Approximate Capacity Region for the Symmetric Gaussian Interference Channel with Noisy Feedback

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Abstract

Recent results have shown that feedback can significantly increase the capacity of interference networks. This paper considers the impact of noise on the gain due to feedback. Specifically, this paper considers the two-user linear deterministic interference channel with partial feedback, as a stepping stone in order to characterize the capacity region for the two-user Gaussian interference channel with noisy feedback. The capacity region for the symmetric linear deterministic interference channel with partial feedback has been obtained. Partial feedback has been shown to increase the capacity region if and only if the amount of feedback level l is greater than a certain threshold l^*, and it is found that l^* is equal to the per-user symmetric capacity without feedback. One of the key ideas is a novel converse outer bound on weighted sum rates $2R_1 + R_2$ and $R_1 + 2R_2$. The novel outer bounds are tightened by specially defined auxiliary random variables. It has been illustrated through numerous examples, that the outer bounds on the sum rate $R_1 + R_2$ alone are not sufficient to characterize the capacity region, and $2R_1 + R_2$ and $R_1 + 2R_2$ bounds are also necessary. The result and the techniques developed for this linear deterministic model are then applied to characterize inner bounds and outer bounds for the symmetric Gaussian IC with noisy feedback. The outer bounds have been shown to be at most 11.7 bits/s/Hz away from the achievable rate region. As a corollary, the generalized-degree-of-freedom region, which approximates the capacity region of the symmetric Gaussian IC at high SNR, is found.

I. Introduction

One of the most important issues for communication networks is that of interference management. Characterizing the capacity region of the two-user Gaussian interference channel (GIC) remains one of the fundamental unresolved problems in information theory. Recent breakthroughs in dealing with the capacity characterization of the GIC have made use of the linear deterministic interference channel (LD-IC) model [1], [2]. The main idea behind these works is that an appropriately defined LD model can serve as a good approximation to the Gaussian channel. By gaining valuable insights from studying the LD-IC, the proof techniques and ideas can be lifted over to the GIC. The capacity region of the GIC has been characterized to within 1-bit in [3].

There are many techniques to manage interference, such as treatment of interference as noise, interference alignment [4], and usage of feedback [5]. In this work, we focus on interference management via feedback. It is well known that, while feedback does not increase the capacity of the discrete memoryless point-to-point channel, it does enlarge the capacity region of multi-user channels. The fact that feedback enlarges the capacity region of the discrete memoryless multiple-access channel (MAC) was shown by Gaarder and Wolf [6]. Afterwards, Ozarow [7] found the capacity region of the two-user Gaussian MAC with noiseless feedback. Recently, Suh and Tse [5] obtained an interesting result that noiseless feedback can provide significant capacity gains for the GIC. To understand the usefulness of feedback for the interference channel, consider the very strong interference regime, in which the direct links are weaker than the cross (interference) links. In such a scenario, feedback can provide a substantial capacity gain by using the alternate path of $\mathrm{Tx}_1 \rightarrow \mathrm{Rx}_2 \rightarrow \mathrm{Tx}_2 \rightarrow \mathrm{Rx}_1$, i.e., the information intended from Tx_1 first reaches Rx_2, which is then received as feedback at Tx_2, which uses the strong cross (interference) link to reach the eventual destination at Rx_1. The approximate capacity region of the GIC with noiseless channel output feedback has been characterized [5] to within 2-bits. The results in [5] have been generalized to the case of the fully connected K-user IC [8], and the cyclic K-user IC [9].

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Full and noiseless feedback is too much to ask for when the feedback link is not reliable. Vahid et al. considered an interesting generalization of [5] by studying the two-user GIC with rate-limited feedback [10]. Rate-limited feedback refers to a setting in which the receiver can utilize all the information it has received so far and feed back information over an orthogonal channel of finite capacity (bit-pipe). Several interesting results for the GIC with rate-limited feedback are obtained in [10].

While rate-limited feedback may be useful in scenarios in which the feedback links have good coding schemes to protect feedback signals from error, it places much complexity on the receiver's side. As a result, this model is not appropriate when the complexity of the feedback design is a concern. In order to take some of these issues into account, this paper aims to investigate the model in which the feedback at transmitter j is a scaled and noisy (additive white Gaussian noise corrupted) version of the channel output at receiver j, for $j = 1, 2$. In particular, if the channel output at receiver j is Y_j, then the feedback to transmitter j is $Y_{F_j} = g_j Y_j + \tilde{Z}_j$, for $j = 1, 2$ (see Figure 1). With the eventual goal of understanding the capacity region of the GIC with noisy feedback, we present a linear deterministic model with partial feedback. We show that the LD-IC with partial feedback serves as a good approximation to the GIC with noisy feedback. First, we consider the linear deterministic interference channel with partial feedback. Subsequently, we consider the Gaussian interference channel with noisy feedback, based on the insights that we gain from the linear deterministic model.

The main contributions of this paper are

- To characterize the capacity region for the symmetric LD-IC with partial feedback. To illustrate through numerous examples, that the sum-rate bounds derived in [11] alone are not sufficient to characterize the capacity region, and $2R_1 + R_2$ and $R_1 + 2R_2$ bounds are also necessary. Note that outer bounds are tightened with the help of specially defined auxiliary random variables. To show that partial feedback increases the capacity region if and only if the amount of feedback level l is greater than a certain threshold l^*, and that l^* is equal to the per-user symmetric capacity without feedback.
- Based on results for the symmetric LD-IC with partial feedback, we derive inner bounds and outer bounds for the symmetric Gaussian interference channel with noisy feedback. The outer bounds are shown to be at most 11.7 bits/s/Hz from the achievable rate region. As a corollary of this result, we also obtain a generalized-degree-of-freedom region for the symmetric Gaussian IC with noisy feedback.

We note here that the sum-capacity of the LD-IC with partial feedback was characterized in our previous works [11] [12], and the capacity region for the symmetric LD-IC was partially presented in [13].

Other related work that studied multi-user channels with feedback includes [14]–[22]. [14] [15] found an achievable rate region for interference channel with generalized feedback, and their model can be reduced to many well-known multi-user channels, including ours. However, further optimization needs to be done to make the inner bound tight for our current problem. In [16], Jiang et al. established an achievable rate region for the interference channel with full noiseless feedback. [17] found outer bounds for interference channel with degraded noisy feedback. AWGN MAC with imperfect feedback was studied in [18], which showed that the achievable rate region for MAC with even imperfect feedback is larger than that without feedback. Tandon and Ulukus in [20] derived outer bounds for the Gaussian MAC with noisy feedback (a.k.a user cooperation) and outer bounds for Gaussian interference channel with user cooperation. In [21], Tuninetti developed outer bounds on $R_1 + R_2$ for interference with generalized feedback. Works done IC with source cooperation [23]–[27] are different but also related to works done IC with feedback. Wang and Tse [23] characterized the capacity region, to within a constant number of bits, of the two-user Gaussian interference channel with conferencing transmitters. In [24], interference channel with source cooperation was studied. In that paper, the source nodes are allowed to transmit and receive in full duplex mode, thus they can overhear information from the other source node. Even though the outer bounds on the sum rate in [24] gives the same result as our outer bound in the linear deterministic IC, the two outer bounds differ in Gaussian IC, especially at low SNR.

The structure of the paper is as follows. In section II, we introduce the system models for the discrete memoryless interference channel with noisy feedback, the Gaussian interference channel with noisy feedback and the LD-IC with partial feedback, then we formally state the problem. In section III, we present the results and discussion for the symmetric linear deterministic interference channel with partial feedback. In the subsequent section, we present the results and discussion for the symmetric Gaussian interference channel with noisy feedback. Finally, the paper ends with a conclusion and the appendix, which contains proofs to results in the paper.

II. System model

A discrete memoryless interference channel with noisy feedback comprises two input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, two output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$, a channel transition probability $P_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2)$, two feedback output alphabets $\mathcal{Y}_{F_1}$ and $\mathcal{Y}_{F_2}$, and two feedback channel transition probabilities $P_{Y_{F_j}|Y_j}(y_{F_j}|y_j)$, for $j = 1, 2$. The channels are discrete in the sense that all the alphabet sets are finite. The channels are memoryless in the sense that the channel outputs in the current time

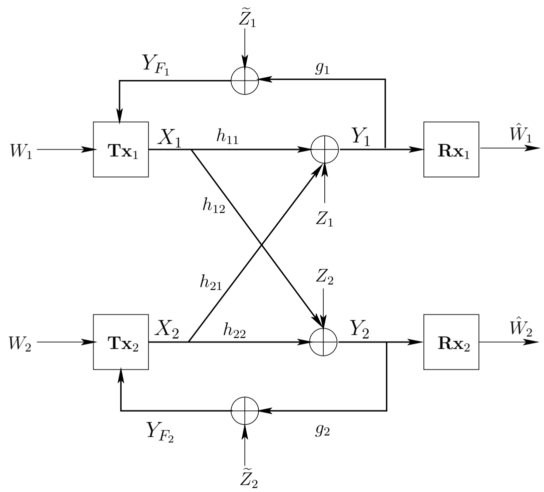

Fig. 1: Gaussian IC with Noisy Feedback.

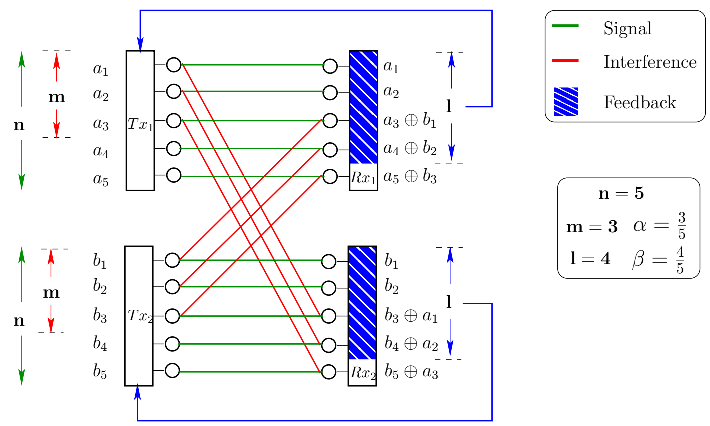

Fig. 2: Symmetric Linear Deterministic IC with Partial Feedback.

slot are dependent on only the channel inputs in the current slot, and are independent of channel inputs in the previous time slots, i.e., $P_{Y_{1i}Y_{2i}|X_{1i}X_{2i}}(y_{1i}y_{2i}|x_{1i}x_{2i}) = P_{Y_{1i}Y_{2i}|X_1^iX_2^i}(y_{1i}y_{2i}|x_1^ix_2^i)$, and $P_{Y_{F_j i}|Y_{ji}}(y_{F_j i}|y_{ji}) = P_{Y_{F_j i}|Y_j^i}(y_{F_j i}|y_j^i)$, for $i \in \{1, 2, ..., T\}$.

The two-user Gaussian interference channel with noisy feedback (see Figure 1), is defined by the following input-output relationships

$$Y_{1i} = h_{11}X_{1i} + h_{21}X_{2i} + Z_{1i}, \tag{1}$$
$$Y_{2i} = h_{12}X_{1i} + h_{22}X_{2i} + Z_{2i}, \tag{2}$$
$$Y_{F_1 i} = g_1 Y_{1i} + \tilde{Z}_{1i}, \tag{3}$$
$$Y_{F_2 i} = g_2 Y_{2i} + \tilde{Z}_{2i}, \tag{4}$$

where X_{ji} denotes the signal sent by transmitter j, Y_{ji} denotes the output at receiver j, $Y_{F_j,i}$ denotes the feedback received at transmitter j, for $j = 1, 2$, at time i, for $i \in \{1, 2, ..., T\}$, and $\{Z_{ji}\}_{i=1}^T$ and $\{\tilde{Z}_{ji}\}_{i=1}^T$ are independent, additive white Gaussian noise processes with zero means and unit variances. The forward channel gains $\{h_{11}, h_{21}, h_{12}, h_{22}\}$ and the feedback channel gains $\{g_1, g_2\}$ are assumed to be constant and known at all terminals. Average unit power constraints are imposed at each transmitter. In other words, for a code of block length T, input sequences must satisfy $\frac{1}{T}\mathbb{E}(\sum_{i=1}^T |X_{ji}|^2) \leq 1$, for $j = 1, 2$.

Transmitter Tx_j, for $j = 1, 2$, wishes to communicate a message $m_j \in \{1, 2, ..., M_j\} = W_j$ to receiver Rx_j. It is assumed that W_1 and W_2 are independent. An (M_1, M_2, T, P_e) feedback code for the interference channel (IC) with noisy feedback consists of a sequence of encoding functions

$$f_j^i : \mathcal{W}_j \times \{\mathcal{Y}_{F_j 1}, \mathcal{Y}_{F_j 2}, ..., \mathcal{Y}_{F_j, i-1}\} \to \mathcal{X}_{ji} \tag{5}$$

for $j = 1, 2$, and $i = 1, 2, ..., T$, and two decoding functions

$$d_{jT} : \{\mathcal{Y}_{j1}, \mathcal{Y}_{j2}, ..., \mathcal{Y}_{jT}\} \to \hat{\mathcal{W}}_j \text{ for } j = 1, 2; \tag{6}$$

such that $\max\{P_{e,1T}, P_{e,2T}\} \leq P_e$, where $P_{e,1T}$ and $P_{e,2T}$ denote the average decoding error probabilities, which are computed as $P_{e,jT} = \mathcal{E}[P(\hat{w}_j \neq w_j|(w_1, w_2) \text{ were sent})]$. A rate pair (R_1, R_2) is achievable for the IC with noisy feedback if there exists an (M_1, M_2, T, P_e)-feedback code such that $P_e \to 0$ as $T \to \infty$ and $\frac{\log(M_1)}{T} \leq R_1$ and $\frac{\log(M_2)}{T} \leq R_2$. The capacity region of the IC with noisy feedback is defined as the closure of the set of all achievable rate pairs. With the goal of understanding the capacity region of the GIC with noisy feedback as defined above, we next describe the linear deterministic interference channel with partial feedback.

