

## Edge-colouring eight-regular planar graphs

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### **Abstract**

It was conjectured by the third author in about 1973 that every  $d$ -regular planar graph (possibly with parallel edges) can be  $d$ -edge-coloured, provided that for every odd set  $X$  of vertices, there are at least  $d$  edges between  $X$  and its complement. For  $d = 3$  this is the four-colour theorem, and the conjecture has been proved for all  $d \leq 7$ , by various authors. Here we prove it for  $d = 8$ .

# 1 Introduction

One form of the four-colour theorem, due to Tait [9], asserts that a 3-regular planar graph can be 3-edge-coloured if and only if it has no cut-edge. But when can  $d$ -regular planar graphs be  $d$ -edge-coloured?

Let  $G$  be a graph. (Graphs in this paper are finite, and may have loops or parallel edges.) If  $X \subseteq V(G)$ ,  $\delta_G(X) = \delta(X)$  denotes the set of all edges of  $G$  with an end in  $X$  and an end in  $V(G) \setminus X$ . We say that  $G$  is *oddly  $d$ -edge-connected* if  $|\delta(X)| \geq d$  for all odd subsets  $X$  of  $V(G)$ . Since every perfect matching contains an edge of  $\delta(X)$  for every odd set  $X \subseteq V(G)$ , it follows that every  $d$ -regular  $d$ -edge-colourable graph is oddly  $d$ -edge-connected. (Note that for a 3-regular graph, being oddly 3-edge-connected is the same as having no cut-edge, because if  $X \subseteq V(G)$ , then  $|\delta(X)| = 1$  if and only if  $|X|$  is odd and  $|\delta(X)| < 3$ .) The converse is false, even for  $d = 3$  (the Petersen graph is a counterexample); but for planar graphs perhaps the converse is true. That is the content of the following conjecture [8], proposed by the third author in about 1973.

**1.1 Conjecture.** *If  $G$  is a  $d$ -regular planar graph, then  $G$  is  $d$ -edge-colourable if and only if  $G$  is oddly  $d$ -edge-connected.*

Some special cases of this conjecture have been proved.

- For  $d = 3$  it is the four-colour theorem, and was proved by Appel and Haken [1, 2, 7];
- for  $d = 4, 5$  it was proved by Guenin [5];
- for  $d = 6$  it was proved by Dvorak, Kawarabayashi and Kral [3];
- for  $d = 7$  it was proved by Kawarabayashi and the second author, and appears in the Master's thesis [4] of the latter. The methods of the present paper can also be applied to the  $d = 7$  case, resulting in a proof somewhat simpler than the original, and this simplified proof for the  $d = 7$  case will be presented in another, four-author paper [6].

Here we prove the next case, namely:

**1.2** *Every 8-regular oddly 8-edge-connected planar graph is 8-edge-colourable.*

All these proofs (for  $d > 3$ ), including ours, proceed by induction on  $d$ . Thus we need to assume the truth of the result for  $d = 7$ .

## 2 An unavoidable list of reducible configurations.

The graph we wish to edge-colour has parallel edges, but it is more convenient to work with the underlying simple graph. If  $H$  is  $d$ -regular and oddly  $d$ -edge-connected, then  $H$  has no loops, because for every vertex  $v$ ,  $v$  has degree  $d$ , and yet  $|\delta_H(v)| \geq d$ . (We write  $\delta(v)$  for  $\delta(\{v\})$ .) Thus to recover  $H$  from the underlying simple graph  $G$  say, we just need to know the number  $m(e)$  of parallel edges of  $H$  that correspond to each edge  $e$  of  $G$ . Let us say a  $d$ -target is a pair  $(G, m)$  with the following properties (where for  $F \subseteq E(G)$ ,  $m(F)$  denotes  $\sum_{e \in F} m(e)$ ):

- $G$  is a simple graph drawn in the plane;

- $m(e) \geq 0$  is an integer for each edge  $e$ ;
- $m(\delta(v)) = d$  for every vertex  $v$ ; and
- $m(\delta(X)) \geq d$  for every odd subset  $X \subseteq V(G)$ .

In this language, 1.1 says that for every  $d$ -target  $(G, m)$ , there is a list of  $d$  perfect matchings of  $G$  such that every edge  $e$  of  $G$  is in exactly  $m(e)$  of them. (The elements of a list need not be distinct.) If there is such a list we call it a  $d$ -edge-colouring, and say that  $(G, m)$  is  $d$ -edge-colourable. For an edge  $e \in E(G)$ , we call  $m(e)$  the *multiplicity* of  $e$ . If  $X \subseteq V(G)$ ,  $G|X$  denotes the subgraph of  $G$  induced on  $X$ . We need:

**2.1** *Let  $(G, m)$  be a  $d$ -target, that is not  $d$ -edge-colourable, but such that every  $d$ -target with fewer vertices is  $d$ -edge-colourable. Then*

- $|V(G)| \geq 6$ ;
- for every  $X \subseteq V(G)$  with  $|X|$  odd, if  $|X|, |V(G) \setminus X| \neq 1$  then  $m(\delta(X)) \geq d + 2$ ; and
- $G$  is three-connected, and  $m(e) \leq d - 2$  for every edge  $e$ .

**Proof.** If  $m(e) = 0$  for some edge  $e$ , we may delete  $e$  without affecting the problem; so we may assume that  $m(e) > 0$  for every edge  $e$ . It is easy to check that  $G$  is connected and  $|V(G)| \geq 6$  and we omit it. For the second assertion let  $X \subseteq V(G)$  with  $|X|$  odd and with  $|X|, |V(G) \setminus X| \neq 1$ . Thus  $m(\delta(X)) \geq d$  since  $(G, m)$  is a  $d$ -target; suppose that  $m(\delta(X)) = d$ . There is a component of  $G|X$  with an odd number of vertices, with vertex set  $X'$  say; and so  $m(\delta(X')) \geq d$  since  $(G, m)$  is a  $d$ -target. But  $\delta(X') \subseteq \delta(X)$ , and  $m(e) > 0$  for every edge  $e$ ; and so  $\delta(X') = \delta(X)$ . Since  $G$  is connected it follows that  $X' = X$ , and so  $G|X$  is connected. Similarly  $G|Y$  is connected, where  $Y = V(G) \setminus X$ . Replace each edge  $e$  of  $G$  by  $m(e)$  parallel edges, forming  $H$ ; and contract all edges of  $H|Y$ , forming a  $d$ -regular oddly  $d$ -edge-connected planar graph  $H_1$  with fewer vertices than  $H$  (because  $|Y| > 1$ ). By hypothesis it follows that  $H_1$  is  $d$ -edge-colourable. Similarly so is the graph obtained from  $H$  by contracting all edges of  $G|X$ . But these colourings can be combined to give a  $d$ -edge-colouring of  $H$ , a contradiction. This proves that  $m(\delta(X)) > d$ . Since  $m(\delta(v)) = d$  for every vertex  $v$ , it follows that  $m(\delta(X))$  has the same parity as  $d|X|$ , and so  $m(\delta(X)) \geq d + 2$ . This proves the second assertion.

For the third assertion, suppose that  $G$  is not three-connected. Since  $|V(G)| > 3$ , there is a partition  $(X, Y, Z)$  of  $V(G)$  where  $X, Y \neq \emptyset$ , with  $|Z| = 2$ , such that there are no edges between  $X$  and  $Y$ . Let  $Z = \{z_1, z_2\}$  say. Either both  $|X|, |Y|$  are odd, or they are both even. If they are both odd, then since  $\delta(X), \delta(Y)$  are disjoint subsets of  $\delta(z_1) \cup \delta(z_2)$ , and

$$m(\delta(X)), m(\delta(Y)) \geq d = m(\delta(z_1)), m(\delta(z_2)),$$

we have equality throughout, and in particular  $m(\delta(X)), m(\delta(Y)) = d$ . But then  $|X| = |Y| = 1$  from the second assertion, contradicting that  $|V(G)| \geq 6$ . Now assume  $|X|, |Y|$  are both even. Since  $\delta(X \cup \{z_1\}), \delta(Y \cup \{z_2\})$  have the same union and intersection as  $\delta(z_1), \delta(z_2)$ , it follows that  $m(\delta(X \cup \{z_1\})) = d$ , contrary to the second assertion. Thus  $G$  is three-connected. Since  $m(e) \geq 1$  for every edge  $e$ , and  $m(\delta(v)) = d$  for every vertex  $v$ , it follows that  $m(e) \leq d - 2$  for every edge  $e$ . This proves the third assertion, and hence proves 2.1. ■

A *triangle* is a region of  $G$  incident with exactly three edges. If a triangle is incident with vertices  $u, v, w$ , for convenience we refer to it as  $uvw$ , and in the same way an edge with ends  $u, v$  is called  $uv$ . Two edges are *disjoint* if they are distinct and no vertex is an end of both of them, and otherwise they *meet*. Let  $r$  be a region of  $G$ , and let  $e \in E(G)$  be incident with  $r$ ; let  $r'$  be the other region incident with  $e$ . We say that  $e$  is  *$i$ -heavy* (for  $r$ ), where  $i \geq 2$ , if either  $m(e) \geq i$  or  $r'$  is a triangle  $uvw$  where  $e = uv$  and

$$m(uv) + \min(m(uw), m(vw)) \geq i.$$

We say  $e$  is a *door* for  $r$  if  $m(e) = 1$  and there is an edge  $f$  incident with  $r'$  and disjoint from  $e$  with  $m(f) = 1$ . We say that  $r$  is *big* if there are at least four doors for  $r$ , and *small* otherwise. A *square* is a region with length four.

Since  $G$  is drawn in the plane and is two-connected, every region  $r$  has boundary some cycle which we denote by  $C_r$ . In what follows we will be studying cases in which certain configurations of regions are present in  $G$ . We will give a list of regions the closure of the union of which is a disc. For convenience, for an edge  $e$  in the boundary of this disc, we call the region outside the disc incident with  $e$  the “second region” for  $e$ ; and we write  $m^+(e) = m(e)$  if the second region is big, and  $m^+(e) = m(e) + 1$  if the second region is small. This notation thus depends not just on  $(G, m)$  but on what regions we have specified, so it is imprecise, and when there is a danger of ambiguity we will specify it more clearly.

Let us say an 8-target  $(G, m)$  is *prime* if

- $m(e) > 0$  for every edge  $e$ ;
- $|V(G)| \geq 6$ ;
- $m(\delta(X)) \geq 10$  for every  $X \subseteq V(G)$  with  $|X|$  odd and  $|X|, |V(G) \setminus X| \neq 1$ ;
- $G$  is three-connected, and  $m(e) \leq 6$  for every edge  $e$ ;

and in addition  $(G, m)$  contains none of the following:

**Conf(1):** A triangle  $uvw$  where  $u, v$  both have degree three.

**Conf(2):** A triangle  $uvw$ , where  $u$  has degree three and its third neighbour  $x$  satisfies

$$m(ux) < m(uw) + m(vw).$$

**Conf(3):** Two triangles  $uvw, uwx$  with  $m(uv) + m(uw) + m(vw) + m(ux) \geq 8$ .

**Conf(4):** A square  $uvw$  where  $m(uv) + m(vw) + m(ux) \geq 8$  and

$$(m(uw), m(vw), m(wx), m(ux)) \neq (4, 2, 1, 2).$$

**Conf(5):** Two triangles  $uvw, uwx$  where  $m^+(uv) + m(uw) + m^+(wx) \geq 7$ .

**Conf(6):** A square  $uvw$  where  $m^+(uv) + m^+(wx) \geq 7$ .

**Conf(7):** A triangle  $uvw$  with  $m^+(uv) + m^+(uw) \geq 7$ .

- Conf(8):** A triangle  $uvw$ , where  $m(uv) = 3$ ,  $m(uw) = 2$ ,  $m(vw) = 2$ , and the second region for one of  $uv, uw, vw$  has no door disjoint from  $uvw$ .
- Conf(9):** A triangle  $uvw$  with  $m(uv), m(uw), m(vw) = 2$ , such that  $u$  has degree at least four, and the second regions for  $uv, uw$  both have at most one door, and no door that is disjoint from  $uvw$ .
- Conf(10):** A square  $uvw$  and a triangle  $wxy$ , where  $m(uv) = m(wx) = m(xy) = 2$ , and  $m(vw) = 4$ .
- Conf(11):** A square  $uvw$  and a triangle  $wxy$ , where  $m(uv) \geq 3$ ,  $m(wy) \geq 3$ ,  $m(wx) = 1$ ,  $m(ux) \leq 3$ , and  $m^+(xy) \geq 3$ .
- Conf(12):** A square  $uvw$  and a triangle  $wxy$ , where  $m^+(uv), m(vw) \geq 2$ ,  $m(wx) = m(wy) = 2$ ,  $m(ux) \leq 3$ , and  $m^+(xy) \geq 3$ .
- Conf(13):** A region  $r$  of length five, with edges  $e_1le_5$  in order, where  $m(e_1) \geq \max(m(e_2), m(e_5))$ ,  $m(e_1) + m(e_2) + m(e_3) \geq 8$  and  $m^+(e_1) + m^+(e_4) \geq 7$ .
- Conf(14):** A region  $r$  and an edge  $e$  of  $C_r$ , such that  $m^+(e) \geq 6$  and at most six edges of  $C_r$  disjoint from  $e$  are doors for  $r$ .
- Conf(15):** A region  $r$  with length at least four, and an edge  $e$  of  $C_r$ , such that  $m^+(e) \geq 4$  and every edge of  $C_r$  disjoint from  $e$  is 3-heavy.
- Conf(16):** A region  $r$  and an edge  $uv$  of  $C_r$ , and a triangle  $uvw$ , such that  $m(uv) + m^+(uv) \geq 4$ , and  $m(vw) \leq m(uw)$ , and the second edge of  $C_r$  incident with  $u$  has multiplicity at most  $m(uw)$ , and every edge of  $C_r$  not incident with  $u$  is 3-heavy.
- Conf(17):** A region  $r$  with length at least five, and an edge  $e$  of  $C_r$ , such that  $m^+(e) \geq 5$ , every edge  $f$  of  $C_r$  disjoint from  $e$  satisfies  $m^+(f) \geq 2$ , and at most one of them is not 3-heavy.
- Conf(18):** A region  $r$  with length at least four and an edge  $uv$  of  $C_r$ , and a triangle  $uvw$ , such that  $m^+(uv) + m(uv) \geq 5$ , and  $m(vw) \leq m(uw)$ , and the second edge of  $C_r$  incident with  $u$  has multiplicity at most  $m(uw)$ , and either
- $m(uv) = 3$  and  $uv$  is 5-heavy, and every edge  $f$  of  $C_r$  disjoint from  $uv$  satisfies  $m^+(f) \geq 2$ , and at most one of them is not 3-heavy, or
  - $m^+(f) \geq 2$  for every edge  $f$  of  $C_r$  not incident with  $u$ , and at most one such edge is not 3-heavy.
- Conf(19):** A region  $r$  with length at least five and an edge  $e$  of  $C_r$ , such that  $m^+(e) \geq 5$ , every edge of  $C_r$  disjoint from  $e$  is 2-heavy, and at most two of them are not 3-heavy.

We will prove these restrictions are too much, that in fact no 8-target is prime (theorem 3.1). To deduce 1.2, we will show that if there is a counterexample, then some counterexample is prime; but for this purpose, just choosing a counterexample with the minimum number of vertices is not enough, and we need a more delicate minimization. If  $(G, m)$  is a  $d$ -target, its *score sequence* is the  $(d+1)$ -tuple  $(n_0, n_1, \dots, n_d)$  where  $n_i$  is the number of edges  $e$  of  $G$  with  $m(e) = i$ . If  $(G, m)$  and  $(G', m')$  are  $d$ -targets, with score sequences  $(n_0, \dots, n_d)$  and  $(n'_0, \dots, n'_d)$  respectively, we say that  $(G', m')$  is *smaller* than  $(G, m)$  if either

- $|V(G')| < |V(G)|$ , or
- $|V(G')| = |V(G)|$  and there exists  $i$  with  $1 \leq i \leq d$  such that  $n'_i > n_i$ , and  $n'_j = n_j$  for all  $j$  with  $i < j \leq d$ , or
- $|V(G')| = |V(G)|$ , and  $n'_j = n_j$  for all  $j$  with  $0 < j \leq d$ , and  $n'_0 < n_0$ .

(The anomalous treatment of  $n_0$  is just a device to allow  $d$ -targets to have edges with  $m(e) = 0$ , while minimum  $d$ -counterexamples have none.) If some  $d$ -target is not  $d$ -edge-colourable, then we can choose a  $d$ -target  $(G, m)$  with the following properties:

- $(G, m)$  is not  $d$ -edge-colourable
- every smaller  $d$ -target is  $d$ -edge-colourable.

Let us call such a pair  $(G, m)$  a *minimum  $d$ -counterexample*. To prove 1.2, we prove two things:

- No 8-target is prime (theorem 3.1), and
- Every minimum 8-counterexample is prime (theorem 4.1).

It will follow that there is no minimum 8-counterexample, and so the theorem is true.

### 3 Discharging and unavoidability

In this section we prove the following, with a discharging argument.

#### 3.1 No 8-target is prime.

The proof is broken into several steps, through this section. Let  $(G, m)$  be a 8-target, where  $G$  is three-connected. For every region  $r$ , we define

$$\alpha(r) = 8 - 4|E(C_r)| + \sum_{e \in E(C_r)} m(e).$$

We observe first:

#### 3.2 The sum of $\alpha(r)$ over all regions $r$ is positive.

**Proof.** Since  $(G, m)$  is a 8-target,  $m(\delta(v)) = 8$  for each vertex  $v$ , and, summing over all  $v$ , we deduce that  $2m(E(G)) = 8|V(G)|$ . By Euler's formula, the number  $R$  of regions of  $G$  satisfies  $|V(G)| - |E(G)| + R = 2$ , and so  $2m(E(G)) - 8|E(G)| + 8R = 16$ . But  $2m(E(G))$  is the sum over all regions  $r$ , of  $\sum_{e \in E(C_r)} m(e)$ , and  $8R - 8|E(G)|$  is the sum over all regions  $r$  of  $8 - 4|E(C_r)|$ . It follows that the sum of  $\alpha(r)$  over all regions  $r$  equals 16. This proves 3.2. ■

We normally wish to pass one unit of charge from every small region to every big region with which it shares an edge; except that in some rare circumstances, sending one unit is too much, and we only send 1/2 or 0. More precisely, for every edge  $e$  of  $G$ , define  $\beta_e(s)$  for each region  $s$  as follows. Let  $r, r'$  be the two regions incident with  $e$ .

