# Impact of User Pairing on 5G Non-Orthogonal Multiple Access 

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#### Abstract

Non-orthogonal multiple access (NOMA) represents a paradigm shift from conventional orthogonal multiple access (MA) concepts, and has been recognized as one of the key enabling technologies for 5 G systems. In this paper, the impact of user pairing on the performance of two NOMA systems, NOMA with fixed power allocation (F-NOMA) and cognitive radio inspired NOMA (CR-NOMA), is characterized. For FNOMA, both analytical and numerical results are provided to demonstrate that F-NOMA can offer a larger sum rate than orthogonal MA, and the performance gain of F-NOMA over conventional MA can be further enlarged by selecting users whose channel conditions are more distinctive. For CR-NOMA, the quality of service ( QoS ) for users with the poorer channel condition can be guaranteed since the transmit power allocated to other users is constrained following the concept of cognitive radio networks. Because of this constraint, CR-NOMA has different behavior compared to F-NOMA. For example, for the user with the best channel condition, CR-NOMA prefers to pair it with the user with the second best channel condition, whereas the user with the worst channel condition is preferred by F-NOMA.


## I. Introduction

Multiple access in 5 G mobile networks is an emerging research topic, since it is key for the next generation network to keep pace with the exponential growth of mobile data and multimedia traffic [1] and [2]. Non-orthogonal multiple access (NOMA) has recently received considerable attention as a promising candidate for 5 G multiple access [3]-6]. Particularly, NOMA uses the power domain for multiple access, where different users are served at different power levels. The users with better channel conditions employ successive interference cancellation (SIC) to remove the messages intended for other users before decoding their own [7]. The benefit of using NOMA can be illustrated by the following example. Consider that there is a user close to the edge of its cell, denoted by $A$, whose channel condition is very poor. For conventional MA, an orthogonal bandwidth channel, e.g., a time slot, will be allocated to this user, and the other users cannot use this time slot. The key idea of NOMA is to squeeze another user with better channel condition, denoted by $B$, into this time slot. Since $A$ 's channel condition is very poor, the interference from $B$ will not cause much performance degradation to $A$, but the overall system throughput can be significantly improved since additional information can be delivered between the base station (BS) and $B$. The design of NOMA for uplink transmissions has been proposed in [4], and the performance of NOMA with randomly deployed mobile stations has been characterized in [5]. The combination of cooperative diversity with NOMA has been considered in [8].

[^0]Since multiple users are admitted at the same time, frequency and spreading code, co-channel interference will be strong in NOMA systems, i.e., a NOMA system is interference limited. As a result, it may not be realistic to ask all the users in the system to perform NOMA jointly. A promising alternative is to build a hybrid MA system, in which NOMA is combined with conventional MA. In particular, the users in the system can be divided into multiple groups, where NOMA is implemented within each group and different groups are allocated with orthogonal bandwidth resources. Obviously the performance of this hybrid MA scheme is very dependent on which users are grouped together, and the aim of this paper is to investigate the effect of this grouping. Particularly, tn this paper, we focus on a downlink communication scenario with one BS and multiple users, where the users are ordered according to their connections to the BS, i.e., the $m$-th user has the $m$-th worst connection to the BS. Consider that two users, the $m$-th user and the $n$-th user, are selected for performing NOMA jointly, where $m<n$. The impact of user pairing on the performance of NOMA will be characterized in this paper, where two types of NOMA will be considered. One is based on fixed power allocation, termed F-NOMA, and the other is cognitive radio inspired NOMA, termed CR-NOMA.

For the F-NOMA scheme, the probability that F-NOMA can achieve a larger sum rate than conventional MA is first studied, where an exact expression for this probability as well as its high signal-to-noise ratio (SNR) approximation are obtained. These developed analytical results demonstrate that it is almost certain for F-NOMA to outperform conventional MA, and the channel quality of the $n$-th user is critical to this probability. In addition, the gap between the sum rates achieved by FNOMA and conventional MA is also studied, and it is shown that this gap is determined by how different the two users' channel conditions are, as initially reported in [8]. For example, if $n=M$, it is preferable to choose $m=1$, i.e., pairing the user with the best channel condition with the user with the worst channel condition. The reason for this phenomenon can be explained as follows. When $m$ is small, the $m$-th user's channel condition is poor, and the data rate supported by this user's channel is also small. Therefore the spectral efficiency of conventional MA is low, since the bandwidth allocated to this user cannot be accessed by other users. The use of F-NOMA ensures that the $n$-th user will have access to the resource allocated to the $m$-th user. If $(n-m)$ is small, the $n$-th user's channel quality is similar to the $m$-th user's, and the benefit to use NOMA is limited. But if $n \gg m$, the $n$-th user can use the bandwidth resource much more efficiently than the $m$-th user, i.e., a larger $(n-m)$ will result in a larger performance gap between F-NOMA and conventional MA.

The key idea of CR-NOMA is to opportunistically serve the $n$-th user on the condition that the $m$-th user's quality of service ( QoS ) is guaranteed. Particularly the transmit power
allocated to the $n$-th user is constrained by the $m$-th user's signal-to-interference-noise ratio (SINR), whereas F-NOMA uses a fixed set of power allocation coefficients. Since the $m$-th user's QoS can be guaranteed, we mainly focus on the performance of the $n$-th user offered by CR-NOMA. An exact expression for the outage probability achieved by CRNOMA is obtained first, and then used for the study of the diversity order. In particular, we show that the diversity order experienced by the $n$-th user is $m$, which means that the $m$-th user's channel quality is critical to the performance of CR-NOMA. This is mainly because of the imposed SINR constraint, where the $n$-th user can be admitted into the bandwidth channel occupied by the $m$-th user, only if the $m$ th user's SINR is guaranteed. As a result, with a fixed $m$, increasing $n$ does not bring much improvement to the $n$-th user's outage probability, which is different from F-NOMA. If the ergodic rate is used as the criterion, a similar difference between F-NOMA and CR-NOMA can be observed. Again take the scenario described in the last paragraph as an example. If $n=M$, in order to yield a large gain over conventional MA, F-NOMA prefers the choice of $m=1$, but CR-NOMA prefers the choice of $m=M-1$, i.e., pairing the user with the best channel condition with the user with the second best channel condition.

## II. NOMA With Fixed Power Allocation

Consider a downlink communication scenario with one BS and $M$ mobile users. Without loss of generality, assume that the users' channels have been ordered as $\left|h_{1}\right|^{2} \leq \cdots \leq\left|h_{M}\right|^{2}$, where $h_{m}$ denotes the Rayleigh fading channel gain between the BS and the ordered $m$-th user. Consider that the $m$-th user and the $n$-th user, $m<n$, are paired to perform NOMA.

In this section, we focus on F-NOMA, where the BS allocates a fixed amount of transmit power to each user. In particular, denote $a_{m}$ and $a_{n}$ as the power allocation coefficients for the two users, where these coefficients are fixed and $a_{m}^{2}+a_{n}^{2}=1$. According to the principle of NOMA, $a_{m} \geq a_{n}$ since $\left|h_{m}\right|^{2} \leq\left|h_{n}\right|^{2}$. The rates achievable to the two users are given by

$$
\begin{equation*}
R_{m}=\log \left(1+\frac{\left|h_{m}\right|^{2} a_{m}^{2}}{\left|h_{m}\right|^{2} a_{n}^{2}+\frac{1}{\rho}}\right) \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{n}=\log \left(1+\rho a_{n}^{2}\left|h_{n}\right|^{2}\right) \tag{2}
\end{equation*}
$$

respectively, where $\rho$ denotes the transmit SNR. Note that the $n$-th user can decode the message intended for the $m$-th user successfully and $R_{n}$ is always achievable at the $n$-th user, since $R_{m} \leq \log \left(1+\frac{\left|h_{n}\right|^{2} a_{m}^{2}}{\left|h_{n}\right|^{2} a_{n}^{2}+\frac{1}{\rho}}\right)$.

On the other hand, an orthogonal MA scheme, such as timedivision multiple-access (TDMA), can support the following data rate:

$$
\begin{equation*}
\bar{R}_{i}=\frac{1}{2} \log \left(1+\rho\left|h_{i}\right|^{2}\right), \tag{3}
\end{equation*}
$$

where $i \in\{m, n\}$. In the following subsections, the impact of user pairing on the sum rate and the individual user rates achieved by F-NOMA is investigated.

## A. Impact of user pairing on the sum rate

In this subsection, we focus on how user pairing affects the probability that NOMA achieves a lower sum rate than conventional MA schemes, which is given by

$$
\begin{equation*}
\mathrm{P}\left(R_{m}+R_{n}<\bar{R}_{m}+\bar{R}_{n}\right) . \tag{4}
\end{equation*}
$$

The following theorem provides an exact expression for the above probability as well as its high SNR approximation.

