

On the mean number of 2-torsion elements in the class groups, narrow class groups, and ideal groups of cubic orders and fields

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Abstract

Given any family of cubic fields defined by local conditions at finitely many primes, we determine the mean number of 2-torsion elements in the class groups and narrow class groups of these cubic fields, when they are ordered by their absolute discriminants.

For an order \mathcal{O} in a cubic field, we study the three groups: $\text{Cl}_2(\mathcal{O})$, the group of ideal classes of \mathcal{O} of order 2; $\text{Cl}_2^+(\mathcal{O})$, the group of narrow ideal classes of \mathcal{O} of order 2; and $\mathcal{I}_2(\mathcal{O})$, the group of ideals of \mathcal{O} of order 2. We prove that the mean value of the difference $|\text{Cl}_2(\mathcal{O})| - \frac{1}{4}|\mathcal{I}_2(\mathcal{O})|$ is always equal to 1, regardless of whether one averages over the maximal orders in real cubic fields, over all orders in real cubic fields, or indeed over *any* family of real cubic orders defined by local conditions. For the narrow class group, we prove that the average value of the difference $|\text{Cl}_2^+(\mathcal{O})| - |\mathcal{I}_2(\mathcal{O})|$ is equal to 1 for any such family. Also, for any family of complex cubic orders defined by local conditions, we prove similarly that the mean value of the difference $|\text{Cl}_2(\mathcal{O})| - \frac{1}{2}|\mathcal{I}_2(\mathcal{O})|$ is always equal to 1, independent of the family.

The determination of these mean numbers allows us to prove a number of further results as by-products. Most notably, we prove—in stark contrast to the case of quadratic fields—that: 1) a positive proportion of cubic fields have *odd* class number; 2) a positive proportion of real cubic fields have isomorphic 2-torsion in the class group and the narrow class group; and 3) a positive proportion of real cubic fields contain units of mixed real signature. We also show that a positive proportion of real cubic fields have narrow class group strictly larger than the class group, and thus a positive proportion of real cubic fields do not possess units of every possible real signature.

1 Introduction

For an order \mathcal{O} in a number field K , let $\text{Cl}(\mathcal{O})$ and $\text{Cl}^+(\mathcal{O})$ denote the class group and the narrow class group¹ of \mathcal{O} respectively. If \mathcal{O} is the maximal order of K , then one defines the class group and narrow class group of K by $\text{Cl}(K) := \text{Cl}(\mathcal{O})$ and $\text{Cl}^+(K) := \text{Cl}^+(\mathcal{O})$. Finally, for any prime p let $\text{Cl}_p(\mathcal{O})$, $\text{Cl}_p^+(\mathcal{O})$, $\text{Cl}_p(K)$, and $\text{Cl}_p^+(K)$ denote the p -torsion subgroups of $\text{Cl}(\mathcal{O})$, $\text{Cl}^+(\mathcal{O})$, $\text{Cl}(K)$, and $\text{Cl}^+(K)$, respectively.

In this article, we begin by proving:

¹Recall that the *class group* $\text{Cl}(\mathcal{O})$ of an order \mathcal{O} in a number field K is defined as $\mathcal{I}(\mathcal{O})/P(\mathcal{O})$, where $\mathcal{I}(\mathcal{O})$ is the group of invertible fractional ideals of \mathcal{O} and $P(\mathcal{O})$ is the group of principal fractional ideals of \mathcal{O} , that is, ideals of the form $a\mathcal{O}$ where $a \in K^\times$. The *narrow class group* $\text{Cl}^+(\mathcal{O})$ is defined as the quotient $\mathcal{I}(\mathcal{O})/P(\mathcal{O})^+$, where now $P(\mathcal{O})^+$ is the group of *totally positive* principal fractional ideals of \mathcal{O} , that is, ideals of the form $a\mathcal{O}$ where a is an element of K^\times such that $\sigma(a)$ is positive for every embedding $\sigma : K \rightarrow \mathbb{R}$.

Theorem 1 *When the set of isomorphism classes of cubic fields satisfying any specified set of local conditions at any finite set of primes are ordered by their absolute discriminants:*

- (a) *The average number of 2-torsion elements in the class groups of such totally real cubic fields is $5/4$.*
- (b) *The average number of 2-torsion elements in the (narrow) class groups of such complex cubic fields is $3/2$.*
- (c) *The average number of 2-torsion elements in the narrow class groups of such totally real cubic fields is 2.*

Thus, the average number of 2-torsion elements in the class groups or narrow class groups of cubic fields of given real signature remains the same, regardless of which family of cubic fields we are averaging over.

In the case where no local specifications are made in Theorem 1 (i.e., where we average over all cubic fields), parts (a) and (b) of Theorem 1 were first proven in [3] by rather indirect methods, and confirmed a special case of the Cohen–Martinet–Malle heuristics (see [8, 15]). In the current paper, we generalize these results using a more direct correspondence (see Theorem 11), which additionally allows us to obtain Theorem 1(c), and indeed Theorems 1(a)–(c) for general families of cubic fields defined by local conditions at finitely many primes.² Our current method is more direct in the sense that, not only does it yield a count of all the 2-torsion elements in class groups and narrow class groups in various families of cubic fields, but it in fact gives an explicit construction of these elements.

This direct correspondence also allows us to study on average how many ideal classes of order 2 in the class groups of cubic orders arise simply because there exists an ideal of order 2 representing it. Recall that the *ideal group* $\mathcal{I}(\mathcal{O})$ of an order \mathcal{O} is the group of invertible fractional ideals of \mathcal{O} , of which the class group $\text{Cl}(\mathcal{O})$ is a quotient. If an order \mathcal{O} is not maximal, then it is possible for an *ideal* I of \mathcal{O} to have order 2, i.e., I can be a fractional ideal of \mathcal{O} satisfying $I^2 = \mathcal{O}$ but $I \neq \mathcal{O}$.³

For an order \mathcal{O} in a number field, let $\mathcal{I}_p(\mathcal{O})$ denote the p -torsion subgroup of the ideal group $\mathcal{I}(\mathcal{O})$ of \mathcal{O} . In [6, Thm. 8], we showed that the mean value of the difference $|\text{Cl}_3(\mathcal{O})| - |\mathcal{I}_3(\mathcal{O})|$ is always equal to 1, regardless of whether one averages over the maximal orders in complex quadratic fields, over all orders in such fields, or even over very general families of complex quadratic orders defined by local conditions. Similarly, for very general families of real quadratic orders defined by local conditions, we showed that the mean value of the difference $|\text{Cl}_3(\mathcal{O})| - \frac{1}{3}|\mathcal{I}_3(\mathcal{O})|$ is always equal to 1, independent of the family.

In this article, we prove the analogues of these results for the 2-torsion elements in the class groups and ideal groups of *cubic* orders. To state the result, we require just a bit of notation. For each prime p , let Σ_p be any set of isomorphism classes of orders in étale cubic algebras over \mathbb{Q}_p . We say that the collection (Σ_p) of local specifications is *acceptable* if, for all sufficiently large p , the set Σ_p contains all maximal cubic rings over \mathbb{Z}_p , or at least all those maximal cubic rings over \mathbb{Z}_p that are not totally ramified at p . Then we prove:

²We are not aware of any extensions of the Cohen–Lenstra–Martinet–Malle heuristics that may have led to the predictions of Theorem 1(c). It would be interesting to have similar Cohen–Lenstra–Martinet–Malle style heuristics that apply more generally to narrow class groups as well as to orders.

³For maximal orders, it is easy to show that any 2-torsion element (indeed, any torsion element) in the ideal group must be the trivial ideal.

Theorem 2 *Let (Σ_p) be any acceptable collection of local specifications as above, and let Σ denote the set of all isomorphism classes of cubic orders \mathcal{O} such that $\mathcal{O} \otimes \mathbb{Z}_p \in \Sigma_p$ for all p . Then, when orders in Σ are ordered by their absolute discriminants:*

- (a) *The average size of $|\text{Cl}_2(\mathcal{O})| - \frac{1}{4}|\mathcal{I}_2(\mathcal{O})|$ for totally real cubic orders \mathcal{O} in Σ is 1.*
- (b) *The average size of $|\text{Cl}_2(\mathcal{O})| - \frac{1}{2}|\mathcal{I}_2(\mathcal{O})| = |\text{Cl}_2^+(\mathcal{O})| - \frac{1}{2}|\mathcal{I}_2(\mathcal{O})|$ for complex cubic orders \mathcal{O} in Σ is 1.*
- (c) *The average size of $|\text{Cl}_2^+(\mathcal{O})| - |\mathcal{I}_2(\mathcal{O})|$ for totally real cubic orders \mathcal{O} in Σ is 1.*

It is rather intriguing that the mean values in Theorem 2 remain the same regardless of the choice (Σ_p) of acceptable local specifications for the cubic orders being averaged over. If (Σ_p) is acceptable and each Σ_p consists only of *maximal* cubic rings over \mathbb{Z}_p , then we recover (a more general version of) Theorem 1, since the only 2-torsion element of the ideal group of a maximal order is the trivial ideal.