- If  $s \neq r, r'$  then  $\beta_e(s) = 0$ .
- If  $r, r'$  are both big or both small then  $\beta_e(r), \beta_e(r') = 0$ .

Henceforth we assume that  $r$  is big and  $r'$  is small; let  $f, f'$  be the edges of  $C_r \setminus e$  that share an end with  $e$ .

- 1: If  $e$  is a door for  $r$  (and hence  $m(e) = 1$ ) then  $\beta_e(r) = \beta_e(r') = 0$ .
- 2: If  $m(e) = 2$  and  $m^+(f) = m^+(f') = 6$  then  $\beta_e(r) = \beta_e(r') = 0$ .
- 3: If  $m(e) = 2$  and  $m^+(f) = 6$  and  $m^+(f') = 5$  or vice versa then  $\beta_e(r) = -\beta_e(r') = 1/2$ .
- 4: If  $m(e) = 3$  and  $m^+(f) = m^+(f') = 5$  then  $\beta_e(r) = \beta_e(r') = 0$ .
- 5: If  $m(e) = 3$  and exactly one of  $m^+(f), m^+(f') = 5$ , then  $\beta_e(r) = -\beta_e(r') = 1/2$ .
- 6: Otherwise  $\beta_e(r) = -\beta_e(r') = 1$ .

(Think of  $\beta_e$  as passing some amount of charge between the two regions incident with  $e$ .) For each region  $r$ , define  $\beta(r)$  to be the sum of  $\beta_e(r)$  over all edges  $e$ . We see that the sum of  $\beta(r)$  over all regions  $r$  is zero.

The effect of  $\beta$  is passing charge from small regions to big regions with which they share an edge. We need another “discharging” function, that passes charge from triangles to small regions with which they share an edge. If  $r$  is a triangle, incident with edges  $e, f, g$ , we define its *multiplicity*  $m(r) = m(e) + m(f) + m(g)$ . A region  $r$  is *tough* if  $r$  is a triangle, its multiplicity is at least five, and if  $r = uvw$  where  $m(uv) = 1$  and  $m(uw) = m(vw) = 2$ , then  $m^+(uw) + m^+(vw) \geq 5$ . For every edge  $e$  of  $G$ , define  $\gamma_e(s)$  for each region  $s$  as follows. Let  $r, r'$  be the two regions incident with  $e$ .

- If  $s \neq r, r'$  then  $\gamma_e(s) = 0$ .
- If one of  $r, r'$  is big, or neither is tough, or they both are tough, then  $\gamma_e(r) = \gamma_e(r') = 0$ .

Henceforth we assume that  $r'$  is tough, and  $r$  is small and not tough. Let  $e, e_1, e_2$  be the edges incident with  $r'$ , and let  $r_1, r_2$  be the regions different from  $r'$  incident with  $e_1, e_2$  respectively.

- 1: If  $m(e) = 1$  and  $m(e_1), m(e_2) \geq 2$ , and  $m^+(e_1) + m^+(e_2) \geq 6$  then  $\gamma_e(r) = -\gamma_e(r') = 1$ .
- 2: If  $m(e) = 1$  and  $m^+(e_1) \geq 4$  and  $m(e_2) = 1$  and  $r_2$  is small, then  $\gamma_e(r) = -\gamma_e(r') = 1/2$ .
- 3: If  $m(e) = 1$  and  $m(e_1) = 3$  and  $m(e_2) = 1$  and  $r_2$  is small, and the edge  $f$  of  $C_r \setminus e$  that shares an end with  $e, e_1$  satisfies  $m(f) = 4$ , then  $\gamma_e(r) = -\gamma_e(r') = 1/2$ .
- 4: If  $m(e) = 2$  and  $m(e_1), m(e_2) \geq 2$  and  $m^+(e_1) + m^+(e_2) \geq 5$ , and either
  - $r$  has more than one door, or



- some door for  $r$  is disjoint from  $e$ , or
- some edge  $f$  of  $C_r$  consecutive with  $e$  has multiplicity four, and  $r_1, r_2$  are both small,

then  $\gamma_e(r) = -\gamma_e(r') = 1$ .

- 5:** If  $m(e) = 2$  and  $m(e_1), m(e_2) = 2$  and some end of  $e$  has degree three, incident with  $e_1$  say, and  $r_1$  is small and  $r_2$  is big, then  $\gamma_e(r) = -\gamma_e(r') = 1/2$ .
- 6:** If  $m(e) = 3$  and  $m(e_1), m(e_2) = 2$  then  $\gamma_e(r) = -\gamma_e(r') = 1$ .
- 7:** Otherwise  $\gamma_e(r) = \gamma_e(r') = 0$ .

For each region  $r$ , define  $\gamma(r)$  to be the sum of  $\gamma_e(r)$  over all edges  $e$ . Again, the sum of  $\gamma(r)$  over all regions  $r$  is zero.

We observe that, immediately from the rules, we have

**3.3** *Let  $e$  be incident with regions  $r, r'$ . Then  $\beta_e(r)$  is non-zero only if exactly one of  $r, r'$  is big; and  $\gamma_e(r)$  is non-zero only if exactly one of  $r, r'$  is tough and neither is big. Thus in all cases, at most one of  $\beta_e(r), \gamma_e(r)$  is non-zero. Moreover  $|\beta_e(r) + \gamma_e(r)| \leq 1$ .*

Let  $\alpha, \beta, \gamma$  be as above. Then the sum over all regions  $r$  of  $\alpha(r) + \beta(r) + \gamma(r)$  is positive, and so there is a region  $r$  with  $\alpha(r) + \beta(r) + \gamma(r) > 0$ . Let us examine the possibilities for such a region. There now begins a long case analysis, and to save writing we just say “by Conf(7)” instead of “since  $(G, m)$  does not contain Conf(7)”, and so on.

**3.4** *If  $r$  is a big region and  $\alpha(r) + \beta(r) + \gamma(r) > 0$ , then  $(G, m)$  is not prime.*

**Proof.** Suppose that  $(G, m)$  is prime. Let  $C = C_r$ . Since  $r$  is big it follows that  $\gamma(r) = 0$ , and so  $\alpha(r) + \beta(r) > 0$ ; that is,

$$\sum_{e \in E(C)} (4 - m(e) - \beta_e(r)) < 8.$$

For  $e \in E(C)$ , define  $\phi(e) = m(e) + \beta_e(r)$ , and let us say  $e$  is *major* if  $\phi(e) > 4$ . If  $e$  is major, then since  $\beta_e(r) \leq 1$ , it follows that  $m(e) \geq 4$ ; and so  $\beta_e(r)$  is an integer, from the  $\beta$ -rules, and therefore  $\phi(e) \geq 5$ . Moreover, no two major edges are consecutive, since  $G$  has minimum degree at least three.

Let  $D$  be the set of doors for  $C$ . Let

- $\xi = 1$  if there are consecutive edges  $e, f$  in  $C$  such that  $\phi(e) > 5$  and  $f$  is a door for  $r$
- $\xi = 2$  if there is no such pair  $e, f$ .

(1) *Let  $e, f, g$  be the edges of a path of  $C$ , in order, where  $e, g$  are major. Then*

$$(4 - \phi(e)) + 2(4 - \phi(f)) + (4 - \phi(g)) \geq 2\xi|\{f\} \cap D|.$$

Let  $r_1, r_2, r_3$  be the regions different from  $r$  incident with  $e, f, g$  respectively. Now  $m(e) \leq 6$  since  $(G, m)$  is prime, and if  $m(e) = 6$  then  $r_1$  is big, by Conf(14), and so  $\beta_e(r) = 0$ ; and so in any case,  $\phi(e) \leq 6$ . Similarly  $\phi(g) \leq 6$ . Also,  $\phi(e), \phi(g) \geq 5$  since  $e, g$  are major. Thus  $\phi(e) + \phi(g) \in \{10, 11, 12\}$ .

Suppose that  $\phi(e) + \phi(g) = 12$ . We must show that  $\phi(f) \leq 2 - \xi|\{f\} \cap D|$ . Now  $m(e) \geq 5$ , and so  $m(f) \leq 2$ , since  $G$  is three-connected. If  $m(f) = 2$  then  $f \notin D$ , and  $\beta_f(r) = 0$  from the  $\beta$ -rules; and so  $\phi(f) \leq 2 - \xi|\{f\} \cap D|$ . If  $m(f) = 1$ , then  $\beta_f(r) \leq 1$ , so we may assume that  $f \in D$ ; but then  $\xi = 1$  and  $\phi(f) = 1 \leq 2 - \xi|\{f\} \cap D|$ .

Next suppose that  $\phi(e) + \phi(g) = 11$ . We must show that  $\phi(f) \leq 5/2 - \xi|\{f\} \cap D|$ . Again one of  $\phi(e), \phi(g) \geq 6$ , say  $\phi(e) = 6$ ; and so  $m^+(e) \geq 6$ . In particular  $m(e) \geq 5$ , and so  $m(f) \leq 2$ . Since  $\phi(g) \geq 5$  we have  $m^+(g) \geq 5$ , and so if  $m(f) = 2$ , then  $\beta_f(r) \leq 1/2$  from the  $\beta$ -rules; and since  $f \notin D$  we have  $\phi(f) \leq 5/2 - \xi|\{f\} \cap D|$ . If  $m(f) = 1$ , then  $\phi(f) \leq 2$ , and so we may assume that  $f \in D$ ; but then  $\xi = 1$  and  $\phi(f) = 1$ , and again  $\phi(f) \leq 5/2 - \xi|\{f\} \cap D|$ .

Finally, suppose that  $\phi(e) + \phi(g) = 10$ . We must show that  $\phi(f) \leq 3 - \xi|\{f\} \cap D|$ . Suppose that  $m(f) \geq 3$ . Since  $m^+(e), m^+(g) \geq 5$  (because  $e, g$  are major), it follows that  $m(f) = 3$ , and  $m(e) = m(g) = 4$  because  $G$  is three-connected; but then  $\beta_f(r) = 0$  from the  $\beta$ -rules, and since  $f \notin D$  we have  $\phi(f) \leq 3 - \xi|\{f\} \cap D|$ . Next suppose that  $m(f) = 2$ . Then  $\phi(f) \leq 3 = 3 - \xi|\{f\} \cap D|$  as required. Lastly if  $m(f) = 1$ , then  $\phi(f) \leq 2$ , so we may assume that  $f \in D$ ; but then  $\xi \leq 2$  and  $\phi(f) = 1 \leq 3 - \xi|\{f\} \cap D|$ . This proves (1).

(2) Let  $e, f$  be consecutive edges of  $C$ , where  $e$  is major. Then

$$(4 - \phi(e)) + 2(4 - \phi(f)) \geq 2\xi|\{f\} \cap D|.$$

We have  $\phi(e) \in \{5, 6\}$ . Suppose that  $\phi(e) = 6$ . We must show that  $\phi(f) \leq 3 - \xi|\{f\} \cap D|$ ; but  $m(f) \leq 2$  since  $m(e) \geq 5$ , and so  $\phi(f) \leq 3$ . We may therefore assume that  $f \in D$ ; but then  $\xi = 1$  and  $\phi(f) = 1 \leq 3 - \xi|\{f\} \cap D|$ . Next, suppose that  $\phi(e) = 5$ ; then we must show that  $\phi(f) \leq 7/2 - \xi|\{f\} \cap D|$ . Since  $m(e) \geq 4$ , it follows that  $m(f) \leq 3$ . If  $m(f) = 3$  then  $m^+(e) = 5$  and so  $\beta_f(r) \leq 1/2$ , from the  $\beta$ -rules; but then  $\phi(f) \leq 7/2 - \xi|\{f\} \cap D|$ . If  $m(f) \leq 2$ , then  $\phi(f) \leq 1$ , so we may assume that  $f \in D$ ; but  $\xi \leq 2$ , and so  $\phi(f) = 1 \leq 7/2 - \xi|\{f\} \cap D|$ . This proves (2).

For  $i = 0, 1, 2$ , let  $E_i$  be the set of edges  $f \in E(C)$  such that  $f$  is not major, and  $f$  meets exactly  $i$  major edges in  $C$ . Let  $D$  be the set of doors for  $C$ . By (1), for each  $f \in E_2$  we have

$$\frac{1}{2}(4 - \phi(e)) + (4 - \phi(f)) + \frac{1}{2}(4 - \phi(g)) \geq \xi|\{f\} \cap D|$$

where  $e, g$  are the major edges meeting  $f$ . By (2), for each  $f \in E_1$  we have

$$\frac{1}{2}(4 - \phi(e)) + (4 - \phi(f)) \geq \xi|\{f\} \cap D|$$

where  $e$  is the major edge consecutive with  $f$ . Finally, for each  $f \in E_0$  we have

$$4 - \phi(f) \geq \xi|\{f\} \cap D|$$

since  $\phi(f) \leq 4$ , and  $\phi(f) = 1$  if  $f \in D$ . Summing these inequalities over all  $f \in E_0 \cup E_1 \cup E_2$ , we deduce that  $\sum_{e \in E(C)} (4 - \phi(e)) \geq \xi|D|$ . Consequently

$$8 > \sum_{e \in E(C)} (4 - m(e) - \beta_e(r)) \geq \xi|D|.$$

But  $|D| \geq 4$  since  $r$  is big, and so  $\xi = 1$  and  $|D| \leq 7$ , a contradiction by Conf(14). This proves 3.4. ■

**3.5** *If  $r$  is a triangle that is not tough, and  $\alpha(r) + \beta(r) + \gamma(r) > 0$ , then  $(G, m)$  is not prime.*

**Proof.** Suppose  $(G, m)$  is prime, and let  $r = uvw$ . Suppose first that  $r$  has multiplicity five; and hence, since it is not tough, we may assume that  $m(uv) = 1$  and  $m(uw) = m(vw) = 2$ , and the second regions for  $uw, vw$  are both big. Thus from the  $\beta$ -rules,  $\beta_{uw}(r), \beta_{vw}(r) = -1$ , and since  $\beta_{uv}(r) + \gamma_{uv}(r) \leq 1$ , we deduce that  $\beta(r) + \gamma(r) \leq -1$ . But

$$\alpha(r) = -4 + m(uv) + m(vw) + m(uw) = 1,$$

contradicting that  $\alpha(r) + \beta(r) + \gamma(r) > 0$ . Thus  $r$  has multiplicity at most four.

Since  $\alpha(r) = -4 + m(uv) + m(vw) + m(uw) \leq 0$ , and  $\beta(r) \leq 0$ , it follows that  $\gamma(r) > 0$ . Hence  $\gamma_e(r) > 0$  for some edge  $e$  incident with  $r$ , say  $e = uv$ .

(1)  *$m(e) = 1$  for every edge  $e$  incident with  $r$  such that  $\gamma_e(r) > 0$ .*

For suppose that  $m(e) > 1$  and  $\gamma_e(r) > 0$ . Since  $r$  has multiplicity at most four it follows that  $m(e) = 2$ . Since  $\gamma_e(r) > 0$ , there is a vertex  $x \neq w$  such that  $uvx$  is a triangle, and  $m(ux), m(vx) \geq 2$ , and one of  $m^+(ux), m^+(vx)$  is at least three, say  $m^+(ux) \geq 3$ ; and  $r$  has two doors. By Conf(5),  $m^+(vw) = 1$ , and so  $\beta_{vw}(r) = -1$  and  $\beta_{uw} \leq 0$ , and hence  $\beta(r) \leq -1$ ; yet  $\gamma(r) \leq 1$ , contradicting that  $\alpha(r) + \beta(r) + \gamma(r) > 0$ . This proves (1).

(2) *There is no edge  $e$  incident with  $r$  and with a big region such that  $m(e) = 1$ .*

Let  $r$  be incident with edges  $e, f, g$ , and suppose that  $m(e) = 1$  and  $e$  is incident with a big region. Thus  $\beta(r) \leq -1$ , and so  $\gamma(r) > 1$ ; and consequently  $\gamma_f(r), \gamma_g(r) > 0$ , and therefore  $m(f) = m(g) = 1$  from (1). But then  $\alpha(r) = -1$ , and yet  $\gamma(r) \leq 2$ , contradicting that  $\alpha(r) + \beta(r) + \gamma(r) > 0$ . This proves (2).

Choose  $e$  with  $\gamma_e(r) > 0$ , say  $e = uv$ . Thus  $m(uv) = 1$ , and there is a tough triangle  $r' = uvx$  say. By Conf(3),  $r'$  has multiplicity at most six.

(3) *We may assume that  $m^+(ux) \leq 3$  and  $m^+(vx) \leq 3$ .*

For suppose that  $m^+(ux) \geq 4$ . By (2),  $m^+(vw) \geq 2$ , contrary to Conf(5). This proves (3).

Now  $\gamma_{uv}(r) > 0$ , and from (1), (3), it follows that  $\gamma_{uv}(r)$  is determined by the first  $\gamma$ -rule. In particular,  $m^+(ux) = 3$ , and  $m^+(vx) = 3$ . By Conf(16),  $uw$  and  $vw$  are not 3-heavy, and so by the same argument  $\gamma_{uw}(r) = 0$  and  $\gamma_{vw}(r) = 0$ ; and so  $\gamma(r) = 1$ . Consequently  $\alpha(r) > -1$ , and so we may assume that  $m(uw) = 2$ . Let  $r_1$  be the second region for  $uw$ . Now  $m(ux) + m(uv) + m(uw) \leq 6$ , and so there is an edge  $f$  incident with  $r_1$  and  $u$  different from  $uw, ux$ . Moreover,  $m(f) \leq 3$ , since  $m(ux) + m(uv) + m(uw) \geq 5$ ; and so if  $r_1$  is big then  $\beta_{uw}(r) = -1$ , a contradiction. Thus  $r_1$  is small, contrary to Conf(5). This proves 3.5. ■

**3.6** *If  $r$  is a tough triangle with  $\alpha(r) + \beta(r) + \gamma(r) > 0$ , then  $(G, m)$  is not prime.*

**Proof.** Suppose  $(G, m)$  is prime, and let  $r = uvw$ . Now  $\alpha(r) = m(uv) + m(vw) + m(uw) - 4$ , so

$$m(uv) + m(vw) + m(uw) + \beta(r) + \gamma(r) > 4.$$

Let  $r_1, r_2, r_3$  be the regions different from  $r$  incident with  $uv, vw, uw$  respectively. It follows that  $\beta_e(r), \gamma_e(r) \leq 0$  for every edge  $e$  of  $r$ .

(1) If  $r_1$  is big then  $\beta_{uv}(r) = -1$ .