Theorem 1. Suppose that the $m$-th and $n$-th ordered users are paired to perform NOMA. The probability that F-NOMA achieves a lower sum rate than conventional MA is given by

$$
\begin{align*}
& \mathrm{P}\left(R_{m}+R_{n}<\bar{R}_{m}+\bar{R}_{n}\right)=  \tag{5}\\
& 1-\sum_{i=0}^{n-1-m}\binom{n-1-m}{i} \frac{(-1)^{i} \varpi_{1}}{m+i} \int_{\varpi_{4}}^{\varpi_{2}} f(y)(F(y))^{n-1-m-i} \\
& \times(1-F(y))^{M-n}\left([F(y)]^{m+i}-\left[F\left(\frac{\varpi_{2}-y}{1+y}\right)\right]^{m+i}\right) d y \\
& -\frac{\varpi_{3}}{\rho} \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j} \frac{\rho}{M-n+j+1} e^{-\frac{(M-n+j+1) \varpi_{2}}{\rho}}, \\
& \text { where } f(x)=\frac{1}{\rho} e^{-\frac{x}{\rho}}, \quad F(x)=1-e^{-\frac{x}{\rho}}, \quad \varpi_{1}= \\
& \frac{M!}{(m-1)!(n-1-m)!(M-n)!}, \varpi_{2}=\frac{1-2 a_{n}^{2}}{a_{n}^{4}}, \varpi_{3}=\frac{M!}{(n-1)!(M-n)!}
\end{align*}
$$ and $\varpi_{4}=\sqrt{1+\varpi_{2}}-1$. At high $S N R$, this probability can be approximated as follows:

$$
\begin{equation*}
\mathrm{P}\left(R_{m}+R_{n}<\bar{R}_{m}+\bar{R}_{n}\right) \approx \frac{1}{\rho^{n}}\left(\frac{\varpi_{3} \varpi_{2}^{n}}{n}-\varpi_{1} \varpi\right) \tag{6}
\end{equation*}
$$

where $\varpi=\sum_{i=0}^{n-1-m}\binom{n-1-m}{i} \frac{(-1)^{i}}{m+i} \int_{\varpi_{4}}^{\varpi_{2}} y^{n-1-m-i}$ $\times\left(y^{m+i}-\left[\frac{\varpi_{2}-y}{(1+y)}\right]^{m+i}\right) d y$, i.e., $\varpi$ is a constant and not a function of $\rho$.

Proof: See the appendix.
Theorem 1 demonstrates that it is almost certain for F NOMA to outperform conventional MA, particularly at high SNR. Furthermore, the decay rate of the probability $\mathrm{P}\left(R_{m}+\right.$ $\left.R_{n}<\bar{R}_{m}+\bar{R}_{n}\right)$ is approximately $\frac{1}{\rho^{n}}$, i.e., the quality of the $n$ th user's channel determines the decay rate of this probability.

## B. Asymptotic studies of the sum rate achieved by NOMA

In addition to the probability $\mathrm{P}\left(R_{m}+R_{n}<\bar{R}_{m}+\bar{R}_{n}\right)$, it is also of interest to study how large of a performance gain F-NOMA offers over conventional MA, i.e.,

$$
\mathrm{P}\left(R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}<R\right),
$$

where $R$ is a targeted performance gain. The probability studied in the previous subsection can be viewed as a special case by setting $R=0$. An interesting observation for the cases with $R>0$ is that there will be an error floor for $\mathrm{P}\left(R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}<R\right)$, regardless of how large the SNR
is. This can be shown by studying the following asymptotic expression of the sum rate gap:

$$
\begin{align*}
& R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}  \tag{7}\\
\underset{\rho \rightarrow}{\rightarrow} & \log \left(\frac{1}{a_{n}^{2}}\right)+\log \left(\rho a_{n}^{2}\left|h_{n}\right|^{2}\right)-\log \left(\rho\left|h_{m}\right|\left|h_{n}\right|\right) \\
= & \log \left|h_{n}\right|-\log \left|h_{m}\right|,
\end{align*}
$$

which is not a function of SNR. Hence the probability can be expressed asymptotically as follows:

$$
\begin{align*}
& \mathrm{P}\left(R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}<R\right)  \tag{8}\\
& \rho \rightarrow \infty \\
& \mathrm{P}\left(\log \left|h_{n}\right|-\log \left|h_{m}\right|<R\right) .
\end{align*}
$$

When $R=0$, $\mathrm{P}\left(R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}<R\right) \rightarrow 0$, which is consistent with Theorem 1 since

$$
\mathrm{P}\left(R_{m}+R_{n}<\bar{R}_{m}+\bar{R}_{n}\right) \sim \frac{1}{\rho^{n}} \underset{\rho \rightarrow \infty}{\rightarrow} 0
$$

When $R \neq 0$, 8 implies that the probability $\mathrm{P}\left(R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}<R\right)$ can be expressed asymptotically as follows:

$$
\begin{equation*}
\mathrm{P}\left(\log \left|h_{n}\right|-\log \left|h_{m}\right|<R\right) \rightarrow \mathrm{P}\left(\frac{\left|h_{n}\right|^{2}}{\left|h_{m}\right|^{2}}<2^{2 R}\right) \tag{9}
\end{equation*}
$$

Directly applying the joint probability density function (pdf) of the users' channels shown in (31), the probability can be rewritten as follows:

$$
\begin{align*}
& \mathrm{P}\left(\log \left|h_{n}\right|-\log \left|h_{m}\right|<R\right)  \tag{10}\\
& =\int_{0}^{\infty} \int_{2^{-2 R} y}^{y} \varpi_{1} f(x) f(y)[F(x)]^{m-1} \\
& \times(F(y)-F(x))^{n-1-m}(1-F(y))^{M-n} d x d y
\end{align*}
$$

which is quite complicated to evaluate. In [9], a simpler pdf for the ratio of two order statistics has been provided as follows:

$$
\begin{array}{r}
f_{\frac{\left|h_{m}\right|^{2}}{\left|h_{n}\right|^{2}}}(z)=\frac{M!}{(m-1)!(n-m-1)!(M-n)!} \sum_{j_{1}=0}^{m-1} \sum_{j_{2}=0}^{n-m-1} \\
(-1)^{j_{1}+j_{2}}\binom{m-1}{j_{1}}\binom{n-m-1}{j_{2}}\left(\tau_{2}+\tau_{1} z\right)^{-2}
\end{array}
$$

where $\tau_{1}=j_{1}-j_{2}+n-m$ and $\tau_{2}=M-n+1+j_{2}$. By using this pdf, the addressed probability can be calculated as follows:
$\mathrm{P}\left(\log \left|h_{n}\right|-\log \left|h_{m}\right|<R\right)$

$$
\begin{align*}
& \rightarrow \frac{M!}{(m-1)!(n-m-1)!(M-n)!} \sum_{j_{1}=0}^{m-1} \sum_{j_{2}=0}^{n-m-1}  \tag{11}\\
& \frac{(-1)^{j_{1}+j_{2}}}{\tau_{1}}\binom{m-1}{j_{1}}\binom{n-m-1}{j_{2}}\left(\frac{1}{\tau_{2}+2^{-2 R} \tau_{1}}-\frac{1}{\tau_{2}+\tau_{1}}\right) .
\end{align*}
$$

## C. Impact of user pairing on individual user rates

Careful user pairing not only improves the sum rate, but also has the potential to improve the individual user rates, as shown in this section. We first focus on the probability that F-NOMA can achieve a larger rate than orthogonal MA for the $m$-th user which is given by

$$
\begin{align*}
& \mathrm{P}\left(R_{m}>\bar{R}_{m}\right)  \tag{12}\\
= & \mathrm{P}\left(\left(1+\frac{\left|h_{m}\right|^{2} a_{m}^{2}}{\left|h_{m}\right|^{2} a_{n}^{2}+\frac{1}{\rho}}\right)^{2}>\left(1+\rho\left|h_{m}\right|^{2}\right)\right)
\end{align*}
$$