Theorem 1 immediately implies that most cubic fields have odd class number. More precisely:

Corollary 3 (1) *A positive proportion (at least 75%) of totally real cubic fields have odd class number.*

(2) *A positive proportion (at least 50%) of complex cubic fields have odd class number.*

We now compare the class group with the narrow class group. For orders \mathcal{O} in number fields with predominantly real places, such as totally real number fields, it is natural to expect the narrow class group to be strictly larger than the class group, since one is forming a quotient of $I_{\mathcal{O}}$ by principal ideals satisfying much more stringent conditions. In the quadratic case, this expectation is supported by:

Theorem 4 *When ordered by their discriminants, a density of 100% of real quadratic fields F satisfy $\text{Cl}(F) \neq \text{Cl}^+(F)$.⁴*

In fact, since in the real quadratic case the size of the narrow class group is either the same as or twice the size of the class group, it is natural to try and distinguish these two groups just based on the sizes of their 2-torsion subgroups. We have:

Theorem 5 *When ordered by their discriminants, a density of 100% of real quadratic fields F satisfy $\text{Cl}_2(F) \neq \text{Cl}_2^+(F)$.*

Now recall that for any number field L with r distinct real embeddings, there is a *signature* homomorphism $\text{sgn} : \mathcal{O}_L^\times \rightarrow \{\pm 1\}^r$ that takes a *unit* of L —i.e., an invertible element of the ring of integers \mathcal{O}_L of L —to its sign (one sign for each of the real embeddings $\sigma : L \rightarrow \mathbb{R}$); a unit u has *mixed signature* if $\text{sgn}(u) \neq \text{sgn}(\pm 1)$, i.e., if there exist two real embeddings $\sigma_+, \sigma_- : L \rightarrow \mathbb{R}$ such that $\sigma_{\pm}(u) = \pm 1$. Then, in terms of units of quadratic fields, we have:

⁴This is not to say that *all* real quadratic fields satisfy $\text{Cl}(F) \neq \text{Cl}^+(F)$; indeed, there are also infinitely many quadratic fields for which $\text{Cl}(F) = \text{Cl}^+(F)$, but asymptotically such fields have density zero in the set of all quadratic fields. The 0% of quadratic fields satisfying $\text{Cl}(F) = \text{Cl}^+(F)$ are actually very interesting; see, for example, the compelling heuristics of Stevenhagen [17] on such fields and the beautiful recent work of Fouvry and Klüners [11] in this regard.

Corollary 6 *When ordered by their discriminants, a density of 0% of real quadratic fields have a unit of mixed signature.*

Thus, at least asymptotically, we completely understand the question of how often the class group and narrow class group differ for real quadratic fields. Namely, 100% of the time, the narrow class group looks like the class group with one extra 2-torsion factor.

Theorems 4 and 5 are well-known and easily proven; they are true primarily because for most d there are local obstructions to having a solution to the negative norm unit equation $x^2 - dy^2 = -4$. Also, the 2-torsion elements in class groups and narrow class groups in the quadratic case are mostly governed by genus theory. One of the reasons Cohen and Lenstra did not formulate heuristics for narrow class groups in the quadratic case is that these groups differ from class groups of quadratic fields only in their 2-parts, which are related to genus theory. From this point of view, when computing narrow class groups vs. class groups, it is natural to try and investigate a case where 2 is unrelated to genus theory—for example, the case of *cubic* fields, to which we now turn.

A classical theorem of Armitage and Frohlich [1] states that

$$\dim_{\mathbb{Z}/2\mathbb{Z}}(\text{Cl}_2^+(K)) - \dim_{\mathbb{Z}/2\mathbb{Z}}(\text{Cl}_2(K)) \leq 1$$

for cubic fields K . Thus Theorems 1(a) and 1(c), when taken together, immediately imply:

Corollary 7 *A positive proportion (at least 50%) of totally real cubic fields K satisfy $\text{Cl}_2(K) \neq \text{Cl}_2^+(K)$.*

Corollary 8 *A positive proportion (at least 25%) of totally real cubic fields K satisfy $\text{Cl}_2(K) = \text{Cl}_2^+(K)$.*

Note that Corollary 8 for cubic fields is in stark contrast to Theorem 5 for quadratic fields.

As far as the existence of units of mixed type in cubic fields, we have:

Corollary 9 *A positive proportion (at least 50%) of totally real cubic fields K do not possess units of every possible signature.*

Corollary 10 *A positive proportion (at least 75%) of totally real cubic fields K possess units of mixed signature.*

Note again the stark contrast between Corollary 10 and the corresponding situation in Corollary 6 for quadratic fields. Corollaries 3 and 7–10 continue to hold, with the same percentages, even when one averages over just those cubic fields satisfying any desired set of local conditions at a finite set of primes, or more generally over *any* acceptable family of maximal cubic orders (allowing us to consider families of cubic fields satisfying suitable infinite sets of local conditions).

We prove Theorems 1 and 2 and their corollaries as an application of a composition law on a prehomogeneous vector space that was investigated in [2]. We lay down the relevant preliminaries about this law of composition, defined on pairs of ternary quadratic forms, in Section 2; more precisely, we give a correspondence between pairs of ternary quadratic forms and 2-torsion elements in class groups of cubic rings over a principal ideal domain. This allows us to describe a composition law on certain integer-matrix pairs of ternary quadratic forms in Section 3, leading to groups that are closely related to both the class groups and (with some additional effort) the narrow class groups of orders in cubic fields. In Section 4, we focus on the composition law on *reducible* pairs of

ternary quadratic forms, which leads to groups that are closely related to the ideal groups of orders in cubic fields. In Section 5, we describe how one can count the appropriate integer points in this prehomogeneous vector space using the results of [3]. By counting these configurations of points, we then complete the proofs of Theorems 1 and 2 in Section 6. Finally, we deduce Corollaries 3 and 7–10 in Section 7.

2 Parametrization of order 2 ideal classes in cubic orders

The key ingredient in the proof of Theorem 1 is a parametrization of ideal classes of order 2 in cubic rings by means of equivalence classes of pairs of integer-matrix ternary quadratic forms, which was obtained in [2]. We begin by describing this orbit space briefly.

If T is a principal ideal domain, then let $V_T = T^2 \otimes \text{Sym}^2 T^3$ denote the space of pairs (A, B) of ternary quadratic forms over T . We write an element $(A, B) \in V_T$ as a pair of 3×3 symmetric matrices with coefficients in T as follows:

$$(A, B) = \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} \right). \quad (1)$$

In particular, the elements of $V_{\mathbb{Z}}$ are called pairs (A, B) of *integer-matrix* ternary quadratic forms.

The group $\text{GL}_2(T) \times \text{GL}_3(T)$ acts naturally on the space V_T . Namely, an element $g_2 \in \text{GL}_2(T)$ acts by changing the basis of the rank 2 T -module of forms spanned by (A, B) ; more precisely, if $g_2 = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$, then we set $g_2 \cdot (A, B) = (rA - sB, -tA + uB)$. Similarly, an element $g_3 \in \text{GL}_3(T)$ changes the basis of the rank 3 T -module on which the forms A and B take values; we have $g_3 \cdot (A, B) = (g_3 A g_3^t, g_3 B g_3^t)$. It is easy to see that these actions of g_2 and g_3 commute, leading to an action of $\text{GL}_2(T) \times \text{GL}_3(T)$. However, this action is not faithful in general. If we let

$$G_T := \text{GL}_2(T) \times \text{GL}_3(T) / \left\{ \left(\begin{bmatrix} \lambda^{-2} & 0 \\ 0 & \lambda^{-2} \end{bmatrix}, \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) : \lambda \in T^\times \right\},$$

then for all principal ideal domains T , the group G_T acts faithfully on V_T .