Using the deterministic model in [1], a non-negative integer n_{kj} is used to represent the channel gain from transmitter Tx_k to receiver Rx_j and it is given by $n_{kj} = \lceil \log h_{kj}^2 \rceil^+$. Note that the effect of the Gaussian noise is captured by these representative numbers. Let q denote the maximum channel gains in the interference channel, i.e., $q = \max(n_{kj})$. Thus, the transmitted signal from transmitter k at the time i will have a maximum of q bits visible to any receiver. Denote $X_{ki} = [X_{ki}^1, ..., X_{ki}^q]^T \in F_2^q$, for $k = 1, 2$, where the leftmost bit is the most significant bit and the rightmost bit is the least significant bit. In this linear model, the effect of interference between various signals is captured as the superposition of those signals. At the time i, the outputs at the receivers are given as

$$Y_{1i} = S^{q-n_{11}} X_{1i} \oplus S^{q-n_{21}} X_{2i}, \tag{7}$$

$$Y_{2i} = S^{q-n_{12}} X_{1i} \oplus S^{q-n_{22}} X_{2i}, \tag{8}$$

where S is the a square shift matrix of size q given by

$$S := \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & ... & 0 & 1 & 0 \end{bmatrix} \tag{9}$$

and the operation is modulo 2 addition in F_2.

Next, we analyze the feedback links in the Gaussian interference channel. The feedback links are effectively equivalent to

$$Y_{F_1 i} = g_1 Y_{1i} + \tilde{Z}_{1i} \tag{10}$$

$$= g_{F_1} \frac{Y_{1i}}{\sqrt{h_{11}^2 + h_{21}^2 + 2h_{11}h_{21} + 1}} + \tilde{Z}_{1i}, \tag{11}$$

$$Y_{F_2 i} = g_2 Y_{2i} + \tilde{Z}_{2i} \tag{12}$$

$$= g_{F_2} \frac{Y_{2i}}{\sqrt{h_{12}^2 + h_{22}^2 + 2h_{12}h_{22} + 1}} + \tilde{Z}_{2i}. \tag{13}$$

Using equations (10-13), we now model the corresponding feedback in the LD-IC model. The channel gains g_{F_j} for the feedback links can be represented by l_j, for $j = 1, 2$, where $l_j = \lceil \log g_{F_j}^2 \rceil^+$, and hence the feedback signals at the transmitters are given as

$$Y_{F_1 i} = S^{q-l_1} Y_{1i}, \quad Y_{F_2 i} = S^{q-l_2} Y_{2i}. \tag{14}$$

Effectively, via the feedback links, the transmitter j sees only the top l_j bits of the received signals R_{ji} (see Figure 2) .

The paper focuses on the symmetric LD-IC in which $m = n_{12} = n_{21}, n = n_{11} = n_{22}$, and $l = l_1 = l_2$, and the symmetric Gaussian IC with noisy feedback. Define

$$\text{SNR} := h_{11}^2 = h_{22}^2, \tag{15}$$

$$\text{INR} := h_{21}^2 = h_{12}^2, \tag{16}$$

$$\text{SNR}_F := g_{F_1}^2 = g_1^2 \cdot \text{var}(Y_1) = g_{F_2}^2 = g_2^2 \cdot \text{var}(Y_2). \tag{17}$$

III. Symmetric deterministic IC with partial feedback

As a stepping stone towards approximating the capacity region for the Gaussian IC with noisy feedback, we first consider the associated symmetric linear deterministic model.

A. *Capacity region*

Given a triple (n, m, l), we denote the capacity region for symmetric LD-IC with partial feedback by $\mathcal{C}^{\text{P}-\text{FB}}(n, m, l)$, which is the set of all achievable rate pairs (R_1, R_2) with partial feedback. We find it useful to define forward and feedback interference parameters respectively as follows

$$\alpha := \frac{m}{n}, \quad \beta := \frac{l}{n}. \tag{18}$$

The forward interference parameter α measures the normalized interference, whereas the feedback interference parameter β measures the normalized feedback. For the purpose of comparison with related work, we also define the normalized rates, with respect to n, as $R_j^* := \frac{R_j}{n}$, for $j = 1, 2$. Equivalent to $\mathcal{C}^{\text{P}-\text{FB}}(n, m, l)$, the normalized capacity region $\mathcal{C}^{\text{P}-\text{FB}}(\alpha, \beta)$ is the set of all achievable normalized rate pairs (R_1^*, R_2^*) with partial feedback.

The capacity region for the symmetric LD-IC with partial feedback is given by the following theorem.

Theorem 1. *The normalized capacity region $\mathcal{C}^{\mathrm{P-FB}}(\alpha, \beta)$ of the symmetric linear deterministic interference channel with partial feedback, is the set of non-negative normalized rate pairs (R_1^*, R_2^*) that satisfy*

$$R_1^* \leq \max(1, \alpha), \tag{19}$$
$$R_2^* \leq \max(1, \alpha), \tag{20}$$
$$R_1^* \leq 1 + (\beta - 1)^+, \tag{21}$$
$$R_2^* \leq 1 + (\beta - 1)^+, \tag{22}$$
$$R_1^* + R_2^* \leq (1-\alpha)^+ + \max(1, \alpha), \tag{23}$$
$$R_1^* + R_2^* \leq 2\max[(1-\alpha)^+, \alpha] + 2\min[(1-\alpha)^+, (\beta - \max(\alpha, (1-\alpha)^+))^+], \tag{24}$$
$$2R_1^* + R_2^* \leq (1-\alpha)^+ + \max(1, \alpha) + \max[\alpha, (1-\alpha)^+] + \min[(1-\alpha)^+, (\beta - \max(\alpha, (1-\alpha)^+))^+], \tag{25}$$
$$R_1^* + 2R_2^* \leq (1-\alpha)^+ + \max(1, \alpha) + \max[\alpha, (1-\alpha)^+] + \min[(1-\alpha)^+, (\beta - \max(\alpha, (1-\alpha)^+))^+]. \tag{26}$$

where $(\alpha)^+ := \max(0, \alpha)$.

Proof: We will present the forward proof in section VI-B. A system with partial feedback can perform no better than a system with full feedback. Thus, any outer bound that is applicable to the full feedback model, is also applicable to the partial feedback model. Thus, for the proof of outer bounds for equations (19), (20) and (23), please refer to [5]. The outer bound for the equation (21) is a simple cut-set bound [28], that follows from the outer bound

$$R_1 \leq H(Y_1, Y_{F_2}|X_2), \tag{27}$$

which can be proved easily. Nevertheless, there is an alternative way to prove this outer bound. In the regime where $\alpha < 1$, this outer bound is inactive due to outer bound in (19); in the strong and very strong interference regimes, i.e. $\alpha \geq 1$, this outer bound follows from an interesting observation. The observation is that, when $\beta \leq 1$, feedback Y_{F_2} does not help as the feedback is a composition of X_2 and the top n bits of X_1, and when $\beta > 1$, feedback starts to help but there is some overlap as the top n of X_1 in this case is a mixture of Y_{F_2} and X_2. Thus, we will present, in the appendix, an alternative, slightly more complicated, proof, which might be of interest to some readers, based on this simple, but intriguing, observation. The outer bound on the equation (22) is proved similarly. In addition, we will present the rest of the converse proof for Theorem 1 in section VI-A. ■

Next, we will compare the result for the partial feedback model with related results for the no feedback model, the rate-limited feedback model and the full feedback model.

B. Comparison with other feedback models

We recall here the capacity regions for the no feedback model, the rate-limited feedback model and the full feedback model. The normalized capacity region $\mathcal{C}^{\mathrm{No-FB}}(\alpha)$ of the symmetric linear deterministic channel with no feedback model [2], in which $\beta = 0$, is given the set of non-negative rate pairs (R_1^*, R_2^*) that satisfy

$$\begin{aligned} R_1^* &\leq 1, \\ R_2^* &\leq 1, \\ R_1^* + R_2^* &\leq (1-\alpha)^+ + \max(1, \alpha), \\ R_1^* + R_2^* &\leq 2\max[(1-\alpha)^+, \alpha], \\ 2R_1^* + R_2^* &\leq (1-\alpha)^+ + \max(1, \alpha) + \max[\alpha, (1-\alpha)^+], \\ R_1^* + 2R_2^* &\leq (1-\alpha)^+ + \max(1, \alpha) + \max[\alpha, (1-\alpha)^+]. \end{aligned} \tag{28}$$

The normalized capacity region $\mathcal{C}^{\mathrm{Full-FB}}(\alpha)$ of the full feedback model [5], in which $\beta = 1$, is given the set of non-negative normalized rate pairs (R_1^*, R_2^*) that satisfy

$$\begin{aligned} R_1^* &\leq \max(1, \alpha), \\ R_2^* &\leq \max(1, \alpha), \\ R_1^* + R_2^* &\leq (1-\alpha)^+ + \max(1, \alpha). \end{aligned} \tag{29}$$

The normalized capacity region $\mathcal{C}^{\mathrm{RL-FB}}(\alpha, \beta')$ of the rate-limited feedback model found in [10], is equivalent to the set of non-negative normalized rate pairs (R_1^*, R_2^*) that satisfy

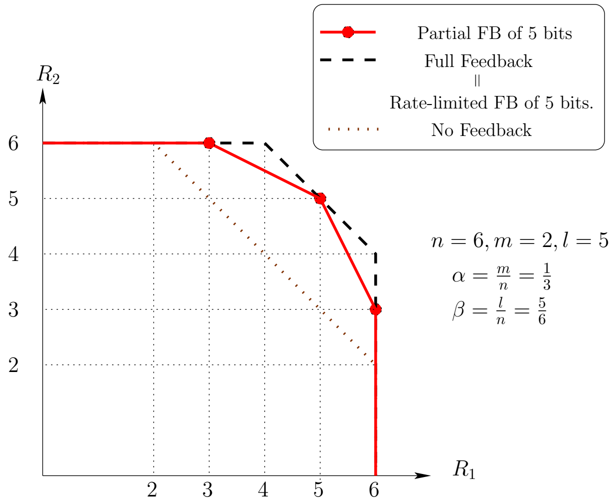

Fig. 3: Capacity regions for $n = 6, m = 2$ and $l = 5$.

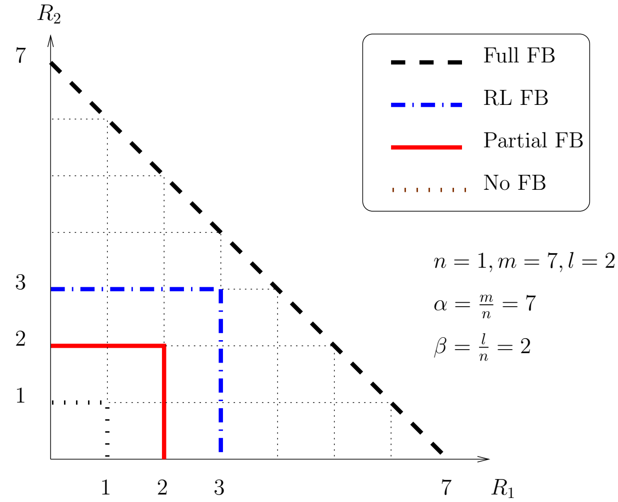

Fig. 4: Capacity regions for $n = 1, m = 7$ and $l = 2$.

$$\begin{aligned}
R_1^* &\le \max(1,\alpha),\\
R_2^* &\le \max(1,\alpha),\\
R_1^* &\le 1+\beta',\\
R_2^* &\le 1+\beta',\\
R_1^*+R_2^* &\le (1-\alpha)^+ + \max(1,\alpha),\\
R_1^*+R_2^* &\le 2\max[(1-\alpha)^+,\alpha] + 2\min[(1-\alpha)^+,\beta'],\\
2R_1^*+R_2^* &\le (1-\alpha)^+ + \max(1,\alpha) + \max[\alpha,(1-\alpha)^+] + \min[(1-\alpha)^+,\beta'],\\
R_1^*+2R_2^* &\le (1-\alpha)^+ + \max(1,\alpha) + \max[\alpha,(1-\alpha)^+] + \min[(1-\alpha)^+,\beta'].
\end{aligned} \tag{30}$$

In constrast to that in the partial-feedback model, the receivers in a rate-limited feedback model, with feedback rate β', can feed back to the transmitters any function of the received outputs, even though β' in the rate-limited feedback model is also a normalized rate of feedback just like β in the partial-feedback model. Clearly, such encoding functions include sending back the top $n\beta'$ bits; and hence the capacity of our model is in general contained within the capacity region with the same amount of rate-limited feedback. Thus, when $\beta = \beta'$, the capacity regions for these four models always satisfy the following rule

$$\mathcal{C}^{\text{No}-\text{FB}}(\alpha) \subseteq \mathcal{C}^{\text{P}-\text{FB}}(\alpha,\beta) \subseteq \mathcal{C}^{\text{RL}-\text{FB}}(\alpha,\beta') \subseteq \mathcal{C}^{\text{Full}-\text{FB}}(\alpha). \tag{31}$$

The set inclusions here can be strict. We illustrate the results through examples.