After some algebraic manipulations, the above probability can be further rewritten as follows:
$\mathrm{P}\left(R_{m}>\bar{R}_{m}\right)=\mathrm{P}\left(\left|h_{m}\right|^{2}<\frac{1-2 a_{n}^{2}}{\rho a_{n}^{4}}\right)$
$=\int_{0}^{\frac{1-2 a_{n}^{2}}{\rho a_{n}^{4}}} \frac{\varpi_{5}}{\rho} e^{-\frac{(M-m+1) y}{\rho}}\left(1-e^{-\frac{y}{\rho}}\right)^{m-1} d y$
$=\sum_{i=0}^{m-1}\binom{m-1}{i} \frac{(-1)^{i} \varpi_{5}}{M-m+i+1}\left(1-e^{-\frac{\left(1-2 a_{n}^{2}\right)(M-m+i+1)}{\rho a_{n}^{4}}}\right)$,
where $\varpi_{5}=\frac{M!}{(m-1)!(M-m)!}$.
By applying a series expansion, the above probability can be rewritten as follows:

$$
\begin{align*}
\mathrm{P}\left(R_{m}>\bar{R}_{m}\right) & =\sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{i+1} \varpi_{5}  \tag{14}\\
& \times \sum_{k=1}^{\infty}(-1)^{k} \frac{\left(1-2 a_{n}^{2}\right)^{k}(M-m+i+1)^{k-1}}{k!\rho^{k} a_{n}^{4 k}}
\end{align*}
$$

Again applying the results in (41) and (42), the above equation can be approximated as follows:

$$
\begin{equation*}
\mathrm{P}\left(R_{m}>\bar{R}_{m}\right) \approx \varpi_{5} \frac{\left(1-2 a_{n}^{2}\right)^{m}}{m \rho^{m} a_{n}^{4 m}} \tag{15}
\end{equation*}
$$

which means that $\mathrm{P}\left(R_{m}>\bar{R}_{m}\right)$ decays at a rate of $\frac{1}{\rho^{m}}$.
On the other hand, the probability that the $n$-th user can experience better performance in a NOMA system than in orthogonal MA systems is given by
$\mathrm{P}\left(R_{n}>\bar{R}_{n}\right)=\mathrm{P}\left(\log \left(1+\rho a_{n}^{2}\left|h_{n}\right|^{2}\right)>\frac{1}{2} \log \left(1+\rho\left|h_{n}\right|^{2}\right)\right.$.
Following similar steps as previously, we obtain the following:

$$
\begin{equation*}
\mathrm{P}\left(R_{n}>\bar{R}_{n}\right)=\mathrm{P}\left(\left|h_{n}\right|^{2}>\frac{1-2 a_{n}^{2}}{\rho a_{n}^{4}}\right) \tag{16}
\end{equation*}
$$

Interestingly $\mathrm{P}\left(R_{n}>\bar{R}_{n}\right)$ in 16) is very much similar to $\mathrm{P}\left(R_{m}>\bar{R}_{m}\right)$ in 13), which yields the following:

$$
\begin{align*}
& \mathrm{P}\left(R_{n}>\bar{R}_{n}\right)=1-\sum_{i=0}^{n-1}\binom{n-1}{i} \frac{(-1)^{i} \varpi_{3}}{M-n+i+1}  \tag{17}\\
& \times\left(1-e^{-\frac{\left(1-2 a_{n}^{2}\right)(M-n+i+1)}{\rho a_{n}^{4}}}\right)
\end{align*}
$$

and its high SNR approximation is given by

$$
\begin{equation*}
\mathrm{P}\left(R_{n}>\bar{R}_{n}\right) \approx 1-\varpi_{3} \frac{\left(1-2 a_{n}^{2}\right)^{n}}{n \rho^{n} a_{n}^{4 n}} \tag{18}
\end{equation*}
$$

As can be seen from (15) and (18), the two users will have totally different experience in NOMA systems. Particularly, a user with a better channel condition is more willing to perform NOMA since $\mathrm{P}\left(R_{n}>\bar{R}_{n}\right) \rightarrow 1$, which is not true for a user with a poor channel condition. Furthermore, it is preferable to pair two users whose channel conditions are significantly distinct, since (15) and (18) implies that $m$ should be as small as possible and $n$ should be as large as possible.

## III. Cognitive Radio Inspired NOMA

NOMA can be also viewed as a special case of cognitive radio systems [10] and [11], in which a user with a strong channel condition, viewed as a secondary user, is squeezed into the spectrum occupied by a user with a poor channel condition, viewed as a primary user. Following the concept of cognitive radio networks, a variation of NOMA, termed as CR-NOMA, can be designed as follows. Suppose that the BS needs to serve the $m$-th user, i.e., a user a with poor channel condition, due to either the high priority of this user's messages or user fairness, e.g., this user has not been served for a long time. This user can be viewed as a primary user in a cognitive radio system. The $n$-th user can be admitted into this channel on the condition that the $n$-th user will not cause too much performance degradation to the $m$-th user.

Consider that the targeted SINR at the $m$-th user is $I$, which means that the choices of the power allocation coefficients, $a_{m}$ and $a_{n}$, need to satisfy the following constraint:

$$
\begin{equation*}
\frac{\left|h_{m}\right|^{2} a_{m}^{2}}{\left|h_{m}\right|^{2} a_{n}^{2}+\frac{1}{\rho}} \geq I \tag{20}
\end{equation*}
$$

This means that the maximal transmit power that can be allocated to the $n$-th user is given by

$$
\begin{equation*}
a_{n}^{2}=\max \left\{0, \frac{\left|h_{m}\right|^{2}-\frac{I}{\rho}}{\left|h_{m}\right|^{2}(1+I)}\right\} \tag{21}
\end{equation*}
$$

which means that $a_{n}=0$ if $\left|h_{m}\right|^{2}<\frac{I}{\rho}$. Note that the choice of $a_{n}$ in (21) is a function of the channel coefficient $h_{m}$, unlike the constant choice of $a_{n}$ used by F-NOMA in the previous section.

Since the $m$-th user's QoS can be guaranteed due to 20, we only need to study the performance experienced by the $n$ th user. Particularly the outage performance of the $n$-th user is defined as follows:

$$
\begin{equation*}
\mathrm{P}_{o}^{n} \triangleq \mathrm{P}\left(\log \left(1+a_{n}^{2} \rho\left|h_{n}\right|^{2}\right)<R\right) \tag{22}
\end{equation*}
$$

and the following theorem provides an exact expression for the above outage probability as well as its approximation.
Theorem 2. Suppose that the transmit power allocated to the $n$-th user can satisfy the predetermined SINR threshold, I, as shown in 21). The $n$-th user's outage probability achieved by CR-NOMA is given by (19), where $g(y)=e^{-y}, G(y)=$
$1-e^{-y}, \epsilon_{1}=\frac{2^{R}-1}{\rho}, b=\frac{I}{\rho}, a=1+I$ and $b \leq a \epsilon_{1}$. The diversity order achieved by $\stackrel{\rho}{C R}-N O M A$ is given by

$$
\lim _{\rho \rightarrow \infty}-\frac{\log P_{o}^{n}}{\log \rho}=m
$$

Proof: See the appendix.
Theorem 2 demonstrates an interesting phenomenon that, in CR-NOMA, the diversity order experienced by the $n$-th user is determined by how good the $m$-th user's channel quality is. This is because the $n$-th user can be admitted to the channel occupied by the $m$-th user only if the $m$-th user's QoS is met. For example, if the $m$-th user's channel is poor and its targeted SINR is high, it is very likely that the BS allocates all the power to the $m$-th user, and the $n$-th user might not even get served.

Recall from the previous section that F-NOMA can achieve a diversity gain of $n$ for the $n$-th user, and therefore the diversity order achieved by CR-NOMA could be much smaller than that achieved by F-NOMA, particularly if $n \gg m$. This performance difference is again due to the imposed power constraint shown in 21.

It is important to point out that CR-NOMA can strictly guarantee the $m$-th user's QoS, and therefore achieve better fairness compared to F-NOMA. In particular, the use of CR-NOMA can ensure that a diversity order of $m$ is achievable to the $n$-th user, and admitting the $n$-th user into the same channel as the $m$-th user will not cause too much performance degradation to the $m$-th user. Particularly the SINR experienced by the $m$-th user is strictly maintained at the predetermined level $I$.

## Sum rate achieved by CR-NOMA

Without sharing the spectrum with the $n$-th user, i.e, all the bandwidth resource is allocated to the $m$-th user, the following rate is achievable:

$$
\begin{equation*}
\tilde{R}_{m}=\log \left(1+\rho\left|h_{m}\right|^{2}\right) \tag{23}
\end{equation*}
$$

It is easy to show that the use of CR-NOMA always achieves a larger sum rate since

$$
\begin{align*}
& R_{m}+R_{n}-\tilde{R}_{m}  \tag{24}\\
= & \log \left(1+\frac{\left|h_{m}\right|^{2} a_{m}^{2}}{\left|h_{m}\right|^{2} a_{n}^{2}+\frac{1}{\rho}}\right)+\log \left(1+\rho a_{n}^{2}\left|h_{n}\right|^{2}\right) \\
& -\log \left(1+\rho\left|h_{m}\right|^{2}\right) \\
= & \log \frac{1+\rho a_{n}^{2}\left|h_{n}\right|^{2}}{1+\rho a_{n}^{2}\left|h_{m}\right|^{2}} \geq 0
\end{align*}
$$

This superior performance gain is not surprising, since the key idea of CR-NOMA is to serve a user with a strong channel condition, without causing too much performance degradation to the user with a poor channel condition.