The action of $\text{SL}_2(\mathbb{C}) \times \text{SL}_3(\mathbb{C})$ on $V_{\mathbb{C}}$ turns out to have a unique polynomial invariant (see, e.g., Sato–Kimura [16]); i.e., the ring of all polynomial functions on $V_{\mathbb{C}}$ that are invariant under the action of $\text{SL}_2(\mathbb{C}) \times \text{SL}_3(\mathbb{C})$ is generated by one element. This generating element can be defined naturally over \mathbb{Z} , or indeed, over any principal ideal domain T .

To construct this fundamental invariant, we notice first that the action of $\text{SL}_3(\mathbb{C})$ on $V_{\mathbb{C}}$ has four independent polynomial invariants, namely the coefficients a, b, c, d of the binary cubic form

$$f(x, y) = f_{(A, B)}(x, y) := \text{Det}(Ax - By), \quad (2)$$

where $(A, B) \in V_{\mathbb{C}}$. For an element $(A, B) \in V_T$, we call $f(x, y)$ as defined by (2) the *binary cubic form invariant* of (A, B) . An element $\gamma \in \text{GL}_2(T)$ then acts on the binary cubic form invariant $f(x, y)$ by $\gamma \cdot f(x, y) := f((x, y)\gamma)$.

Now it is well-known that the action of $\text{SL}_2(\mathbb{C})$ on the space of complex binary cubic forms has a unique polynomial invariant (up to scaling), namely its discriminant $\text{Disc}(f)$. The unique polynomial invariant for the action of $\text{SL}_2(\mathbb{C}) \times \text{SL}_3(\mathbb{C})$ on $V_{\mathbb{C}}$ is then given by $\text{Disc}(A, B) := \text{Disc}(\text{Det}(Ax - By))$, which is an integer polynomial of degree 12 in the entries of A and B .

For $(A, B) \in V_T$, we call this fundamental invariant $\text{Disc}(A, B)$ the *discriminant* of (A, B) . Note that for any $(g_2, g_3) \in \text{GL}_2(T) \times \text{GL}_3(T)$ and $(A, B) \in V_T$, we have

$$\text{Disc}((g_2, g_3) \cdot (A, B)) = \det(g_2)^6 \det(g_3)^8 \text{Disc}(A, B),$$

and thus $\text{Disc}(A, B)$ is also a *relative invariant* for the action of G_T on V_T . We say that $(A, B) \in V_T$ is *nondegenerate* if it has nonzero discriminant.

The orbits of G_T on V_T have an important arithmetic significance, namely they essentially parametrize order 2 ideal classes in cubic rings over T . Recall that a *cubic ring* over T is any ring with unit that is free of rank 3 as a T -module; for example, an order in a cubic number field is a cubic ring over \mathbb{Z} . In 1964, Delone and Faddeev [10] showed that orders in cubic fields are parametrized by irreducible integral binary cubic forms; this was recently extended to general cubic rings and general binary cubic forms by Gan, Gross, and Savin [12] (see also the work of Gross and Lucianovic [14]).

We have observed above that a pair $(A, B) \in V_T$ of ternary quadratic forms yields a binary cubic form invariant f over T , and thus we naturally obtain a cubic ring $R(f)$ over T from an element $(A, B) \in V_T$ via the Gan–Gross–Savin parametrization described above. We say that a cubic ring over T is *nondegenerate* if it has nonzero discriminant.

We now describe a precise correspondence between nondegenerate pairs of ternary quadratic forms in V_T and ideal classes “of order 2” in the corresponding cubic rings over a principal ideal domain T . To simplify the statement, we assume that we have a set S of representatives in $T \setminus \{0\}$ for the left action of T^\times on $T \setminus \{0\}$. For example, if $T = \mathbb{Z}$, then we will always take S to be the set of positive elements of \mathbb{Z} ; similarly, if $T = \mathbb{Z}_p$, then S will always be chosen to consist of the powers of p in \mathbb{Z}_p ; and if T is a field, then we will always take $S = \{1\}$. For an ideal I of a cubic ring R over T , we use $N_S(I)$ to denote the unique generator in S of the usual ideal norm of I viewed as a T -ideal. (When the choice of S is clear, as in the case of \mathbb{Z} , \mathbb{Z}_p , or a field, then we usually drop it from the notation.) Then we have the following generalization of [2, Thm. 4]:

Theorem 11 *Let T and S be as above. Then there is a bijection between the set of nondegenerate G_T -orbits on the space $V_T = T^2 \otimes \text{Sym}^2 T^3$ and the set of equivalence classes of triples (R, I, δ) , where R is a nondegenerate cubic ring over T , I is an ideal of R having rank 3 as a T -module, and δ is an invertible element of $R \otimes \text{Frac}(T)$ such that $I^2 \subseteq (\delta)$ and $N_S(I)^2 = N(\delta)$. (Here two triples (R, I, δ) and (R', I', δ') are equivalent if there exists an isomorphism $\phi : R \rightarrow R'$ and an element $\kappa \in R' \otimes \text{Frac}(T)$ such that $I' = \kappa\phi(I)$ and $\delta' = \kappa^2\phi(\delta)$.)*

The proof of Theorem 11 is identical to that given in [2] (with a suitable change to the definition of orientation, described below). We briefly sketch the bijective map for general T .

Given a triple (R, I, δ) over T , we construct the corresponding pair of ternary quadratic forms over T as follows. First, we may write $R = T + T\omega + T\theta$ where $\langle 1, \omega, \theta \rangle$ is a *normal basis*, i.e., $\omega\theta \in T$. Furthermore, we can write $I = T\alpha_1 + T\alpha_2 + T\alpha_3$, where $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ has the *same orientation* as $\langle 1, \omega, \theta \rangle$, i.e., the change-of-basis matrix from $\langle 1, \omega, \theta \rangle$ to $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ has determinant lying in our fixed choice S of representatives in $T \setminus \{0\}$ for the left action of T^\times on $T \setminus \{0\}$. Since $I^2 \subseteq \delta R$, we have for all $i, j \in \{1, 2, 3\}$ that

$$\alpha_i\alpha_j = \delta(c_{ij} + b_{ij}\omega + a_{ij}\theta) \tag{3}$$

where $a_{ij}, b_{ij}, c_{ij} \in T$. Then $(A, B) = ((a_{ij}), (b_{ij}))$ yields the desired pair of symmetric 3×3 matrices

over T . In basis-free terms, (A, B) corresponds to the symmetric bilinear map of T -modules

$$\varphi : I \otimes I \rightarrow R/T \quad \alpha \otimes \beta \mapsto \left(\frac{\alpha\beta}{\delta} \right) \bmod T. \quad (4)$$

On the other hand, if $(A, B) \in V_T$ is a pair of ternary quadratic forms over T , and $f_{(A,B)}(x, y) = \text{Det}(Ax - By) = ax^3 + bx^2y + cxy^2 + dy^3$ is the associated binary cubic form with coefficients $a, b, c, d \in T$, then $R = R(f_{(A,B)})$ is the cubic ring with basis $\langle 1, \omega, \theta \rangle$ satisfying

$$\begin{aligned} \omega\theta &= -ad \\ \omega^2 &= -ac - b\omega + a\theta \\ \theta^2 &= -bd - d\omega + c\theta. \end{aligned} \quad (5)$$

In order to describe I in terms of a basis $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, it suffices to show that all c_{ij} and δ in (3) are uniquely determined by (A, B) . First, one shows using Equation (3), together with the associative law on triple products $(\delta^{-1}\alpha_i)(\delta^{-1}\alpha_j)(\delta^{-1}\alpha_k)$, that the c_{ij} can be written explicitly as

$$c_{ij} = \sum_{\substack{i' < i'' \\ j' < j''}} \begin{pmatrix} i & i' & i'' \\ 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} j & j' & j'' \\ 1 & 2 & 3 \end{pmatrix} \cdot \begin{vmatrix} a_{ij} & a_{ij'} \\ a_{i'j} & a_{i'j'} \end{vmatrix} \cdot \begin{vmatrix} b_{ij} & b_{ij''} \\ b_{i''j} & b_{i''j''} \end{vmatrix},$$

where $\begin{pmatrix} i & i' & i'' \\ 1 & 2 & 3 \end{pmatrix}$ denotes the sign of the permutation of (123) described by the top row. The system of equations given in (3) shows that for each $j \in \{1, 2, 3\}$, we have

$$\alpha_1 : \alpha_2 : \alpha_3 = c_{1j} + b_{1j}\omega + a_{1j}\theta : c_{2j} + b_{2j}\omega + a_{2j}\theta : c_{3j} + b_{3j}\omega + a_{3j}\theta.$$

The ratios are independent of the choice of j , and so this determines $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ uniquely up to a scalar factor in $R \otimes \text{Frac}(T)$. Once we fix a choice of $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, Equation (3) then uniquely determines $\delta \in R \otimes \text{Frac}(T)$. The R -ideal structure of the T -module I is given by [2, Eqn. (11)]. This completely and explicitly determines the triple (R, I, δ) from the pair (A, B) of ternary quadratic forms.