For let us examine the  $\beta$ -rules. Certainly  $uv$  is not a door for  $r_1$ , since  $r$  is a triangle; so the first rule does not apply. Let  $f, f'$  be the edges incident with  $r_1$  different from  $uv$  that are incident with  $u, v$  respectively. If the second  $\beta$ -rule applies then  $m(uv) = 2$  and  $m(f), m(f') \geq 5$ , which implies that  $m(uw), m(vw) = 1$ , contradicting that  $uvw$  has multiplicity at least five. If the third rule applies, then  $m(uv) = 2$  and  $m^+(f) = 6$  and  $m^+(f') = 5$  say; but then  $m(uw) = 1$  and  $m(vw) = 2$ , contrary to Conf(1). The fourth rule does not apply, by Conf(1). Thus we assume that the fifth rule applies. Let  $m(uv) = 3$ ,  $m^+(f) = 5$ , and  $m^+(f') < 5$ . Hence  $m(f) = 4$ , and so  $u$  has degree three, and  $m(vw) = 1$  by Conf(2), and  $r_3$  is small, and  $\beta_{uv}(r) = -1/2$ . Since

$$m(uv) + m(vw) + m(uw) + \beta(r) + \gamma(r) > 4$$

it follows that

$$\beta_{uv}(r) + \beta_{vw}(r) + \gamma_{uw}(r) + \gamma_{vw}(r) \geq 0,$$

and since all the terms on the left are non-positive it follows that they are all zero. Now  $r_2$  is not big since  $\beta_{vw}(r) = 0$ , and  $r_3$  is not a triangle by Conf(2), so the third  $\gamma$ -rule applies to  $uw$ , a contradiction since  $\gamma_{uw}(r) = 0$ . This proves (1).

Let  $X = \{u, v, w\}$ . Since  $(G, m)$  is prime, it follows that  $|V(G) \setminus X| \geq 3$ , and  $m(\delta(X)) \geq 10$ . But

$$m(\delta(X)) = m(\delta(u)) + m(\delta(v)) + m(\delta(w)) - 2m(uv) - 2m(uw) - 2m(vw),$$

and so  $10 \leq 8 + 8 + 8 - 2m(uv) - 2m(uw) - 2m(vw)$ , that is,  $r$  has multiplicity at most seven. Suppose first that  $r$  has multiplicity seven. By Conf(3), none of  $r_1, r_2, r_3$  is a triangle. Now  $\beta(r) + \gamma(r) > -3$ . Consequently we may assume that  $\beta_{uv}(r) + \gamma_{uv}(r) > -1$ , and hence  $r_1$  is small by (1). By Conf(7),  $m(uv) + m(uw) < 6$  and hence  $m(vw) \geq 2$ ; and similarly  $m(uw) \geq 2$ . Now  $\gamma_{uv}(r) > -1/2$ , and so the first, fourth and sixth  $\gamma$ -rules do not apply to  $uv$ . Since the first  $\gamma$ -rule does not apply,  $m(uv) > 1$ . Since the sixth  $\gamma$ -rule does not apply, one of  $m(uw), m(vw) > 2$ , say  $m(uw) \geq 3$ , and so  $m(uv) = 2$ ,  $m(uw) = 3$  and  $m(vw) = 2$ . Since the fourth  $\gamma$ -rule does not apply,  $r_1$  has no door disjoint from  $uv$ , contrary to Conf(8).

Next, suppose that  $r$  has multiplicity six. Thus  $\beta(r) + \gamma(r) > -2$ , and so by (1), at most one of  $r_1, r_2, r_3$  is big. Suppose that  $m(uv) = 4$ ; then  $m(vw), m(uw) = 1$ . Since at most one of  $r_1, r_2, r_3$  is big, it follows from Conf(7) that  $r_1$  is big, and hence  $r_2, r_3$  are small. By Conf(3),  $r_2, r_3$  are not tough. By the second  $\gamma$ -rule,  $\gamma_{vw}(r) = \gamma_{uw}(r) = -1/2$ , and since  $\beta_{uv}(r) = -1$  by (1), this contradicts  $\beta(r) + \gamma(r) > -2$ . Thus  $m(uv) \leq 3$ . Suppose next that  $m(uv) = 3$ ; then from the symmetry we may assume that  $m(uw) = 2$  and  $m(vw) = 1$ . Since one of  $r_1, r_2$  is small, and  $r_3$  is not tough by Conf(3), the first  $\gamma$ -rule implies that  $\beta_{vw}(r) + \gamma_{vw}(r) \leq -1$ . Since  $\beta(r) + \gamma(r) > -2$ , it follows from (1) that neither of  $r_1, r_3$  is big, contrary to Conf(7). Thus  $m(uv) \leq 2$ , and similarly  $m(uw), m(vw) \leq 2$ , and so

$m(uv), m(uw), m(vw) = 2$ . Since  $\beta(r) + \gamma(r) > -2$ , it follows that  $\beta_e(r) + \gamma_e(r) \leq -1$  for at most one edge  $e$  incident with  $r$ ; and so we may assume that  $\beta_{uv}(r) + \gamma_{uv}(r) > -1$  and  $\beta_{uw}(r) + \gamma_{uw}(r) > -1$ . By (1),  $r_1, r_3$  are both small. By Conf(3),  $r_1, r_3$  are not tough, and since the fourth  $\gamma$ -rule does not apply, it follows that  $r_1$  has at most one door, and no door disjoint from  $uv$ , and  $r_3$  has at most one door, and no door disjoint from  $uw$ , and  $u$  has degree at least four, contrary to Conf(9).

Finally, suppose that  $r$  has multiplicity five. Now  $\beta(r) + \gamma(r) > -1$ , and hence  $\beta_e(r) + \gamma_e(r) > -1$  for every edge  $e$  incident with  $r$ ; and so by (1)  $r_1, r_2, r_3$  are all small. Suppose that  $m(uv) = 3$ , and hence  $m(uw), m(vw) = 1$ . If neither of  $r_2, r_3$  is tough, then by the second  $\gamma$ -rule,  $\gamma_{uw}(r) = \gamma_{vw}(r) = -1/2$ , a contradiction. Thus we may assume that  $r_3$  is a tough triangle  $uwx$ . By Conf(5),  $m(wx) = 1$ , and so  $m(ux) \geq 3$  since  $r_3$  is tough, contrary to Conf(3). Thus we may assume that  $m(uv) \leq 2$ ; and so from the symmetry we may assume that  $m(uv) = m(uw) = 2$  and  $m(vw) = 1$ . The first  $\gamma$ -rule does not apply to  $vw$ , and so  $r_2$  is a tough triangle  $vwx$ . By Conf(3),  $m(vx), m(wx) \leq 2$ , and so  $m(vx), m(wx) = 2$ . Since  $r_2$  is tough, one of  $vx, wx$  is incident with a small region different from  $uvx$ , contrary to Conf(5). This proves 3.6.  $\blacksquare$

**3.7** *If  $r$  is a small region with length at least four and with  $\alpha(r) + \beta(r) + \gamma(r) > 0$ , then  $(G, m)$  is not prime.*

**Proof.** Suppose that  $(G, m)$  is prime. Let  $C = C_r$ . Note that for each  $e \in E(C)$ ,  $-1 \leq \beta_e(r) \leq 0$  and  $0 \leq \gamma_e(r) \leq 1$ . Since  $\alpha(r) = 8 - 4|E(C)| + \sum_{e \in E(C)} m(e)$ , it follows that

$$8 - 4|E(C)| + \sum_{e \in E(C)} m(e) + \sum_{e \in E(C)} (\beta_e(r) + \gamma_e(r)) > 0,$$

that is,

$$\sum_{e \in E(C)} (m(e) + \beta_e(r) + \gamma_e(r) - 4) > -8.$$

For each  $e \in E(C)$ , let

$$\phi(e) = m(e) + \beta_e(r) + \gamma_e(r).$$

It follows that  $|\phi(e) - m(e)| \leq 1$  for each  $e$  by 3.3. For each integer  $i$ , let  $E_i$  be the set of edges of  $C$  such that  $\phi(e) \in \{i, i - \frac{1}{2}\}$ .

(1) *For every  $e \in E(C)$ ,  $\phi(e)$  is one of  $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4$ , and hence  $E(C)$  is the union of  $E_0, E_1, E_2, E_3, E_4$ .*

For let  $e \in E(C)$ . Since  $m(e) \geq 1$  and  $\beta_e(r) \geq -1$  it follows that  $\phi(e) \geq 0$ . Next we show that  $\phi(e) \leq 4$ . Now  $m(e) < 6$  by Conf(14). Suppose that  $m(e) = 5$ . Then the second region incident with  $e$  is big, by Conf(14); and hence  $\beta_e(r) = -1$  from the  $\beta$ -rules, and  $\gamma_e(r) = 0$  and so  $\phi(e) \leq 4$ . Now suppose that  $m(e) = 4$ . Then by the  $\gamma$ -rules,  $\gamma_e(r) = 0$ , and so  $\phi(e) \leq 4$ . Finally, if  $m(e) \leq 3$  then  $\phi(e) \leq 4$  since  $\gamma_e(r) \leq 1$ . Thus  $\phi(e) \leq 4$  in all cases. Finally, suppose that  $\phi(e) = \frac{7}{2}$ , and hence  $m(e) = 3$  or  $4$ . If  $m(e) = 3$  then  $\gamma_e(r) = 1/2$ , contrary to the  $\gamma$ -rules; while if  $m(e) = 4$  then  $\beta_e(r) = -1/2$ , contrary to the  $\beta$ -rules. This proves (1).

(2) *Let  $e \in E(C)$ ; then  $e \in E_4$  if and only if either  $m^+(e) \geq 5$ , or  $m(e) = 3$  and  $e$  is 5-heavy.*

Moreover, no two edges in  $E_4$  are consecutive in  $C$ .

The first assertion is immediate from the  $\beta$ - and  $\gamma$ -rules. For the second, suppose that  $e, f \in E_4$  share an end  $v$ . Since  $v$  has degree at least three, it follows that  $m(e) + m(f) \leq 7$  and so we may assume that  $m(e) = 3$ . Let  $e$  have ends  $u, v$ ; then from the first assertion there is a triangle  $uvw$  where  $m(uw), m(vw) = 2$ . Hence  $m(f) = 3$ , and so there is similarly a triangle containing  $f$ , with third vertex  $x$ . Consequently  $w = x$ ; but this contradicts Conf(3) and hence proves (2).

(3) *If  $e \in E_4$ , and  $f \in E(C)$  is disjoint from  $e$ , and every edge in  $E(C) \setminus \{f\}$  disjoint from  $e$  is 3-heavy, and there is no edge of  $C$  with multiplicity one disjoint from  $f$ , then  $f \in E_0$ .*

For by Conf(6) if  $|E(C)| = 4$  and  $m^+(e) \geq 5$ , or by Conf(17) or Conf(18) otherwise, it follows that  $m^+(f) = 1$ . Since there is no edge of  $C$  with multiplicity one disjoint from  $f$ , it follows that  $\beta_f(r) = -1$  from the  $\beta$ -rules, and so  $f \in E_0$ . This proves (3).

For  $0 \leq i \leq 4$ , let  $n_i = |E_i|$ .

(4) *If  $e \in E(C)$  satisfies  $m(e) = 2$ , and  $n_4 = 0$ , and  $r$  has at most one door, and no door disjoint from  $e$ , then  $\phi(e) \leq 2$ .*

For if not, then  $\gamma_e(r) > 0$ , and so from the  $\gamma$ -rules, there is a triangle  $uvw$  with  $e = uv$ , and some edge  $f$  of  $C$  consecutive with  $e$  satisfies  $m^+(f) = 5$ ; but then  $f \in E_4$ , contradicting that  $n_4 = 0$ . This proves (4).

(5) *If  $u, v, w$  are consecutive vertices in  $C$ , and  $uv \in E_4$  and  $m(uv) = 3$ , then  $\phi(vw) \leq 2$ .*

For since  $uv \in E_4$ , by (2) there is a triangle  $uvx$  with  $m(ux) = m(vx) = 2$ . From Conf(2) it follows that  $m(vw) \leq 2$ ; and since  $w$  is not adjacent to  $x$  by Conf(3), and hence  $vw$  is not 4-heavy, the  $\gamma$ -rules imply that  $\phi(vw) \leq 2$ . This proves (5).

Let  $C$  have vertices  $v_1lv_k$  in order, and let  $v_{k+1}$  mean  $v_1$ . For  $1 \leq i \leq k$  let  $e_i$  be the edge  $v_i v_{i+1}$ , and let  $r_i$  be the region incident with  $e_i$  different from  $r$ .

Since

$$\sum_{e \in E(C)} (\phi(e) - 4) > -8,$$

we have  $4n_0 + 3n_1 + 2n_2 + n_3 \leq 7$ , that is,

$$3n_0 + 2n_1 + n_2 + k - n_4 \leq 7,$$

since  $n_0 + n_1 + n_2 + n_3 + n_4 = k$ . But by (1),  $n_4 \leq k/2$  and so

$$3n_0 + 2n_1 + n_2 + k/2 \leq 7.$$

Since  $k \geq 4$  it follows that  $3n_0 + 2n_1 + n_2 \leq 5$ , and hence  $n_0 + n_1 \leq 2$ .

**Case 1:**  $n_0 + n_1 = 2$ .

Since  $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$ , we have  $n_4 \geq n_0 + n_2 + k - 3$ . Thus  $n_4 > 0$ . If  $k = 4$ , let  $e \in E_4$ ; then by (3) the edge  $f$  of  $C$  disjoint from  $e$  belongs to  $E_0$ , and so by (2),  $n_4 = 1$ ; but this contradicts  $n_0 + n_2 + k - 3 \leq n_4$ .

Thus  $k \geq 5$ . Since

$$3n_0 + 2n_1 + n_2 + k/2 \leq 7,$$

and  $2n_0 + 2n_1 = 4$  and  $k/2 \geq 5/2$ , it follows that  $n_0 = n_2 = 0$  and  $n_1 = 2$  and  $k \leq 6$ .

Suppose that  $k = 6$ ; then  $n_4 = 3$  since  $n_4 \geq n_0 + n_2 + k - 3$ , so we may assume that  $e_1, e_3, e_5 \in E_4$ . By Conf(17) and Conf(18), it follows that  $m^+(e_4) = 1$ , and hence  $e_4 \in E_0 \cup E_1$ , and similarly  $e_6, e_2 \in E_0 \cup E_1$ , a contradiction since  $n_0 + n_1 = 2$ . Thus  $k = 5$ , and so  $n_4 \geq 2$ , and by (2)  $n_4 = 2$  and we may assume that  $e_1, e_3 \in E_4$ . By Conf(17) and Conf(18),  $m^+(e_4) = 1$ , and similarly  $m^+(e_5) = 1$ . Since  $n_1 = 2$ , and  $n_0, n_2 = 0$ , it follows that  $m(e_2) > 1$ . But then  $e_4 \in E_0$  by (3), contradicting that  $n_0 = 0$ .

**Case 2:**  $k = 4$  and  $n_0 + n_1 = 1$  and  $n_4 > 0$ .

Let  $e_4 \in E_4$ ; by (3),  $e_2 \in E_0$  and so  $m(e_2) = 1$ . By (2) and Conf(2) and Conf(4), it follows that  $m(e_1), m(e_3) \leq 2$ . Now  $e_2$  is the only edge of  $C$  that is not 2-heavy, since  $n_0 + n_1 = 1$ , and in particular  $r$  has at most one door. Since  $4n_0 + 3n_1 + 2n_2 + n_3 \leq 7$  and  $n_0 = 1$ , it follows that  $n_2 \leq 1$ , so we may assume that  $e_1 \notin E_2$ . Thus  $\phi(e_1) > 2$ , and hence  $m(e_1) = 2$ . By (2) and (5),  $m^+(e_4) \geq 5$ , so by Conf(4),  $m(e_4) = 4$ . Since  $\phi(e_1) > 2$ , it follows from the  $\gamma$ -rules that  $r_1$  is a triangle  $v_1v_2w$  say, where  $m(v_1w), m(v_2w) \geq 2$ . Consequently  $m(v_1w) = 2$ . Since  $e_3 \notin E_1$ , it follows that  $m^+(e_3) \geq 2$ ; so  $m(v_2w) = m^+(v_2w) = 2$  by Conf(18) (taking  $v_2, v_1, w$  to be the vertices called  $u, v, w$  in Conf(18) respectively). From Conf(10) it follows that  $m(e_3) = 1$ . From the  $\gamma$ -rules it follows that  $\phi(e_1) = 5/2$ . Since  $\sum_{e \in E(C)} \phi(e) > 8$  and  $\phi(e_2) + \phi(e_4) \leq 4$ , it follows that  $\phi(e_3) \geq 2$ . Since  $m(e_3) = 1$ , the  $\gamma$ -rules imply that  $e_3$  is 3-heavy, contrary to Conf(16) (taking  $v_2, v_1, w$  to be the vertices called  $u, v, w$  in Conf(16) respectively).

**Case 3:**  $k = 4$  and  $n_0 + n_1 = 1$  and  $n_4 = 0$ .

Let  $e_4 \in E_0 \cup E_1$ , and so  $m(e_4) \leq 2$ . Since every edge of  $C$  that is not 2-heavy belongs to  $E_0 \cup E_1$ , it follows that  $e_1, e_2, e_3$  are 2-heavy. Since  $n_4 = 0$ , it follows that  $m^+(e_i) \leq 4$  for  $i = 1, 2, 3, 4$ .

Suppose that  $\phi(e_1) \geq 3$ , and hence  $\phi(e_1) = 3$  by (1) since  $n_4 = 0$ . By (4) it follows that  $m(e_1) \geq 3$ . If  $m^+(e_1) = 3$ , then from the  $\beta$ -rules, the edge  $xv_2$  of  $r_1$  incident with  $v_2$  and different from  $e_1$  has multiplicity four and hence  $m(e_2) = 1$ ; and since  $x, v_3$  are non-adjacent by Conf(2), this contradicts that  $e_2$  is 2-heavy. Thus  $m^+(e_1) \geq 4$ . By Conf(6),  $m^+(e_3) \leq 2$ , and so  $\phi(e_3) \leq 2$  by (4). Since  $\phi(e_2) \leq 3$ , and  $\phi(e_4) \leq 1$ , and  $\sum_{e \in E(C)} \phi(e) > 8$ , it follows that  $\phi(e_2) \geq 5/2$  (and so  $e_2$  is 3-heavy), and  $\phi(e_3) \geq 3/2$ , and  $\phi(e_4) \geq 1/2$  (and so  $m^+(e_4) \geq 2$ ). By Conf(2), it is not the case that  $m(e_3) = 2$  and the edge of  $r_3$  consecutive with  $e_3$  and incident with  $v_3$  has multiplicity four; and so, since  $\phi(e_3) \geq 3/2$ , the  $\beta$ -rules imply that  $m(e_3) = 1$  and  $r_3$  is a triangle  $v_3v_4y$  say. Now by Conf(15), not both  $m(v_3y), m(v_4y) \geq 2$ ; and  $m(e_2) \leq 3$  by Conf(4), so by Conf(18),  $m^+(v_3y), m^+(v_4y) \leq 3$ . But then the  $\gamma$ -rules imply that  $\phi(e_3) \leq 1$ , a contradiction. This proves that  $\phi(e_1) \leq 5/2$ ; and similarly  $\phi(e_3) \leq 5/2$ .