In addition, it is of interest to study how much the averaged rate gain CR-NOMA can yield, i.e., $\mathcal{E}\left\{R_{n}\right\}$. This averaged rate gain can be calculated as follows:

$$
\begin{aligned}
\mathcal{E}\left\{R_{n}\right\} & =\int_{b}^{\infty} \int_{x}^{\infty} \log \left(1+\frac{x-b}{x a} \rho y\right) \\
& \times f_{\left|h_{m}\right|^{2},\left|h_{n}\right|^{2}}(x, y) d y d x
\end{aligned}
$$

$$
\begin{align*}
& \mathrm{P}_{n}^{o}=\varpi_{5} \sum_{i=0}^{M-n}\binom{M-n}{i}(-1)^{i} \frac{[G(b)]^{m+i}}{m+i}+\sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \int_{b}^{a \epsilon_{1}} g(y)(1-G(y))^{M-n} G(y)^{n-1-m-i} \varpi_{1}  \tag{19}\\
& \times \frac{\left(G(y)^{m+i}-G(b)^{m+i}\right)}{m+i} d y+\sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \int_{a \epsilon_{1}}^{b+a \epsilon_{1}}(1-G(y))^{M-n} G(y)^{n-1-m-i} \frac{\left(G(y)^{m+i}-G(b)^{m+i}\right)}{m+i} \\
& \left.\times \varpi_{1} g(y) d y+\sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \int_{b+a \epsilon_{1}}^{\infty} g(y)(1-G(y))^{M-n} G(y)^{n-1-m-i} \varpi_{1} \frac{\left(G\left(\frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}\right)^{m+i}-G(b)^{m+i}\right.}{m+i}\right)
\end{align*}
$$

In general, the evaluation of the above equation is difficult, and in the following we provide a case study when $n-m=1$. Particularly, the joint pdf of the channels for this special case can be simplified and the averaged rate gain can calculated as follows:

$$
\begin{align*}
\mathcal{E}\left\{R_{n}\right\} & =\varpi_{1} \int_{b}^{\infty} f(x)[F(x)]^{m-1} \int_{x}^{\infty} \log \left(1+\frac{x-b}{x a} \rho y\right) \\
& \times f(y)(1-F(y))^{M-n} d y d x  \tag{26}\\
& =\frac{-\varpi_{1}}{M-n+1} \int_{b}^{\infty} f(x)[F(x)]^{m-1} \\
& \times \int_{x}^{\infty} \log \left(1+\frac{x-b}{x a} \rho y\right) d(1-F(y))^{M-n+1} d x
\end{align*}
$$

After some algebraic manipulations, the above equation can be rewritten as follows:

$$
\begin{aligned}
\mathcal{E}\left\{R_{n}\right\} & =\frac{\varpi_{1}}{M-n+1} \int_{b}^{\infty} f(x)[F(x)]^{m-1} \\
& \times\left(\log \left(1+\frac{x-b}{a} \rho\right)(1-F(x))^{M-n+1}\right. \\
& \left.+\frac{1}{\ln 2} \int_{x}^{\infty}(1-F(y))^{M-n+1} \frac{\frac{x-b}{x a} \rho}{1+\frac{x-b}{x a} \rho y} d y\right) d x .
\end{aligned}
$$

Now applying Eq. (3.352.2) in [12], the average rate gain can be expressed as follows:

$$
\begin{aligned}
\mathcal{E}\left\{R_{n}\right\} & =\frac{\varpi_{1}}{M-n+1} \int_{b}^{\infty} f(x)[F(x)]^{m-1} \\
& \times\left(\log \left(1+\frac{x-b}{a} \rho\right)(1-F(x))^{M-n+1}-\frac{e^{\frac{x^{2} a}{\rho(x-b)}}}{\ln 2}\right. \\
& \left.\times \operatorname{Ei}\left(-(M-n+1) x-\frac{(M-n+1) x a}{\rho(x-b)}\right)\right) d x,
\end{aligned}
$$

where $\operatorname{Ei}(\cdot)$ denotes the exponential integral.

## IV. Numerical Studies

In this section, computer simulations are used to evaluate the performance of two NOMA schemes as well as the accuracy of the developed analytical results.

(a) $m=1$

(b) $m=2$

Fig. 1. The probability that F-NOMA realizes a lower sum rate than conventional MA. $M=5$. The analytical results are based on Theorem 1

## A. NOMA with fixed power allocation

In Fig. 11 the probability that F-NOMA realizes a lower sum rate than conventional MA, i.e., $\mathrm{P}\left(R_{m}+R_{n}<\bar{R}_{m}+\right.$ $\left.\bar{R}_{n}\right)$, is shown as a function of SNR. $a_{m}^{2}=\frac{4}{5}$ and $a_{n}^{2}=\frac{1}{5}$. As can be seen from both figures, F-NOMA almost always outperforms conventional MA, particularly at high SNR. The simulation results in Fig. 1 also demonstrate the accuracy of the analytical results provided in Theorem 1 For example, the exact expression of $\mathrm{P}\left(R_{m}+R_{n}<\bar{R}_{m}+\bar{R}_{n}\right)$ shown in Theorem 1 matches perfectly with the simulation results, whereas the developed approximation results become accurate at high SNR.

Another important observation from Fig. 1 is that increasing $n$, i.e., scheduling a user with a better channel condition, will make the probability decrease at a faster rate. This observation is consistent to the high SNR approximation results provided in Theorem 11 which show that the slope of the curve for the probability $\mathrm{P}\left(R_{m}+R_{n}<\bar{R}_{m}+\bar{R}_{n}\right)$ is a function of $n$. In Fig. 2) the probability $\mathrm{P}\left(R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}<R\right)$ is shown with different choices of $R$. Comparing Fig. 11 to Fig. 2] one can observe that $\mathrm{P}\left(R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}<R\right)$ never approaches zero, regardless of how large the SNR is. This observation confirms the analytical results developed in (11) which show that the probability $\mathrm{P}\left(R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}<R\right)$ is no longer a function of SNR, when $\rho \rightarrow 0$. It is interesting to observe that the choice of a smaller $m$ is preferable to reduce $\mathrm{P}\left(R_{m}+R_{n}-\bar{R}_{m}-\bar{R}_{n}<R\right)$, a phenomenon previously reported in [8].


Fig. 2. The probability that the sum rate gap between F-NOMA and conventional MA is larger than $R . M=5$ and $n=M$. The analytical results are based on 11 .

In Fig. 3, two different but related probabilities are shown together. One is $\mathrm{P}\left(R_{m}>\bar{R}_{m}\right)$, i.e., the probability that it is beneficial for the user with a poor channel condition to perform F-NOMA, and the other is $\mathrm{P}\left(R_{n}<\bar{R}_{n}\right)$, i.e., the probability that the user with a strong channel condition prefers conventional MA. In Section (I) C, analytical results
have been developed to show that both $\mathrm{P}\left(R_{m}>\bar{R}_{m}\right)$ and $\mathrm{P}\left(R_{n}<\bar{R}_{n}\right)$ are decreasing with increasing SNR, which is confirmed by the simulation results in Fig. 3. The reason that $\mathrm{P}\left(R_{m}>\bar{R}_{m}\right)$ is reduced at a higher SNR is that the $m$ th user's rate in an F-NOMA system becomes a constant, i.e., $\log \left(1+\frac{\left|h_{m}\right|^{2} a_{m}^{2}}{\left|h_{m}\right|^{2} a_{n}^{2}+\frac{1}{\rho}}\right) \underset{\rho \rightarrow \infty}{\rightarrow} \log \left(1+\frac{a_{m}^{2}}{a_{n}^{2}}\right)$, which is much smaller than $\bar{R}_{m}$, at high SNR. On the other hand, it is more likely for $R_{n}$ to be larger than $\bar{R}_{n}$ since there is a factor of $\frac{1}{2}$ outside of the logarithm of $\bar{R}_{n}$.