Finally, we describe the action of $\text{GL}_2(T) \times \text{GL}_3(T)$ on the bases of R/T and I corresponding to the action of the same group on pairs (A, B) of ternary quadratic forms over T described previously. If $g_2 \in \text{GL}_2(T)$ is the matrix $g_2 = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$, then it acts on the normal basis of $R/T = \langle \omega, \theta \rangle$ and the T -basis $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ of the ideal I by

$$\begin{aligned} \langle \omega, \theta \rangle &\mapsto \det(g_2) \cdot \langle \omega, \theta \rangle \cdot g_2^t = \det(g_2) \cdot \langle r\omega + s\theta, t\omega + u\theta \rangle \\ \langle \alpha_1, \alpha_2, \alpha_3 \rangle &\mapsto \det(g_2) \cdot \langle \alpha_1, \alpha_2, \alpha_3 \rangle. \end{aligned} \quad (6)$$

On the other hand, $g_3 \in \text{GL}_3(T)$ acts on the bases $\langle \omega, \theta \rangle$ and $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ by

$$\begin{aligned} \langle \omega, \theta \rangle &\mapsto \det(g_3)^2 \cdot \langle \omega, \theta \rangle \\ \langle \alpha_1, \alpha_2, \alpha_3 \rangle &\mapsto \langle \alpha_1, \alpha_2, \alpha_3 \rangle \cdot g_3^t. \end{aligned} \quad (7)$$

One now easily checks that the equivalence defined on triples (R, I, δ) in the statement of the theorem exactly corresponds to G_T -equivalence on pairs $(A, B) \in V_T$ of ternary quadratic forms over T .

Remark 12 Note that $G_{\mathbb{Z}} \cong \text{GL}_2(\mathbb{Z}) \times \text{SL}_3(\mathbb{Z})$, and so we recover [2, Thm. 4].

The following corollary of Theorem 11 describes the stabilizer in G_T of an element $(A, B) \in V_T$:

Corollary 13 *The stabilizer in G_T of a nondegenerate element $(A, B) \in V_T$ is given by the semidirect product*

$$\text{Aut}(R; I, \delta) \ltimes U_2^+(R_0),$$

where: (R, I, δ) is the triple corresponding to (A, B) as in Theorem 11; $\text{Aut}(R; I, \delta)$ is the group $\{\phi \in \text{Aut}(R) : \exists \kappa \in R \otimes \text{Frac}(T) \text{ s.t. } \phi(I) = \kappa I \text{ and } \phi(\delta) = \kappa^2 \delta\}$; $R_0 = \text{End}_R(I)$ is the endomorphism ring of I ; and $U_2^+(R_0)$ denotes the group of units of R_0 having order dividing 2 and norm 1.

Proof: First, note that the elements of $\text{GL}_2(T) \times \text{GL}_3(T)$ that preserve the bases of R/T and I under the action described by (6) and (7) are exactly the elements of

$$C(T) := \left\{ \left(\begin{bmatrix} \lambda^{-2} & 0 \\ 0 & \lambda^{-2} \end{bmatrix}, \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) : \lambda \in T^\times \right\}.$$

That is, $C(T)$ acts pointwise trivially on V_T . This implies that if $g \in \text{GL}_2(T) \times \text{GL}_3(T)$, then any element of the coset $g \cdot C(T)$ acts on the bases of R/T and I in the same way that g acts. Furthermore, if the actions of $g, g' \in \text{GL}_2(T) \times \text{GL}_3(T)$ are distinct on the bases of R/T and I , then $g' \notin g \cdot C(T)$. Thus, it suffices to compute the number of distinct actions on the bases of R/T and I by elements $g \in \text{GL}_2(T) \times \text{GL}_3(T)$ that preserve $A = (a_{ij})$ and $B = (b_{ij})$ in (3), in order to compute the number of cosets $g \cdot C(T)$ in the stabilizer in $\text{GL}_2(T) \times \text{GL}_3(T)$ of an element $(A, B) \in V_T$.

Now suppose $g = (g_2, g_3) \in \text{GL}_2(T) \times \text{GL}_3(T)$ preserves a pair (A, B) . In terms of the corresponding triple (R, I, δ) , g acts on the basis of R/T as described in Equations (6) and (7), which induces an automorphism ϕ of R . As g must preserve the triple (R, I, δ) up to equivalence, we see that $\phi \in \text{Aut}(R; I, \delta)$, i.e., there exists $\kappa \in R \otimes \text{Frac}(T)$ such that $\phi(I) = \kappa I$ and $\phi(\delta) = \kappa^2 \delta$. To preserve the structure coefficients b_{ij} and a_{ij} in (3), the change of basis of R/T corresponding to the element $\phi \in \text{Aut}(R; I, \delta)$ then uniquely determines the change of basis for I (namely, $\alpha \mapsto \kappa^{-1} \phi(\alpha)$ for $\alpha \in I$, assuming we keep δ the same in (3)), up to transformations on the basis of I that act via multiplication by an element of $U_2^+(\text{End}_R(I))$. This is the desired conclusion. \square

3 Composition of pairs of integer-matrix forms and (narrow) 2-class groups

Let us now restrict ourselves to those cubic rings R over the base ring \mathbb{Z} that are orders in S_3 -cubic fields. It is known that, when ordered by absolute discriminant, such S_3 -cubic orders constitute a proportion of 100% of all orders in cubic fields (since orders in abelian cubic fields are negligible in number, in comparison). Note that the automorphism group $\text{Aut}(R)$ of an S_3 -cubic order R is trivial.

Let us say that a triple (R, I, δ) is *projective* if I is projective as an R -module (i.e., if I is invertible as a fractional ideal of R); in such a case we have $I^2 = (\delta)$. For a fixed S_3 -cubic order R , we therefore obtain a natural law of composition on equivalence classes of projective triples (R, I, δ) , defined by

$$(R, I, \delta) \circ (R, I', \delta') = (R, II', \delta \delta').$$

The equivalence classes of projective triples (R, I, δ) then form a group under this composition law, which we denote by $H(R)$.

Let us say that a pair $(A, B) \in V_{\mathbb{Z}}$ is *projective* if the corresponding (R, I, δ) , under the bijection of Theorem 11, is projective. We then also obtain a composition law on projective equivalence classes of pairs (A, B) of ternary quadratic forms having binary cubic form invariant equal

to a given f , where $R(f) \cong R$ (a higher degree analogue of Gauss composition). We denote the corresponding group on such equivalence classes of (A, B) also by $H(R)$.

Fix an S_3 -cubic order R . Let U denote the unit group of R , and let $U^{\text{pos norm}}$ denote the subgroup of those units having positive norm. Then we have an exact sequence

$$1 \rightarrow \frac{U^{\text{pos norm}}}{U^2} \rightarrow H(R) \rightarrow \text{Cl}_2(R) \rightarrow 1, \quad (8)$$

where U^2 denotes the subgroup of U consisting of square units.

If R is an order in a totally real cubic field, then the group $U^{\text{pos norm}}/U^2$ has order 4. Meanwhile, if R is an order in a complex cubic field, then the latter group has order 2. We thus obtain:

Lemma 14 *We have $|H(R)| = 4 \cdot |\text{Cl}_2(R)|$ when R is an order in a totally real S_3 -cubic field, and $|H(R)| = 2 \cdot |\text{Cl}_2(R)|$ when R is an order in a complex cubic field.*

Equation (8) and Lemma 14 thus make precise the relationship between $H(R)$ and $\text{Cl}_2(R)$.