Since  $\sum_{e \in E(C)} \phi(e) > 8$ , and  $\phi(e_2) \leq 3$  (because  $n_4 = 0$ ) it follows that  $\phi(e_1) + \phi(e_3) \geq 9/2$ , and  $\phi(e_4) \geq 1/2$ ; and from the symmetry we may assume that  $\phi(e_1) = 5/2$  and  $\phi(e_3) \geq 2$ . The  $\beta$ -

and  $\gamma$ -rules imply that  $m(e_1) = 3$  (since  $m^+(e_2) \leq 4$ ). Since  $\phi(e_2) + \phi(e_3) \geq 5$ , and  $\phi(e_3) \leq 5/2$ , it follows that  $\phi(e_2) \geq 5/2$  (and hence  $m(e_2) \geq 2$ ).

Suppose that  $m(e_3) = 1$ . Since  $\phi(e_3) \geq 2$ , the first  $\gamma$ -rule applies, and so  $r_3$  is a triangle  $v_3v_4y$ , and  $m(v_3y), m(v_4y) \geq 2$ , and  $m^+(v_3y) + m^+(v_4y) \geq 6$ . By Conf(4),  $m(e_2) \leq 3$ , so by Conf(18),  $m^+(v_3y), m^+(v_4y) \leq 3$ , and hence equality holds for both. By Conf(11),  $m(v_3y), m(v_4y) = 2$ ; but this is contrary to Conf(16).

So  $m(e_3) \geq 2$ , and by Conf(4),  $m(e_2) = m(e_3) = 2$ . If  $m^+(e_3) = 2$ , then from the  $\beta$ -rules it follows that both edges of  $r_3$  consecutive with  $e_3$  have multiplicity five; but this is impossible since  $m(e_2) = 2$ . So  $m^+(e_3) = 3$ . Since  $\phi(e_2) \geq 5/2$  it follows that  $r_2$  is a triangle  $v_2v_3x$ ,  $m(v_2x), m(v_3x) \geq 2$ , and one of  $m^+(v_2x), m^+(v_3x) \geq 3$ , and  $e_4$  is a door for  $r$ . Since  $\phi(e_4) > 0$ , we deduce that  $m^+(e_4) \geq 2$ . By Conf(2),  $m(v_2x) = 2$ . By Conf(12),  $m^+(v_3x) = 2$  and  $m^+(v_2x) = 2$ , a contradiction.

**Case 4:**  $k = 4$  and  $n_0 + n_1 = 0$ .

Since  $n_0, n_1 = 0$ , it follows that  $\phi(e_i) \geq 3/2$  and hence  $e_i$  is 2-heavy, for  $1 \leq i \leq 4$ . Consequently  $n_4 = 0$ , from (3). Since  $\sum_{e \in E(C)} \phi(e) > 8$ , we may assume because of the symmetries of the square that  $\phi(e_1) + \phi(e_3) \geq 9/2$ , and  $\phi(e_1) \geq \phi(e_3)$ , and therefore  $\phi(e_1) \geq 5/2$ . Thus  $m(e_1) \geq 3$  from (4). If some edge  $f$  of the boundary of  $r_1$  consecutive with  $e_1$  satisfies  $m(f) = 4$ , say  $f = v_1x$ , then  $m(e_4) = 1$  and  $v_1$  has degree three; but since  $e_4$  is 2-heavy, it follows that  $x, v_4$  are adjacent, contrary to Conf(2). Thus there is no such  $f$ , and so by the  $\beta$ -rules,  $m^+(e_1) \geq 4$ .

Suppose that  $m(e_3) \geq 2$ . By Conf(6) it follows that  $m^+(e_3) = 2$ , and in particular  $r_3$  is big. Since  $\phi(e_3) \geq 3/2$ , the  $\beta$ -rules imply that some edge  $f$  of the boundary of  $r_3$  consecutive with  $e_3$  satisfies  $m(f) = 5$ , say  $f = v_4x$ ; and since  $x, v_1$  are nonadjacent by Conf(2) it follows that  $e_4 \in E_0 \cup E_1$ , a contradiction. Thus  $m(e_3) = 1$ . Since  $e_3$  is 2-heavy it follows that  $r_3$  is a triangle  $v_3v_4x$  say.

By Conf(4),  $m(e_2), m(e_4) \leq 3$ . By Conf(15), we may assume that  $m(v_3x) = 1$ ; and by Conf(18),  $m^+(v_4x) \leq 3$ . Since  $m(e_4) \leq 3$ , the  $\gamma$ -rules imply that  $\phi(e_3) \leq 1$ , a contradiction.

**Case 5:**  $k \geq 5$  and  $n_0 + n_1 = 1$ .

Since  $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$ , we have  $n_4 \geq n_0 + n_2 + k - 5$ . Let  $E_0 \cup E_1 = \{e_k\}$ .

Suppose that  $n_4 = 0$ . Then since  $n_4 \geq n_0 + n_2 + k - 5$  it follows that  $k = 5$ . Since

$$\sum_{e \in E(C)} \phi(e) > 4k - 8 = 12,$$

and  $\phi(e_5) \leq 1$ , and  $\phi(e_i) \leq 3$  for  $i = 1, 2, 3, 4$  (by (1), since  $n_4 = 0$ ) it follows that  $\phi(e_i) \geq 5/2$  for  $i = 1, 2, 3, 4$ , and hence  $e_1le_4$  are 3-heavy. If  $m(e_1) \leq 2$ , then since  $\phi(e_1) \geq 5/2$  it follows from the  $\gamma$ -rules that  $m(e_2) = 4$  and  $r_2$  is small; but then  $e_2 \in E_4$ , a contradiction. Thus  $m(e_1) \geq 3$ ; so  $m(e_1) = m^+(e_1) = 3$  by Conf(15). Since  $m(e_2) \geq 2$ , it follows that not both edges of  $r_1$  consecutive with  $e_1$  have multiplicity four, and so from the  $\beta$ -rules,  $\phi(e_1) \leq 5/2$ . Similarly  $\phi(e_4) \leq 5/2$ , contradicting that  $\sum_{e \in E(C)} \phi(e) > 12$ . This proves that  $n_4 > 0$ .

Suppose that  $n_2 = 0$ . Thus  $e_1le_4$  are 3-heavy. Since  $n_4 > 0$ , (3) implies that  $n_0 = 1$ . Since  $\phi(e_1) > 2$ , the  $\beta$ - and  $\gamma$ -rules imply that either:

- $m(e_1) = 2$  and  $r_1$  is a triangle  $v_1v_2w$  say; and  $m(v_1w), m(v_2w) \geq 2$ , and  $m(e_2) = 4$ . Consequently  $m(v_2w) = 2$ , contrary to Conf(16).



- $m(e_1) = 3$  and  $r_1$  is big, and, if  $u_1v_1v_2u_2$  is the three-edge path of  $C_{r_1}$  with middle edge  $e_1$ , then one of  $m(u_1v_1), m(u_2v_2) = 4$  and is incident with a small region. But if  $m(u_1v_1) = 4$  then the second region incident with it is  $r_k$ , and this is not small since  $n_0 = 1$ ; and if  $m(u_2v_2) = 4$  then  $v_2$  has degree three and  $m(e_2) = 1$ , and since  $e_2$  is 3-heavy it follows that  $u_2, v_3$  are adjacent, and  $m(u_2v_3) \geq 2$ , contrary to Conf(2).
- $m^+(e_1) \geq 4$ ; but this is contrary to Conf(15).

This proves that  $n_2 \geq 1$ .

Since  $3n_0 + 2n_1 + n_2 + k/2 \leq 7$ , we have  $n_0 + n_2 + k/2 \leq 5$ , and in particular  $n_2 \leq 2$ . If  $e \in E(C)$  is not 3-heavy, then  $\phi(e) \leq 2$  from the  $\gamma$ -rules, and so at most two edges of  $E(C)$  not in  $E_0 \cup E_1$  are not 3-heavy. By Conf(8) and Conf(19) it follows that  $e_1, e_{k-1} \notin E_4$ , so every edge in  $E_4$  is disjoint from  $e_k$ . Since there are three consecutive edges of  $C$  not in  $E_4$ , and no two edges in  $E_4$  are consecutive by (2), it follows that  $n_4 \leq k/2 - 1$ ; and since  $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$ , it follows that  $n_0 + n_2 + k/2 \leq 4$ , and so  $n_2 = 1$ , and  $n_0 = 0$ , and  $k \leq 6$ . In particular, from (5) every edge  $e \in E_4$  has  $m(e) \geq 4$ .

Suppose that  $k = 6$ . Since  $n_4 \geq n_0 + n_2 + k - 5$  and  $n_4 \leq k/2 - 1$ , it follows that  $n_4 = 2$ ; and so  $E_4 = \{e_2, e_4\}$ , since the members of  $E_4$  are disjoint from  $e_6$  and from each other. Since  $e_2 \in E_4$ , (3) implies that  $e_5$  is not 3-heavy, and so  $e_5 \in E_2$ ; and similarly  $e_1 \in E_2$ , a contradiction since  $n_2 = 1$ .

Thus  $k = 5$ . Since  $n_4 \leq k/2 - 1$  it follows that  $n_4 = 1$ , so we may assume that  $E_4 = \{e_2\}$ . By (3),  $e_4$  is not 3-heavy, and so  $\phi(e_4) \leq 2$ . Consequently  $E_2 = \{e_4\}$ , and  $\phi(e_1) + \phi(e_3) \geq 11/2$ . Since  $\phi(e_4), \phi(e_5) > 0$ , it follows that  $m^+(e_4), m^+(e_5) \geq 2$ , and since  $m^+(e_2) \geq 5$ , two applications of Conf(13) imply that  $m(e_3) + m(e_4) \leq 3$  and  $m(e_1) + m(e_5) \leq 3$ . Since  $m(e_1), m(e_3) \geq 2$  (because  $\phi(e_1), \phi(e_3) > 2$ ) it follows that  $m(e_1), m(e_3) = 2$  and  $e_1, e_3$  are 4-heavy; and  $m(e_4), m(e_5) = 1$ . Since  $\phi(e_4) > 1$ ,  $r_4$  is a triangle  $v_4v_5x$  say. Since  $e_4$  is not 3-heavy, one of  $m(v_4x), m(v_5x) = 1$ . If  $m(v_4x) = 1$  then by Conf(16),  $m(xv_5) \leq 2$ ; but then  $\phi(e_4) = 1$  from the  $\gamma$ -rules, a contradiction. So  $m(v_5x) = 1$ . Since  $\phi(e_4) > 1$ , the  $\gamma$ -rules imply that  $m^+(v_4x) \geq 4$ . But this contradicts Conf(18).

**Case 6:**  $k \geq 5$  and  $n_0 + n_1 = 0$ .

Since  $n_0, n_1 = 0$ , it follows that  $\phi(e_i) \geq 3/2$  and hence  $e_i$  is 2-heavy, for  $1 \leq i \leq k$ . Since  $3n_0 + 2n_1 + n_2 + k - n_4 \leq 7$ , we have  $n_4 \geq n_2 + k - 7$ .

Suppose first that  $n_4 > 0$ . By (2) and Conf(8) and Conf(19), every edge in  $E_4$  is disjoint from at least three edges that are not 3-heavy and that therefore belong to  $E_2$ . In particular  $n_2 \geq 3$ . Let  $e \in E_4$ ; then  $e$  is disjoint from all the other edges in  $E_4$ , and from at least three edges in  $E_2$ , so  $k - 3 \geq n_4 - 1 + 3$ , that is,  $k \geq n_4 + 5$ . But  $n_4 \geq n_2 + k - 7 \geq k - 4$ , a contradiction.

This proves that  $n_4 = 0$ , and so  $E(C) = E_2 \cup E_3$ . Since  $n_4 \geq n_2 + k - 7$ , it follows that  $n_2 + k \leq 7$ . In particular,  $k \in \{5, 6, 7\}$ . From (4), every edge  $e \in E(C)$  with  $m(e) = 2$  belongs to  $E_2$ , since  $n_4 = 0$  and there are no doors for  $r$ . Consequently every  $e \in E_3$  satisfies  $m(e) \geq 3$ . Suppose that  $m^+(e) = 3$  for some  $e \in E_3$ , say  $e = e_1$ . Thus  $r_1$  is big, and  $\beta_e(r) > -1$  since  $\phi(e) > 2$ . Hence from the  $\beta$ -rules, some edge of  $C_{r_1}$  consecutive with  $e_1$  has multiplicity four, say  $v_1x$ . Hence  $m(e_k) = 1$ , and since  $n_0, n_1 = 0$ , it follows that  $r_k$  is a triangle, and therefore  $x, v_k$  are adjacent, contrary to Conf(2). This proves that  $m^+(e) \geq 4$  for every  $e \in E_3$ .

By Conf(15), every edge in  $E_3$  is disjoint from some edge in  $E_2$ , and in particular  $n_2 \geq 2$ . Since  $n_2 + k \leq 7$ , we have  $k = 5$  and  $n_2 = 2$ . Every edge in  $E_3$  is disjoint from one of the edges in  $E_2$ , so we may assume that  $e_1, e_2 \in E_2$ , and  $e_3, e_4, e_5 \in E_3$ . Since  $m^+(e_3), m^+(e_4), m^+(e_5) \geq 4$ , Conf(13)

implies that  $m^+(e_1) \leq 2$ ; and by Conf(15),  $e_1$  is not 3-heavy. From the  $\gamma$ -rules,  $\phi(e_1) \leq 3/2$ , and similarly  $\phi(e_2) \leq 3/2$ . But for  $i = 3, 4, 5$ ,  $\phi(e_i) \leq 3$  since  $n_4 = 0$ ; and so  $\sum_{e \in E(C)} \phi(e) \leq 12$ , contradicting our initial assumption that

$$\sum_{e \in E(C)} (\phi(e) - 4) > -8.$$

This completes the proof of 3.7. ■

**Proof of 3.1.** Suppose that  $(G, m)$  is a prime 8-target, and let  $\alpha, \beta, \gamma$  be as before. Since the sum over all regions  $r$  of  $\alpha(r) + \beta(r) + \gamma(r)$  is positive, there is a region  $r$  with  $\alpha(r) + \beta(r) + \gamma(r) > 0$ . But this is contrary to one of 3.4, 3.5, 3.6, 3.7. This proves 3.1. ■

## 4 Reducibility

Now we begin the second half of the paper, devoted to proving the following.

**4.1** *Every minimum 8-counterexample is prime.*

Again, the proof is broken into several steps. Clearly no minimum 8-counterexample  $(G, m)$  has an edge  $e$  with  $m(e) = 0$ , because deleting  $e$  would give a smaller 8-counterexample; and by 2.1, every minimum 8-counterexample satisfies the conclusions of 2.1. Thus, it remains to check that  $(G, m)$  contains none of Conf(1)–Conf(19). Sometimes it is just as easy to prove a result for general  $d$  instead of  $d = 8$ , and so we do so.

**4.2** *If  $(G, m)$  is a minimum  $d$ -counterexample, then every triangle has multiplicity less than  $d$ .*

**Proof.** Let  $uvw$  be a triangle of  $G$ , and let  $X = \{u, v, w\}$ . Since  $|V(G)| \geq 6$ , 2.1 implies that  $m(\delta(X)) \geq d + 2$ . But

$$m(\delta(X)) = m(\delta(u)) + m(\delta(v)) + m(\delta(w)) - 2m(uv) - 2m(uw) - 2m(vw),$$

and so  $d + 2 \leq d + d + d - 2m(uv) - 2m(uw) - 2m(vw)$ , that is,  $m(uv) + m(uw) + m(vw) \leq d - 1$ . This proves 4.2. ■

If  $C$  is a cycle of length four in  $G$ , say with vertices  $u, v, w, x$  in order, let  $m'$  be defined as follows:  $m'(uv) = m(uv) - 1$ ,  $m'(vw) = m(vw) + 1$ ,  $m'(wx) = m(wx) - 1$ ,  $m'(ux) = m(ux) + 1$ , and  $m'(e) = m(e)$  for all other edges  $e$ . If  $(G, m)$  is a minimum  $d$ -counterexample, then because of the second statement of 2.1, it follows that  $(G, m')$  is a  $d$ -target. (Note that possibly  $m'(uv), m'(wx)$  are zero; this is the reason to permit  $m(e) = 0$  in a  $d$ -target.) We say that  $(G, m')$  is obtained from  $(G, m)$  by *switching on the sequence  $u\bar{v}w\bar{x}u$* . If  $(G, m')$  is smaller than  $(G, m)$ , we say that the sequence  $u\bar{v}w\bar{x}u$  is *switchable*.

**4.3** *No minimum  $d$ -counterexample contains Conf(1).*

**Proof.** Suppose that  $(G, m)$  is a minimum  $d$ -counterexample, with a triangle  $uvw$ , where  $u, v$  have degree three. Let the neighbours of  $u, v$  not in  $\{u, v, w\}$  be  $x, y$  respectively. Let  $H$  be a simple graph obtained from  $G$  by adding new edges if necessary to make  $w, x, y$  pairwise adjacent, and extend  $m$  to  $E(H)$  by setting  $m(e) = 0$  for every new edge. Thus  $(H, m)$  is not  $d$ -edge-colourable, and although it may not be a minimum  $d$ -counterexample, no  $d$ -counterexample has fewer vertices.