Example 1. Consider a channel in which $n = 6, m = 2$ and $l = 5$. Figure 3 shows the capacity regions with no feedback, with full feedback, with rate-limited feedback of $l = 5$ bits, and with partial feedback of $l = 5$ bits. Several interesting observations are worth making:

- The capacity region with full feedback coincides with that of rate-limited feedback of $l = 5$ bits.
- The sum capacity is 10 bits/channel-use for full, rate-limited and partial feedback settings.
- Most importantly, the capacity region with partial feedback is strictly contained in the capacity region with full feedback and rate-limited feedback. Previously, the capacity region for the model with full feedback did not require the bounds on $2R_1 + R_2$ and $R_1 + 2R_2$. On the other hand, it is here that we can clearly see the necessity of $2R_1 + R_2$ and $R_1 + 2R_2$ bounds in characterizing the exact capacity region when the feedback links are noisy.

Example 2. Consider another channel in which $n = 1, m = 7$ and $l = 2$. Figure 4 shows the capacity regions with no feedback, with full feedback, with rate-limited feedback of $l = 2$ bits, and with partial feedback of $l = 2$ bits. Several interesting observations are worth making:

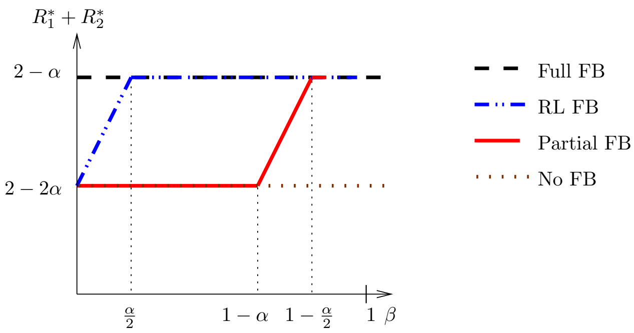

Fig. 5: Normalized sum rate for $0 \leq \alpha < \frac{1}{2}$.

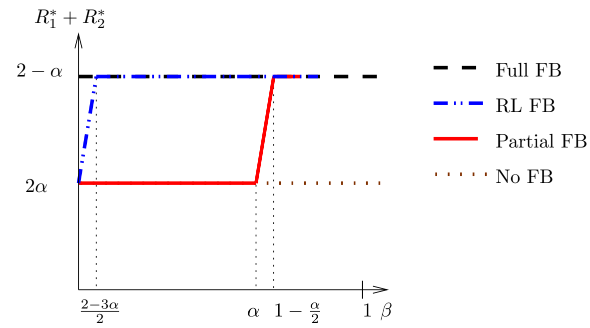

Fig. 6: Normalized sum rate for $\frac{1}{2} \leq \alpha < \frac{2}{3}$.

- All the set inclusions in (31) are strict. In other words, the capacity regions of the no feedback model, the partial feedback model and the rate-limited feedback model are strictly included in that of the partial feedback model, the rate-limited feedback model, and the full feedback model respectively.
- When $l = 2$, the capacity region of the partial feedback model is strictly larger than that of the no feedback model. In fact, this holds as long as $l > 1$. Thus, we can partially observe the role the partial feedback link plays in enlarging the capacity region. The capacity region of the partial feedback model are characterized by not only the direct link strength n and the cross interference link strength m, but also the feedback link strength l.

As a direct result of Theorem 1, we have the following corollary.

Corollary 1. *The normalized sum rate $R_1^* + R_2^*$ of the partial feedback model is the same as that of the no feedback model when $\beta \leq \beta_1^*$, where*

$$\beta_1^* = \begin{cases} \max(\alpha, (1-\alpha)^+) & \text{if } \alpha \leq 1, \\ 1 & \text{if } 1 < \alpha. \end{cases} \tag{32}$$

The normalized sum rate $R_1^* + R_2^*$ of the partial feedback model is the same as that of the full feedback model when $\beta \geq \beta_2^*$, where

$$\beta_2^* = \begin{cases} 1 - \frac{\alpha}{2} & \text{if } \alpha \leq 1, \\ \frac{\alpha}{2} & \text{if } 1 < \alpha. \end{cases} \tag{33}$$

The normalized sum rate $R_1^* + R_2^*$ as a function of β, for a fixed value of α, in different regimes, is illustrated in Figures 5, 6, 7, and 8. The normalized sum rate $R_1^* + R_2^*$ of the partial feedback model is the same as that of the no feedback model when $\beta \leq \beta_1^*$, which is defined in Corollary 1. Notice that β_1^* is the per-user symmetric capacity for the no feedback model. The normalized sum rate $R_1^* + R_2^*$ of the partial feedback model is strictly smaller than that of the rate-limited feedback model. The normalized sum rate $R_1^* + R_2^*$ for the partial feedback model reaches saturation and achieves the same performance as that of the full feedback model when $\beta \geq \beta_2^*$. Notice that β_2^* is the per-user symmetric capacity of the full feedback model.

Note that the normalized sum rate $R_1^* + R_2^*$ is not increased by any amount of feedback in the moderately strong interference regime, where $\frac{2}{3} \leq \alpha \leq 1$, and the strong interference regime, where $1 \leq \alpha \leq 2$.

As a direct result of Theorem 1, we have another corollary.

Corollary 2. *The capacity region of the partial feedback model is increased by the partial feedback if and only if $\beta \geq \beta_1^*$, where β_1^* is defined as in Corollary 1.*

In the following sub-section, we present discussion on ideas of the achievability proof and the converse proof for Theorem 1.

C. Achievability

In the classical interference channel without feedback, the HK encoding scheme currently gives the best achievable rate region [29] [30]. It was proved in [3] [2] that the HK encoding scheme can achieve the capacity region of the linear

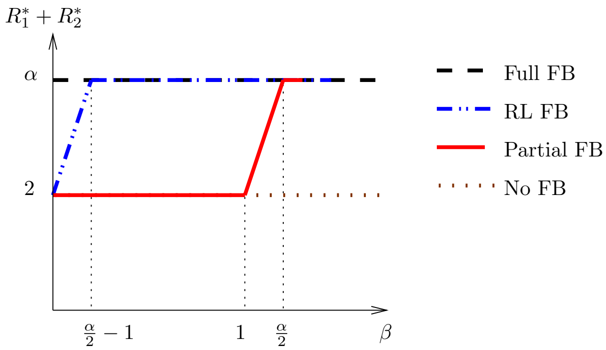

Fig. 7: Normalized sum rate for $2 \leq \alpha$ and $\alpha < 4$.

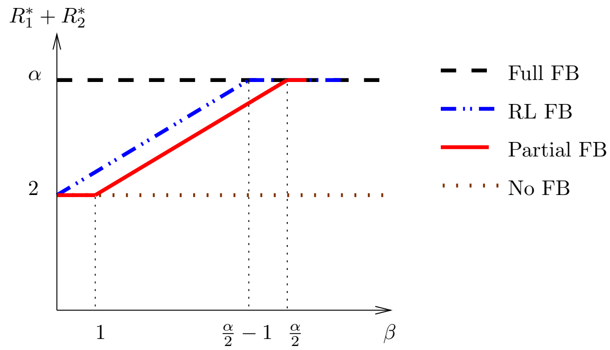

Fig. 8: Normalized sum rate for $2 \leq \alpha$, and $4 \leq \alpha$.

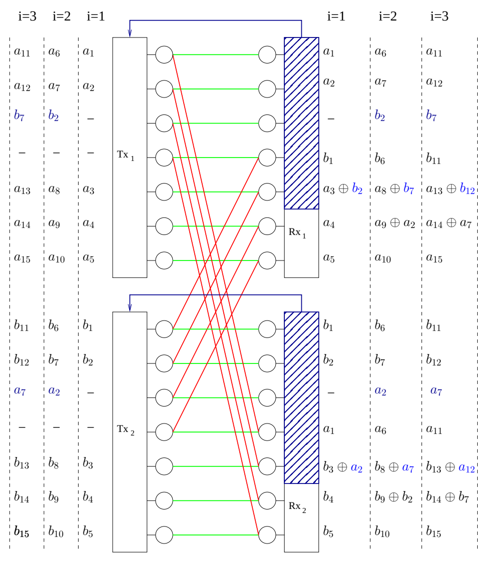

Fig. 9: Encoding example for $(n = 7; m = 4; l = 5)$

deterministic interference channel with no feedback. In the HK encoding scheme, messages are split into two parts: common information and private information. However, splitting messages into two parts is not sufficient to account for the effect of feedback links on the capacity region of the interference channel with noisy feedback. Previous works have made used of more-than-two message splitting [14] [15] [10] [24]. Tuninetti [14] in 2007 developed an achievability scheme for a very generic model: IC with generalized feedback. It is true that IC with noisy feedback is a special case of IC with generalized feedback. Thus, any achievable scheme developed for IC with generalized feedback is also applicable for IC with noisy feedback. We are going to prove that this achievable scheme for IC with generalized feedback is optimal for IC with noisy feedback. The remaining question is which choice of auxiliary random variables will obtain the optimal achievable rate region. Before answering this question, we will consider an example.

Example 3. Consider an example, in which $n = 7, m = 4$ and $l = 5$. In this example, we show an encoding scheme to achieve the point $(R_1, R_2) = (5, 5)$ in the achievable rate region. Without feedback, the maximum achievable sum rate is 8 bits per channe use. Here, we manage to obtain a sum rate of 10 bits per channel use through feedback. The encoding scheme is shown in Figure 9.

In the first time slot $t = 1$, each transmitter sends 5 fresh information bits as shown in the figure. With a feedback channel gain $l = 4$, Tx_1 sees only the top 5 bits, which are $a_1, a_2, -, b_1, a_3 \oplus b_2$, and hence it can recover b_2. In the second time slot $i = 2$, transmitter Tx_1 sends 5 new fresh information bits again and encodes b_2 at the third topmost signal level as shown in the figure. The third topmost signal level is chosen to ensure that the resolving signal bit b_2 is received cleanly at Tx_1. With the help of b_2, Rx_1 can resolve the interference in the previous time slot and decode a_3 successfully. Due to symmetry, the same encoding operation is carried out at Tx_2 and Rx_2. We can repeat this encoding scheme again for a duration of B time slots. It is easy to see that this scheme asymptotically achieves a sum rate $R_1 + R_2 = 10$ bits/channel

use. Thus, the bound $R_1 + R_2 \leq 2m + 2(l-m)^+$ is active in this example and the encoding scheme has achieved the sum capacity in this regime.
A careful observation suggests, in each channel use, the message bits from a transmitter is categorized into three parts. For example, transmitter 1, in the second time slot when $t = 2$, has 3 private bits a_8, a_9, a_{10}, 2 cooperative common bit $(b_1, -)$, and the remaining 2 bits as non-cooperative common bits. This example suggests the size of the cooperate common message in general to be $(l - (n-m)^+)^+$ and the position of the cooperative common message to be within the top m bits of each transmitter.

A detailed choice of auxiliary random variables are shown in the proof in section VI-B.

Remark 1. Apart from the generic achievable scheme shown in section VI-B, we developed an alternative, more elementary achievable scheme, which was presented in [13]. That alternative scheme gives certain alternative points of view, which are not captured by the generic achievable scheme here.

D. *Outer bounds*

Consider the same example in Figure 9. Notice that for $l \leq 4$, the feedback link does not show any advantage over the situation without feedback. For example, in the first slot $t = 1$ when $l = 4$, even though transmitter 1 sees 4 bits $(a_1, a_2, -, b_1)$ via the feedback link, the knowledge of b_1 is redundant as no interference has appeared at receiver 1 yet. However, when $l = 5$, there is interference at $a_3 \oplus b_2$. Thus, we start to see the benefit of the feedback link. Notice that transmitter 1 always knows the top $n - m = 3$ bits of receiver 1. However, the benefit of feedback does not occur when l exceeds $n - m$. It only occurs when l exceeds m. This motivates us to define X_{top1} and X_{top2} in the converse. For more details, please refer to section VI-A.

Remark 2. The equation (24) can be proved using the equation (1) of Theorem 1 in the paper [24]. Even though our outer bound for equation (24) and the outer bound for that equation (1) give the same result for the linear deterministic model, they give different results for the Gaussian IC at low SNR.

IV. Symmetric Gaussian interference channel with noisy feedback

With the results and techniques developed for the symmetric linear deterministic model, we are one step closer to approximating the capacity region for the symmetric Gaussian IC with noisy feedback. First, we derive the outer bounds, next we derive the inner bounds. Then, we show that the gap between the outer bounds and the inner bounds is a constant.

A. *Outer bounds*

Define

$$\alpha_G := \frac{\log \mathrm{INR}}{\log \mathrm{SNR}}. \tag{34}$$

The outer bounds for the symmetric Gaussian interference channel with noisy feedback is given by the following theorem.