If we further assume that R is an order in a totally real S_3 -cubic field, we can restrict to the group $H^+(R) \subset H(R)$ consisting of those triples (R, I, δ) in which δ is totally positive⁵. The group $H^+(R)$ turns out to be closely related to the group $\text{Cl}_2^+(R)$; in particular, we find:

Lemma 15 *Let R be an order in a totally real S_3 -cubic field, and let $H^+(R)$ be the subgroup of $H(R) = \{(R, I, \delta)\}$ where δ is totally positive. Then $|H^+(R)| = |\text{Cl}_2^+(R)|$.*

Proof: Let U again denote the unit group of R , and let $U^{\gg 0} \subset U$ denote the subgroup of totally positive units. Let $\text{sgn} : U \rightarrow \{\pm 1\}^3$ denote the signature homomorphism that takes a unit to its sign (one sign for each of the three real embeddings $R \rightarrow \mathbb{R}$). Then we have the following commutative triangle of exact sequences:

$$\begin{array}{ccccccc}
 & & 1 & & & & 1 \\
 & & \searrow & & & & \swarrow \\
 1 & \longrightarrow & U^{\gg 0}/U^2 & \longrightarrow & U/U^2 & \xrightarrow{\text{sgn}} & \{\pm 1\}^3 & \longrightarrow & \{\pm 1\}^3/\{\text{sgn}(U)\} & \longrightarrow & 1 \\
 & & \searrow & & & & \swarrow & & & & \\
 & & & & H^+(R) & & \text{Cl}_2^+(R) & & & & \\
 & & & & \searrow & & \swarrow & & & & \\
 & & & & & \{(R, I) : \delta \gg 0\} & & & & & \\
 & & & & \searrow & & \swarrow & & & & \\
 & & & & 1 & & 1 & & & &
 \end{array}$$

Here $\{(R, I) : \delta \gg 0\}$ denotes the group of all equivalence classes of ideals I of R for which there exists a totally positive $\delta \in R$ with $I^2 = (\delta)$. Since $|U/U^2| = |\{\pm 1\}^3| = 8$, we then also have $|U^{\gg 0}/U^2| = |\{\pm 1\}^3/\{\text{sgn}(U)\}|$, and therefore $|H^+(R)| = |\text{Cl}_2^+(R)|$, as desired. \square

⁵An element $\delta \in R \otimes \mathbb{Q}$ is *totally positive* if for every embedding $\sigma : R \otimes \mathbb{Q} \rightarrow \mathbb{R}$, $\sigma(\delta)$ is positive.

4 Reducible pairs of ternary quadratic forms

The bijection given in Theorem 11 includes all pairs $(A, B) \in V_{\mathbb{Z}}$ of integer-matrix ternary quadratic forms—even *reducible* pairs, i.e., those that share a common rational zero in \mathbb{P}^2 . The question then arises: which elements in $H(R)$ correspond to reducible pairs (A, B) of ternary quadratic forms?

Lemma 16 *Let (A, B) be a projective element of $V_{\mathbb{Z}}^{(i)}$ whose binary cubic form invariant $f_{(A,B)}$ is irreducible over \mathbb{Q} , and let (R, I, δ) denote the corresponding triple as given by Theorem 11. Then (A, B) has a rational zero in \mathbb{P}^2 if and only if δ is a square in $(R \otimes \mathbb{Q})^{\times}$.*

Proof: Suppose $\delta = r^2$ for some invertible $r \in R \otimes \mathbb{Q}$. Then by replacing I by $r^{-1}I$ and δ by $r^{-2}\delta$ if necessary, we may assume that $\delta = 1$. Let α_1 be the smallest positive element of $I \cap \mathbb{Z}$, and extend to a basis $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ of I . By (3), we have that $b_{11} = 0$ and $a_{11} = 0$, which implies that the associated pair of ternary quadratic forms has a rational zero, namely $(1 : 0 : 0)$, in \mathbb{P}^2 .

Conversely, suppose that A and B have a common rational zero in \mathbb{P}^2 . If (R, I, δ) is the corresponding triple, then this means that there exists $\alpha \in I$ such that $\phi(\alpha \otimes \alpha) = \alpha^2/\delta \in \mathbb{Z}$, where ϕ is defined as in (4). Setting $\alpha^2 = n\delta$ for some $n \in \mathbb{Z}$, and taking norms to \mathbb{Z} on both sides, reveals that $N(\alpha)^2 = n^3 N(\delta) = n^3 N(I)^2$. Thus $n = m^2$ is a square. This implies that δ must be a square in $(R \otimes \mathbb{Q})^{\times}$, namely, $\delta = (\alpha/m)^2$. \square

For a cubic order R , let $H_{\text{red}}(R)$ be the subgroup of (equivalence classes of) projective triples $(R, I, \delta) \in H(R)$ where δ is a square in $(R \otimes \mathbb{Q})^{\times}$. By the previous lemma, $H_{\text{red}}(R)$ is the subgroup of equivalence classes of projective pairs $(A, B) \in V_{\mathbb{Z}}$ for which $R(f_{(A,B)}) = R$ in their corresponding triple, and such that (A, B) has a rational zero in \mathbb{P}^2 . As in the introduction, let $\mathcal{I}_2(R)$ denote the 2-torsion subgroup of the ideal group of R , i.e., the subgroup of invertible fractional ideals I of R such that $I^2 = R$. We can then define a map

$$\varphi : \mathcal{I}_2(R) \rightarrow H(R) \quad I \mapsto (R, I, 1).$$

It is evident that $\text{im}(\mathcal{I}_2(R)) \subseteq H_{\text{red}}(R)$. We show that φ in fact defines an isomorphism between $\mathcal{I}_2(R)$ and $H_{\text{red}}(R)$:

Theorem 17 *The map φ yields an isomorphism of $\mathcal{I}_2(R)$ with $H_{\text{red}}(R)$.*

Proof: The preimage of the identity $(R, R, 1) \in H(R)$ can only contain 2-torsion ideals of the form $\kappa \cdot R$ where $\kappa \in (R \otimes \mathbb{Q})^{\times}$. To be a 2-torsion ideal, we must have $(\kappa R)^2 = R$; thus $\kappa^2 \in R^{\times}$ and so $\kappa \in R^{\times}$. Therefore, the preimage of the identity is simply the ideal R and the map is injective. To show surjectivity, let $(R, I, \delta) \in H_{\text{red}}(R)$. Since δ is a square by definition, let $\delta = \xi^2$ and recall that $(R, I, \delta) \sim (R, \xi^{-1}I, 1)$; thus $\xi^{-1}I \in \mathcal{I}_2(R)$. \square

Corollary 18 *Assume that R is maximal. Then $H_{\text{red}}(R)$ contains only the identity element of $H(R)$, which can be represented by $(R, R, 1)$.*

Proof: Since maximal orders are Dedekind domains, the only ideal that is 2-torsion in the ideal group of R is R . \square

5 Class numbers of pairs of ternary quadratic forms

To prove Theorem 1, we would like to restrict the elements of $V_{\mathbb{Z}}$ under consideration to those that are “irreducible” in an appropriate sense. More precisely, we call a pair (A, B) of integral ternary quadratic forms in $V_{\mathbb{Z}}$ *absolutely irreducible* if

- (i) A and B do not possess a common \mathbb{Q} -rational zero as conics in \mathbb{P}^2 ; and
- (ii) the binary cubic form $f(x, y) = \text{Det}(Ax - By)$ is irreducible over \mathbb{Q} .

Next, we note that a pair $(A, B) \in V_{\mathbb{R}}$ of real ternary quadratic forms gives a pair of conics in $\mathbb{P}^2(\mathbb{C})$ which, at least in nondegenerate cases, intersect in four points in $\mathbb{P}^2(\mathbb{C})$. We call these four points the *zeroes* of (A, B) . The action of $G_{\mathbb{R}}$ on $V_{\mathbb{R}}$ is seen to have three distinct orbits of nondegenerate elements, namely the orbits $V^{(i)} \subset V_{\mathbb{R}}$ for $i = 0, 1, 2$, where $V^{(i)}$ consists of the elements in $V_{\mathbb{R}}$ having $4 - 2i$ real zeroes and i pairs of complex zeroes in $\mathbb{P}^2(\mathbb{C})$.