Define  $f(w) = m(uw) + m(vw)$ ,  $f(x) = m(ux)$ , and  $f(y) = m(vy)$ . Since  $m(\delta(\{u, v\}))$  is even, it follows that  $f(w) + f(x) + f(y)$  is even. Define

$$\begin{aligned} n(wx) &= \frac{1}{2}(f(x) + f(w) - f(y)) \\ n(wy) &= \frac{1}{2}(f(y) + f(w) - f(x)) \\ n(xy) &= \frac{1}{2}(f(x) + f(y) - f(w)). \end{aligned}$$

It follows that  $n(wx), n(wy), n(xy)$  are integers. Since  $m(\delta(\{u, v, w\})) \geq d$  and  $m(\delta(w)) = d$ , it follows that  $m(ux) + m(vy) \geq m(uw) + m(vw)$  and hence  $n(xy) \geq 0$ . Similarly, since  $m(\delta(\{u, v, x\})) \geq d$  and  $m(\delta(x)) = d$ , it follows that  $n(wy) \geq 0$ , and similarly  $n(wx) \geq 0$ .

Let  $G' = H \setminus \{u, v\}$ . For each edge  $e$  of  $G'$ , define  $m'(e)$  as follows. If  $e$  is incident with a vertex different from  $x, y, w$  let  $m'(e) = m(e)$ . For  $e = xy, wx, wy$  let  $m'(e) = m(e) + n(e)$ . We claim that  $(G', m')$  is a  $d$ -target. To show this, let  $X \subseteq V(G')$  with  $|X|$  odd; we must show that  $m'(\delta_{G'}(X)) \geq d$ . By replacing  $X$  by its complement if necessary (which also is odd, since  $|V(G)|$  is even), we may assume that  $X$  contains at most one of  $w, x, y$ . But then from the choice of  $f(w), f(x), f(y)$ , it follows that  $m'(\delta_{G'}(X)) = m(\delta_G(X)) \geq d$  as required. Thus  $(G', m')$  is a  $d$ -target. Since  $|V(G')| < |V(G)|$ , there are  $d$  perfect matchings  $F'_1 \sqcup \dots \sqcup F'_d$  of  $G'$  such that every edge  $e \in E(G')$  is in exactly  $m'(e)$  of them. Now each of  $F'_i \sqcup \dots \sqcup F'_d$  contains at most one of the edges  $wx, wy, xy$ . Let  $I_1, I_2, I_3, I_0$  be the sets of  $i \in \{1, \dots, d\}$  such that  $F'_i$  contains  $wx, wy, xy$  or none of the three, respectively. Thus  $|I_1| = m'(wx) = m(wx) + n(wx)$ . For  $n(wx)$  values of  $i \in I_1$  let  $F_i = (F'_i \setminus \{wx\}) \cup \{ux, vw\}$ , and for the remaining  $m(wx)$  values let  $F_i = F'_i \cup \{uv\}$ . Thus  $F_i$  is a perfect matching of  $G$  for each  $i \in I_1$ . Define  $F_i$  ( $i \in I_2$ ) similarly. For  $n(xy)$  values of  $i \in I_3$  let  $F_i = (F'_i \setminus \{xy\}) \cup \{ux, vy\}$ , and for the others let  $F_i = F'_i \cup \{uv\}$ . For  $i \in I_0$  let  $F_i = F'_i \cup \{uv\}$ . Then  $F_1 \sqcup \dots \sqcup F_d$  are perfect matchings of  $G$ , and we claim that every edge  $e$  is in exactly  $m(e)$  of them. This is clear if  $e$  has an end different from  $u, v, w, x, y$ ; and true from the construction if both ends of  $e$  are in  $\{w, x, y\}$ . From the symmetry we may therefore assume that  $e$  is incident with  $u$ . If  $e = ux$ , then  $e$  belongs to  $n(wx) + n(xy)$  of  $F_1 \sqcup \dots \sqcup F_d$ ; but

$$n(wx) + n(xy) = \frac{1}{2}(f(x) + f(w) - f(y)) + \frac{1}{2}(f(x) + f(y) - f(w)) = f(x) = m(ux)$$

as required. The other two cases are similar. This is a contradiction, since  $(G, m)$  is a minimum  $d$ -counterexample, and so there is no such triangle  $uvw$ . This proves 4.3.  $\blacksquare$

Incidentally, a similar proof would show that  $G$  is four-connected except for cutsets of size three that cut off just one vertex, but we do not need this.

If  $(G, m)$  is a  $d$ -target, and  $x, y$  are distinct vertices both incident with some common region  $r$ , we define  $(G, m) + xy$  to be the  $d$ -target  $(G', m')$  obtained as follows:

- If  $x, y$  are adjacent in  $G$ , let  $(G', m') = (G, m)$ .

- If  $x, y$  are non-adjacent in  $G$ , let  $G'$  be obtained from  $G$  by adding a new edge  $xy$ , extending the drawing of  $G$  to one of  $G'$  and setting  $m'(e) = m(e)$  for every  $e \in E(G)$  and  $m'(xy) = 0$ .

#### 4.4 No minimum $d$ -counterexample contains Conf(2).

**Proof.** Let  $(G, m)$  be a minimum  $d$ -counterexample, with a triangle  $uvw$ , and suppose that  $u$  has only one other neighbour  $x$ , and  $m(ux) < m(uw) + m(vw)$ . Let  $(G', m'') = ((G, m) + vx) + wx$ . For each  $e \in E(G')$ , define  $m'(e)$  as follows. If  $e \neq ux, uw, vw, vx$  let  $m'(e) = m(e)$ . Let

$$\begin{aligned} m'(vx) &= m''(vx) + m(vw) \\ m'(vw) &= 0 \\ m'(ux) &= m(ux) - m(vw) \\ m'(uw) &= m(uw) + m(vw). \end{aligned}$$

Since  $m(uv) + m(uw) + m(ux) = d$  and  $m(uv) + m(uw) + m(vw) \leq d$  since  $m(\delta(\{u, v, w\})) \geq d$ , it follows that  $m(ux) \geq m(vw)$ , and so  $m'(e) \geq 0$  for every edge  $e$ . Moreover,  $m'(\delta(z)) = d$  for every vertex  $z$ , from the construction. We claim that  $(G', m')$  is a  $d$ -target. For let  $X \subseteq V(G')$  with  $|X|$  odd; and we may assume that  $u \notin X$ . We must show that  $m'(\delta(X)) \geq d$ . If  $X$  contains at most one of  $v, w, x$  then  $m'(\delta(X)) = m(\delta(X)) \geq d$  as required, so we may assume that  $X$  contains at least two of  $v, w, x$ . If  $v, w, x \in X$  then  $m'(\delta(X)) \geq m'(\delta(u)) = d$  as required. If  $X \cap \{v, w, x\} = \{v, w\}$  then  $m'(\delta(X)) = m(\delta(X)) + 2m(vw) \geq d$ , and if  $X \cap \{v, w, x\} = \{w, x\}$  then  $m'(\delta(X)) = m(\delta(X)) \geq d$ , so we may assume that  $X \cap \{v, w, x\} = \{v, x\}$ , and hence  $m'(\delta(X)) = m(\delta(X)) - 2m(vw)$ . We must therefore show that in this case,  $m(\delta(X)) \geq 2m(vw) + d$ . To see this, note that

$$\begin{aligned} m(\delta(X \cup \{u, w\})) &= m(\delta(X)) - m(ux) - m(uv) - m(vw) - m''(xw) \\ &\quad + (d - m(uw) - m(vw) - m''(xw)) \leq m(\delta(X)) - 2m(vw) \end{aligned}$$

since  $m''(xw) \geq 0$  and  $m(ux) + m(uv) + m(uw) = d$ . Since  $m(\delta(X \cup \{u, w\})) \geq d$ , it follows that  $m(\delta(X)) \geq 2m(vw) + d$  as required. This proves that  $(G', m')$  is a  $d$ -target. Since  $m'(uw) > m(ux), m(vw)$  (the first from the hypothesis), it follows that  $(G', m')$  is smaller than  $(G, m)$ , and so is  $d$ -edge-colourable; let  $F'_1 l F'_d$  be a  $d$ -edge-colouring. Now every perfect matching containing  $vx$  also contains  $uw$ , since  $vx$  is not disjoint from any other edge incident with  $u$ . Hence there are at least  $m(vw)$  of  $F'_1 l F'_d$  that contain both  $vx$  and  $uw$ . Choose  $m(vw)$  of them, say  $F'_1 l F'_{m(vw)}$ ; and for  $1 \leq i \leq m(vw)$  define  $F_i = (F'_i \setminus \{vx, uw\}) \cup \{vw, ux\}$ . Define  $F_i = F'_i$  for  $m(vw) + 1 \leq i \leq d$ . Then every edge  $e$  of  $G$  is in  $m(e)$  of  $F_1 l F_d$ , a contradiction. Thus there is no such triangle  $uvw$ . This proves 4.4. ■

#### 4.5 No minimum 8-counterexample contains Conf(3) or Conf(4).

**Proof.** To handle both cases at once, let us assume that  $(G, m)$  is an 8-target, and  $uvw, uwx$  are triangles with  $m(uv) + m(uw) + m(vw) + m(ux) \geq 8$ , (where possibly  $m(uw) = 0$ ); and either  $(G, m)$  is a minimum 8-counterexample, or  $m(uw) = 0$  and deleting  $uw$  gives a minimum 8-counterexample  $(G_0, m_0)$  say. We must show that  $m(uw) = 0$  and  $(m(uv), m(vw), m(wx), m(ux)) = (4, 2, 1, 2)$ . Let  $(G, m')$  be obtained by switching  $(G, m)$  on  $u\dot{y}\dot{w}\dot{x}u$ .

(1)  $(G, m')$  is not smaller than  $(G, m)$ .

Because suppose it is. Then it admits an 8-edge-colouring; because if  $(G, m)$  is a minimum 8-counterexample this is clear, and otherwise  $m(uw) = 0$ , and  $(G', m')$  is smaller than  $(G_0, m_0)$ . Let  $F'_1 l F'_8$  be an 8-edge-colouring of  $(G', m')$ . Since

$$m'(uv) + m'(uw) + m'(vw) + m'(ux) \geq 9,$$

one of  $F'_1 l F'_8$ , say  $F'_1$ , contains two of  $uv, uw, vw, ux$  and hence contains  $vw, ux$ . Then

$$(F'_1 \setminus \{vw, ux\}) \cup \{uv, wx\}$$

is a perfect matching, and it together with  $F'_2 l F'_8$  provide an 8-edge-colouring of  $(G, m)$ , a contradiction. This proves (1).

From (1) we deduce that  $\max(m(ux), m(vw)) < \max(m(uv), m(wx))$ . It follows that

$$m(uv) + m(uw) + m(vw) + m(wx) \leq 7,$$

by (1) applied with  $u, w$  exchanged; and

$$m(uv) + m(ux) + m(wx) + m(uw) \leq 7,$$

by (1) applied with  $v, x$  exchanged. Consequently  $m(ux) > m(wx)$ , and hence  $m(ux) \geq 2$ ; and  $m(vw) > m(wx)$ , and so  $m(vw) \geq 2$ . Suppose that  $m(uv) \leq 3$ . Since

$$\max(m(ux), m(vw)) < \max(m(uv), m(wx)),$$

it follows that  $m(uv) = 3$  and  $m(vw) = m(ux) = 2$ ; and therefore  $m(wx) = 1$ , since  $m(ux) > m(wx)$ . But this is contrary to (1).

We deduce that  $m(uv) \geq 4$ . Since  $m(vw) \geq 2$  and  $m(uv) + m(uw) + m(vw) + m(wx) \leq 7$ , it follows that  $m(uw) + m(wx) \leq 1$ ; so  $m(uw) = 0$  and  $m(wx) = 1$ . But then

$$(m(uv), m(vw), m(wx), m(ux)) = (4, 2, 1, 2).$$

This proves 4.5. ■

## 5 Guenin's cuts

We still have many configurations to handle, to finish the proof of 4.1, but all the others are handled by a method of Guenin [5], which we introduce in this section. In particular, nothing so far has assumed the truth of 1.1 for  $d = 7$ , but now we will need to use that.

Let  $(G, m)$  be a  $d$ -target, and let  $xuyy$  be a three-edge path of  $G$ , where  $x, y$  are incident with a common region. Let  $(G', m')$  be obtained from  $(G, m) + xy$  by switching on the cycle  $xuyy\bar{x}$ . We say that  $(G', m')$  is obtained from  $(G, m)$  by *switching on  $xuyy$* . If  $(G', m')$  is smaller than  $(G, m)$ , we say that the path  $xuyy$  is *switchable*.

Let  $G$  be a three-connected graph drawn in the plane, and let  $G^*$  be its dual graph; let us identify  $E(G^*)$  with  $E(G)$  in the natural way. A *cocycle* means the edge-set of a cycle of the dual graph; thus,  $Q \subseteq E(G)$  is a cocycle of  $G$  if and only if  $Q$  can be numbered  $\{e_1le_k\}$  for some  $k \geq 3$  and there are distinct regions  $r_1lr_k$  of  $G$  such that  $1 \leq i \leq k$ ,  $e_i$  is incident with  $r_i$  and with  $r_{i+1}$  (where  $r_{k+1}$  means  $r_1$ ).

Guenin's method is the use of the following:

**5.1** *Suppose that  $d \geq 1$  is an integer such that every  $(d-1)$ -regular oddly  $(d-1)$ -edge-connected planar graph is  $(d-1)$ -edge-colourable. Let  $(G, m)$  be a minimum  $d$ -counterexample, and let  $xuyy$  be a path of  $G$  with  $x, y$  on a common region. Let  $(G', m')$  be obtained by switching on  $xuyy$ , and let  $F_1lF_d$  be a  $d$ -edge-colouring of  $(G', m')$ , where  $xy \in F_k$ . Let  $I = \{1ld\} \setminus \{k\}$  if  $xy \notin E(G)$ , and  $I = \{1ld\}$  if  $xy \in E(G)$ . Then for each  $i \in I$ , there is a cocycle  $Q_i$  of  $G'$  with the following properties:*

- for  $1 \leq j \leq d$  with  $j \neq i$ ,  $|F_j \cap Q_i| = 1$ ;
- $|F_i \cap Q_i| \geq 5$ ;
- there is a set  $X \subseteq V(G)$  with  $|X|$  odd such that  $\delta_{G'}(X) = Q_i$ ; and
- $uv, xy \in Q_i$  and  $ux, vy \notin Q_i$ .

**Proof.** Let  $i \in I$ . If  $i \neq k$  and  $xy \in F_i$ , it follows that  $m'(xy) \geq 2$  since  $xy \in F_k$ ; and so  $xy \in E(G)$ . Thus in either case  $F_i$  is a perfect matching of  $G$ . For each edge  $e$  of  $G'$ , let  $p(e) = 1$  if  $e \in F_i$ , and  $p(e) = 0$  otherwise; and for each edge  $e$  of  $G$ , let  $n(e) = m(e) - p(e)$ . Thus  $(G, n)$  has the property that for each vertex  $z$ ,  $n(\delta_G(z)) = d - 1$ . If there is a list of  $d - 1$  perfect matchings of  $G$  such that every edge  $e$  is in  $n(e)$  of them, then adding  $F_i$  to this list gives a  $d$ -edge-colouring of  $(G, m)$ , a contradiction. Thus by hypothesis, there exists  $Y \subseteq V(G)$  with  $|Y|$  odd and with  $n(\delta_G(Y)) < d - 1$ . Since  $|Y|$  and  $n(\delta_G(Y))$  have the same parity, it follows that  $n(\delta_G(Y)) \leq d - 3$ . Since  $\delta_G(Y)$  is an edge-cut of the connected graph  $G$ , it can be partitioned into "bonds" (edge-cuts  $\delta_G(X)$  such that  $G[X]$ ,  $G \setminus X$  are both connected), and hence one of these bonds  $\delta_G(X)$  has  $n(\delta_G(X))$  odd, and consequently  $|X|$  also odd. Since  $\delta_G(X)$  is a bond of  $G$  and hence  $\delta_{G'}(X)$  is a bond of  $G'$ , there is a cocycle  $Q_i$  of  $G'$  with  $Q_i = \delta_{G'}(X)$ . We claim that  $Q_i$  satisfies the theorem. For we have seen the third assertion; we must check the other three.

From the choice of  $X$  we have  $n(\delta_G(X)) \leq d - 3$ . Since  $|X|, |V(G) \setminus X| \geq 3$  (because  $n(\delta_G(z)) = d - 1$  for each vertex  $z$ ), it follows from 2.1 that  $m(\delta_G(X)) \geq d + 2$ , and so  $p(\delta_G(X)) \geq 5$ , that is,  $|F_i \cap Q_i| \geq 5$ . This proves the second assertion. We recall that  $F_1lF_d$  is a  $d$ -edge-colouring of  $(G', m')$ ; and so for  $1 \leq j \leq d$  with  $j \neq i$ , some edge of  $\delta_{G'}(X)$  belongs to  $F_j$ , and so

$$\sum_{1 \leq j \leq d, j \neq i} |F_j \cap Q_i| \geq d - 1.$$

On the other hand, every edge  $e$  of  $G'$  belongs to  $m'(e)$  of  $F_1lF_d$ , and hence to  $m'(e) - p(e)$  of the  $d - 1$  perfect matchings in this list without  $F_i$ . Consequently

$$\sum_{1 \leq j \leq d, j \neq i} |F_j \cap Q_i| = \sum_{e \in Q_i} m'(e) - p(e).$$

It follows that  $\sum_{e \in Q_i} m'(e) - p(e) \geq d - 1$ ; but  $m'(e) - p(e) = n(e)$  for all edges of  $G'$  except  $xu, uv, vy, xy$ , and so

$$|\{uv, xy\} \cap Q_i| - |\{ux, vy\} \cap Q_i| + \sum_{e \in Q_i} n(e) \geq d - 1.$$

Since  $\sum_{e \in Q_i} n(e) \leq d - 3$ , it follows that  $uv, xy \in Q_i$  and  $ux, vy \notin Q_i$ . This proves the fourth assertion. Moreover, since

$$\sum_{1 \leq j \leq d, j \neq i} |F_j \cap Q_i| = d - 1,$$

it follows that  $|F_j \cap Q_i| = 1$  for all  $j \in \{1, \dots, d\}$  with  $j \neq i$ . This proves the first assertion, and so proves 5.1. ■

By the result of [6], every 7-regular oddly 7-edge-connected planar graph is 7-edge-colourable, so we can apply 5.1 when  $d = 8$ .