Theorem 2. *The capacity region of the symmetric Gaussian interference channel with noisy feedback, is included by the set of non-negative pairs (R_1, R_2), for some $0 \leq \rho \leq 1$, satisfying*

$$R_1 \leq \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right) := \psi_1 \tag{35}$$

$$R_2 \leq \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right) \tag{36}$$

$$R_1 \leq \frac{1}{2}\log\left(\mathrm{SNR}+1\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+1} + 1\right) := \psi_2 \tag{37}$$

$$R_2 \leq \frac{1}{2}\log\left(\mathrm{SNR}+1\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+1} + 1\right) \tag{38}$$

$$R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1} + 1\right) + \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right) := \psi_3 \tag{39}$$

$$R_1 + R_2 \leq \psi_4 \tag{40}$$

$$2R_1 + R_2 \leq \psi_5 \tag{41}$$

$$R_1 + 2R_2 \leq \psi_5, \tag{42}$$

where

$$\psi_4 := \begin{cases} \log\left(\frac{\mathrm{INR}^3+\mathrm{INR}^2}{\mathrm{SNR}(\mathrm{INR}+1)}+1\right)+\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right)+\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\log 3 & \text{if } \frac{1}{2}\leq\alpha_G<1, \\ \log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}.\mathrm{INR}}+1}{\mathrm{INR}+1}\right)+\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right) & \text{otherwise}, \end{cases} \tag{43}$$

$$\psi_5 := \begin{cases} \frac{1}{2}\log\left(\frac{\mathrm{INR}^3+\mathrm{INR}^2}{\mathrm{SNR}(\mathrm{INR}+1)}+1\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log 3 & \\ \quad +\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)+\frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right) & \text{if } \frac{1}{2}\leq\alpha_G<1, \\ \frac{1}{2}\log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}{\mathrm{INR}+1}\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right) & \\ \quad +\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)+\frac{1}{2}\log\left(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1\right) & \text{otherwise}. \end{cases} \tag{44}$$

Proof: The bounds of (35),(36) and (39) were derived in [5]. Thus, it suffices to prove the bounds of (37),(40) and (41). These bounds will be proved in Appendix, section VI-C. The proof of (38) and (42) follow by symmetry. ■

Remark 3. When $\mathrm{SNR}_F \to \infty$, the outer bounds of (37), (38), (41) and (42) are redundant. At high SNR, the outer bounds here are equivalent to that in the full-feedback model [5]. At low SNR, the outer bounds here are slightly looser than that in the full-feedback model as we do not include the following cut-set outer bound [5]

$$R_1 \leq h(Y_2|X_2) - h(Z_2) + h(Y_1|X_2,S_1) - H(Z_1) \tag{45}$$

$$\leq \frac{1}{2}\log\left(1+(1-\rho^2)\mathrm{INR}\right)+\frac{1}{2}\log\left(1+\frac{(1-\rho^2)\mathrm{SNR}}{1+(1-\rho^2)\mathrm{INR}}\right). \tag{46}$$

However, the absence of the bound (46) will not affect our constant-gap analysis much, so we do not consider it here.

Remark 4. The symmetric Gaussian IC with noisy feedback and the symmetric Gaussian IC with rate-limited feedback [10] share the same bounds of (35), (36) and (39).

Remark 5. Theorem II.2 in the paper [21] gives a generic outer bound on the sum rate $R_1 + R_2$ for IC with generalized feedback, that involves 6 auxiliary random variables. We are not sure whether this generic outer bound can be used to prove our outer bound of (40), and we are not sure which realizations of auxiliary random variables will give us a tight outer bound on the sum rate.

Remark 6. None of the outer bounds on the sum rate $R_1 + R_2$, for IC with source cooperation, in Theorem 2 of the paper [24], is equivalent to our outer bound (40), especially at low SNR.

B. Inner bounds

The inner bounds for the symmetric Gaussian interference channel with noisy feedback is given by the following theorem.

Theorem 3. *The capacity region of the two-user symmetric Gaussian interference channel with noisy feedback includes the set of all non-negative pairs of* (R_1, R_2) *satisfying*

$$R_1 \leq \min(\tau_6, \tau_4+\tau_1, \tau_1+\tau_2+\tau_3) \tag{47}$$

$$R_2 \leq \min(\tau_6, \tau_4+\tau_1, \tau_1+\tau_2+\tau_3) \tag{48}$$

$$R_1+R_2 \leq \min(\tau_2+\tau_6, 2\tau_1+\tau_5+\tau_2, 2\tau_1+2\tau_3) \tag{49}$$

$$2R_1+R_2 \leq \min(\tau_6+\tau_2+\tau_3+\tau_1, 3\tau_1+\tau_5+\tau_2+\tau_3) \tag{50}$$

$$R_1+2R_2 \leq \min(\tau_6+\tau_2+\tau_3+\tau_1, 3\tau_1+\tau_5+\tau_2+\tau_3) \tag{51}$$

where

$$\tau_6 := \frac{1}{2}\log\frac{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}{\mathrm{INR}\cdot P_p+1} \tag{52}$$

$$\tau_5 := \frac{1}{2}\log\frac{\mathrm{SNR}(P_{nc}+P_p)+\mathrm{INR}(P_{nc}+P_p)+1}{\mathrm{INR}\cdot P_p+1} \tag{53}$$

$$\tau_4 := \frac{1}{2}\log\frac{\mathrm{SNR}(P_{nc}+P_p)+\mathrm{INR}\cdot P_p+1}{\mathrm{INR}\cdot P_p+1} \tag{54}$$

$$\tau_3 := \frac{1}{2}\log\frac{\mathrm{SNR}\cdot P_p+\mathrm{INR}(P_{nc}+P_p)+1}{\mathrm{INR}\cdot P_p+1} \tag{55}$$

$$\tau_2 := \frac{1}{2}\log\frac{\mathrm{SNR}\cdot P_p + \mathrm{INR}\cdot P_p + 1}{\mathrm{INR}\cdot P_p + 1} \tag{56}$$

$$\tau_1 := \frac{1}{2}\log\frac{\tau_{1n}}{\tau_{1d}} \tag{57}$$

$$\tau_{1n} := \frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}\times[\mathrm{INR}(P_{cc}+P_{nc}+P_p)+1]+1$$

$$\tau_{1d} := \frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}\times[\mathrm{INR}(P_{nc}+P_p)+1]+1,$$

for all power allocation schemes that satisfy

$$P_p + P_{cc} + P_{nc} = 1. \tag{58}$$

Proof: Refer to section VI-E. ■

C. A constant gap between inner and outer bounds

Define

$$\delta_R := \delta_{R_1} := \delta_{R_2} := \min(\psi_1,\psi_2) - \min(\tau_6, \tau_4+\tau_1, \tau_1+\tau_2+\tau_3, \tau) \tag{59}$$

$$\delta_{2R} := \delta_{R_1+R_2} := \min(\psi_3,\psi_4,2\psi_1,2\psi_2,\psi_1+\psi_2) - \min(\tau_2+\tau_6, 2\tau_1+\tau_2+\tau_5, 2\tau_1+2\tau_3, 2\tau) \tag{60}$$

$$\delta_{3R} := \delta_{2R_1+R_2} := \delta_{R_1+2R_2}$$

$$:= \min(\psi_5, \psi_1+\psi_3, \psi_1+\psi_4, \psi_2+\psi_3, \psi_2+\psi_4, 3\psi_1, 3\psi_2) - \min(\tau_1+\tau_2+\tau_3+\tau_6, 3\tau_1+\tau_2+\tau_3+\tau_5, 3\tau). \tag{61}$$

$$\delta := \max(\delta_R, \delta_{2R}, \delta_{3R}) \tag{62}$$

where τ is any achievable rate for any transmitter, using some achievability scheme. In words, δ_R, δ_{2R} and δ_{3R} are the possible gaps between the minimum of the set of the derived outer bounds and and the minimum of the set of the derived inner bounds for the individual rate, the sum rate R_1+R_2, and the weighted sum rates $2R_1+R_2$ and R_1+2R_2 respectively.

This is the main result in this paper.

Theorem 4. *Outer bounds in Theorem 2 are no more than 11.7 bits/s/Hz away from the achievable rate region. More precisely, we have*

$$\delta \le 11.7 \tag{63}$$

Proof: Refer to section VI-F. ■

Remark 7. Now we show how the Gaussian IC with noisy feedback is also related to the Gaussian IC with source cooperation. Note

$$Y_{F_1 i} = g_1 Y_{1i} + \tilde{Z}_{1i} \tag{64}$$

$$= g_1 h_{11} X_{1i} + g_1 h_{21} X_{2i} + g_1 Z_{1i} + \tilde{Z}_{1i}. \tag{65}$$

Transmitter have access to its own codewords, so we will subtract the contribution from X_{1i}, and define a scaled version of the remaining part

$$Y'_{F_1 i} = \frac{1}{\sqrt{g_1^2+1}} g_1 h_{21} X_{2i} + \frac{1}{\sqrt{g_1^2+1}}(g_1 Z_{1i} + \tilde{Z}_{1i}). \tag{66}$$

Thus, the Gaussian IC with noisy feedback is also closely related to the Gaussian IC with source cooperation considered by Prabhakaran and Wiswanath [24] and others.

Remark 8. HK [29] scheme used two-message splitting, which was proved to be at most 1 bit/s/Hz away from the outer bounds [3]. The achievability scheme here makes use of three-message splitting. Prabhakaran and Wiswanath [24] proposed three different encoding schemes, which are based on three-message splitting (which is the same as ours here), four-message splitting and mixture of these two schemes. Other advanced achievibility schemes developed in [15] for the IC with generalized feedback is also applicable to our model. We are not sure if our current inner bounds are sufficiently strict so it might be advantageous to use any alternative achievability schemes found in these related works to reduce the gap further. The large gap may be also due to the outer bounds. It might be neccessary to use techniques similar to that in [3], and to develop new techniques to tighten the outer bounds further. In addition, particular attention should be paid to the outer bounds, such as the bounds of (37), (38), (40 - 42), which are functions of the feedback link strength SNR_F. Furthermore, the gap is estimated based on a crude estimation method. A more refined technique should be employed to reduce the gap further. The work in [24] considered bounds for the symmetric Gaussian IC with source cooperation and obtained a gap of 10 bits/s/Hz for the sum rate $R_1 + R_2$, for real random variables. From the proof of this theorem, our gap for the sum rate only is $\delta_{2R} = 9.3$ bits/s/Hz. At high SNR, bounds in [24] and the bounds here will give the same result. However, at low SNR, ignoring the differences in estimation of the gap, our outer bounds on the sum rate seem to be slightly better for symmetric Gaussian IC with symmetric noisy feedback.

Define

$$\beta_G := \frac{\log \mathrm{SNR_F}}{\log \mathrm{SNR}}. \tag{67}$$

$$\tag{68}$$

Next, define the generalized degrees of freedom as

$$d_1(\alpha_G, \beta_G) := \lim_{\mathrm{SNR}\to\infty} \frac{R_1(\mathrm{SNR}, \mathrm{INR}, \mathrm{SNR_F})}{\frac{1}{2}\log(1+\mathrm{SNR})}, \tag{69}$$

$$d_2(\alpha_G, \beta_G) := \lim_{\mathrm{SNR}\to\infty} \frac{R_2(\mathrm{SNR}, \mathrm{INR}, \mathrm{SNR_F})}{\frac{1}{2}\log(1+\mathrm{SNR})}. \tag{70}$$

As a result of Theorem 4, we obtain the following corollary, which gives the generalized-degree-of-freedom region of the symmetric Gaussian interference channel with noisy feedback.

Corollary 3. *For the symmetric Gaussian interference channel with noisy feedback, the generalized-degrees-of-freedom region is the set of non-negative pairs* (d_1, d_2) *that satisfy*

$$d_1 \le \max(1, \alpha_G), \tag{71}$$

$$d_2 \le \max(1, \alpha_G), \tag{72}$$

$$d_1 \le \max(1, \beta_G), \tag{73}$$

$$d_2 \le \max(1, \beta_G), \tag{74}$$

$$d_1 + d_2 \le \max(1, \alpha_G) + (1-\alpha_G)^+, \tag{75}$$

$$d_1 + d_2 \le 2\max(1-\alpha_G, \alpha_G) + 2(\beta_G - \max(1-\alpha_G, \alpha_G))^+, \tag{76}$$

$$2d_1 + d_2 \le (\beta_G - \max(1-\alpha_G, \alpha_G))^+ + \max(1-\alpha_G, \alpha_G) + (1-\alpha_G)^+ + \max(1, \alpha_G), \tag{77}$$

$$d_1 + 2d_2 \le (\beta_G - \max(1-\alpha_G, \alpha_G))^+ + \max(1-\alpha_G, \alpha_G) + (1-\alpha_G)^+ + \max(1, \alpha_G). \tag{78}$$

Remark 9. The generalized-degrees-of-freedom region for the symmetric Gaussian IC is similar to the capacity region for the symmetric LD-IC. Therefore, any set of remarks and observations that are applicable to the symmetric LD-IC, also applies directly to the generalized-degree-of-freedom region of the Gaussian IC.

D. Discussion on the asymmetric Gaussian interference channel with noisy feedback

The symmetric Gaussian interference channel with symmetric noisy feedback is only a special case of the asymmetric Gaussian interference channel with asymmetric noisy feedback. To approximate the asymmetric Gaussian interference channel with noisy feedback directly is a challenging task. Thus, it is beneficial to first find the capacity region for the asymmetric LD-IC with asymmetric partial feedback. A keen reader would have noticed that the inner bounds and outer bounds developed in the proof of Theorem 1 (sections VI-A and VI-B) are also applicable to the asymmetric LD-IC with asymmetric partial feedback. However, in the outer bounds, we relied on carefully-defined auxiliary random variables X_{top1} and X_{top2} to optimally tighten the outer bounds. Similarly, in the inner bounds, we relied on the carefully-chosen random variables U_1 and U_2, in terms of the size and the location of the bits assigned to these two random variables with respect to X_1 and X_2 respectively, so that we can optimally maximize the inner bounds to the extent that the inner bounds match the outer bounds exactly. We are not sure if the current outer bounds and inner bounds are sufficient to determine the capacity region for the asymmetric LD-IC with partial feedback. To choose optimal sets of random variables X_{topj}, U_j, for $j \in \{1, 2\}$, which enable us to determine the capacity region for asymmetric LD-IC with asymmetric partial feedback, for different values of n_{ij} and l_j, for $i, j \in \{1, 2\}$, is a non-trivial problem. Therefore, to approximate the capacity region for the asymmetric Gaussian IC with asymmetric noisy feedback remains an open problem for now.