Fig. 3. The behavior of individual data rates achieved by F-NOMA, $\mathrm{P}\left(R_{n}<\right.$ $\left.\bar{R}_{n}\right)$ and $\mathrm{P}\left(R_{m}>\bar{R}_{m}\right) . M=5$. The analytical results are based on 14 and 17 .

## B. Cognitive radio inspired NOMA

In Fig. 4 the $n$-th user's outage probability achieved by CR-NOMA is shown as a function of SNR. As can be seen from the figure, the exact expression for the outage probability $\mathrm{P}_{n}^{o} \triangleq \mathrm{P}\left(R_{n}<R\right)$ developed in Theorem 2 matches the simulation results perfectly. Recall from Theorem 2 that the diversity order achievable for the $n$-th user is $m$. Or in other words, the slope of the outage probability is determined by the channel quality of the $m$-th user, which is also confirmed by Fig. 4 For example, when increasing $m$ from 1 to 2 , the outage probability is significantly reduced, and its slope is also increased. To clearly demonstrate the diversity order, we have provided an auxiliary curve in the figure which is proportional to $\frac{1}{\rho^{m}}$. As can be observed in the figure, this auxiliary curve is parallel to the one for $\mathrm{P}\left(R_{n}<R\right)$, which confirms that the diversity order achieved by CR-NOMA is $m$.

Since Theorem 2 states that the diversity order of $\mathrm{P}\left(R_{n}<\right.$ $R$ ) is not a function of $n$, an interesting question is whether a different choice of $n$ matters. Fig. 5 is provided to answer this question. While the use of a larger $n$ does bring some reduction of $\mathrm{P}\left(R_{n}<R\right)$, the performance gain of increasing $n$ is negligible, particularly at high SNR. This is because the


Fig. 4. The outage probability for the $n$-th user achieved by CR-NOMA, when $n=M . M=5, R=1$ bit per channel use (BPCU) and $I=5$. The analytical results are Theorem 2


Fig. 5. The outage probability for the $n$-th user achieved by CR-NOMA. $m=1, M=5$, and $R=1 \mathrm{BPCU}$.
channel quality of the $m$-th user becomes a bottleneck for admitting the $n$-th user into the same channel.

In Fig. 6 the performance of CR-NOMA is evaluated by using the ergodic data rate as the criterion. Due to the use of (21), the $m$-th user's QoS can be satisfied, and therefore we only focus on the $n$-th user's data rate, which is the performance gain of CR-NOMA over conventional MA. Fig. 6 demonstrates that, by fixing $(n-m)$, it is beneficial to select two users with better channel conditions. While Fig. 5 shows that changing $n$ with a fixed $m$ does not affect the outage probability, Fig. 6 demonstrates that user pairing has a significant impact on the ergodic rate. Specifically, when fixing the choice of $m$, pairing it with a user with a better


Fig. 6. The ergodic data rate for the $n$-th user achieved by CR-NOMA. $M=5$ and $I=5$. Analytical results are based on (27.
channel condition can yield a gain of more than 1 bit per channel use (BPCU) at 30dB. Another interesting observation from Fig. 6 is that with a fixed $n$, increasing $m$ will improve the performance of CR-NOMA, which is different from FNOMA. For example, when $n=M$, Fig. 2 shows that the user with the worst channel condition, $m=1$, is the best partner, whereas Fig. 6 shows that the user with the second best channel condition, i.e., $m=M-1$, is the best choice.

## V. Conclusions

In this paper the impact of user pairing on the performance of two NOMA systems, NOMA with fixed power allocation (F-NOMA) and cognitive radio inspired NOMA (CR-NOMA),
has been studied. For F-NOMA, both analytical and numerical results have been provided to demonstrate that F-NOMA can offer a larger sum rate than orthogonal MA, and the performance gain of F-NOMA over conventional MA can be further enlarged by selecting users whose channel conditions are more distinctive. For CR-NOMA, the channel quality of the user with a poor channel condition is critical, since the transmit power allocated to the other user is constrained following the concept of cognitive radio networks. One promising future direction of this paper is that the analytical results can be used as criteria designing distributed approaches for dynamic user pairing/grouping.

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## AppENDIX

Proof for Theorem 1. Observe that the sum rate achieved by NOMA can be expressed as follows:

$$
R_{m}+R_{n}=\log \left(\frac{1+\rho\left|h_{m}\right|^{2}}{\rho\left|h_{m}\right|^{2} a_{n}^{2}+1}\right)\left(1+\rho a_{n}^{2}\left|h_{n}\right|^{2}\right)
$$

On the other hand the sum rate achieved by conventional MA is given by

$$
\begin{equation*}
\bar{R}_{m}+\bar{R}_{n}=\log \left(1+\rho\left|h_{m}\right|^{2}\right)^{\frac{1}{2}}\left(1+\rho\left|h_{n}\right|^{2}\right)^{\frac{1}{2}} \tag{28}
\end{equation*}
$$

Now the addressed probability can be written as follows:

$$
\begin{align*}
& \mathrm{P}\left(R_{m}+R_{n}>\bar{R}_{m}+\bar{R}_{n}\right)  \tag{29}\\
& =\mathrm{P}\left(\left(\frac{1+\rho\left|h_{m}\right|^{2}}{\rho\left|h_{m}\right|^{2} a_{n}^{2}+1}\right)\left(1+\rho a_{n}^{2}\left|h_{n}\right|^{2}\right)>\left(1+\rho\left|h_{m}\right|^{2}\right)^{\frac{1}{2}}\right. \\
& \left.\times\left(1+\rho\left|h_{n}\right|^{2}\right)^{\frac{1}{2}}\right) \\
& =\mathrm{P}\left(\frac{1+\rho\left|h_{m}\right|^{2}}{\left(1+\rho a_{n}^{2}\left|h_{m}\right|^{2}\right)^{2}}>\frac{1+\rho\left|h_{n}\right|^{2}}{\left(1+\rho a_{n}^{2}\left|h_{n}\right|^{2}\right)^{2}}\right) .
\end{align*}
$$

After some algebraic manipulations, this probability can be rewritten as follows:

$$
\begin{align*}
& \mathrm{P}\left(R_{m}+R_{n}>\bar{R}_{m}+\bar{R}_{n}\right)  \tag{30}\\
& =\mathrm{P}\left(\rho\left(\left|h_{m}\right|^{2}+\left|h_{n}\right|^{2}\right)+\rho^{2}\left|h_{m}\right|^{2}\left|h_{n}\right|^{2}>\frac{1-2 a_{n}^{2}}{a_{n}^{4}}\right)
\end{align*}
$$

The right-hand side of the above inequality is non-negative since the $n$-th user will get less power than the $m$-th user, i.e., $a_{n}^{2} \leq \frac{1}{2}$. Note that the joint pdf of $\rho\left|h_{m}\right|^{2}$ and $\rho\left|h_{n}\right|^{2}$ is given by 13]

$$
\begin{align*}
& f_{\left|h_{m}\right|^{2},\left|h_{n}\right|^{2}}(x, y)=\varpi_{1} f(x) f(y)[F(x)]^{m-1}(1-F(y))^{M-n} \\
& \times(F(y)-F(x))^{n-1-m} \tag{31}
\end{align*}
$$

In addition, the marginal pdf of $\left|h_{n}\right|^{2}$ is given by

$$
\begin{equation*}
f_{\left|h_{n}\right|^{2}}(y)=\varpi_{3} f(y)(F(y))^{n-1}(1-F(y))^{M-n} \tag{32}
\end{equation*}
$$

By applying the above density functions, the addressed probability can be expressed as follows:

$$
\begin{align*}
& \mathrm{P}\left(R_{m}+R_{n}>\bar{R}_{m}+\bar{R}_{n}\right)=\underbrace{\int_{\varpi_{2}}^{\infty} f_{\left|h_{n}\right|^{2}}(y) d y}_{Q_{2}}  \tag{33}\\
& +\underbrace{\int_{(x+y)+x y>\varpi_{2}, x<y<\varpi_{2}} \int_{\left|h_{m}\right|^{2},\left|h_{n}\right|^{2}}(x, y) d x d y}_{Q_{1}}
\end{align*}
$$

Note that the integral range for $x$ in $Q_{1}$ is $\frac{\varpi_{2}-y}{1+y}<x<$ $y$, and this range implies that $\frac{\varpi_{2}-y}{1+y}<y$, which causes an additional constraint on $y$, i.e., $y>\sqrt{1+\varpi_{2}}-1$. By applying
the binomial expansion, the joint pdf can be further written as follows:

$$
\begin{aligned}
& f_{\left|h_{m}\right|^{2},\left|h_{n}\right|^{2}}(x, y)=\varpi_{1} \sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} f(x) \\
& \times f(y)[F(x)]^{m-1+i}(1-F(y))^{M-n}(F(y))^{n-1-m-i}
\end{aligned}
$$