## 5.2 No minimum 8-counterexample contains Conf(5) or Conf(6).

**Proof.** To handle both at once, let us assume that  $(G, m)$  is an 8-target, and  $uvw, uwx$  are two triangles with  $m^+(uv) + m(uw) + m^+(wx) \geq 7$ ; and either  $(G, m)$  is a minimum 8-counterexample, or  $m(uw) = 0$  and deleting  $uw$  gives a minimum 8-counterexample. We claim that  $u\bar{x}w\bar{v}y\bar{u}$  is switchable. For suppose not; then we may assume that  $m(vw) > \max(m(uv), m(wx))$  and  $m(vw) \geq m(ux)$ . Now since one of  $m(uv), m(wx) \geq 3$ , and 4.5 implies that we do not have Conf(3) or Conf(4), it follows that

$$m(uv) + m(uw) + m(vw) + m(wx) \leq 7.$$

Yet  $m(uv) + m(uw) + m(wx) \geq 5$  since  $m^+(uv) + m(uw) + m^+(wx) \geq 7$ ; and so  $m(vw) \leq 2$ . Consequently  $m(uv), m(wx) = 1$ , and  $m(ux) \leq 2$ . Since  $u\bar{x}w\bar{v}y\bar{u}$  is not switchable, it follows that  $m(ux) = 2$ ; and since  $m^+(uv) + m(uw) + m^+(wx) \geq 7$ , it follows that  $m(uw) \geq 3$ , giving Conf(3), contrary to 4.5. This proves that  $u\bar{x}w\bar{v}y\bar{u}$  is switchable.

Let  $r_1, r_2$  be the second regions incident with  $uv, wx$  respectively, and for  $i = 1, 2$  let  $D_i$  be the set of doors for  $r_i$ . Let  $k = m(uv) + m(uw) + m(wx) + 2$ . Let  $(G, m')$  be obtained by switching, and let  $F_1, \dots, F_k$  be an 8-edge-colouring of  $(G, m')$ , where  $F_i$  contains one of  $uv, uw, wx$  for  $1 \leq i \leq k$ . For  $1 \leq i \leq 8$ , let  $Q_i$  be as in 5.1.

(1) For  $1 \leq i \leq 8$ , either  $F_i \cap Q_i \cap D_1 \neq \emptyset$ , or  $F_i \cap Q_i \cap D_2 \neq \emptyset$ ; and both are nonempty if either  $k = 8$  or  $i = 8$ .

For let the edges of  $Q_i$  in order be  $e_1, \dots, e_n, e_1$ , where  $e_1 = wx, e_2 = uw$ , and  $e_3 = uv$ . Since  $F_j$  contains one of  $e_1, e_2, e_3$  for  $1 \leq j \leq k$ , it follows that none of  $e_4, \dots, e_n$  belongs to any  $F_j$  with  $j \leq k$  and  $j \neq i$ , and, if  $k = 7$  and  $i \neq 8$ , that only one of them is in  $F_8$ . But since at most one of  $e_1, e_2, e_3$  is in  $F_i$  and  $|F_i \cap Q_i| \geq 5$ , it follows that  $n \geq 7$ ; so either  $e_4, e_5$  belong only to  $F_i$ , or  $e_n, e_{n-1}$  belong only to  $F_i$ , and both if  $k = 8$  or  $i = 8$ . But if  $e_4, e_5$  are only contained in  $F_i$ , then they both have multiplicity one, and are disjoint, so  $e_4$  is a door for  $r_1$  and hence  $e_4 \in F_i \cap Q_i \cap D_1$ . Similarly if  $e_n, e_{n-1}$  are only contained in  $F_i$  then  $e_n \in F_i \cap Q_i \cap D_2$ . This proves (1).

Now  $k \leq 8$ , so one of  $r_1, r_2$  is small since  $m^+(uv) + m(uw) + m^+(wx) \geq 7$ ; and if  $k = 8$  then by (1)  $|D_1|, |D_2| \geq 8$ , a contradiction. Thus  $k = 7$ , so both  $r_1, r_2$  are small, but from (1)  $|D_1| + |D_2| \geq 9$ , again a contradiction. This proves 5.2.  $\blacksquare$

### 5.3 No minimum 8-counterexample contains Conf(7).

**Proof.** Let  $(G, m)$  be a minimum 8-counterexample, and suppose that  $uvw$  is a triangle with  $m^+(uv) + m^+(uw) \geq 7$ . Let  $r_1, r_2$  be the second regions for  $uv, uw$  respectively, and for  $i = 1, 2$  let  $D_i$  be the set of doors for  $r_i$ . By 5.2, we do not have Conf(5), so neither of  $r_1, r_2$  is a triangle. Since  $m(uv) + m(uw) \geq 5$ , one of  $m(uv), m(uw) \geq 3$ , so we may assume that  $m(uv) \geq 3$ . Let  $tu$  be the edge incident with  $r_2$  different from  $uw$ . Since  $m(uv) + m(uw) \geq 5$ , it follows that  $m(tu) \leq 3$ , and by 4.2,  $m(vw) \leq 2$ . Thus the path  $tuyw$  is switchable. Note that  $t, w$  are non-adjacent in  $G$ , since  $r_2$  is not a triangle. Let  $(G', m')$  be obtained by switching on this path, and let  $F_1lF_8$  be an 8-edge-colouring of it. Let  $k = m(uv) + m(uw) + 2$ ; thus  $k \geq 7$ , since  $m(uv) + m(uw) \geq 5$ , and we may assume that for  $1 \leq j < k$ ,  $F_j$  contains one of  $uv, uw$ , and  $tw \in F_k$ .

Let  $I = \{1l8\} \setminus \{k\}$ , and for each  $i \in I$ , let  $Q_i$  be as in 5.1. Now let  $i \in I$ , and let the edges of  $Q_i$  in order be  $e_1le_n, e_1$ , where  $e_1 = uv, e_2 = uw$ , and  $e_3 = tw$ . Since  $F_j$  contains one of  $e_1, e_2, e_3$  for  $1 \leq j \leq k$  it follows that none of  $e_4le_n$  belong to any  $F_j$  with  $j \leq k$ ; and if  $k = 7$  and  $i \neq 8$ , only one of them belongs to  $F_8$ . Since  $F_i$  contains at most one of  $e_1, e_2, e_3$  and  $|F_i \cap Q_i| \geq 5$ , it follows that  $n \geq 7$ , and so either  $e_4, e_5$  are only contained in  $F_i$ , or  $e_n, e_{n-1}$  are only contained in  $F_i$ ; and both if either  $k = 8$  or  $i = 8$ . Thus either  $e_4 \in F_i \cap Q_i \cap D_2$  or  $e_n \in F_i \cap Q_i \cap D_1$ , and both if  $k = 8$  or  $i = 8$ . Since  $k \leq 8$ , one of  $r_1, r_2$  is small since  $m^+(uv) + m^+(uw) \geq 7$ ; and yet if  $k = 8$  then  $|D_1|, |D_2| \geq |I| = 7$ , a contradiction. Thus  $k = 7$ , so  $r_1, r_2$  are both small, and yet  $|D_1| + |D_2| \geq 8$ , a contradiction. This proves 5.3.  $\blacksquare$

### 5.4 No minimum 8-counterexample contains Conf(8).

**Proof.** Let  $(G, m)$  be a minimum 8-counterexample, and suppose that  $uvw$  is a triangle, and its edges have multiplicities 3, 2, 2 (in some order). We will show that the second region  $r$  for  $uw$  has a door disjoint from  $uw$ . By 4.5, we do not have Conf(3), so  $r$  is not a triangle. By exchanging  $u, w$  if necessary we may assume that  $m(vw) = 2$ . Let  $tu$  be the edge incident with  $r$  different from  $uw$ . We claim that the path  $tuyw$  is switchable. For certainly  $m(uv) \geq m(vw)$ , so it suffices to check that  $m(uv) \geq m(tu)$ . If not, then since  $m(uv) \geq 2$  and  $m(uv) + m(uw) \geq 5$ , it follows that  $m(uv) = 2, m(tu) = 3$  and  $m(uw) = 3$ , and we have Conf(2), contrary to 4.4. Thus  $tuyw$  is switchable. Let  $(G', m')$  be obtained by switching, and let  $F_1lF_8$  be an 8-edge-colouring of  $(G', m')$ . Since  $m'(uv) + m'(uw) = 6$ , we may assume that  $F_1lF_6$  each contain one of  $uv, uw$ ; and  $tw \in F_7$ , and therefore  $vw \in F_8$ . Let  $I = \{1l6, 8\}$ ; and for  $i \in I$ , let  $Q_i$  be as in 5.1. Since  $Q_8$  contains  $uv, uw, tw$  and  $F_1lF_7$  each contain one of  $uv, uw, tw$ , it follows that no other edge of  $Q_8$  belongs to any of  $F_1lF_7$ , and so  $Q_8 \cap F_8$  contains a door for  $r$ , say  $e$ . Moreover  $e \neq tu$  since  $tu \notin Q_8$ ; and  $e$  is not incident with  $w$  since  $vw \in F_8$ . Consequently  $e$  is disjoint from  $uw$ . This proves 5.4.  $\blacksquare$

### 5.5 No minimum 8-counterexample contains Conf(9).



**Proof.** Let  $(G, m)$  be a minimum 8-counterexample, and suppose that  $uv_1v_2$  is a triangle, with  $m(uv_1), m(uv_2), m(v_1v_2) = 2$ , such that the second regions  $r_1, r_2$  for  $uv_1, uv_2$  respectively both have at most one door, and no door that is disjoint from  $uv_1v_2$ . For  $i = 1, 2$ , let  $D_i$  be the set of doors for  $r_i$ . For  $i = 1, 2$ , let  $ux_i$  and  $v_iy_i$  be edges incident with  $r_i$  different from  $uv_i$ .

Now  $x_1 \neq x_2$  since  $u$  has degree at least four; and so  $m(ux_1) + m(ux_2) \leq 4$  and we may assume that  $m(ux_1) \leq 2$ . Consequently the path  $x_1uv_2v_1$  is switchable. Note that  $v_1, x_1$  may be adjacent, but if so then  $m(v_1x_1) = 1$  from 4.2. Let  $(G', m')$  be obtained by switching, and let  $F_1lF_8$  be an 8-edge-colouring, where  $uv_2 \in F_1, F_2, F_3$ , and  $uv_1 \in F_4, F_5$  and  $v_1x_1 \in F_6$ , and  $v_1x_1 \in F_7$  if  $v_1x_1 \in E(G)$ . Since  $v_1v_2$  belongs to some  $F_i$ , and  $v_1v_2$  meets all of  $uv_2, uv_1, v_1x_1$ , we may assume that  $v_1v_2 \in F_8$ . Let  $I = \{1l5, 7, 8\}$  if  $x_1v_1 \notin E(G)$ , and  $I = \{1l8\}$  otherwise. For  $i \in I$ , let  $Q_i$  be as in 5.1.

We claim that  $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$  for  $i = 7, 8$ . First suppose that  $v_1x_1 \notin E(G)$ . Then for  $1 \leq j \leq 6$  and for  $i = 7, 8$ ,  $F_j \cap Q_i \cap \{uv_2, uv_1, v_1x_1\} \neq \emptyset$ , and so no other edges of  $Q_i$  belong to any  $F_j$  with  $j \in \{1l6\}$ . Since only one edge of  $Q_i \setminus \{uv_2, uv_1, v_1x_1\}$  belongs to the  $F_j$  with  $j \in \{7, 8\} \setminus \{i\}$ , it follows that  $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$  as required. Now suppose that  $v_1x_1 \in E(G)$ . Then for  $1 \leq j \leq 7$  and for  $i = 7, 8$ ,  $F_j \cap Q_i \cap \{uv_2, uv_1, v_1x_1\} \neq \emptyset$ . and so no other edges of  $Q_i$  belong to any  $F_j$  with  $j \in \{1l7\}$  and  $j \neq i$ . For  $i = 7$ , as before it follows that  $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$ ; for  $i = 8$  we find that  $F_i \cap Q_i \cap D_1, F_i \cap Q_i \cap D_2 \neq \emptyset$ . Thus in any case, we have  $F_i \cap Q_i \cap (D_1 \cup D_2) \neq \emptyset$  for  $j = 7, 8$ .

Now by hypothesis,  $D_1 \cup D_2 \subseteq \{ux_1, ux_2, v_1y_1, v_2y_2\}$ ; and  $ux_1 \notin Q_7, Q_8$  from the choice of switchable path, and  $v_1y_1, v_2y_2 \notin F_8$  since  $v_1v_2 \in F_8$ . Thus  $ux_2 \in F_8 \cap D_2$ . Since  $|D_2| \leq 1$  by hypothesis, it follows that  $v_2y_2 \notin D_2$ , and  $ux_2 \notin F_7$  since  $ux_2 \in F_8$  and  $m(ux_2) = 1$ . Thus  $v_1y_1 \in D_1$ . Now  $m(ux_2) = 1$ , and so the path  $x_2uv_1v_2$  is switchable; so by the same argument with  $v_1, v_2$  exchanged, it follows that  $ux_1 \in D_1$  and  $v_2y_2 \in D_2$ , contrary to the hypothesis. This proves 5.5.

### 5.6 No minimum 8-counterexample contains Conf(10).

**Proof.** For suppose that  $(G, m)$  is a minimum counterexample, with a square  $uvwx$  and a triangle  $wxy$ , where  $m(uv) = m(wx) = m(xy) = 2$ , and  $m(vw) = 4$ . By 4.5, we do not have Conf(4), and it follows that  $m(ux) = 1$ . Since  $m(\delta(w)) = 8$  it follows that  $m(wy) \leq 2$ , and so  $uxyw$  is switchable. Let  $(G', m')$  be obtained by switching on this path, and let  $F_1lF_8$  be an 8-edge-colouring of it. We may assume that  $xy \in F_1, F_2, F_3$ , and  $xw \in F_4, F_5$ , and  $uw \in F_6$ . Let  $I = \{1l8\} \setminus \{6\}$ , and let  $Q_i$  ( $i \in I$ ) be as in 5.1. Now  $vw \notin F_4, F_5, F_6$ , so there are four values of  $i \in \{1, 2, 3, 7, 8\}$  such that  $vw \in F_i$ , and from the symmetry we may assume that  $F_1, F_2, F_7$  contain  $vw$  (and so does one of  $F_3, F_8$ ). It follows that  $vw \notin Q_i$  for  $i \in I$ , and so  $uv \in Q_i$  for each  $i \in I$ . Since  $uv$  belongs to two of  $F_1lF_8$ , there exists  $j \neq 8$  with  $uv \in F_j$ . Moreover,  $F_j$  does not contain  $vw$ , and so  $j \neq 1, 2, 7$ ; so  $j \in \{3, 4, 5, 6\}$ . But  $|Q_1 \cap F_j| \geq 2$ , since one of  $xy, xw, vw \in Q_1 \cap F_j$ , a contradiction. This proves 5.6. ■

### 5.7 No minimum 8-counterexample contains Conf(11), Conf(12) or Conf(13).

**Proof.** To handle all these cases simultaneously, let us assume that  $(G, m)$  is a 8-target, and  $v_1v_2v_3v_4v_5v_1$  are the vertices in order of some cycle of  $G$ , and this cycle bounds a disc which is

the union of three triangles of  $G$ , namely  $v_1v_2v_3$ ,  $v_1v_3v_5$  and  $v_3v_4v_5$ . Moreover, there is a subset  $Z \subseteq \{v_1v_3, v_3v_5\}$  such that  $m(e) = 0$  for all  $e \in Z$  and deleting the edges in  $Z$  gives a minimum 8-counterexample. Finally, we assume that

$$m(v_1v_2) + m(v_1v_3) + m(v_2v_3) + m(v_3v_4) + m(v_3v_5) \geq 8,$$

and

$$m^+(v_1v_2) + m(v_1v_3) + m(v_3v_5) + m^+(v_4v_5) \geq 7.$$

To obtain the subcases Conf(11), Conf(12) and Conf(13), we set, respectively,

- $Z = \{v_1v_3\}$ ,  $m(v_1v_2) \geq 3$ ,  $m(v_3v_4) \geq 3$ ,  $m(v_3v_5) = 1$ ,  $m^+(v_4v_5) \geq 3$ , and  $m(v_1v_5) \leq 3$
- $Z = \{v_3v_5\}$ ,  $m^+(v_1v_2) \geq 3$ ,  $m(v_2v_3) = 2$ ,  $m(v_3v_4) \geq 2$ ,  $m(v_1v_3) = 2$ ,  $m(v_1v_5) \leq 3$  and  $m^+(v_4v_5) \geq 2$
- $Z = \{v_1v_3, v_3v_5\}$ ,  $m(v_1v_2) \geq \max(m(v_2v_3), m(v_1v_5))$ .

(Edges not mentioned are unrestricted.) Let  $(G, m')$  be obtained by switching on the sequence  $v_2v_3v_5v_1v_2$ . (We postpone for the moment the question of whether this sequence is switchable.) Let us suppose (for a contradiction) that  $(G, m')$  admits an 8-edge-colouring  $F_1lF_8$ . Let  $k = m(v_1v_2) + m(v_1v_3) + m(v_3v_5) + 2$ ; then we may assume that  $F_1lF_k$  each contain exactly one of  $v_1v_2, v_1v_3, v_3v_5$ , and  $v_3v_5 \in F_k$ . Hence  $k \leq 8$ . Let  $I = \{1l8\}$  if  $m(v_3v_5) \geq 1$ , and  $I = \{1l8\} \setminus \{k\}$  otherwise. Since  $v_2v_3$  meets all the edges  $v_1v_2, v_1v_3, v_3v_5$ , it follows that none of  $F_1lF_k$  contain  $v_2v_3$ , and so  $k + m(v_2v_3) - 1 \leq 8$  and we may assume that  $v_2v_3 \in F_j$  for  $k + 1 \leq j \leq k + m(v_2v_3) - 1$ . Thus there are exactly  $9 - k - m(v_2v_3)$  values of  $j \in \{1l8\}$  such that  $F_j$  contains none of  $v_1v_2, v_1v_3, v_3v_5, v_2v_3$ . Since by hypothesis

$$m(v_1v_2) + m(v_1v_3) + m(v_2v_3) + m(v_3v_4) + m(v_3v_5) \geq 8,$$

and so  $m(v_3v_4) > 9 - k - m(v_2v_3)$ , there exists  $h \leq k + m(v_2v_3) - 1$  such that  $v_3v_4 \in F_h$ ; since  $v_3v_4$  meets each of  $v_1v_3, v_2v_3$  and  $v_3v_5$ , it follows that  $v_1v_2 \in F_h$ , and so  $h < k$ ; and from the symmetry we may assume that  $h = 1$ .