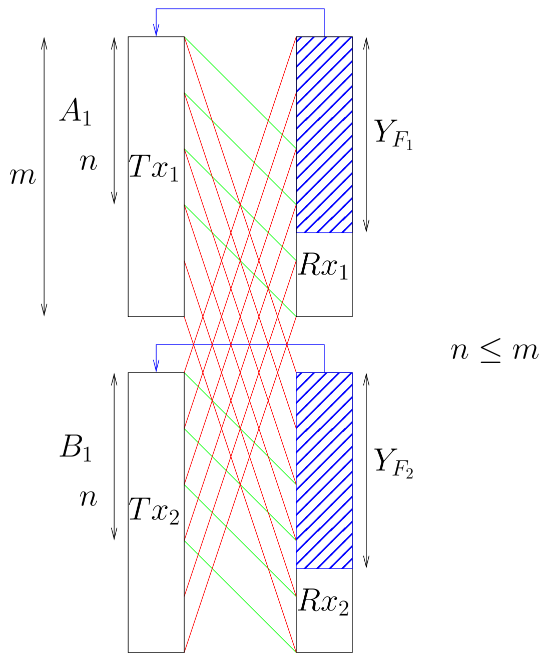

Fig. 10: Illustration of A_1 and B_1 when $n \le m$.

V. Conclusions

In this paper, we have obtained the capacity region for the symmetric linear deterministic interference channel with partial feedback. We have shown that partial feedback increases the capacity region if and only if the amount of feedback level l is greater than a certain threshold l^*, and it is found that l^* is equal to the per-user symmetric capacity without feedback. One of the key ideas is a novel converse proof which includes outer bound on weighted sum rates $2R_1 + R_2$ and $R_1 + 2R_2$. Our novel outer bounds are tightened by specially defined auxiliary random variables. We have also illustrated through numerous examples, that the outer bounds on the sum rate $R_1 + R_2$ derived in [11] alone are not sufficient to characterize the capacity region, and $2R_1 + R_2$ and $R_1 + 2R_2$ bounds are also necessary. The result and the techniques developed for this linear deterministic model are then applied to characterize inner bounds and outer bounds for the symmetric Gaussian IC with noisy feedback. The outer bounds are shown to be at most 11.7 bits/s/Hz away from the achievable rate region. As a corollary, the generalized-degree-of-freedom region, which approximates the capacity region of the symmetric Gaussian IC at high SNR, is found.

VI. Appendix

A. Converse proof

1) Bounds on R_1 and R_2: Now, the outer bounds of (21) and (22) on R_1 and R_2 respectively, are proved. When $0 \le m < n$, we always have $R_j \le n$, for $j = 1, 2$, as proved above. Thus, we only need to consider the case $n \le m$. Consider Figure 10. Let A_1 denote the top n bits of transmitter 1, and let B_1 denote the top n bits of transmitter 2.

We have

$$\begin{aligned}
TR_1 &= H(W_1) \\
&\overset{(a)}{=} H(W_1|W_2) \\
&= I(W_1; A_1^T, Y_{F_2}^T|W_2) + H(W_1|W_2, A_1^T, Y_{F_2}^T) \\
&\overset{(b)}{=} I(W_1; A_1^T, Y_{F_2}^T|W_2) + H(W_1|W_2, A_1^T, Y_{F_2}^T, X_2^T, Y_1^T) \\
&\overset{(c)}{\le} H(A_1^T, Y_{F_2}^T|W_2) + H(W_1|Y_1^T) \\
&\overset{(d)}{\le} \sum_{i=1}^{T} H(A_{1i}, Y_{F_2 i}|A_1^{i-1}, Y_{F_2}^{i-1}, W_2) + 1 + TP_e^T \\
&\overset{(e)}{\le} \sum_{i=1}^{T} H(A_{1i}, Y_{F_2 i}|X_{2i}) + 1 + TP_e^T,
\end{aligned} \tag{79}$$

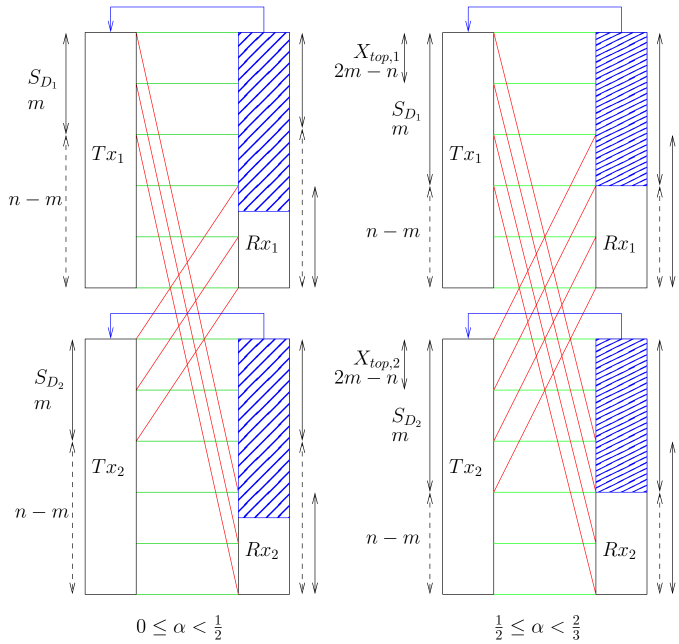

Fig. 11: Illustration of S_{D_j} and $X_{top,j}$.

where
(a) follows from the independence between W_1 and W_2;
(b) follows from the fact X_2^T is a function of $(W_2, Y_{F_2}^T)$, and Y_1^T is a function of (A_1^T, X_2^T);
(c) follows from the facts that $H(A_1^T, Y_{F_2}^T | W_2, W_1) = 0$ and that conditioning reduces the entropy;
(d) follows from Fano's inequality; and
(e) follows from the fact that $X_{2,i}$ is a function of $(W_2, Y_{F_2}^{i-1})$.

We next bound the term $\sum_{i=1}^T H(A_{1i}, Y_{F_2 i}|X_{2i})$ in (79).

Case 1: $0 \le l \le n$. For this case, $Y_{F_2 i}$ is a function of (A_{1i}, X_{2i}). Thus, we have

$$\sum_{i=1}^T H(A_{1i}, Y_{F_2 i}|X_{2i}) = \sum_{i=1}^T H(Y_{F_2 i}|X_{2i}, A_{1i}) + \sum_{i=1}^T H(A_{1i}|X_{2i}) \le 0 + nT. \tag{80}$$

Case 2: $n \le l \le m$. In this case, $A_{1,i}$ is a function of $(Y_{F_2,i}, X_{2,i})$. We have

$$\sum_{i=1}^T H(A_{1i}, Y_{F_2 i}|X_{2i}) = \sum_{i=1}^T H(Y_{F_2 i}|X_{2i}) + \sum_{i=1}^T H(A_{1i}|X_{2i}, Y_{F_2 i}) \le lT + 0. \tag{81}$$

From both these cases, we conclude that $R_1 \le n + (l-n)^+$. The inequality for R_2 can be proved in a similar manner.

2) Bound on $R_1 + R_2$: Let S_{D_1} represent the top m bits of the first transmitter. When $m < n$, it will be the top m bits out of n bits. When $n < m$, it will represent all the bits from the first transmitter. Intuitively, S_{D_1} represents the m information bits that are visible at both receivers. Similarly, let S_{D_2} represent the top m bits for the second transmitter.

Furthermore, define X_{topj} as the top $\min(m, (2m-n)^+)$ bits of transmitter j. In other words, X_{topj} is the top $(2m-n)^+$ bits of transmitter j when $\frac{n}{2} \le m \le n$, the top m bits when $m \ge n$. No equivalent variable is defined in the case when $m \le \frac{n}{2}$. These two random variables mainly serve to explain the bounds in the weak interference regime and the moderately strong interference regime.

It is worthwhile to give examples on these four random variables for ease of reading. Consider the case of the very weak interference regime where $0 < m \le \frac{n}{2}$. X_{topj} and S_{D_j}, for $j = 1, 2$, are illustrated in Figure 11. In this regime, S_{D_1} represents the top m bits of transmitter 1, and X_{top1} is a null region in this regime.

Consider a second example. Consider the case of the weak interference regime where $\frac{n}{2} \le m \le \frac{2n}{3}$. Again, X_{topj} and S_{D_j}, for $j = 1, 2$, are also illustrated in Figure 11. In this regime, S_{D_1} also represents the top m bits of transmitter 1. X_{top1} is the top $2m - n$ bits of transmitter 1.

In the proofs below, we make use of the following lemma.

Lemma 1.

$$I(S_{D_2}^T, X_{top1}^T, Y_{F_2}^T, W_2; Y_{F_1}^T, W_1) \le H(Y_{F_1}^T|W_1) + \sum_{i=1}^{T}[H(Y_{F_2 i}|X_{2i}, X_{top1,i}) + H(X_{top1,i}|S_{D_2 i})] \quad (82)$$

$$I(S_{D_1}^T, X_{top2}^T, Y_{F_1}^T, W_1; Y_{F_2}^T, W_2) \le H(Y_{F_2}^T|W_2) + \sum_{i=1}^{T}[H(Y_{F_1 i}|X_{1i}, X_{top2,i}) + H(X_{top2,i}|S_{D_1 i})]. \quad (83)$$

Proof:

$$\begin{aligned}
&I(S_{D_2}^T, X_{top1}^T, Y_{F_2}^T, W_2; Y_{F_1}^T, W_1) \\
&= I(W_2; Y_{F_1}^T, W_1) + I(S_{D_2}^T, X_{top1}^T, Y_{F_2}^T; Y_{F_1}^T, W_1|W_2) \\
&\overset{(a)}{=} H(Y_{F_1}^T|W_1) + H(Y_{F_2}^T, S_{D_2}^T, X_{top1}^T|W_2) \\
&\overset{(b)}{\le} H(Y_{F_1}^T|W_1) + \sum_{i=1}^{T}[H(Y_{F_2 i}|Y_{F_2}^{i-1}, S_{D_2}^i, X_{top1}^i, W_2, X_{2i}) + H(X_{top1,i}|Y_{F_2}^{i-1}, S_{D_2}^i, W_2, X_{2i}) + H(S_{D_2 i}|Y_{F_2}^{i-1}, W_2, X_{2i})] \\
&\overset{(c)}{=} H(Y_{F_1}^T|W_1) + \sum_{i=1}^{T}[H(Y_{F_2 i}|X_{2i}, X_{top1,i}) + H(X_{top1,i}|S_{D_2 i})]
\end{aligned}$$

where

(a) follows from the fact that, given (W_1, W_2), the entropy of any random variable is 0; and that W_1 is independent of W_2;
(b) comes from the fact that X_{2i} is a function of $(Y_{F_2}^{i-1}, W_2)$; and
(c) follows from the fact $S_{D_2 i}$ is a function of X_{2i}.

The second part of the lemma is proved similarly to the above. ■

We have

$$\begin{aligned}
&T(R_1 + R_2 - p_e^T) \\
&\le I(W_1; Y_1^T) + I(W_2; Y_2^T) \\
&\le I(W_1; Y_1^T, Y_{F_1}^T) + I(W_2; Y_2^T, Y_{F_2}^T) \\
&= H(Y_1^T) + H(Y_{F_1}^T|Y_1^T) - H(Y_{F_1}^T|W_1) - H(Y_1^T|Y_{F_1}^T, W_1) + H(Y_2^T) + H(Y_{F_2}^T|Y_2^T) - H(Y_{F_2}^T|W_2) - H(Y_2^T|Y_{F_2}^T, W_2) \\
&\overset{(a)}{=} H(Y_1^T) - H(S_{D_2}^T|Y_{F_1}^T, W_1) + H(Y_2^T) - H(S_{D_1}^T|Y_{F_2}^T, W_2) - H(Y_{F_1}^T|W_1) - H(Y_{F_2}^T|W_2) \\
&\overset{(b)}{=} H(Y_1^T) - H(S_{D_2}^T, X_{top1}^T|Y_{F_1}^T, W_1) + H(Y_2^T) - H(S_{D_1}^T, X_{top2}^T|Y_{F_2}^T, W_2) - H(Y_{F_1}^T|W_1) - H(Y_{F_2}^T|W_2) \\
&\le H(Y_1^T) + [I(S_{D_2}^T, X_{top1}^T; Y_{F_1}^T, W_1) - H(S_{D_2}^T, X_{top1}^T)] + [H(S_{D_2}^T, X_{top1}^T|Y_2^T) - H(S_{D_2}^T, X_{top1}^T|Y_2^T, X_2^T)] \\
&\quad + H(Y_2^T) + [I(S_{D_1}^T, X_{top2}^T; Y_{F_2}^T, W_2) - H(S_{D_1}^T, X_{top2}^T)] + [H(S_{D_1}^T, X_{top2}^T|Y_1^T) - H(S_{D_1}^T, X_{top2}^T|Y_1^T, X_1^T)] \\
&\quad - H(Y_{F_1}^T|W_1) - H(Y_{F_2}^T|W_2) \\
&= I(S_{D_2}^T, X_{top1}^T; Y_{F_1}^T, W_1) + H(Y_2^T|S_{D_2}^T, X_{top1}^T) - H(S_{D_2}^T, X_{top1}^T|Y_2^T, X_2^T) \\
&\quad + I(S_{D_1}^T, X_{top2}^T; Y_{F_2}^T, W_2) + H(Y_1^T|S_{D_1}^T, X_{top2}^T) - H(S_{D_1}^T, X_{top2}^T|Y_1^T, X_1^T) - H(Y_{F_1}^T|W_1) - H(Y_{F_2}^T|W_2) \\
&\overset{(c)}{=} I(S_{D_2}^T, X_{top1}^T; Y_{F_1}^T, W_1) + H(Y_2^T|S_{D_2}^T, X_{top1}^T) + I(S_{D_1}^T X_{top2}^T; Y_{F_2}^T, W_2) + H(Y_1^T|S_{D_1}^T, X_{top2}^T) \\
&\quad - H(Y_{F_1}^T|W_1) - H(Y_{F_2}^T|W_2) \\
&\overset{(d)}{\le} I(S_{D_2}^T, X_{top1}^T, Y_{F_2}^T, W_2; Y_{F_1}^T, W_1) + H(Y_2^T|S_{D_2}^T, X_{top1}^T) + I(S_{D_1}^T, X_{top2}^T, Y_{F_1}^T, W_1; Y_{F_2}^T, W_2) + H(Y_1^T|S_{D_1}^T, X_{top2}^T) \\
&\quad - H(Y_{F_1}^T|W_1) - H(Y_{F_2}^T|W_2) \\
&\overset{(e)}{\le} \sum_{i=1}^{T}[H(Y_{2i}|S_{D_2 i}, X_{top1,i}) + H(Y_{1i}|S_{D_1 i}, X_{top2,i}) + H(Y_{F_2 i}|X_{2i}, X_{top1,i}) + H(Y_{F_1 i}|X_{1i}, X_{top2,i}) + H(X_{top1,i}|S_{D_2 i}) \\
&\quad + H(X_{top2,i}|S_{D_1 i}),
\end{aligned}$$