Therefore the probability $Q_{1}$ can now be evaluated as follows:

$$
\begin{align*}
Q_{1} & =\varpi_{1} \sum_{i=0}^{n-1-m}\binom{n-1-m}{i} \frac{(-1)^{i}}{m+i} \int_{\varpi_{4}}^{\varpi_{2}} f(y)(F(y))^{n-1-m-i} \\
& \times(1-F(y))^{M-n}\left([F(y)]^{m+i}-\left[F\left(\frac{\varpi_{2}-y}{1+y}\right)\right]^{m+i}\right) d y . \tag{34}
\end{align*}
$$

On the other hand, $Q_{2}$ can be calculated as follows:

$$
\begin{align*}
Q_{2} & =\int_{\varpi_{2}}^{\infty} \varpi_{3} f(y)(F(y))^{n-1}(1-F(y))^{M-n} d y  \tag{35}\\
& =\int_{\varpi_{2}}^{\infty} \varpi_{3} \frac{1}{\rho} e^{-\frac{(M-n+1) y}{\rho}}\left(1-e^{-\frac{y}{\rho}}\right)^{n-1} d y
\end{align*}
$$

By applying the binomial expansion, $Q_{2}$ can be written as follows:

$$
\begin{aligned}
Q_{2} & =\frac{\varpi_{3}}{\rho} \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j} \int_{\varpi_{2}}^{\infty} e^{-\frac{(M-n+j+1) y}{\rho}} d y \\
& =\frac{\varpi_{3}}{\rho} \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j} \frac{\rho}{M-n+j+1} e^{-\frac{(M-n+j+1) \varpi_{2}}{\rho}} .
\end{aligned}
$$

Combining (34) with 36, the first part of the theorem is proved.

To find high SNR approximations for $Q_{1}$ and $Q_{2}$, first observe that the integral in (34) is calculated for the range of $0 \leq y<\varpi_{2}$. At high SNR, the two functions $f(y)$ and $F(y)$ can be approximated as follows: $f(y)=\frac{1}{\rho} e^{-\frac{y}{\rho}} \approx \frac{1}{\rho}$ and $F(y)=1-e^{-\frac{y}{\rho}} \approx \frac{y}{\rho}$, since $0 \leq y \leq \varpi_{2}$ and $\rho \rightarrow \infty$.

Define $u(y)=\frac{\varpi_{2}-y}{1+y}$. It is straightforward to show

$$
0 \leq u(y) \leq \varpi_{2}
$$

for $0 \leq y \leq \varpi_{2}$, since $\frac{d g(y)}{d y}<0$. Therefore at high SNR, we can have the following approximation:

$$
F\left(\frac{\varpi_{2}-y}{1+y}\right)=1-e^{-\frac{\varpi_{2}-y}{\rho(1+y)}} \approx \frac{\varpi_{2}-y}{\rho(1+y)}
$$

Now the probability $Q_{1}$ can be approximated as follows:

$$
\begin{align*}
Q_{1} & \approx \varpi_{1} \sum_{i=0}^{n-1-m}\binom{n-1-m}{i} \frac{(-1)^{i}}{m+i} \int_{\varpi_{4}}^{\varpi_{2}} \frac{1}{\rho}\left(\frac{y}{\rho}\right)^{n-1-m-i} \\
& \times\left(\left[\frac{y}{\rho}\right]^{m+i}-\left[\frac{\varpi_{2}-y}{\rho(1+y)}\right]^{m+i}\right) d y \\
& \approx \frac{\varpi_{1}}{\rho^{n}} \sum_{i=0}^{n-1-m}\binom{n-1-m}{i} \frac{(-1)^{i}}{m+i} \int_{\varpi_{4}}^{\varpi_{2}} y^{n-1-m-i} \\
& \times\left(y^{m+i}-\left[\frac{\varpi_{2}-y}{(1+y)}\right]^{m+i}\right) d y \tag{37}
\end{align*}
$$

The high SNR approximation for $Q_{2}$ is more complicated. After applying the series expansion of the exponential functions in (36), we have

$$
\begin{align*}
& Q_{2}=\sum_{i=0}^{\infty} \frac{\varpi_{3}}{\rho} \sum_{j=0}^{n-1}\binom{n-1}{j} \frac{(-1)^{i+j} \frac{(M-n+j+1)^{i-1} \varpi_{2}^{i}}{\rho^{i-1}}}{i!}  \tag{38}\\
& =\sum_{i=0}^{\infty} \frac{(-1)^{i} \varpi_{3} \varpi_{2}^{i}}{i!\rho^{i}} \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j}(M-n+j+1)^{i-1} .
\end{align*}
$$

Consider $Q_{2}$ as a function of $\varpi_{2}$, and $Q_{2}=1$ is true for $\varpi_{2}=0$, as can be seen from the definition of $Q_{2}$ in (33). On the other hand by letting $\varpi_{2}=0$ in (36), we obtain the following equality:

$$
\begin{equation*}
\varpi_{3} \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j} \frac{1}{M-n+j+1}=1 \tag{39}
\end{equation*}
$$

Consequently $Q_{2}$ can be rewritten as follows:

$$
\begin{align*}
Q_{2} & =1+\sum_{i=1}^{\infty} \frac{(-1)^{i} \varpi_{3} \varpi_{2}^{i}}{i!\rho^{i}} \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j}  \tag{40}\\
& \times \sum_{l=0}^{i-1}\binom{i-1}{l}(M-n+1)^{i-1-l} j^{l}
\end{align*}
$$

Recall the following sums of the binomial coefficients (Eq. ( 0.154 .3 ) in [12]):

$$
\begin{equation*}
\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j} j^{l}=0 \tag{41}
\end{equation*}
$$

for $n-2 \geq l \geq 1$ and

$$
\begin{equation*}
\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j} j^{n-1}=(-1)^{n-1}(n-1)! \tag{42}
\end{equation*}
$$

Therefore all the components in (40) containing $j^{l}, l<(n-1)$, can be removed, since they are equal to zero by using 41). Furthermore, all the components containing $j^{l}, l>(n-1)$ can also be ignored, since the one with $j=n-1$ is the dominant
factor. With these steps, the probability can be approximated as follows:

$$
\begin{align*}
Q_{2} & \approx 1+\frac{(-1)^{n} \varpi_{3} \varpi_{2}^{n}}{n!\rho^{n}}(-1)^{n-1}(n-1)!  \tag{43}\\
& =1-\frac{\varpi_{3} \varpi_{2}^{n}}{n \rho^{n}}
\end{align*}
$$

Combining (37) and (43), the second part of the theorem is also proved.

Proof for Theorem 2;
Recall that the outage performance of the $n$-th user is given by

$$
\begin{aligned}
& \mathrm{P}\left(\log \left(1+a_{n}^{2} \rho\left|h_{n}\right|^{2}\right)<R\right) \\
= & \underbrace{\left(\operatorname{Pog}\left(1+\frac{\left|h_{m}\right|^{2}-\frac{I}{\rho}}{\left|h_{m}\right|^{2}(1+I)} \rho\left|h_{n}\right|^{2}\right)<R,\left|h_{m}\right|^{2}>\frac{I}{\rho}\right)}_{Q_{3}} \\
& +\underbrace{\mathrm{P}\left(\left|h_{m}\right|^{2}<\frac{I}{\rho}\right)}_{Q_{4}} .
\end{aligned}
$$

The first factor in the above equation can be calculated as follows:

$$
\begin{equation*}
Q_{3}=\mathrm{P}\left(\frac{\left|h_{m}\right|^{2}-\frac{I}{\rho}}{\left|h_{m}\right|^{2}(1+I)}\left|h_{n}\right|^{2}<\epsilon_{1},\left|h_{m}\right|^{2}>\frac{I}{\rho}\right) . \tag{45}
\end{equation*}
$$

Recall that the users' channels are ordered, i.e., $\left|h_{m}\right|^{2}<\left|h_{n}\right|^{2}$, which brings additional constraints to the integral range in the above equation. The constraints can be written as follows:

$$
\begin{equation*}
b<\left|h_{m}\right|^{2}<\min \left\{\left|h_{n}\right|^{2}, \frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}\right\} . \tag{46}
\end{equation*}
$$

The outage events due to these constraints can be classified as follows:

1) If $\left|h_{n}\right|^{2}<a \epsilon_{1}$, we have the following:

$$
\begin{align*}
& \mathrm{P}\left(\frac{\left|h_{m}\right|^{2}-\frac{I}{\rho}}{\left|h_{m}\right|^{2}(1+I)}\left|h_{n}\right|^{2}<\epsilon_{1}\right)  \tag{47}\\
& =\mathrm{P}\left(\left|h_{m}\right|^{2}\left(\left|h_{n}\right|^{2}-\epsilon_{1} a\right)<b\left|h_{n}\right|^{2}\right)=1 .
\end{align*}
$$

Therefore the probability $Q_{3}$ can be expressed as follows: ${ }^{17}$

$$
Q_{3}=\mathrm{P}\left(b \leq\left|h_{n}\right|^{2}<a \epsilon_{1},\left|h_{n}\right|^{2}>\left|h_{m}\right|^{2}>b\right) .
$$

2) If $\left|h_{n}\right|^{2}>a \epsilon_{1}$, there are two possible events:
a) If $\left|h_{n}\right|^{2}>b+a \epsilon_{1}$, we have $\frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}<\left|h_{n}\right|^{2}$, and $Q_{3}$ can be written as follows:

$$
Q_{3}=\mathrm{P}\left(\left|h_{n}\right|^{2}>b+a \epsilon_{1}, b<\left|h_{m}\right|^{2}<\frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}\right) .
$$

[^1]b) If $\left|h_{n}\right|^{2}<b+a \epsilon_{1}$, we have $\frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}>\left|h_{n}\right|^{2}$, and $Q_{3}$ can be written as follows:
$$
Q_{3}=\mathrm{P}\left(a \epsilon_{1}<\left|h_{n}\right|^{2}<b+a \epsilon_{1}, b<\left|h_{m}\right|^{2}<\left|h_{n}\right|^{2}\right),
$$
which is again conditioned on $b<a \epsilon_{1}$.
Therefore, the probability $Q_{3}$ can be written as follows:
\[

$$
\begin{align*}
Q_{3} & =\mathrm{P}\left(b \leq\left|h_{n}\right|^{2}<a \epsilon_{1},\left|h_{n}\right|^{2}>\left|h_{m}\right|^{2}>b\right)  \tag{48}\\
& +\mathrm{P}\left(\left|h_{n}\right|^{2}<b+a \epsilon_{1}, b<\left|h_{m}\right|^{2}<\left|h_{n}\right|^{2}\right) \\
& +\mathrm{P}\left(\left|h_{n}\right|^{2}>b+a \epsilon_{1}, b<\left|h_{m}\right|^{2}<\frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}\right)
\end{align*}
$$
\]

The first probability in (48) can be calculated by applying (32) as follows:

$$
\begin{aligned}
& \mathrm{P}\left(b \leq\left|h_{n}\right|^{2}<a \epsilon_{1},\left|h_{n}\right|^{2}>\left|h_{m}\right|^{2}>b\right) \\
& =\sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \int_{b}^{a \epsilon_{1}} g(y)(1-G(y))^{M-n} \\
& \times G(y)^{n-1-m-i} \int_{b}^{y} \varpi_{1} g(x)[G(x)]^{m-1+i} d x d y \\
& =\sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \int_{b}^{a \epsilon_{1}} g(y)(1-G(y))^{M-n} \\
& \times G(y)^{n-1-m-i} \varpi_{1} \frac{\left(G(y)^{m+i}-G(b)^{m+i}\right)}{m+i} d y .
\end{aligned}
$$

Following similar steps, the second probability in 48) can be expressed as

$$
\begin{aligned}
& \mathrm{P}\left(a \epsilon_{1}<\left|h_{n}\right|^{2}<b+a \epsilon_{1}, b<\left|h_{m}\right|^{2}<\left|h_{n}\right|^{2}\right) \\
& =\sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \int_{a \epsilon_{1}}^{b+a \epsilon_{1}} g(y)(1-G(y))^{M-n} \\
& \times G(y)^{n-1-m-i} \varpi_{1} \frac{\left(G(y)^{m+i}-G(b)^{m+i}\right)}{m+i} d y .
\end{aligned}
$$

The third probability in (48) can be calculated as follows:

$$
\begin{align*}
& \mathrm{P}\left(\left|h_{n}\right|^{2}>b+a \epsilon_{1}, b<\left|h_{m}\right|^{2}<\frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}\right)  \tag{50}\\
& =\sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \int_{b+a \epsilon_{1}}^{\infty} g(y)(1-G(y))^{M-n} \\
& \times G(y)^{n-1-m-i} \int_{b}^{\frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}} \varpi_{1} g(x)[G(x)]^{m-1+i} d x d y \\
& =\sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \int_{b+a \epsilon_{1}}^{\infty} g(y)(1-G(y))^{M-n} \\
& \times G(y)^{n-1-m-i} \varpi_{1} \frac{\left(G\left(\frac{b}{1-\frac{a \epsilon_{1}}{\left|n_{n}\right|^{2}}}\right)^{m+i}-G(b)^{m+i}\right)}{m+i} d y .
\end{align*}
$$

Note that $Q_{4}$ can be obtained easily by applying (32) and the first part of the theorem is proved.

Recall that the first probability in (48) can be expressed as follows:

$$
\begin{aligned}
& \mathrm{P}\left(b \leq\left|h_{n}\right|^{2}<a \epsilon_{1},\left|h_{n}\right|^{2}>\left|h_{m}\right|^{2}>b\right) \\
& =\varpi_{1} \sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \int_{b}^{a \epsilon_{1}} g(y) \\
& \times(1-G(y))^{M-n} G(y)^{n-1-m-i} \frac{\left(G(y)^{m+i}-G(b)^{m+i}\right)}{m+i} d y,
\end{aligned}
$$

where the integral range is $0 \leq y \leq\left(a \epsilon_{1}\right)$. Note that when $\rho \rightarrow \infty, \epsilon_{1}$ approaches zero, which means $y \rightarrow 0, g(y) \approx 1$ and $G(y) \approx 1-y$. Therefore the above probability can be approximated as follows:

$$
\begin{align*}
& \mathrm{P}\left(b \leq\left|h_{n}\right|^{2}<a \epsilon_{1},\left|h_{n}\right|^{2}>\left|h_{m}\right|^{2}>b\right)  \tag{51}\\
& \approx \varpi_{1} \sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \\
& \times \int_{b}^{a \epsilon_{1}} y^{n-1-m-i} \frac{\left(y^{m+i}-b^{m+i}\right)}{m+i} d y \\
& \approx \varpi_{1} \sum_{i=0}^{n-1-m}\binom{n-1-m}{i}(-1)^{i} \\
& \times \frac{\left(\frac{\left(a \epsilon_{1}\right)^{n}-b^{n}}{m+i+1}-b^{m+i}\left(\frac{\left(a \epsilon_{1}\right)^{n-m-i}-b^{n-m-i}}{n-m-i}\right)\right)}{m+i} \rightarrow \rho^{-n}
\end{align*}
$$

Following similar steps, the second probability in (48) can be approximated as follows:

$$
\begin{equation*}
\mathrm{P}\left(a \epsilon_{1}<\left|h_{n}\right|^{2}<b+a \epsilon_{1}, b<\left|h_{m}\right|^{2}<\left|h_{n}\right|^{2}\right) \rightarrow \rho^{-n} \tag{52}
\end{equation*}
$$

The exact diversity order of the third probability in (48) is difficult to obtain. Particularly the expression in (50) is difficult to use for asymptotic studies, since the range of $y$ is not limited and those manipulations related to high SNR approximations cannot be applied here. We first rewrite (50) in an alternative form as follows:

$$
\begin{aligned}
& \mathrm{P}\left(\left|h_{n}\right|^{2}>b+a \epsilon_{1}, b<\left|h_{m}\right|^{2}<\frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}\right) \\
& =\mathrm{P}\left(b<\left|h_{m}\right|^{2}<b+a \epsilon_{1}, b+a \epsilon_{1}<\left|h_{n}\right|^{2}<\frac{\left|h_{m}\right|^{2} a \epsilon_{1}}{\left|h_{m}\right|^{2}-b}\right) .
\end{aligned}
$$

Note that $b+a \epsilon_{1}<\frac{\left|h_{m}\right|^{2} a \epsilon_{1}}{\left|h_{m}\right|^{2}-b}$ always holds since $\left|h_{m}\right|^{2}<$ $\frac{b}{1-\frac{b a_{1}}{\left|h_{n}\right|^{2}}}$.