For each  $i \in I$  let  $Q_i$  as in 5.1. Now  $|F_j \cap Q_i| = 1$  for  $1 \leq j \leq 8$  with  $j \neq i$ ; and since  $F_1$  contains  $v_1v_2, v_3v_4$  it follows that for  $i \neq 1$   $v_3v_4 \notin Q_i$ . Consequently  $v_4v_5 \in Q_i$  for all  $i \in I \setminus \{1\}$ . Let  $r_1, r_2$  be the second regions for  $v_1v_2, v_4v_5$  respectively, and let their sets of doors be  $D_1, D_2$ . Hence for each  $j \in \{1l8\}$ , since there exists  $i \in I \setminus \{1\}$  with  $i \neq j$ , it follows that  $F_j$  contains at most one of  $v_1v_2, v_1v_3, v_3v_5, v_4v_5$ , and so we may assume that  $v_4v_5 \in F_j$  for  $k + 1 \leq j \leq k'$  where  $k' = k + m(v_4v_5)$ , and in particular  $k' \leq 8$ . From the hypothesis,  $k' \geq 7$ .

(1) For  $i \in I \setminus \{1\}$ , one of  $F_i \cap D_1, F_i \cap D_2$  is non-empty, and both if  $k' = 8$  or  $i = 8$ .

Let  $e_1le_n, e_1$  be the edges of  $Q_i$  in order, where  $e_1 = v_1v_2$ ,  $e_2 = v_1v_3$ ,  $e_3 = v_3v_5$  and  $e_4 = v_4v_5$ . Thus for  $1 \leq j \leq k'$ ,  $F_j$  contains one of  $e_1, e_2, e_3, e_4$ , and hence contains none of  $e_5le_n$  if  $j \neq i$ . Now since  $F_i$  contains at most one of  $e_1, e_2, e_3, e_4$  and  $|F_i \cap Q_i| \geq 5$ , it follows that  $n \geq 8$ . Hence  $e_5le_n$  belong only to  $F_i$ , except that one belongs to  $F_8$  if  $i, k < 8$ . This proves (1) as usual.

Since  $k' \leq 8$ , one of  $r_1, r_2$  is small since  $m^+(v_1v_2) + m(v_1v_3) + m(v_3v_5) + m^+(v_4v_5) \geq 7$ . Consequently, (1) implies that  $k' = 7$ ; and so  $r_1, r_2$  are both small, again a contradiction to (1).

This proves that  $(G, m')$  is not 8-edge-colourable, and in particular the sequence  $v_2v_3v_5v_1v_2$  is not switchable. Let us look at the subcases for Conf(11), Conf(12), Conf(13) listed above. In the Conf(11) subcase,  $m(v_1v_2) \geq 3 \geq m(v_1v_5)$ , so we only need to check that  $m(v_1v_2) \geq m(v_2v_3)$ . If not, then  $m(v_2v_3) = 4$ , contrary to Conf(2). In the Conf(13) subcase, the condition that  $m(v_1v_2) \geq \max(m(v_2v_3), m(v_1v_5))$  is explicitly given. In the Conf(12) subcase,  $m(v_1v_2) \geq 2 \geq m(v_2v_3)$ , so we only need to check that  $m(v_1v_2) \geq m(v_1v_5)$ . Suppose not; then  $m(v_1v_5) = 3$  and  $m(v_1v_2) = 2$ . In this case the sequence  $v_2v_3v_5v_1v_2$  is not switchable, so we need a different approach.

Since  $(G, m')$  given above is not 8-colourable, it follows from 2.1 that  $m'(\delta(X)) \geq 10$  for every subset  $X \subseteq V(G)$  with  $|X|$  odd and  $|X|, |V(G) \setminus X| \geq 3$ . Let  $(G, m'')$  be obtained from  $(G, m')$  by switching again on the same sequence. Now  $(G, m'')$  is a 8-target, since  $m(v_2v_3), m(v_1v_5) \geq 2$ ; and it is smaller than  $(G, m)$ , and therefore admits an 8-edge-colouring, say  $F_1lF_8$ . Since  $m''(v_1v_2) + m''(v_1v_3) + m''(v_3v_5) + m''(v_1v_5) > 8$ , some  $F_i$  contains two of  $v_1v_2, v_1v_3, v_3v_5, v_1v_5$ , and therefore contains  $v_1v_2$  and  $v_3v_5$ . By replacing  $F_i$  by  $(F_i \setminus \{v_1v_2, v_3v_5\}) \cup \{v_2v_3, v_1v_5\}$  we therefore obtain an 8-edge-colouring of  $(G, m')$ , a contradiction. This proves 5.7.  $\blacksquare$

### 5.8 No minimum 8-counterexample contains Conf(14).

**Proof.** Let  $(G, m)$  be a minimum 8-counterexample, and suppose that some edge  $uv$  is incident with regions  $r_1, r_2$  where  $r_1$  has at most six doors disjoint from  $uv$ , and  $m(uv) \geq 5$ , and either  $m(uv) \geq 6$  or  $r_2$  is small. By exchanging  $r_1, r_2$  if necessary, we may assume that if  $r_1, r_2$  are both small, then the length of  $r_1$  is at least the length of  $r_2$ . By 4.5, we do not have Conf(3), so not both  $r_1, r_2$  are triangles, and by 4.2, if  $m(uv) \geq 6$  then neither of  $r_1, r_2$  is a triangle; so  $r_1$  is not a triangle. Let  $xuyy$  be a path of  $C_{r_1}$ . Since  $m(e) \geq 5$ , this path is switchable; let  $(G', m')$  be obtained from  $(G, m)$  by switching on it, and let  $F_1lF_8$  be an 8-edge-colouring of  $(G', m')$ . Let  $k = m'(uv) + m'(xy) \geq 7$ . Let  $I = \{1l8\} \setminus \{k\}$  if  $x, y$  are non-adjacent in  $G$ , and  $I = \{1l8\}$  if  $xy \in E(G)$ . For  $i \in I$ , let  $Q_i$  be as in 5.1. Since  $Q_i$  contains both  $uv, xy$  for each  $i \in I$ , it follows that for  $1 \leq j \leq 8$ ,  $F_j$  contains at most one of  $uv, xy$ . Thus we may assume that  $uv \in F_i$  for  $1 \leq i \leq m'(uv)$ , and  $xy \in F_i$  for  $m'(uv) < i \leq k$ . Thus  $k \leq 8$ . Let  $D_1$  be the set of doors for  $r_1$  that are disjoint from  $e$ , and let  $D_2$  be the set of doors for  $r_2$ .

(1) For each  $i \in I$ , one of  $F_i \cap Q_i \cap D_1, F_i \cap Q_i \cap D_2$  is nonempty, and if  $k = 8$  or  $i > k$  then both are nonempty.

Let  $i \in I$ , and let the edges of  $Q_i$  in order be  $e_1le_n, e_1$ , where  $e_1 = uv$  and  $e_2 = xy$ . Since  $|F_i \cap Q_i| \geq 5$  and  $F_i$  contains at most one of  $e_1, e_2$ , it follows that  $n \geq 6$ . Suppose that  $k = 8$ . Then for  $1 \leq j \leq 8$ ,  $F_j$  contains one of  $e_1, e_2$ ; and hence for all  $j \in \{1l8\}$  with  $j \neq i$ ,  $e_3le_n \notin F_j$ . It follows that  $e_n, e_{n-1}$  belong only to  $F_i$  and hence  $e_n \in F_i \cap Q_i \cap D_2$ . Since this holds for all  $i \in I$ , it follows that  $|D_2| \geq |I| \geq 7$ . Hence  $r_2$  is big, and so by hypothesis,  $m(uv) \geq 6$ . Since  $k = 8$  it follows that  $xy \notin E(G)$ . Consequently  $e_3$  is an edge of  $C_{r_1}$ , and since  $e_3, e_4$  belong only to  $F_i$ , it follows that  $e_3$  is a door for  $r_1$ . But  $e_3 \neq ux, vy$  from the choice of the switchable path, and so  $e_3 \in F_i \cap Q_i \cap D_1$ . Hence in this case (1) holds.

Thus we may assume that  $k = 7$ ; and so  $m(e) = 5$ , and  $r_2$  is small, and  $xy \notin E(G)$ , and  $uv \in F_1lF_6$ , and  $xy \in F_7$ . Thus  $I = \{1l6, 8\}$ . If  $i = 8$ , then since  $uv, xy \in Q_i$  and  $F_j$  contains one of  $e_1, e_2$  for all  $j \in \{1l7\}$ , it follows as before that  $e_3 \in F_i \cap Q_i \cap D_1$  and  $e_n \in F_i \cap Q_i \cap D_2$ . Thus we

may assume that  $i \leq 6$ . For  $1 \leq j \leq 8$  with  $j \neq i$ ,  $|F_j \cap Q_i| = 1$ , and for  $1 \leq j \leq 7$ ,  $F_j$  contains one of  $e_1, e_2$ . Hence  $e_3, e_n$  belong only to  $F_i$  and to  $F_8$ , and only one of them belongs to  $F_8$ . If neither of  $e_n, e_{n-1}$  belong to  $F_8$  then  $e_n \in F_i \cap Q_i \cap D_2$  as required; so we assume that  $F_8$  contains one of  $e_n, e_{n-1}$ ; and so  $e_3, e_{n-2}$  belong only to  $F_i$ . Since  $n \geq 6$ , it follows that  $e_3 \in F_i \cap Q_i \cap D_1$  as required. This proves (1).

If  $k = 8$ , then (1) implies that  $|D_1| \geq 7$  as required. So we may assume that  $k = 7$  and hence  $m(e) = 5$  and  $xy \notin E(G)$ ; and  $r_2$  is small. Suppose that there are three values of  $i \in \{1, 6\}$  such that  $|F_i \cap D_1| = 1$  and  $F_i \cap D_2 = \emptyset$ , say  $i = 1, 2, 3$ . Let  $f_i \in F_i \cap D_1$  for  $i = 1, 2, 3$ , and we may assume that  $f_3$  is between  $f_1$  and  $f_2$  in the path  $C_{r_1} \setminus \{uv\}$ . Choose  $X \subseteq V(G')$  such that  $\delta_{G'}(X) = Q_3$ . Since only one edge of  $C_{r_1} \setminus \{e\}$  belongs to  $Q_3$ , one of  $f_1, f_2$  has both ends in  $X$  and the other has both ends in  $V(G') \setminus X$ ; say  $f_1$  has both ends in  $X$ . Let  $Z$  be the set of edges of  $G'$  with both ends in  $X$ . Thus  $(F_1 \cap Z) \cup (F_2 \setminus Z)$  is a perfect matching, since  $e \in F_1 \cap F_2$ , and no other edge of  $\delta_{G'}(X)$  belongs to  $F_1 \cup F_2$ ; and similarly  $(F_2 \cap Z) \cup (F_1 \setminus Z)$  is a perfect matching. Call them  $F'_1, F'_2$  respectively. Then  $F'_1, F'_2, F_3, F_4, F_8$  form an 8-edge-colouring of  $(G', m')$ , yet  $f_1, f_2$  are the only edges of  $D_1 \cup D_2$  included in  $F'_1 \cup F'_2$ , and neither of them is in  $F'_2$ , contrary to (1). Thus there are no three such values of  $i$ ; and similarly there are at most two such that  $|F_i \cap D_2| = 1$  and  $F_i \cap D_1 = \emptyset$ . Thus there are at least three values of  $i \in I$  such that  $|F_i \cap D_1| + |F_i \cap D_2| \geq 2$  (counting  $i = 8$ ), and so  $|D_1| + |D_2| \geq 10$ . But  $|D_1| \leq 6$  by hypothesis and  $|D_2| \leq 3$  since  $r_2$  is small, a contradiction. This proves 5.8.  $\blacksquare$

### 5.9 No minimum 8-counterexample contains *Conf(15)* or *Conf(16)*.

**Proof.** To handle both at once, we assume that  $(G, m)$  is an 8-target with a region  $r$ , and  $uv \in E(C_r)$ , and  $uvw$  is another region, satisfying:

- either  $(G, m)$  is a minimum 8-counterexample, or  $m(uv) = 0$  and deleting  $uv$  gives a minimum 8-counterexample
- $m(uv) + m^+(uw) \geq 4$
- every edge of  $C_r$  not incident with  $u$  is 3-heavy
- $m(vw) \leq m(uw)$ , and the second edge of  $C_r$  incident with  $u$  has multiplicity at most  $m(uw)$ .

Note that while *Conf(16)* fits these conditions, some instances of *Conf(15)* may not, and we will handle them later. Let the second neighbour of  $u$  in  $C$  be  $t$ .

By hypothesis, the path  $tuvw$  is switchable; let  $(G', m')$  be obtained from it by switching, and let  $F_1, F_8$  be an 8-edge-colouring of it. Let  $k = m(uw) + m(uv) + 2 \geq 5$ ; then we may assume that  $F_1, F_{k-1}$  contain one of  $uw, uv$ , and  $tv \in F_k$ . Let  $I = \{1, 8\}$  if  $tv \in E(G)$ , and  $I = \{1, 8\} \setminus \{k\}$  otherwise. For each  $i \in I$  let  $Q_i$  be as in 5.1. Thus each  $Q_i$  contains all of  $uw, uv, tv$ , and so no edge of  $Q_i \setminus \{uw, uv, tv\}$  belongs to  $F_j$  for any  $j \neq i$  with  $j \leq k$ .

(1)  $k = 5$ .

For suppose that  $k \geq 6$ . Choose  $i \in I \cap \{7, 8\}$ . Since  $Q_i$  contains  $uv, uw, tv$ , it follows that  $F_1, F_6$  all

contain an edge in  $\{uv, uw, tv\} \cap Q_i$ ; and hence no edge of  $Q_i \setminus \{uv, uw, tv\}$  belongs to any of  $F_1 l F_6$ . Choose an edge  $f$  of  $C_r \setminus \{u, v\}$  with  $f \in Q_i$ . Now  $f \neq tu$  by the choice of switchable path, and so  $f$  is 3-heavy (with respect to  $(G, m)$ ), and if  $f = tv$  then  $m'(f) > m(f)$ . Consequently there are three values of  $j \in \{1l8\} \setminus \{k\}$  such that  $F_j \cap Q_i$  contains an edge different from  $uv, uw$ , and hence some such  $j$  belongs to  $\{1l5\}$ , a contradiction. This proves (1).

Let  $r_1$  be the second region for  $uw$ , and let  $D_1$  be the set of doors for  $r_1$ . From (1) it follows that  $r_1$  is small, and so  $|D_1| \leq 3$ .

(2) For  $i = 6, 7, 8$ ,  $|Q_i \cap F_i \cap D_1| = 1$ ; and the edges of  $F_6$  and  $F_8$  in  $Q_7$  have a common end (they may be the same).

For let  $i \in \{6, 7, 8\}$ ; then  $i \in I$ . Let the edges of  $Q_i$  be  $e_1 l e_n, e_1$  in order, where  $e_1 = uw$ ,  $e_2 = uv$  and  $e_3 = tv$ . Then  $n \geq 7$ , since  $|F_i \cap Q_i| \geq 5$ . Let  $h = 3$  if  $tv \in E(G)$ , and  $h = 4$  otherwise. Then  $e_h$  is an edge of  $C_r$  not incident with  $u$ , and so it is 3-heavy; and hence either  $m(e_h) \geq 3$ , or the second region for  $e_h$  is a triangle and  $e_{h+1}$  is an edge of it, and  $m(e_h) + m(e_{h+1}) \geq 3$ . Moreover, if  $e_h = tv$  then  $m'(e_h) > m(e_h)$ . Thus in all cases it follows that there are three values of  $j \neq 5$  with  $1 \leq j \leq 8$  such that  $F_j \cap Q_i$  contains one of  $e_h, e_{h+1}$ . We deduce that these three values of  $j$  are 6, 7, 8, since  $F_j \cap Q_i \subseteq \{uv, uw\}$  for  $1 \leq j \leq 4$ . Consequently for  $1 \leq j \leq 8$ ,  $F_j \cap Q_i$  includes one of  $e_1, e_2, e_3, e_4, e_5$ . It follows that only  $F_i$  contains  $e_n, e_{n-1}$ , and consequently  $e_n \in Q_i \cap F_i \cap D_1$ . Since  $|D_1| = 3$ , this proves the first assertion of (2). The second follows since, taking  $i = 7$  and defining  $e_h$  as before,  $F_6$  and  $F_8$  each contain one of  $e_h, e_{h+1}$ , and these edges have a common end. This proves (2).

Let  $Q_i \cap F_i \cap D_1 = \{f_i\}$  for  $i = 6, 7, 8$ . We may assume that  $f_6, f_7, f_8$  are in order in the path  $C_{r_1} \setminus \{uw\}$ . Choose  $X \subseteq V(G)$  with  $\delta_{G'}(X) = Q_7$ . Let  $H$  be the subgraph of  $G'$  with vertex set  $V(G)$  and edge set  $(F_6 \setminus F_8) \cap (F_8 \setminus F_6)$ . Thus each component of  $H$  is either a single vertex or a cycle of even length. Now there are either no edges, or two edges, of  $H$  that belong to  $\delta_{G'}(X)$ ; and if there are two then they have a common end by (2). It follows that the component of  $H$ , say  $C$ , that contains  $f_6$  does not contain  $f_8$ . Let  $F'_6 = (F_8 \cap E(C)) \cup (F_6 \setminus E(C))$  and  $F'_8 = (F_6 \cap E(C)) \cup (F_8 \setminus E(C))$ ; then  $F'_6, F'_8$  are perfect matchings of  $G'$ , and  $F_1 l F_5, F'_6, F_7, F'_8$  is an 8-edge-colouring of  $(G, m')$ . On the other hand both  $f_6, f_8$  belong to  $F'_8$ , so this 8-edge-colouring does not satisfy (2), a contradiction.

It remains to deal with the case of Conf(15) when the path  $t \dot{u} \dot{v} \dot{y}$  is not switchable. Thus, now we assume that

- $(G, m)$  is a minimum 8-counterexample
- $r$  is a region of length at least four, and  $e$  is an edge of  $C_r$
- $m^+(e) \geq 4$ , and every edge of  $C_r$  disjoint from  $e$  is 3-heavy
- one of the edges of  $C_r$  incident with  $e$  has multiplicity more than  $m(e)$ .