Now let $S \subset V_{\mathbb{Z}}$ be any $G_{\mathbb{Z}}$ -invariant subset defined by finitely many congruence conditions. For each prime p , let S_p denote the p -adic closure of S in $V_{\mathbb{Z}_p} = V_{\mathbb{Z}} \otimes \mathbb{Z}_p$, and let $M_p(S)$ denote the “ p -adic mass” of S_p in $V_{\mathbb{Z}_p}$, defined by

$$M_p(S) := \sum_{x \in G_{\mathbb{Z}_p} \backslash S_p} \frac{1}{\text{Disc}_p(x)} \cdot \frac{1}{|\text{Stab}_{G_{\mathbb{Z}_p}}(x)|}, \quad (9)$$

where $\text{Disc}_p(x)$ denotes the discriminant of $x \in V_{\mathbb{Z}_p}$ as a power of p , and $\text{Stab}_{G_{\mathbb{Z}_p}}(x)$ denotes the stabilizer of x in $G_{\mathbb{Z}_p}$.

Setting $n_i = 24, 4, 8$ for $i = 0, 1, 2$ respectively, we may now state the following result counting the number of absolutely irreducible elements $(A, B) \in S \cap V_{\mathbb{Z}}^{(i)}$, up to $G_{\mathbb{Z}}$ -equivalence, having absolute discriminant at most X .

Theorem 19 *For any $G_{\mathbb{Z}}$ -invariant subset $S \subset V_{\mathbb{Z}}$ defined by finitely many congruence conditions, let $N^{(i)}(S; X)$ denote the number of $G_{\mathbb{Z}}$ -equivalence classes of absolutely irreducible elements $(A, B) \in S \cap V_{\mathbb{Z}}^{(i)}$ satisfying $|\text{Disc}(A, B)| < X$. Then*

$$\lim_{X \rightarrow \infty} \frac{N^{(i)}(S; X)}{X} = \frac{1}{2n_i} \cdot \prod_p \left(\frac{p-1}{p} \cdot M_p(S) \right). \quad (10)$$

Proof: Let $\mu_p(S)$ denote the p -adic density of the p -adic closure S_p of S in $V_{\mathbb{Z}_p}$, where we normalize the additive measure μ on $V_{\mathbb{Z}_p}$ so that $\mu(V_{\mathbb{Z}_p}) = 1$. Then [3, Eqn. (32)] implies⁶

$$\lim_{X \rightarrow \infty} \frac{N^{(i)}(S; X)}{X} = \frac{2 \cdot \zeta(2)^2 \cdot \zeta(3)}{n_i} \prod_p \mu_p(S). \quad (11)$$

Theorem 19 is thus reduced to re-expressing the p -adic density $\mu_p(S)$ in terms of the p -adic mass $M_p(S)$. This is accomplished by the following lemma:

⁶Note that in [3], $V_{\mathbb{Z}}$ is defined as the set of all *integer-coefficient* pairs of ternary quadratic forms (see [3, Eqn. (3)]). The lattice of all *integer-coefficient* pairs of ternary quadratic forms contains the lattice of all integer-matrix pairs of ternary quadratic forms as an index 64 sublattice. However, the notion of discriminant in [3] also differs from the one used in this paper; namely, an integer-matrix pair $(A, B) \in V_{\mathbb{Z}}$ has discriminant equal to $\text{Disc}(\text{Det}(Ax - By))$ in the current paper, but has discriminant equal to $256 \cdot \text{Disc}(\text{Det}(Ax - By))$ according to [3, §2]. This implies that the constant in Equation (11) changes from $\frac{\zeta(2)^2 \zeta(3)}{2n_i}$ as in [3] to $\frac{2 \cdot \zeta(2)^2 \cdot \zeta(3)}{n_i}$ here.

Lemma 20 *We have*

$$\mu_p(S) = |4|_p \cdot \frac{\#G_{\mathbb{F}_p}}{p^{12}} \cdot M_p(S).$$

Proof: We normalize the Haar measure dg on the p -adic group $G_{\mathbb{Z}_p}$ so that $\int_{g \in G_{\mathbb{Z}_p}} dg = \#G_{\mathbb{F}_p} \cdot p^{-12}$. Since $|\text{Disc}(x)|_p^{-1} \cdot dx$ is a $G_{\mathbb{Q}_p}$ -invariant measure on $V_{\mathbb{Q}_p}$, we must have for any $v_0 \in V_{\mathbb{Z}_p}$ of nonzero discriminant that

$$\int_{x \in G_{\mathbb{Z}_p} \cdot v_0} dx = c \cdot \int_{g \in G_{\mathbb{Z}_p} / \text{Stab}_{G_{\mathbb{Z}_p}}(v_0)} |\text{Disc}(gv_0)|_p \cdot dg = c \cdot \frac{\#G_{\mathbb{F}_p} \cdot p^{-12}}{\text{Disc}_p(v_0) \cdot \#\text{Stab}_{G_{\mathbb{Z}_p}}(v_0)},$$

for some constant c . The constant $c = c(v_0)$ can be determined by calculating the Jacobian of the change of variable $g \mapsto g \cdot v_0$ from $G_{\mathbb{Z}_p}$ to $V_{\mathbb{Z}_p}$. Now this is an algebraic calculation, involving polynomials in the coordinates on $G_{\mathbb{Z}_p}$ and $V_{\mathbb{Z}_p}$. Since any two choices of v_0 are G_K -equivalent for some finite extension K of \mathbb{Q}_p (since $V_{\mathbb{Z}_p}$ is a prehomogeneous vector space [16], there is one open orbit over the algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p), and since the Haar measure dg naturally extends to a Haar measure on G_K , we conclude that our constant c cannot depend on v_0 .

It thus suffices to compute the constant c for $v_0 = \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$, which corresponds to the triple $(\mathbb{Z}_p^3, \mathbb{Z}_p^3, 1)$. When $p \neq 2$, it can be shown that $G_{\mathbb{Z}_p} \cdot v_0 \subset V_{\mathbb{Z}_p}$ is defined simply by conditions modulo p . Indeed, the only cubic ring over \mathbb{Z}_p whose reduction modulo p is isomorphic to \mathbb{F}_p^3 is \mathbb{Z}_p^3 , and the only ideal class of \mathbb{Z}_p^3 having rank 3 over \mathbb{Z}_p is \mathbb{Z}_p^3 . If $\delta \in \mathbb{Z}_p^3$ reduces to $1 = (1, 1, 1)$ modulo p , then δ is in the same squareclass as $1 \in (\mathbb{Z}_p^3)^\times$, and thus the triple $(\mathbb{Z}_p^3, \mathbb{Z}_p^3, \delta)$ is equivalent to $(\mathbb{Z}_p^3, \mathbb{Z}_p^3, 1)$ for odd primes p . When $p = 2$, if $\delta \in \mathbb{Z}_2^3$ reduces to $1 = (1, 1, 1)$ modulo p , then there are 16 choices for the squareclass of $\delta \in \mathbb{Z}_2^3$. The group $\text{Aut}(\mathbb{Z}_2^3) \cong S_3$ acts on these 16 squareclasses, and the squareclass of $\delta = (1, 1, 1)$ lies in a single orbit. Furthermore, $\text{Aut}(\mathbb{Z}_2^3; \mathbb{Z}_2^3, \delta)$, for δ in any one of these 16 squareclasses, is simply the stabilizer of that squareclass under this action of S_3 . It follows by Corollary 13 and the orbit-stabilizer theorem that the 2-adic mass of $G_{\mathbb{Z}_2} \cdot v_0$ —i.e., the 2-adic mass of the $G_{\mathbb{Z}_2}$ -orbit of all $(A, B) \in V_{\mathbb{Z}_2}$ corresponding to the triple $(\mathbb{Z}_2^3, \mathbb{Z}_2^3, 1)$ —is exactly $1/16$ of the 2-adic mass of the set of all $(A, B) \in V_{\mathbb{Z}_2}$ whose reduction modulo 2 corresponds to the triple $(\mathbb{F}_2^3, \mathbb{F}_2^3, 1)$.