Now applying the joint pdf of the two channel coefficients, we obtain the following expression:

$$
\begin{aligned}
& \mathrm{P}\left(b<\left|h_{m}\right|^{2}<b+a \epsilon_{1}, b+a \epsilon_{1}<\left|h_{n}\right|^{2}<\frac{\left|h_{m}\right|^{2} a \epsilon_{1}}{\left|h_{m}\right|^{2}-b}\right) \\
& =\sum_{i=0}^{n-1-m} \varpi_{1}\binom{n-1-m}{i}(-1)^{i} \int_{b}^{b+a \epsilon_{1}} g(x)[G(x)]^{m-1+i} \\
& \times \int_{G\left(b+a \epsilon_{1}\right)}^{G\left(\frac{x a \epsilon_{1}}{x-b}\right)}[G(y)]^{n-1-m-i}(1-G(y))^{M-n} d G(y) d x .
\end{aligned}
$$

Again applying the binomial expansion, the above probability can be further expanded as follows:

$$
\begin{align*}
& \mathrm{P}\left(b<\left|h_{m}\right|^{2}<b+a \epsilon_{1}, b+a \epsilon_{1}<\left|h_{n}\right|^{2}<\frac{\left|h_{m}\right|^{2} a \epsilon_{1}}{\left|h_{m}\right|^{2}-b}\right) \\
& =\sum_{i=0}^{n-1-m} \varpi_{1}\binom{n-1-m}{i}(-1)^{i} \int_{b}^{b+a \epsilon_{1}} g(x)[G(x)]^{m-1+i} \\
& \times \sum_{j=0}^{M-n}\binom{M-n}{j}(-1)^{j} \int_{G\left(b+a \epsilon_{1}\right)}^{G\left(\frac{x a \epsilon_{1}}{x-b}\right)}[G(y)]^{n-1-m-i+j} d G(y) d x \\
& =\sum_{i=0}^{n-1-m} \varpi_{1}\binom{n-1-m}{i}(-1)^{i} \int_{b}^{b+a \epsilon_{1}} g(x)[G(x)]^{m-1+i} \\
& \times \sum_{j=0}^{M-n}\binom{M-n}{j} \frac{(-1)^{j}}{n-1-m-i+j}  \tag{53}\\
& \times\left(\left[G\left(\frac{x a \epsilon_{1}}{x-b}\right)\right]^{n-m-i+j}-\left[G\left(b+a \epsilon_{1}\right)\right]^{n-m-i+j}\right) d x .
\end{align*}
$$

Compared to (50), the above equation is more complicated; however, this expression is more suitable for asymptotic studies, as explained in the following.
Recall that the integral range in (53) is $b<x<b+a \epsilon_{1}$. When $\rho \rightarrow 0$, we have $b \rightarrow 0$ and $b+a \epsilon_{1} \rightarrow 0$, which implies $x \rightarrow 0$. Therefore the following approximation can be obtained:

$$
\begin{align*}
& \mathrm{P}\left(b<\left|h_{m}\right|^{2}<b+a \epsilon_{1}, b+a \epsilon_{1}<\left|h_{n}\right|^{2}<\frac{\left|h_{m}\right|^{2} a \epsilon_{1}}{\left|h_{m}\right|^{2}-b}\right) \\
& \approx \sum_{i=0}^{n-1-m} \varpi_{1}\binom{n-1-m}{i}(-1)^{i}  \tag{54}\\
& \times \sum_{j=0}^{M-n}\binom{M-n}{j} \frac{(-1)^{j}}{n-1-m-i+j} \int_{b}^{b+a \epsilon_{1}} x^{m-1+i} \\
& \times\left(\left[G\left(\frac{x a \epsilon_{1}}{x-b}\right)\right]^{n-m-i+j}-\left[b+a \epsilon_{1}\right]^{n-m-i+j}\right) d x .
\end{align*}
$$

First focus on the following integral which is from the above equation:

$$
\begin{align*}
& \int_{b}^{b+a \epsilon_{1}} x^{m-1+i}\left[G\left(\frac{x a \epsilon_{1}}{x-b}\right)\right]^{n-m-i+j} d x  \tag{55}\\
& \approx \int_{0}^{a \epsilon_{1}}(b+z)^{m-1+i}\left[1-e^{-\frac{a b \epsilon_{1}}{z}}\right]^{n-m-i+j} d z
\end{align*}
$$

We can find the following bounds for the above integral:

$$
\begin{align*}
& \int_{b}^{b+a \epsilon_{1}} x^{m-1+i} d x  \tag{56}\\
& \geq \int_{b}^{b+a \epsilon_{1}} x^{m-1+i}\left[G\left(\frac{x a \epsilon_{1}}{x-b}\right)\right]^{n-m-i+j} d x \\
& \geq \int_{0}^{a \epsilon_{1}}(b+z)^{m-1+i}\left[1-\frac{1}{1+\frac{a b \epsilon_{1}}{z}}\right]^{n-m-i+j} d z
\end{align*}
$$

where the lower bound is obtained due to the inequality

$$
e^{-\frac{a b \epsilon_{1}}{z}} \leq \frac{1}{1+\frac{a b \epsilon_{1}}{z}}
$$

when $0 \leq z \leq a \epsilon_{1}$. The upper bound in (56) can be approximated at high SNR as follows:

$$
\begin{align*}
& \int_{b}^{b+a \epsilon_{1}} x^{m-1+i} d x  \tag{57}\\
& =\frac{\left(b+a \epsilon_{1}\right)^{m+i}-b^{m+i}}{m+i} \rightarrow \frac{1}{\rho^{m+i}} .
\end{align*}
$$

On the other hand, the lower bound in (56) can be approximated as follows:

$$
\begin{align*}
& \int_{0}^{a \epsilon_{1}}(b+z)^{m-1+i}\left[1-\frac{1}{1+\frac{a b \epsilon_{1}}{z}}\right]^{n-m-i+j} d z  \tag{58}\\
& =\left(a b \epsilon_{1}\right)^{n-m-i+j} \int_{a b \epsilon_{1}}^{a \epsilon_{1}+a b \epsilon_{1}} \frac{\left(w+b-a b \epsilon_{1}\right)^{m-1+i}}{w^{n-m-i+j}} d z \\
& =\left(a b \epsilon_{1}\right)^{n-m-i+j} \sum_{k=0}^{m-1+i}\left(b-a b \epsilon_{1}\right)^{m-1+i-k} \\
& \times \int_{a b \epsilon_{1}}^{a \epsilon_{1}+a b \epsilon_{1}} w^{k-(n-m-i+j)} d z \triangleq \sum_{k=0}^{m-1+i} \xi_{k}
\end{align*}
$$

At high SNR, we can show that

$$
\xi_{k} \rightarrow \begin{cases}\rho^{-(n+j)} \ln \rho, & \text { for } k+1=n-m-i+j  \tag{59}\\ \rho^{-(n+j)}, & \text { otherwise }\end{cases}
$$

Since $\frac{\log \log \rho}{\log \rho} \rightarrow 0$ for $\rho \rightarrow \infty$, the lower bound in (56) can be approximated as follows:

$$
\begin{equation*}
\int_{0}^{a \epsilon_{1}}(b+z)^{m-1+i}\left[\frac{a b \epsilon_{1}}{z+a b \epsilon_{1}}\right]^{n-m-i+j} d z \rightarrow \rho^{-(n+j)} \tag{60}
\end{equation*}
$$

Based on the upper and lower bounds in (57) and 60) and after some algebraic manipulation, we have the following inequality:

$$
\rho^{-n} \dot{\leq} \mathrm{P}\left(\left|h_{n}\right|^{2}>b+a \epsilon_{1}, b<\left|h_{m}\right|^{2}<\frac{b}{1-\frac{a \epsilon_{1}}{\left|h_{n}\right|^{2}}}\right) \dot{\leq}^{-m}
$$

where $a \dot{\leq} b$ denotes $\left(-\frac{\log a}{\log \rho}\right) \leq\left(-\frac{\log b}{\log \rho}\right)$ when $\rho \rightarrow \infty$ [14].
Combining (51), (52) and (61), we can obtain the following asymptotic bounds:

$$
\begin{equation*}
\rho^{-n} \dot{\leq} Q_{2} \dot{\leq} \rho^{-m} \tag{62}
\end{equation*}
$$

Following similar steps as above, we can also find that $Q_{3} \doteq \rho^{-m}$, which is dominant in $\mathrm{P}_{o}^{n}$, and the proof for the second part of the theorem is completed.


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[^1]:    ${ }^{1}$ It is assumed that $b \leq a \epsilon_{1}$ here. For the case of $b>a \epsilon_{1}$, the outage probability can be calculated in a straightforward way, since there will be fewer events to analyze. Note that the same diversity order will be obtained regardless of the choice of $b$ and $a \epsilon_{1}$.