Let  $C_r$  have vertices  $v_1 l v_p$  in order, where  $p \geq 4$ ,  $e = v_1 v_2$ , and  $m(v_2 v_3) > m(e)$ . It follows that  $m(v_1 v_2) = 3$  and  $m(v_2 v_3) = 4$ . By 4.5, we do not have Conf(4) so  $p \geq 5$ . The path  $v_1 \dot{v}_2 \dot{v}_3 \dot{v}_4$  is switchable; let  $(G, m')$  be obtained by switching on it. We may assume that  $v_2 v_3 \in F_i$  for  $1 \leq i \leq 5$

and  $v_1v_4 \in F_6$ . Since  $m'(v_1v_2) = 2$  and  $v_1v_2$  meets both  $v_2v_3$  and  $v_1v_4$ , it follows that  $v_1v_2 \in F_7, F_8$ . Consequently  $v_pv_1 \in F_h$  for some  $h$  with  $1 \leq h \leq 5$ . Let  $I = \{1l8\} \setminus \{6\}$ . For each  $i \in I$  let  $Q_i$  be as in 5.1. Now  $Q_7$  contains  $v_2v_3, v_1v_4$ , and so for  $1 \leq j \leq 6$ ,  $F_j \cap Q_7 \subseteq \{v_2v_3, v_1v_4\}$ . In particular  $v_pv_1 \notin Q_7$ . But  $Q_7$  contains an edge  $f$  of  $C_r$ , different from  $v_1v_2$ , and this edge is 3-heavy, since it is different from  $v_pv_1$  and hence disjoint from  $e$ ; and so  $F_j \cap Q_i \setminus \{v_2v_3, v_1v_4\} \neq \emptyset$  for three values of  $j \in \{1l8\}$ , a contradiction. This proves 5.9.

**5.10** *No minimum 8-counterexample contains Conf(17) or Conf(18).*

**Proof.** To handle both at once, we assume that  $(G, m)$  is an 8-target with a region  $r$  with length at least four, and  $uv \in E(C_r)$ , and  $uvw$  is another region, satisfying:

- either  $(G, m)$  is a minimum 8-counterexample, or  $m(uv) = 0$  and deleting  $uv$  gives a minimum 8-counterexample
- $m(uv) + m^+(uv) \geq 5$
- let  $t, x$  be the second neighbours of  $u, v$  in  $C_r$  respectively; if  $m(uv) = 3$  and  $e$  is 5-heavy let  $P = C_r \setminus \{u, v\}$ , and otherwise let  $P = C_r \setminus \{u\}$ ; then every edge  $f$  of  $P$  satisfies  $m^+(f) \geq 2$ , and at most one edge of  $P$  is not 3-heavy
- $m(tu), m(vw) \leq m(uw)$ .

The path  $tuvw$  is switchable; let  $(G', m')$  be obtained by switching on it, and let  $F_1lF_8$  be an 8-edge-colouring of  $(G', m')$ . Since  $r$  has length at least four,  $tv \notin E(G)$ . Let  $k = m(uw) + (uv) + 2 \geq 6$ ; we may assume that  $F_i$  contains one of  $uv, uw$  for  $1 \leq i < k$ , and  $F_k$  contains  $tv$ . Let  $I = \{1l8\} \setminus \{k\}$ ; and for each  $i \in I$  let  $Q_i$  be as in 5.1.

(1) *There is at most one value of  $i \in I$  such that  $Q_i \cap E(P) = \emptyset$ , and if  $i$  is such a value then  $k = 7$  and  $m(uv) = 3$  and  $m(uw), m(vw) = 2$  and  $uw \in F_i$ .*

For suppose that  $i \in I$  and  $Q_i \cap E(P) = \emptyset$ . It follows that  $P = C_r \setminus \{u, v\}$ , and so  $m(uv) = 3$  and  $m(uw), m(vw) = 2$ , and  $k = 7$ . Now for  $1 \leq i \leq 7$ ,  $F_i$  contains one of  $uw, uv, tv$ , and since  $vw$  meets all of these edges it follows that  $vw \in F_8$ . But  $vx$  belongs to some  $F_j$  such that  $F_j$  contains none of  $tv, uv, vw$ , and so  $uw \in F_j$ . Then  $|F_j \cap Q_i| \geq 2$ , so  $j = i$  and hence  $uw \in F_i$ . This proves (1).

Let  $I'$  be the set of  $i \in I$  such that  $Q_i \cap E(P) \neq \emptyset$ . By (1),  $|I'| \geq 6$ . Let  $r_1$  be the second region for  $uw$ , and let its set of doors be  $D_1$ . Thus  $|D_1| \leq 3$  if  $k = 6$ , since  $m(uv) + m^+(uv) \geq 5$ . Let  $I''$  be the set of  $i \in I'$  such that the edge in  $Q_i \cap E(P)$  is not 3-heavy.

(2) *There is a unique edge  $f \in E(P)$  that is not 3-heavy, and it belongs to none of  $F_1lF_k$ . Moreover, if  $i \in I' \setminus I''$  then  $k = 6$  and  $i \leq 5$  and  $F_i \cap Q_i \cap D_1 \neq \emptyset$ .*

Suppose that  $i \in I' \setminus I''$ . There are therefore three values of  $j \in \{1l8\}$  such that  $F_j \cap Q_i \not\subseteq \{uw, uv, tv\}$ , and so at least two that are also different from  $i$ . Consequently, for those two values of  $j$ , it follows that  $uw, uv, tv \notin F_j$  and hence  $k = 6$  and  $j \in \{7, 8\}$ . Thus  $i \leq 5$ . Let the edges of  $Q_i$  in order be  $e_1le_n, e_1$ , where  $e_1 = uw$ ,  $e_2 = uv$  and  $e_3 = tv$ ; then  $n \geq 7$ , since  $|F_i \cap Q_i| \geq 5$ . But  $F_1lF_8$

each contain one of  $e_1le_5$ , so  $e_n \in F_i \cap Q_i \cap D_1$ . This proves the second assertion of (2). For the first assertion, since  $|D_1| \leq 3$ , it follows that  $|I' \setminus I''| \leq 3$ . Since  $|I'| \geq 6$ , it follows that  $|I''| \geq 3$ . But by hypothesis, there is at most one edge in  $P$  that is not 3-heavy, and so this edge exists, say  $f$ . It follows that  $f \in Q_i$ , for all  $i \in I''$ . Now let  $j \in \{1lk\}$ . Choose  $i \in I''$  with  $i \neq j$ ; then  $F_j \cap Q_i \subseteq \{uw, uv, tv\}$ , and so  $F_j$  does not contain  $f$ . This proves (2).

By (2) we may assume that  $f \in F_{k+1}$ . Let  $r_2$  be the second region at  $f$ , and let  $D_2$  be its set of doors. By hypothesis, if  $m(f) = 1$  then  $|D_2| \leq 3$ .

Suppose that  $k \geq 7$ . By (2),  $I'' = I'$  and  $m(f) = 1$ . Let  $i \in I'$ , and let the edges of  $Q_i$  in order be  $e_1le_n$ , where  $e_1 = uw$ ,  $e_2 = uv$ ,  $e_3 = tv$ , and  $e_4 = f$ . Since only one of  $e_1le_4$  belongs to  $F_i$ , and  $|F_i \cap Q_i| \geq 5$ , it follows that  $n \geq 8$ . But  $F_1lF_8$  each contain one of  $e_1le_4$ , and so  $e_5le_n$  only belong to  $F_i$ ; and hence  $e_5 \in F_i \cap Q_i \cap D_2$ . Consequently  $|D_2| \geq |I'| \geq 6$ , a contradiction.

This proves that  $k = 6$ , and hence  $|D_1| \leq 3$ , and  $I' = I$  by (1), and  $7, 8 \in I''$  by (2). Now let  $i \in I''$ . Let the edges of  $Q_i$  in order be  $e_1le_n, e_1$ , where  $e_1 = uw$ ,  $e_2 = uv$ ,  $e_3 = tv$ , and  $e_4 = f$ . Again  $n \geq 8$ .

Suppose that  $m(f) \geq 2$ ; then  $m(f) = 2$  by (2), and  $f \in F_7, F_8$ , and so  $F_1lF_8$  each contain one of  $e_1le_4$ , and therefore  $e_5le_n$  belong to no  $F_j$  with  $j \neq i$ . Since  $n \geq 8$ , it follows that  $e_n \in D_1$ , and so  $F_i \cap Q_i \cap D_1 \neq \emptyset$ . By (2), it follows that  $F_i \cap Q_i \cap D_1 \neq \emptyset$  for all  $i \in I'$ , and so  $|D_1| \geq |I'| = 7$ , a contradiction. Thus  $m(f) = 1$ , and so  $|D_2| \leq 3$ .

Again, let  $i \in I''$ , and let  $e_1le_n, e_1$  be as before. Now  $F_1lF_7$  each contain one of  $e_1le_4$ , and so  $e_5le_n$  belong to no  $F_j$  with  $1 \leq j \leq 7$  and  $j \neq i$ , and only one of them belongs to  $F_8$  if  $i \neq 8$ . We assume first that  $i \neq 8$ . Since  $n \geq 8$ , either  $e_5, e_6 \notin F_8$ , or  $e_n, e_{n-1} \notin F_8$ , and so either  $e_5 \in D_2$  or  $e_n \in D_1$ . Now we assume  $i = 8$ . Then  $e_3le_n$  belong to no  $F_j$  with  $1 \leq j \leq 7$ , and so  $e_5 \in D_2$  and  $e_n \in D_1$ .

In summary, we have shown that for each  $i \in I''$ , either  $F_i \cap D_1 \neq \emptyset$ , or  $F_i \cap D_2 \neq \emptyset$  (both if  $i = 8$ ); and  $8 \in I''$ . By (2), if  $i \in I' \setminus I''$  then either  $F_i \cap D_1 \neq \emptyset$ , or  $F_i \cap D_2 \neq \emptyset$ ; and so  $|D_1| + |D_2| \geq |I'| + 1 \geq 7$ , a contradiction. This proves 5.10.  $\blacksquare$

### 5.11 No minimum 8-counterexample contains $\text{Conf}(19)$ .

**Proof.** Let  $(G, m)$  be a minimum 8-counterexample, and suppose that  $r$  is a region with length at least five, and  $e$  is an edge of  $C_r$ , such that  $m^+(e) \geq 5$ , and every edge of  $C_r$  disjoint from  $e$  is 2-heavy, and at most two of them are not 3-heavy. By 5.10, we do not have  $\text{Conf}(17)$ , so there are at least two edges in  $C_r$  disjoint from  $e$  that are not 3-heavy, and so by hypothesis, there are exactly two, say  $g_1, g_2$ . Thus  $m(g_1), m(g_2) \leq 2$ . By hypothesis,  $g_1, g_2$  are 2-heavy.

Let  $e = uv$ , and let the second neighbours of  $u, v$  in  $C_r$  be  $t, w$  respectively. Since  $m(e) \geq 4$ , it follows that  $m(tu), m(vw) \leq m(uv)$  and so the path  $tuvw$  is switchable. Let  $(G', m')$  be obtained by switching on this path, and let  $F_1lF_8$  be an 8-edge-colouring of it. Let  $k = m(e) + 2$ . We may assume that  $tw \in F_k$ . Let  $I = \{1l8\} \setminus \{k\}$ , and for each  $i \in I$  let  $Q_i$  be as in 5.1. Let  $I_1, I_2, I_3$  be the sets of  $i \in I$  such that  $g_1 \in Q_i$ ,  $g_2 \in Q_i$ , and  $g_1, g_2 \notin Q_i$  respectively.

(1)  $k = 6$ .

For suppose that  $k > 6$ . Let  $i \in I$ , and let the edges of  $Q_i$  in order be  $e_1le_n, e_1$ , where  $e_1 = uv$  and

$e_2 = tw$ . Thus  $e_3$  is an edge of  $C_r$  disjoint from  $e$ . Since  $|F_i \cap Q_i| \geq 5$  and  $|F_i \cap \{e_1, e_2\}| \leq 1$ , it follows that  $n \geq 6$ . Now there are  $k \geq 7$  values of  $j \in \{1l8\}$  such that  $F_j$  contains one of  $e_1, e_2$ ; and so there is at most value of  $j \neq i$  such that  $F_j$  contains one of  $e_3, e_4$ . It follows that  $e_3$  is not 3-heavy and so  $i \in I_1 \cup I_2$ . Since this holds for all  $i \in I$ , we may assume that  $|I_1| \geq 4$ . Let  $i \in I_1$ ; as before, there is at most one value of  $j \neq i$  such that  $F_j$  contains one of  $e_3, e_4$ . Now  $m(g_1) \leq 2$ . If  $m(g_1) = 2$ , then  $g_1 \in F_i$ , and since this holds for all  $i \in I_1$  it follows that  $g_1$  is contained in  $F_i$  for four different values of  $i$ , a contradiction. Thus  $m(g_1) = 1$ . Since  $g_1$  is 2-heavy, the second region for  $g_1$  is a triangle with edge set  $\{g_1, p, q\}$  say, where  $e_4 = p$ . Hence one of  $f_1, p, q$  has multiplicity one and is contained in  $F_i$ . Since this holds for all  $i \in I_1$  and  $|I_1| \geq 4$ , this is impossible. This proves (1).

We may therefore assume that  $uv \in F_i$  for  $1 \leq i \leq 5$  and  $tw \in F_6$ . Since  $k = 6$ , it follows that  $m(e) = 4$  and since  $m^+(e) \geq 5$ , the second region  $r_1$  for  $uv$  is small. Let  $D_1$  be its set of doors.

(2) If  $i \in I_3$  then  $i \leq 5$  and  $F_i \cap Q_i \cap D_1 \neq \emptyset$ .

For let the edges of  $Q_i$  in order be  $e_1le_n, e_1$ , where  $e_1 = uv$  and  $e_2 = tw$ . Then  $F_1lF_6$  each contain an edge in  $\{e_1, e_2\}$ , and so for  $1 \leq j \leq 6$  with  $j \neq i$ , none of  $e_3le_n$  belongs to  $F_j$ . Now  $e_3$  is 3-heavy, and so there are three values of  $j$  such that  $F_j$  contains one of  $e_3, e_4$ ; and so these three values are  $i, 7, 8$ , and  $i \neq 7, 8$ . (Thus  $i \leq 5$  since  $6 \notin I$ .) Hence for  $1 \leq j \leq 8$ ,  $F_j$  contains one of  $e_1le_4$ ; and so  $e_n, e_{n-1}$  belong only to  $F_i$ . Hence  $e_n \in D_1$ . This proves (2).

For  $j = 1, 2$ , let  $I'_j$  be the set of all  $i \in I_j$  such that  $F_i \cap Q_i \cap D_1 = \emptyset$ .

(3) For  $j = 1, 2$ ,  $|I'_j| \leq 2$ , and  $7, 8 \notin I'_j$ , and if  $|I'_j| = 2$  then  $7, 8 \notin I_j$ .

For let  $j = 1$  say. Suppose first that  $m(g_1) = 2$ , and let  $g_1 \in F_a, F_b$  where  $1 \leq a < b \leq 8$ . Let  $i \in I'_1$ , and let  $e_1le_n$  be as before; then  $e_3 = g_1$ . Again, for  $1 \leq j \leq 6$  with  $j \neq i$ , none of  $e_3le_n$  belong to  $F_j$ , and consequently  $a, b \in \{i, 7, 8\}$ . In particular,  $b \geq 7$ , and  $a \in \{i, 7\}$ . Thus if  $a \leq 6$  then  $i = a$  and so  $|I'_1| = 1$  and the claim holds. We assume then that  $(a, b) = (7, 8)$ . But then  $F_1lF_8$  each contain one of  $e_1, e_2, e_3$ , and so  $e_n \in D_1$ , contradicting that  $i \in I'_1$ . So the claim holds if  $m(g_1) = 2$ .

Next we assume that  $m(g_1) = 1$ . Since  $g_1$  is 2-heavy, the second region at  $g_1$  is a triangle with edge set  $\{g_1, p, q\}$  say. Let  $g_1 \in F_a$ . Let  $i \in I'_1$ , and let  $e_1le_n$  be as before; then  $e_3 = g_1$ . Again, for  $1 \leq j \leq 6$  with  $j \neq i$ , none of  $e_3le_n$  belongs to  $F_j$ , and consequently  $a \in \{i, 7, 8\}$ . Thus if  $a \neq 7, 8$  then  $i = a$  and  $|I'_1| = 1$  and the claim holds. We assume then that  $a = 7$ . Thus each of  $F_1lF_7$  contains one of  $e_1, e_2, e_3$ , and for  $1 \leq j \leq 7$  with  $j \neq i$ ,  $F_j$  contains none of  $e_4le_n$ . Since  $F_i \cap Q_i \cap D_1 = \emptyset$ , there exists  $j \in \{1l8\}$  with  $j \neq i$  such that  $F_j$  contains one of  $e_n, e_{n-1}$ ; and hence  $j = 8$ , and so  $i \neq 8$ . (Also,  $i \neq 7$  since  $g_1 \in F_7$  and  $g_1$  meets  $e_4$ . Consequently,  $7, 8 \notin I'_j$ .) Thus  $F_1lF_8$  each contain one of  $e_1, e_2, e_3, e_{n-1}, e_n$ , and so  $e_4$  is only contained in  $F_i$ . Consequently,  $i$  has the property that one of  $p, q$  has multiplicity one, and  $F_i$  contains it. Thus there are at most two such values of  $i$ , and so  $|I'_j| \leq 2$ . Moreover, if there are two such values, say  $c, d$ , then  $c, d \leq 5$  and  $F_c$  contains one of  $p, q$  and  $F_d$  contains the other. Consequently if  $7 \in I_1$ , then one of  $F_c, F_d$  contains two edges of  $Q_7$ , a contradiction. So if  $|I'_j| = 2$  then  $7, 8 \notin I_j$ . This proves (3).

From (2), we may assume that  $7 \in I_1$ , and so  $|I'_1| + |I'_2| \leq 3$  by (3). Consequently there are at



least four values of  $i \in I$  such that  $F_i \cap Q_i \cap D_1 \neq \emptyset$ , and so  $|D_1| \geq 4$ , a contradiction. This proves 5.11. ■

This completes the proof of 4.1 and hence of 1.2. Perhaps despite appearances, there was some system to our choice of the  $\beta$ - and  $\gamma$ -rules. We started with the idea that we would normally pass a charge of one from each small region to each big region sharing an edge with it, and made the minimum modifications we could to the  $\beta$ -rules so that the proof of 3.4 worked. Then we experimented with the  $\gamma$ -rules to make 3.5, 3.6 and 3.7 work out.

It is to be hoped that solving these special cases of the main conjecture 1.1 will lead us to a proof of the general case, but that seems far away at the moment. The same approach does indeed work (more simply) for seven-regular planar graphs, and this gives an alternative proof of the result of [4], to appear in [6]. We tried the same again for nine-regular graphs, but there appeared to be some serious difficulties. Maybe more perseverance will bring it through, but it seems much harder than the eight-regular case.

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