where

(a) follows from the fact that Y_{F_j} is a function of Y_j for $j = 1, 2$;
(b) follows from the fact that X_{topj}^T is a function of X_j^T, which is in turn a function of $(Y_{F_j}^T, W_j)$, for $j = 1, 2$;

(c) follows from the facts that $S_{D_j}^T$ is a function of X_j^T for $j = 1,2$; and X_{top1}^T is a function of X_1^T, which is in turn a function of (Y_2^T, X_2^T), and vice versa;
(d) follows from the fact that side information increases the mutual information; and
(e) follows from Lemma 1.

Case 1: $0 \le m \le \frac{n}{2}$
We have

$$H(Y_{2i}|S_{D_2 i}, X_{top1,i}) = H(Y_{1i}|S_{D_1 i}, X_{top2,i}) \le n - m, \tag{84}$$
$$H(Y_{F_2 i}|X_{2i}, X_{top1,i}) = H(Y_{F_1 i}|X_{1i}, X_{top2,i}) \le (l - (n-m))^+, \tag{85}$$
$$H(X_{top1,i}|S_{D_2 i}) = H(X_{top2,i}|S_{D_1 i}) = 0. \tag{86}$$

Thus, we have $R_1 + R_2 \le 2(n-m) + 2[l - (n-m)]^+$.

Case 2: $\frac{n}{2} \le m \le n$
We have

$$H(Y_{2i}|S_{D_2 i}, X_{top1,i}) = H(Y_{1i}|S_{D_1 i}, X_{top2,i}) \le n - m, \tag{87}$$
$$H(Y_{F_2 i}|X_{2i}, X_{top1,i}) = H(Y_{F_1 i}|X_{1i}, X_{top2,i}) \le (l - m)^+, \tag{88}$$
$$H(X_{top1,i}|S_{D_2 i}) = H(X_{top2,i}|S_{D_1 i}) \le (2m - n)^+. \tag{89}$$

Thus, we have $R_1 + R_2 \le 2m + 2[l - m]^+$.

Case 3: $n \le m$ We have

$$H(Y_{2i}|S_{D_2 i}, X_{top1,i}) = H(Y_{1i}|S_{D_1 i}, X_{top2,i}) = H(Y_{F_2 i}|X_{2i}, X_{top1,i}) = H(Y_{F_1 i}|X_{1i}, X_{top2,i}) = 0, \tag{90}$$
$$H(X_{top1,i}|S_{D_2 i}) = H(X_{top2,i}|S_{D_1 i}) \le m. \tag{91}$$

Thus, we have $R_1 + R_2 \le 2m$.

Combining the three cases, we have proved the fourth outer bound for $R_1 + R_2$.

3) Bound on $2R_1 + R_2$ and $R_1 + 2R_2$: In this sub-section, we focus on the proof for the upper bound on $2R_1 + R_2$. The proof for the bound on $R_1 + 2R_2$ follows in a similar manner.

We have

$$\begin{aligned}
&T(2R_1 + R_2 - p_e^T) \\
&\le 2I(W_1; Y_1^T) + I(W_2; Y_2^T) \\
&\le I(W_1; Y_1^T, Y_{F_1}^T) + I(W_1; Y_1^T, Y_{F_2}^T|W_2) + I(W_2; Y_2^T, Y_{F_2}^T) \\
&= H(Y_1^T) + H(Y_{F_1}^T|Y_1^T) - H(Y_{F_1}^T|W_1) - H(Y_1^T|Y_{F_1}^T, W_1) + H(Y_{F_2}^T|W_2) + H(Y_1^T|Y_{F_2}^T, W_2) - H(Y_1^T, Y_{F_2}^T|W_2, W_1) \\
&\quad + H(Y_2^T) + H(Y_{F_2}^T|Y_2^T) - H(Y_{F_2}^T|W_2) - H(Y_2^T|Y_{F_2}^T, W_2) \\
&\overset{(a)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) - H(Y_1^T|Y_{F_1}^T, W_1) + H(Y_1^T|Y_{F_2}^T, W_2) + H(Y_2^T) - H(Y_2^T|Y_{F_2}^T, W_2) \\
&\overset{(b)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) - H(S_{D_2}^T|Y_{F_1}^T, W_1) + H(Y_1^T|Y_{F_2}^T, W_2) + H(Y_2^T) - H(S_{D_1}^T|Y_{F_2}^T, W_2) \\
&\overset{(c)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) - H(S_{D_2}^T, X_{top1}^T|Y_{F_1}^T, W_1) + H(Y_1^T|Y_{F_2}^T, W_2) + H(Y_2^T) - H(S_{D_1}^T|Y_{F_2}^T, W_2) \\
&\overset{(d)}{\le} H(Y_1^T) - H(Y_{F_1}^T|W_1) - H(S_{D_2}^T, X_{top1}^T|Y_{F_1}^T, W_1) + H(Y_1^T, S_{D_1}^T|Y_{F_2}^T, W_2) + H(Y_2^T) - H(S_{D_1}^T|Y_{F_2}^T, W_2) \\
&\overset{(e)}{\le} H(Y_1^T) - H(Y_{F_1}^T|W_1) - [H(S_{D_2}^T, X_{top1}^T) - I(S_{D_2}^T, X_{top1}^T; Y_{F_1}^T, W_1)] + [H(S_{D_1}^T|Y_{F_2}^T, W_2) \\
&\quad + H(Y_1^T|S_{D_1}^T, Y_{F_2}^T, W_2)] + H(Y_2^T, S_{D_2}^T, X_{top1}^T) - H(S_{D_1}^T|Y_{F_2}^T, W_2) \\
&= H(Y_1^T) - H(Y_{F_1}^T|W_1) + I(S_{D_2}^T, X_{top1}^T; Y_{F_1}^T, W_1) + H(Y_1^T|S_{D_1}^T, Y_{F_2}^T, W_2) + H(Y_2^T|S_{D_2}^T, X_{top1}^T) \\
&\overset{(f)}{\le} H(Y_1^T) - H(Y_{F_1}^T|W_1) + I(S_{D_2}^T, X_{top1}^T, Y_{F_2}^T, W_2; Y_{F_1}^T, W_1) + H(Y_1^T|S_{D_1}^T, Y_{F_2}^T, W_2) + H(Y_2^T|S_{D_2}^T, X_{top1}^T) \\
&\overset{(g)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) + [I(W_2; Y_{F_1}^T, W_1) + I(S_{D_2}^T, X_{top1}^T, Y_{F_2}^T; Y_{F_1}^T, W_1|W_2)] \\
&\quad + H(Y_1^T|S_{D_1}^T, Y_{F_2}^T, W_2) + H(Y_2^T|S_{D_2}^T, X_{top1}^T)
\end{aligned}$$

$$\overset{(h)}{=} H(Y_1^T) - H(Y_{F_1}^T|W_1) + [(H(Y_{F_1}^T|W_1) + H(Y_{F_2}^T, S_{D_2}^T, X_{top1}^T|W_2)] + H(Y_1^T|S_{D_1}^T, Y_{F_2}^T, W_2) + H(Y_2^T|S_{D_2}^T, X_{top1}^T)$$

$$\overset{(i)}{\leq} \sum_{i=1}^{T}[H(Y_{1i}) + H(Y_{F_2i}|Y_{F_2}^{i-1}, S_{D_2}^i, X_{top1}^i, W_2, X_{2i}) + H(S_{D_2i}|Y_{F_2}^{i-1}, W_2, X_{2i}) + H(X_{top1,i}|Y_{F_2}^{i-1}, S_{D_2}^i, W_2)$$
$$+ H(Y_{1i}|S_{D_1i}, S_{D_2i}) + H(Y_{2i}|S_{D_2i}, X_{top1,i})]$$

$$\overset{(j)}{=} \sum_{i=1}^{T}[H(Y_{1i}) + H(Y_{F_2i}|X_{2i}, X_{top1,i}) + H(X_{top1,i}|S_{D_2i}) + H(Y_{1i}|S_{D_1i}, S_{D_2i}) + H(Y_{2i}|S_{D_2i}, X_{top1,i})],$$

where

(a) follows from the facts that $H(Y_1^T, Y_{F_2}^T|W_2, W_1) = 0$, $H(Y_{F_1}^T|Y_1^T) = 0$, and $H(Y_{F_2}^T|Y_2^T) = 0$;
(b) follows from the fact that X_{ji} is a function of $(Y_{F_j}^{i-1}, W_j)$, for $j = 1, 2$;
(c) follows from the fact that X_{top1}^T is a function of X_1^T, which is in turn a function of $(W_1, Y_{F_1}^{T-1})$.This is the crucial step;
(d) follows from the fact that side information increases the entropy;
(e) follows from the fact that side information increases the entropy;
(f) follows from the fact side information increases the mutual information;
(g) follows from the fact that $S_{D_2}^T$ is a function of $(Y_{F_2}^T, W_2)$;
(h) follows from the fact given (W_1, W_2), the entropy of any random variable is 0;
(i) follows from the fact that X_{2i} is a function of $(Y_{F_2}^{i-1}, W_2)$; and
(j) follows from the fact S_{D_2i} is a function of X_{2i}.

Case 1: $0 \leq m \leq \frac{n}{2}$
We have

$$H(Y_{1i}) \leq n, \tag{92}$$
$$H(Y_{F_2i}|X_{2i}, X_{top1,i}) \leq (l - (n-m))^+, \tag{93}$$
$$H(X_{top1,i}|S_{D_2i}) = 0, \tag{94}$$
$$H(Y_{1i}|S_{D_1i}, S_{D_2i}) = H(Y_{2i}|S_{D_2i}, X_{top1,i}) \leq n - m. \tag{95}$$

Thus, we have $2R_1 + R_2 \leq 3n - 2m + [l - (n-m)]^+$.

Case 2: $\frac{n}{2} \leq m \leq n$
We have

$$H(Y_{1i}) \leq n, \tag{96}$$
$$H(Y_{F_2i}|X_{2i}, X_{top1,i}) \leq (l - m)^+, \tag{97}$$
$$H(X_{top1,i}|S_{D_2i}) \leq (2m - n)^+, \tag{98}$$
$$H(Y_{1i}|S_{D_1i}, S_{D_2i}) = H(Y_{2i}|S_{D_2i}, X_{top1,i}) \leq n - m. \tag{99}$$

Thus, we have $2R_1 + R_2 \leq 2n + [l - m]^+$.

Case 3: $n \leq m$
We have

$$H(Y_{1i}) = H(Y_{2i}|S_{D_2i}, X_{top1,i}) \leq m \tag{100}$$
$$H(Y_{F_2i}|X_{2i}, X_{top1,i}) = H(Y_{1i}|S_{D_1i}, S_{D_2i}) = H(Y_{2i}|S_{D_2i}, X_{top1,i}) = 0. \tag{101}$$

Thus, we have $2R_1 + R_2 \leq 2m$.
Combining the three cases, we have proved the bound on $2R_1 + R_2$.

B. Forward proof

In this section, we present an encoding scheme, for the symmetric linear deterministic interference channel with partial feedback.

1) Achievable rate region for discrete memoryless interference channel with noisy feedback: Before we derive the achievable rate region for the symmetric linear deterministic interference channel with partial feedback, we start with a result, which gives an achievable rate region for the general two-user discrete memoryless interference channel with noisy feedback.

Remark 10. Throughout this section, we shall often use the notation convenience $p_{V|U}(v|u) = p(v|u)$ and $p_V(v) = p(v)$, where the dropped subscripts are obvious by observation of the arguments used in the functions.

