}, \mathbf{t}, P)$  be generated by those monomials that are divisible by  $x(P)$ . According to the Filtration rule of Section 2, we see that  $\widehat{C}(\mathcal{D}, \mathbf{t}, P)$  is a sub-

complex of  $\widehat{C}(\mathcal{D}, \mathfrak{t})$ .

Since the edge homotopies  $H_m$  map the subcomplex to itself, the proof of invariance in Theorem 7.2 carries through for the reduced theory as well, and we get  $\widehat{H}_{red}(L)$ . Note that here the  $q$  and  $\delta$  gradings are slightly modified, in particular both of them are shifted up by 1, so that the reduced Khovanov homology and the reduced  $\widehat{H}$  of the unknot  $U$  are supported in  $\delta$  grading 0.

Computations for  $\widehat{H}(K)$  and  $\widehat{H}_{red}(K)$  are given in [17] for large families of knots. A particularly interesting case is the  $(3, 5)$  torus knot for which  $\widehat{H}(T_{3,5}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$  supported in  $\delta$  gradings 1 and 3, and  $\widehat{H}_{red}(T_{3,5}) = \mathbb{Z}_2$  supported in  $\delta$  grading 2, compare also with [1] and [4]. The computations of Seed and the known computations for  $\widehat{HF}$  for branched double covers, see [5], raise the natural question.

**Question 8.1** For knots  $K$  in  $S^3$  is  $\widehat{H}_{red}(K)$  always isomorphic to  $\widehat{HF}(\Sigma_K)$  as mod 2 graded vector spaces over  $\mathbb{Z}_2$ ?

Another natural problem is the following

**Conjecture 8.2** Let  $U$  denote the unknot. If  $K$  is a prime knot so that  $\widehat{H}_{red}(K)$  and  $\widehat{H}_{red}(U)$  are isomorphic as  $\delta$  graded vector spaces, then

$$K = U.$$

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