Now the cardinality of $V_{\mathbb{F}_p}$ is p^{12} . Therefore, if $p \neq 2$, then we see that the measure of $G_{\mathbb{Z}_p} \cdot v_0$ in $V_{\mathbb{Z}_p}$ is equal to p^{-12} times the cardinality of $G_{\mathbb{F}_p} \cdot \bar{v}_0$ in $V_{\mathbb{F}_p}$, where \bar{v}_0 denotes the reduction of v_0 modulo p ; if $p = 2$, then the measure of $G_{\mathbb{Z}_p} \cdot v_0$ is equal to p^{-12} times the cardinality of $G_{\mathbb{F}_p} \cdot \bar{v}_0$ in $V_{\mathbb{F}_p}$ times $1/16$. Since for our choice of v_0 , we have $U_2^+(\mathbb{Z}_p^3) = 4 = U_2^+(\mathbb{F}_p^3)$ for odd primes p , and $U_2^+(\mathbb{Z}_2^3) = 4 = 4 \cdot U_2^+(\mathbb{F}_2^3)$, we conclude using Corollary 13 that $\#\text{Stab}_{G_{\mathbb{Z}_p}}(v_0) = \#\text{Stab}_{G_{\mathbb{F}_p}}(\bar{v}_0) = 24$ when $p \neq 2$, and $\#\text{Stab}_{G_{\mathbb{Z}_2}}(v_0) = 4 \cdot \#\text{Stab}_{G_{\mathbb{F}_2}}(\bar{v}_0) = 24$. Thus, by the orbit-stabilizer theorem, we obtain

$$\int_{x \in G_{\mathbb{Z}_p} \cdot v_0} dx = |16|_p \cdot p^{-12} \cdot \#(G_{\mathbb{F}_p} \cdot \bar{v}_0) = |16|_p \cdot \frac{\#G_{\mathbb{F}_p} \cdot p^{-12}}{\#\text{Stab}_{G_{\mathbb{F}_p}}(\bar{v}_0)} = |4|_p \cdot \frac{\#G_{\mathbb{F}_p} \cdot p^{-12}}{\text{Disc}_p(v_0) \cdot \#\text{Stab}_{G_{\mathbb{Z}_p}}(v_0)},$$

since $\text{Disc}_p(v_0) = 1$ for our choice of v_0 . Thus we must have $c = |4|_p$ for any choice of v_0 satisfying $\text{Disc}(v_0) \neq 0$, and the lemma follows. \square

Returning to the proof of Theorem 19, we observe that $\#G_{\mathbb{F}_p} = (p^2 - 1)(p^2 - p) \cdot (p^3 - 1)(p^3 - p)(p^3 - p^2)/(p - 1)$, and so by Lemma 20, we have

$$\frac{2 \cdot \zeta(2)^2 \cdot \zeta(3)}{n_i} \cdot \prod_p \mu_p(S) = \frac{\zeta(2)^2 \zeta(3)}{2n_i} \cdot \prod_p \left(1 - \frac{1}{p^2}\right) \cdot \left(1 - \frac{1}{p^3}\right) \cdot \left(1 - \frac{1}{p^2}\right) \cdot \left(\frac{p-1}{p} \cdot M_p(S)\right),$$

yielding Theorem 19. \square

To prove Theorems 1 and 2, we choose appropriate sets S on which to apply Theorem 19. This is carried out in the next section.

6 Proof of Theorems 1 and 2

Our goal is to count the $G_{\mathbb{Z}}$ -orbits of pairs of forms in $V_{\mathbb{Z}}^{(i)}$ of bounded absolute discriminant that correspond, under the bijection described in Theorem 11, to equivalence classes of triples (R, I, δ) where R lies in some acceptable family Σ of cubic orders and I is projective. More precisely, for each prime p , let Σ_p be a set of isomorphism classes of nondegenerate cubic rings over \mathbb{Z}_p . We denote the collection (Σ_p) of these local specifications over all p by Σ . As in the introduction, we say that the collection $\Sigma = (\Sigma_p)$ is *acceptable* if, for all sufficiently large p , the set Σ_p contains all maximal cubic rings over \mathbb{Z}_p that are not totally ramified. Finally, for a cubic order R over \mathbb{Z} , we write “ $R \in \Sigma$ ” (or say that “ R is a Σ -order”) if $R \otimes \mathbb{Z}_p \in \Sigma_p$ for all p .

Now fix any acceptable collection $\Sigma = (\Sigma_p)$ of local specifications. Let $S = S(\Sigma)$ denote the set of all absolutely irreducible elements $(A, B) \in V_{\mathbb{Z}}$ such that, in the corresponding triple (R, I, δ) , we have that $R \in \Sigma$ and I is invertible as an ideal class of R (implying that $I \otimes \mathbb{Z}_p$ is the trivial ideal class of $R \otimes \mathbb{Z}_p$). Then we wish to count $N^{(i)}(S(\Sigma); X)$, the number of $G_{\mathbb{Z}}$ -equivalence classes of absolutely irreducible elements $(A, B) \in S(\Sigma) \cap V_{\mathbb{Z}}^{(i)}$ satisfying $|\text{Disc}(A, B)| < X$.

Although S might be defined by infinitely many congruence conditions, the estimate provided in [3, Prop. 23] and [4, Lem. 3.7] (and the fact that Σ is acceptable) implies that Equation (10) continues to hold for the set S (this follows from an identical argument as in [3, §3.3] or [4, §3.4]). More precisely,

$$\lim_{X \rightarrow \infty} \frac{N^{(i)}(S(\Sigma); X)}{X} = \frac{1}{2n_i} \cdot \prod_p \left(\frac{p-1}{p} \cdot M_p(S(\Sigma)) \right). \quad (12)$$

We also have the following algebraic lemma which describes when a pair (A, B) lies in $V_{\mathbb{Z}}^{(i)}$ ($i \in \{0, 1, 2\}$) in terms of the associated triple (R, I, δ) :

Lemma 21 *Let (A, B) be an element of $V_{\mathbb{Z}}^{(i)}$ whose binary cubic form invariant f is irreducible over \mathbb{Q} , and let (R, I, δ) be the corresponding triple as given by Theorem 11.*

- (a) *If $i = 0$, then R is an order in a totally real cubic field, and δ is a totally positive element of $R \otimes \mathbb{Q}$.*
- (b) *If $i = 1$, then R is an order in a complex cubic field.*
- (c) *If $i = 2$, then R is an order in a totally real cubic field, and $\delta \in R \otimes \mathbb{Q}$ is not a totally positive element.*

Proof: If the binary cubic form invariant f of (A, B) is irreducible over \mathbb{Q} , then the cubic ring $R = R(f)$ is a domain [10] (see also [5, Prop. 11]), and thus an order in a cubic field. To check the assertions about δ , it suffices to base change to the real numbers \mathbb{R} , where one can simply check the assertion at one point of each of the three orbits $V^{(i)}$ for $i = 0, 1$, and 2 . \square

Combining Theorem 17 with Lemmas 14, 15, and 21, we obtain

$$\begin{aligned}
N^{(0)}(S(\Sigma), X) + N^{(2)}(S(\Sigma), X) &= \sum_{\substack{R \in \Sigma, \\ 0 < \text{Disc}(R) < X}} 4 \cdot |\text{Cl}_2(R)| - |\mathcal{I}_2(R)|, \\
N^{(1)}(S(\Sigma), X) &= \sum_{\substack{R \in \Sigma, \\ -X < \text{Disc}(R) < 0}} 2 \cdot |\text{Cl}_2(R)| - |\mathcal{I}_2(R)|, \\
N^{(0)}(S(\Sigma), X) &= \sum_{\substack{R \in \Sigma, \\ 0 < \text{Disc}(R) < X}} |\text{Cl}_2^+(R)| - |\mathcal{I}_2(R)|.
\end{aligned} \tag{13}$$

By Theorem 11 and Corollary 13, the p -adic masses of $S = S(\Sigma)$ defined in (9) can be expressed as

$$M_p(S(\Sigma)) = \sum \frac{1}{\text{Disc}_p(R) \cdot |\text{Aut}(R; I, \delta)| \cdot |U_2^+(R_0)|}, \tag{14}$$

where the sum is over all equivalence classes of triples (R, I, δ) over \mathbb{Z}_p represented in S_p . If $R \in \Sigma_p$ is a nondegenerate cubic ring over \mathbb{Z}_p , then in a corresponding triple (R, I, δ) we can always choose $I = R$, since I is a principal ideal (recall that invertible means locally principal). The number of elements δ yielding distinct valid triples (R, R, δ) over \mathbb{Z}_p (in the sense of Theorem 11), up to equivalence, is equal to the number of $\text{Aut}(R)$ -orbits on the set $U^+(R)/U^+(R)^{\times 2}$, where $U^+(R)$ denotes the group of units of R having norm 1. If $\delta \in U^+(R)$ has image $\bar{\delta} \in U^+(R)/U^+(R)^{\times 2}$, then the stabilizer of $\bar{\delta}$ under this action of $\text{Aut}(R)$ is given by $\text{Aut}(R; R, \delta)$ for any lift δ of $\bar{\delta}$. By the orbit-stabilizer theorem, we then obtain that

$$M_p(S(\Sigma)) = \sum \frac{|U^+(R)/U^+(R)^{\times 2}|}{\text{Disc}_p(R) \cdot |\text{Aut}(R)| \cdot |U_2^+(R)|} \tag{15}$$

where the sum is over all isomorphism classes of cubic rings R over \mathbb{Z}_p lying in Σ_p .