Lemma 2. *The capacity region of the two-user discrete memoryless interference channel with noisy feedback as defined above includes the set of* (R_1, R_2) *such that*

$$R_1 \le \rho_1 + \kappa_2 + \rho_3 \tag{102}$$
$$R_2 \le \kappa_1 + \rho_2 + \kappa_3 \tag{103}$$
$$R_1 \le \kappa_6 \tag{104}$$
$$R_1 \le \kappa_4 + \rho_1 \tag{105}$$
$$R_2 \le \rho_6 \tag{106}$$
$$R_2 \le \rho_4 + \kappa_1 \tag{107}$$
$$R_1 + R_2 \le \kappa_2 + \rho_6 \tag{108}$$
$$R_1 + R_2 \le \rho_2 + \kappa_6 \tag{109}$$
$$R_1 + R_2 \le \kappa_1 + \rho_1 + \kappa_5 + \rho_2 \tag{110}$$
$$R_1 + R_2 \le \kappa_1 + \rho_1 + \rho_5 + \kappa_2 \tag{111}$$
$$R_1 + R_2 \le \kappa_1 + \rho_1 + \kappa_3 + \rho_3 \tag{112}$$
$$2R_1 + R_2 \le \kappa_6 + \kappa_2 + \rho_3 + \rho_1 \tag{113}$$
$$2R_1 + R_2 \le 2\rho_1 + \kappa_1 + \kappa_5 + \kappa_2 + \rho_3 \tag{114}$$
$$R_1 + 2R_2 \le \rho_6 + \rho_2 + \kappa_3 + \kappa_1 \tag{115}$$
$$R_1 + 2R_2 \le 2\kappa_1 + \rho_1 + \rho_5 + \rho_2 + \kappa_3, \tag{116}$$

over all joint distributions

$$p(u)p(u_1|u)p(u_2|u)p(v_1|u,u_1)p(v_2|u,u_2)p(x_1|u,u_1,v_1)p(x_2|u,u_2,v_2)p(y_1y_2|x_1x_2)p(y_{F_1}|y_1)p(y_{F_2}|y_2), \tag{117}$$

where

$$\kappa_1 = I(U_2; Y_{F_1}|X_1, V_1, U_1, U) \tag{118}$$
$$\kappa_2 = I(X_1; Y_1|U, U_1, U_2, V_1, V_2) \tag{119}$$
$$\kappa_3 = I(X_1, V_2; Y_1|U, U_1, V_1, U_2) \tag{120}$$
$$\kappa_4 = I(X_1; Y_1|U, U_1, U_2, V_2) \tag{121}$$
$$\kappa_5 = I(X_1, V_2; Y_1|U, U_1, U_2) \tag{122}$$
$$\kappa_6 = I(U, U_2, V_2, X_1; Y_1) \tag{123}$$
$$\rho_1 = I(U_1; Y_{F_2}|UV_2U_2X_2) \tag{124}$$
$$\rho_2 = I(X_2; Y_2|U, U_1, U_2, V_1, V_2) \tag{125}$$
$$\rho_3 = I(X_2, V_1; Y_2|U, U_2, V_2, U_1) \tag{126}$$
$$\rho_4 = I(X_2; Y_2|U, U_2, U_1, V_1) \tag{127}$$
$$\rho_5 = I(X_2, V_1; Y_2|U, U_1, U_2) \tag{128}$$
$$\rho_6 = I(U, U_1, V_1, X_2; Y_2). \tag{129}$$

Proof: This lemma is a corollary of Theorem 1 in Tuninetti's work [14]. By interpreting Y_1, Y_2, Y_3, Y_4 in [14] as $Y_{F_1}, Y_{F_2}, Y_1, Y_2$ in our work respectively, and applying Fourier-Motzkin elemination to the result in that Theorem 1, we obtain this lemma.

■

Remark 11. The lemma, just as related works in [14] [15] [5] [10], uses standard methods which combine three techniques: *block Markov encoding* [31], *backward decoding* [32], and HK message splitting [29]. A message from each transmitter is split into three parts: private message, cooperative common message and non-cooperative common message.

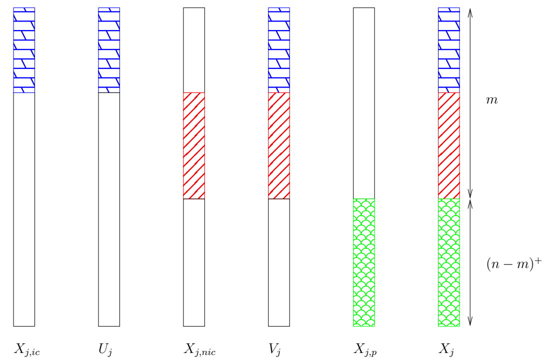

Fig. 12: Generic encoding

2) Achievable rate region for the symmetric interference channel with partial feedback: Now, we apply this lemma to construct a generic encoding scheme, and to find the corresponding achievable rate region for the symmetric deterministic interference channel with partial feedback. Denote $X_{j,CC}, X_{j,NCC}$ and $X_{j,P}$ as column vectors of size $\min(n, m)$ bits, for $j, k \in \{1, 2\}$ and $j \neq k$. We let

$$U = \emptyset \tag{130}$$
$$U_j = U \oplus X_{j,CC} \tag{131}$$
$$V_j = U_j \oplus X_{j,NCC} \tag{132}$$
$$X_j = V_j \oplus X_{j,P}. \tag{133}$$

$X_{j,CC}$, $X_{j,NCC}$, $X_{j,P}$ contain the interfering common message, the non-interfering common message and private message, respectively, for the transmitter j. Consider Figure 12, which illustrates the generic encoding scheme. The interfering common message and the non-interfering common message are restricted to the top area of m bits, and the private message is restricted to the bottom area of $(n-m)^+$ bits. That means in the strong and very strong interference regimes when $n < m$, no private message is encoded. Intuitively, this should be the case as any transmitted signal from any transmitter j will be received by both receivers anyway. As interfering common message causes interference to the non-intended receiver, it needs to be fed back via the feedback link so that interference can be resolved. Thus, the achievable rate of the interfering common message depends directly on the feedback link strength. Hence, we propose an adaptive encoding scheme that varies according the strength of the feedback link. Here, we choose the size of the interfering common message of the transmitter j to be upper-bounded by m . Once the codeword U_j for the cooperative common message $X_{j,CC}$ has been constructed, we construct the codeword V_j which depends on the non-interfering common message $X_{j,NCC}$ and U_j. The non-interfering common message can either contain fresh information bits, or fedback signals, which needs to be relayed again for resolving interference, or null information. Furthermore, the non-interfering common message only occupies positions in the top area of m bits, which has not been taken by the interfering common information. Finally, the codeword X_j for the transmitter j depends on the private message $X_{j,P}$ and V_j. We will show that the optimal achievable rate region matches the outer bound region.

With the encoding scheme shown above, the following equalities, where $f'_i(.)$ are deterministic functions, for $i = 1, 2, 3, 4$,

$$H(Y_1|X_1, V_2) = 0, \tag{134}$$
$$H(Y_2|X_2, V_1) = 0, \tag{135}$$
$$U_1 = f'_1(V_1), \tag{136}$$
$$V_1 = f'_2(X_1), \tag{137}$$
$$U_2 = f'_3(V_2), \tag{138}$$
$$V_2 = f'_4(X_2), \tag{139}$$

always hold. Thus, we have

$$\kappa_1 = I(U_2; Y_{F_1}|X_1, V_1, U_1, U) = I(U_2; Y_{F_1}|X_1) \tag{140}$$
$$\kappa_2 = H(Y_1|U, U_1, U_2, V_1, V_2) = H(Y_1|V_1, V_2) \tag{141}$$
$$\kappa_3 = H(Y_1|U, U_1, V_1, U_2) = H(Y_1|V_1, U_2) \tag{142}$$

$$\kappa_4 = H(Y_1|U, U_1, U_2, V_2) = H(Y_1|U_1, V_2) \quad (143)$$
$$\kappa_5 = H(Y_1|U, U_1, U_2) = H(Y_1|U_1, U_2) \quad (144)$$
$$\kappa_6 = H(Y_1) \quad (145)$$
$$\rho_1 = I(U_1; Y_{F_2}|U, V_2, U_2, X_2) = I(U_1; Y_{F_2}|X_2) \quad (146)$$
$$\rho_2 = H(Y_2|U, U_1, U_2, V_1, V_2) = H(Y_2|V_1, V_2) \quad (147)$$
$$\rho_3 = H(Y_2|U, U_2, V_2, U_1) = H(Y_2|V_2, U_1) \quad (148)$$
$$\rho_4 = H(Y_2|U, U_2, U_1, V_1) = H(Y_2|U_2, V_1) \quad (149)$$
$$\rho_5 = H(Y_2|U, U_1, U_2) = H(Y_2|U_1, U_2) \quad (150)$$
$$\rho_6 = H(Y_2). \quad (151)$$

For readers' convenience and for ease of calculation, we illustrate the encoding schemes case by case.
3) Very weak interference $m \leq \frac{1}{2}n$: We consider 2 cases.

Case 1: $l \leq n - m$.
Set
- $X_{1CC} = X_{2CC} = 0$;
- $X_{1NCC} = X_{2NCC} = m$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1P} = X_{2P} = n - m$ Bernoulli $(\frac{1}{2})$ random bits at the bottom area.

We have

$$\rho_1 = \kappa_1 = I(U_2; Y_{F_1}|X_1) = 0, \quad (152)$$
$$\rho_2 = \kappa_2 = H(Y_1|V_1, V_2) = n - m, \quad (153)$$
$$\rho_3 = \kappa_3 = H(Y_1|V_1, U_2) = n - m, \quad (154)$$
$$\rho_4 = \kappa_4 = H(Y_1|U_1, V_2) = n, \quad (155)$$
$$\rho_5 = \kappa_5 = H(Y_1|U_1, U_2) = n, \quad (156)$$
$$\rho_6 = \kappa_6 = H(Y_1) = n. \quad (157)$$

Applying Lemma 2, the following region is achievable

$$R_1 \leq n, \quad (158)$$
$$R_2 \leq n, \quad (159)$$
$$R_1 + R_2 \leq 2n - m, \quad (160)$$
$$R_1 + R_2 \leq 2(n - m), \quad (161)$$
$$2R_1 + R_2 \leq 3n - 2m, \quad (162)$$
$$R_1 + 2R_2 \leq 3n - 2m. \quad (163)$$

Case 2: $n - m \leq l$.
In this case, the feedback link helps to increase the rate of interfering common message. Set
- $X_{1CC} = X_{2CC} = (l - (n - m))^+$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1NCC} = X_{2NCC} = m - (l - (n - m))^+$ Bernoulli $(\frac{1}{2})$ random bits, right below the interfering common message's region;
- $X_{1P} = X_{2P} = n - m$ Bernoulli $(\frac{1}{2})$ random bits at the bottom area.

Applying Lemma 2, the following region is achievable

$$R_1 \leq n, \quad (164)$$
$$R_2 \leq n, \quad (165)$$
$$R_1 + R_2 \leq 2n - m, \quad (166)$$
$$R_1 + R_2 \leq 2(n - m) + 2(l - (n - m)^+))^+, \quad (167)$$
$$2R_1 + R_2 \leq 3n - 2m + (l - (n - m)^+)^+, \quad (168)$$
$$R_1 + 2R_2 \leq 3n - 2m + (l - (n - m)^+)^+. \quad (169)$$

Thus, we have shown the achievability of the capacity region in Theorem 1 in the very weak interference regime.

The calculation in other regimes are similar. Thus, we will only show the assignment of bits to the random variables, and leave it the readers that these bit assignments allow us to achieve the capacity region in Theorem 1.

4) Weak and moderately strong interference regimes $\frac{1}{2}n \leq m \leq n$: We consider 2 cases.

Case 1: $l \leq m$.

This is the case of weak feedback link, thus, feedback link cannot help to resolve interference at receivers. No interfering common message should be sent. Set

- $X_{1CC} = X_{2CC} = 0$;
- $X_{1NCC} = X_{2NCC} = m$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1P} = X_{2P} = n - m$ Bernoulli $(\frac{1}{2})$ random bits at the bottom area.

Case 2: $m \leq l$. The rate of interfering common message should be chosen carefully to make use of the strong feedback links. Set

- $X_{1CC} = X_{2CC} = (l - (n - m))^+$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1NCC} = X_{2NCC} = m - (l - (n - m))^+$ Bernoulli $(\frac{1}{2})$ random bits, right below the interfering common message's region;
- $X_{1P} = X_{2P} = n - m$ Bernoulli $(\frac{1}{2})$ random bits at the bottom area.

5) Strong and very strong interference regimes: $n \leq m$: Note that no private information is sent in these regimes. We consider two cases.

Case 1: $l \leq n$. Set

- $X_{1CC} = X_{2CC} = 0$;
- $X_{1NCC} = X_{2NCC} = m$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1P} = X_{2P} = 0$.

Case 2: $n \leq l$. Set

- $X_{1CC} = X_{2CC} = m$ Bernoulli $(\frac{1}{2})$ random bits at the top region;
- $X_{1NCC} = X_{2NCC} = 0$;
- $X_{1P} = X_{2P} = 0$.

C. Proof of Theorem 2

Inspired by the LD-IC-PF, we define

$$S_{2G} := \sqrt{\text{INR}} X_2 + Z_1, \tag{170}$$

$$S_{1G} := \sqrt{\text{INR}} X_1 + Z_2, \tag{171}$$

$$X_{top1G} := \begin{cases} \frac{\text{INR}}{\sqrt{\text{SNR}}} X_1 + Z_2, & \frac{1}{2} \leq \alpha_G \leq 1 \\ 0, & \text{otherwise,} \end{cases} \tag{172}$$

$$X_{top2G} := \begin{cases} \frac{\text{INR}}{\sqrt{\text{SNR}}} X_2 + Z_1, & \frac{1}{2} \leq \alpha_G \leq 1 \\ 0, & \text{otherwise.} \end{cases} \tag{173}$$

Almost similarly to the proof of the outer bounds in Theorem 1, with some subtle difference, we can show the following lemma.