Lemma 22 *Let R be a nondegenerate cubic ring over \mathbb{Z}_p . Then*

$$\frac{|U^+(R)/U^+(R)^{\times 2}|}{|U_2^+(R)|}$$

is 1 if $p \neq 2$, and 4 if $p = 2$.

Proof: The unit group of R , as a multiplicative group, is isomorphic to $F' \times G'$, where F' is a finite abelian group and G' is torsion-free and may naturally be viewed as a free rank 3 \mathbb{Z}_p -module. Hence the submodule $U^+(R)$, consisting of those units having norm 1, is isomorphic to $F \times G$, where F is a finite abelian group and G is free of rank 2 as a \mathbb{Z}_p -module.

Let F_2 denote the 2-torsion subgroup of F . Since F_2 is the kernel of the multiplication-by-2 map on F , it is clear that $|F/(2 \cdot F)|/|F_2| = 1$. Therefore, it suffices to check the lemma on the “free” part G of $U^+(R)$, namely, on the \mathbb{Z}_p -module \mathbb{Z}_p^2 , where the result is clear. (The case $p = 2$ differs because, while $2 \cdot \mathbb{Z}_p^2 = \mathbb{Z}_p^2$ for $p \neq 2$, the \mathbb{Z}_2 -module $2 \cdot \mathbb{Z}_2^2$ has index 4 in \mathbb{Z}_2^2 .) \square

Combining (12), (15), and Lemma 22, we obtain

$$\lim_{X \rightarrow \infty} \frac{N^{(i)}(S(\Sigma); X)}{X} = \frac{2}{n_i} \cdot \prod_p \left(\frac{p-1}{p} \cdot \sum_{R \in \Sigma_p} \frac{1}{\text{Disc}_p(R)} \cdot \frac{1}{|\text{Aut}(R)|} \right). \tag{16}$$

Let c_Σ denote the Euler product occurring in (16), i.e.,

$$c_\Sigma := \prod_p \left(\frac{p-1}{p} \cdot \sum_{R \in \Sigma_p} \frac{1}{\text{Disc}_p(R)} \cdot \frac{1}{|\text{Aut}(R)|} \right).$$

We now recall the following result from [5, Thm. 8]:

Lemma 23

- (a) *The number of totally real Σ -orders \mathcal{O} with $|\text{Disc}(\mathcal{O})| < X$ is $\frac{1}{12}c_\Sigma \cdot X + o(X)$.*
- (b) *The number of complex Σ -orders \mathcal{O} with $|\text{Disc}(\mathcal{O})| < X$ is $\frac{1}{4}c_\Sigma \cdot X + o(X)$.*

Thus, we may conclude from (13), (16), and Lemma 23 that

$$\begin{aligned} \frac{\sum_{\substack{R \in \Sigma, \\ 0 < \text{Disc}(R) < X}} |\text{Cl}_2(R)| - \frac{1}{4} \cdot |\mathcal{I}_2(R)|}{\sum_{\substack{R \in \Sigma, \\ 0 < \text{Disc}(R) < X}} 1} &= \frac{1}{4} \cdot \frac{\frac{2}{n_0} \cdot c_\Sigma + \frac{2}{n_2} \cdot c_\Sigma}{\frac{1}{12} \cdot c_\Sigma} = 1, \\ \frac{\sum_{\substack{R \in \Sigma, \\ -X < \text{Disc}(R) < 0}} |\text{Cl}_2(R)| - \frac{1}{2} |\mathcal{I}_2(R)|}{\sum_{\substack{R \in \Sigma, \\ -X < \text{Disc}(R) < 0}} 1} &= \frac{1}{2} \cdot \frac{\frac{2}{n_1} \cdot c_\Sigma}{\frac{1}{4} c_\Sigma} = 1, \quad \text{and} \quad (17) \\ \frac{\sum_{\substack{R \in \Sigma, \\ 0 < \text{Disc}(R) < X}} |\text{Cl}_2^+(R)| - |\mathcal{I}_2(R)|}{\sum_{\substack{R \in \Sigma, \\ 0 < \text{Disc}(R) < X}} 1} &= \frac{\frac{2}{n_0} \cdot c_\Sigma}{\frac{1}{12} c_\Sigma} = 1. \end{aligned}$$

This proves Theorem 2. Furthermore, when Σ is an acceptable collection of maximal cubic orders, then Theorem 2 and Corollary 18 imply Theorem 1, since $|\mathcal{I}_2(R)| = 1$ when R is maximal.

7 Proofs of Corollaries

We conclude by proving Corollaries 3, 7, 8, 9, and 10. As remarked in the introduction, we in fact prove a generalization of each of these corollaries. Namely, let Σ be any acceptable collection of local specifications of maximal cubic orders (thus possibly defined by infinitely many local conditions). Then we prove that Corollaries 3 and 7–10 continue to hold, with the same percentages, for the family of maximal orders defined by any such Σ .

We begin by demonstrating that in the family of all real (resp. complex) cubic fields defined by Σ , a proportion of at least 75% (resp. 50%) have odd class number.

Proof of Corollary 3: Indeed, suppose a lower density of less than 75% (resp. 50%) of totally real (resp. complex) Σ -orders have odd class number. Then an upper density of more than $1/4$ (resp. $1/2$) of these Σ -orders R would satisfy $|\text{Cl}_2(R)| \geq 2$. Thus the limsup of the average size of $|\text{Cl}_2(R)|$ would be strictly larger than $1 + (1/4) = 5/4$ (resp. $1 + (1/2) = 3/2$), contradicting Theorem 1(a) (resp. Theorem 1(b)). \square

Next, we show that at least 50% of real cubic fields K in the family of cubic fields defined by Σ satisfy $\text{Cl}_2(K) \neq \text{Cl}_2^+(K)$, and at least 25% of such K satisfy $\text{Cl}_2(K) = \text{Cl}_2^+(K)$.

Proof of Corollary 7: By the theorem of Armitage and Frohlich, for a cubic field K we have either $|\text{Cl}_2^+(K)| = |\text{Cl}_2(K)|$ or $|\text{Cl}_2^+(K)| = 2 \cdot |\text{Cl}_2(K)|$. Suppose that a lower density of strictly less than 50% of the totally real cubic fields K defined by Σ satisfy $|\text{Cl}_2^+(K)| = 2 \cdot |\text{Cl}_2(K)|$. Then, since $|\text{Cl}_2(K)| \geq 1$ for all K , the liminf of the average size of $|\text{Cl}_2^+(K)|$ would be strictly less than the average size of $2 \cdot |\text{Cl}_2(K)| - (1/2)$. However, the average size of $2 \cdot |\text{Cl}_2(K)| - (1/2)$ is 2 by Theorem 1(a), contradicting Theorem 1(c). \square

Proof of Corollary 8: Suppose that an upper density of strictly more than 75% of totally real cubic fields K defined by Σ satisfy $|\text{Cl}_2^+(K)| \geq 2 \cdot |\text{Cl}_2(K)| \geq |\text{Cl}_2(K)| + 1$. Then the limsup of the average size of $|\text{Cl}_2^+(K)|$ would be strictly larger than the average size of $|\text{Cl}_2(K)| + (3/4)$, which is 2. This again contradicts Theorem 1(c). \square

Finally, we prove that at least 50% of real cubic fields K in the family of cubic fields defined by Σ do not possess units of every possible signature; in addition, at least 75% of such cubic fields K possess units of mixed signature.

Proof of Corollary 9: We note that if the class group and narrow class group of a number field K are not isomorphic, then K cannot possess a unit of every possible real signature. Corollary 7 now implies the result. \square

Proof of Corollary 10: By Corollary 3, a lower density of at least 75% of cubic fields K have odd class number. For any such cubic field K , the 2-Sylow subgroup of $\text{Cl}^+(K)/\text{Cl}(K)$ will be either trivial or a cyclic group of order 2, by Armitage and Frohlich's theorem. Thus any such cubic field K will contain a unit of mixed signature. \square

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