Lemma 3. *Consider the Gaussian IC with noisy feedback as defined in Section II. The capacity region of the symmetric Gaussian IC with noisy feedback, is included by the set of non-negative pairs (R_1, R_2) satisfying*

$$R_1 \leq h(Y_1, Y_{F_2}|X_2) - h(Z_1) - h(\tilde{Z}_2), \tag{174}$$

$$\begin{aligned} R_1 + R_2 \leq\ & h(X_{top1G}|S_{2G}) + h(Y_{F_2}|X_2, X_{top1G}) + h(Y_2|S_{2G}, X_{top1G}) \\ & + h(X_{top2G}|S_{1G}) + h(Y_{F_1}|X_1, X_{top2G}) + h(Y_1|S_{1G}, X_{top2G}) \\ & - h(\tilde{Z}_2) - h(\tilde{Z}_1) - 2h(Z_2) - 2h(Z_1) \end{aligned} \tag{175}$$

$$\begin{aligned} 2R_1 + R_2 \leq\ & h(X_{top1G}|S_{2G}) + h(Y_{F_2}|X_2, X_{top1G}) + h(Y_2|S_{2G}, X_{top1G}) \\ & + h(Y_1|S_{1G}, X_2) + h(Y_1) - 2h(Z_1) - 2h(Z_2) - h(\tilde{Z}_2). \end{aligned} \tag{176}$$

The proof of Lemma 3 is presented in the appendix, section VI-D.

We are going to calculate the mutual informations in the bounds mentioned above and simplify them. We have

$$\begin{aligned} h(Y_1|X_2) - h(Z_1) &= \frac{1}{2}\log[\mathrm{SNR}(1-\rho^2)+1] \\ &\leq \frac{1}{2}\log(\mathrm{SNR}+1). \end{aligned} \tag{177}$$

We can also show that

$$\begin{aligned} h(Y_{F_2}|Y_1, X_2) - h(\tilde{Z}_2) &= h(g_2(h_{12}X_1 + Z_2) + \tilde{Z}_2 | h_{11}X_1 + Z_1, X_2) - h(\tilde{Z}_2) && (178)\\ &\leq h(g_2(h_{12}X_1 + Z_2) + \tilde{Z}_2 | h_{11}X_1 + Z_1) - h(\tilde{Z}_2) && (179)\\ &= \frac{1}{2}\log\left(\frac{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR.INR}}+1}\cdot(\mathrm{SNR}+\mathrm{INR}+1)}{\mathrm{SNR}+1}+1\right) && (180)\\ &\leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+1}+1\right). && (181) \end{aligned}$$

From equations (174), (177) and (181), we can prove the validity of the bound (37).

Next, we are going to prove the bound (40).
Case 1: $\frac{1}{2} \leq \alpha_G < 1$
We have

$$\begin{aligned} h(X_{top1G}|S_{2G}) - h(Z_2) &= \frac{1}{2}\log\left(\frac{\frac{\mathrm{INR}^3}{\mathrm{SNR}}(1-\rho^2)+\mathrm{INR}+\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1}{\mathrm{INR}+1}\right) && (182)\\ &\leq \frac{1}{2}\log\left(\frac{\mathrm{INR}^3+\mathrm{INR}^2}{\mathrm{SNR}\cdot(\mathrm{INR}+1)}+1\right). && (183) \end{aligned}$$

Next, we have

$$\begin{aligned} & h(Y_2|S_2, X_{top1G}) - h(Z_2) && (184)\\ &= h\left(h_{21}X_1 + h_{22}X_2 + Z_2 | \sqrt{\mathrm{INR}}X_2 + Z_1, \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + X_2\right) - h(Z_2) && (185)\\ &= \log\sqrt{\frac{\mathrm{SNR}}{\mathrm{INR}}} + h\left(\frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + \sqrt{\mathrm{INR}}X_2 + \frac{\sqrt{\mathrm{INR}}}{\sqrt{\mathrm{SNR}}}Z_2 | \sqrt{\mathrm{INR}}X_2 + Z_1, \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + X_2\right) - h(Z_2) && (186)\\ &= \frac{1}{2}\log\frac{\mathrm{SNR}}{\mathrm{INR}} + h\left(\frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + \frac{\sqrt{\mathrm{INR}}}{\sqrt{\mathrm{SNR}}}Z_2 - Z_1 | \sqrt{\mathrm{INR}}X_2 + Z_1, \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + X_2\right) - h(Z_2) && (187)\\ &\leq \frac{1}{2}\log\frac{\mathrm{SNR}}{\mathrm{INR}} + h\left(\frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + \frac{\sqrt{\mathrm{INR}}}{\sqrt{\mathrm{SNR}}}Z_2 - Z_1 | \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + X_2\right) - h(Z_2) && (188)\\ &= \frac{1}{2}\log\frac{\mathrm{SNR}}{\mathrm{INR}} + \frac{1}{2}\log\left(\frac{\frac{\mathrm{INR}^3}{\mathrm{SNR}^2}+2\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1-2\frac{\mathrm{INR}^2\sqrt{\mathrm{INR}}}{\mathrm{SNR}\sqrt{\mathrm{SNR}}}}{\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1}\right) && (189)\\ &\leq \frac{1}{2}\log\frac{\mathrm{SNR}}{\mathrm{INR}} + \frac{1}{2}\log 3. && (190) \end{aligned}$$

Next, we have

$$\begin{aligned} h(Y_{F_2}|X_2, X_{top1G}) - h(\tilde{Z}_2) &= h\left(g_2(h_{21}X_1 + Z_2) + \tilde{Z}_2 | X_2, \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + Z_2\right) - h(\tilde{Z}_2) && (191)\\ &\leq h\left(g_2\sqrt{\mathrm{INR}}\cdot X_1 + g_2 Z_2 + \tilde{Z}_2 | \frac{\mathrm{INR}}{\sqrt{\mathrm{SNR}}}X_1 + Z_2\right) - h(\tilde{Z}_2) && (192)\\ &= \frac{1}{2}\log\left(g_2^2\frac{\mathrm{INR}+\frac{\mathrm{INR}^2}{\mathrm{SNR}}-2\frac{\mathrm{INR}\sqrt{\mathrm{INR}}}{\sqrt{\mathrm{SNR}}}}{\frac{\mathrm{INR}^2}{\mathrm{SNR}}+1}+1\right) && (193)\\ &\leq \frac{1}{2}\log\left(g_2^2\frac{\mathrm{INR}+\frac{\mathrm{INR}^2}{\mathrm{SNR}}}{\frac{\mathrm{INR}^2}{\mathrm{SNR}}}+1\right) && (194) \end{aligned}$$

$$= \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNRINR}}+1}\cdot\frac{\mathrm{SNR}+\mathrm{INR}}{\mathrm{INR}}+1\right) \tag{195}$$

$$\leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right) \tag{196}$$

Combining equations (175), (183),(190) and (196), and using symmetry, we have proved the first half of the bound (40).

Case 2: $\alpha_G \notin [\frac{1}{2}, 1]$

In this case, due to the definition of X_{topjG}, the equation (175) is equivalent to

$$R_1 + R_2 \leq h(Y_{F_2}|X_2) + h(Y_2|S_{2G}) + h(Y_{F_1}|X_1) + h(Y_1|S_{1G}) - h(\tilde{Z}_2) - h(\tilde{Z}_1) - h(Z_2) - h(Z_1). \tag{197}$$

Next, we have

$$h(Y_{F_2}|X_2) - h(\tilde{Z}_2) = \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}\cdot[\mathrm{INR}(1-\rho^2)+1]+1\right) \tag{198}$$

$$\leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+1}\cdot(\mathrm{INR}+1)+1\right). \tag{199}$$

Next, we have

$$h(Y_1|S_{1G}) - h(Z_1) \tag{200}$$

$$= \frac{1}{2}\log\left([\mathrm{INR}^2(1-\rho^2)+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1]\cdot\frac{1}{\mathrm{INR}+1}\right) \tag{201}$$

$$\leq \frac{1}{2}\log\left([\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1]\cdot\frac{1}{\mathrm{INR}+1}\right). \tag{202}$$

Combining equations (197), (199), and (202), and using symmetry, we have proved the last half of the bound (40).

Next, we are going to prove the validity of the bound (41).

We have

$$h(Y_1|X_2, S_1) - h(Z_1) \leq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right) \tag{203}$$

$$h(Y_1) - h(Z_1) \leq \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1). \tag{204}$$

Case 1: $\frac{1}{2} \leq \alpha_G < 1$

Combining equations (176), (203), (204), (183), (190) and (196), we have proved the bound (41) for this case.

Case 2: $\alpha_G \notin [\frac{1}{2}, 1]$

Combining equations (176), (203), (204), (197), (199) and (202), we have proved the bound (41) for this remaining case.

D. Proof of Lemma 3

The proof of the bound (174) in Lemma 3 is trivial. In this sub-section, we will only prove the bound (176). The proof of the bound (175) contains no new ideas and can be proved similarly to the proof in Theorem 1 and the proof of (176). Before proving the bound of (176), we need to prove two lemmas.

Lemma 4.

$$h(S_{2G}^T|W_1, Y_{F_1}^T) = h(Y_1^T|W_1, Y_{F_1}^T) \tag{205}$$

$$h(S_{1G}^T|W_2, Y_{F_2}^T) = h(Y_2^T|W_2, Y_{F_2}^T) \tag{206}$$

Proof: We have

$$\begin{aligned}h(Y_1^T|W_1,Y_{F_1}^T) &\overset{(a)}{=} \sum_{i=1}^T h(Y_{1i}|W_1,Y_{F_1}^T,Y_1^{i-1},X_1^i)\\ &= \sum_{i=1}^T h(S_{2G,i}|W_1,Y_{F_1}^T,S_{2G}^{i-1},X_1^i)\\ &\overset{(b)}{=} \sum_{i=1}^T h(S_{2G,i}|W_1,Y_{F_1}^T,S_{2G}^{i-1})\\ &= h(S_{2G}^T|W_1,Y_{F_1}^T)\end{aligned}$$

where (a),(b) comes from the fact X_1^i is a function of $(W_1,Y_{F_1}^{i-1})$.

The other equality is proven in a similar way.

∎

Lemma 5.

$$\begin{aligned}I(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T,W_2;Y_{F_1}^T,W_1) \le \sum_{i=1}^T &[h(X_{top1G,i}|S_{2G,i})-h(Z_{2i})\\ &+h(Y_{F_2i}|X_{2i},X_{top1G,i})-h(\tilde{Z}_{1i})-h(\tilde{Z}_{2i})]+h(Y_{F_1}^T|W_1), \end{aligned}\tag{207}$$

$$\begin{aligned}I(S_{1G}^T,X_{top2G}^T,Y_{F_1}^T,W_1;Y_{F_2}^T,W_2) \le \sum_{i=1}^T &[h(X_{top2G,i}|S_{1G,i})-h(Z_{1i})\\ &+h(Y_{F_1i}|X_{1i},X_{top2G,i})-h(\tilde{Z}_{2i})-h(\tilde{Z}_{1i})]+h(Y_{F_2}^T|W_2). \end{aligned}\tag{208}$$

Proof:

$$\begin{aligned}&I(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T,W_2;Y_{F_1}^T,W_1)\\ &= I(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T;Y_{F_1}^T,W_1|W_2)+I(W_2;Y_{F_1}^T,W_1)\\ &= h(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T|W_2)-h(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T|Y_{F_1}^T,W_1,W_2)+h(Y_{F_1}^T|W_1)+h(W_1)-h(W_1|W_2)-h(Y_{F_1}^T|W_1,W_2)\\ &\overset{(a)}{=} h(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T|W_2)-h(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T|Y_{F_1}^T,W_1,W_2,X_1^T)+h(Y_{F_1}^T|W_1)-h(Y_{F_1}^T|W_1,W_2)\\ &\overset{(b)}{=} h(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T|W_2)-h(S_{2G}^T,Z_2^T,Y_{F_2}^T|Y_{F_1}^T,W_1,W_2,X_1^T)+h(Y_{F_1}^T|W_1)-h(Y_{F_1}^T|W_1,W_2)\\ &\overset{(c)}{\le} h(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T|W_2)-h(S_{2G}^T,Z_2^T,Y_{F_2}^T|Y_{F_1}^T,W_1,W_2,X_1^T,X_2^T)+h(Y_{F_1}^T|W_1)-h(Y_{F_1}^T|W_1,W_2)\\ &\overset{(d)}{\le} h(S_{2G}^T,X_{top1G}^T,Y_{F_2}^T|W_2)-h(Z_1^T,Z_2^T,\tilde{Z}_2^T|Y_{F_1}^T,W_1,W_2,X_1^T,X_2^T)+h(Y_{F_1}^T|W_1)-h(Y_{F_1}^T|W_1,W_2,Y_1^T)\\ &= \sum_{i=1}^T[h(S_{2G,i},X_{top1G,i},Y_{F_2i}|S_{2G}^{i-1},X_{top1G}^{i-1},Y_{F_2}^{i-1},W_2)]-h(Z_1^T)-h(Z_2^T)-h(\tilde{Z}_2^T)+h(Y_{F_1}^T|W_1)-h(\tilde{Z}_1^T|W_1,W_2,Y_1^T)\\ &\overset{(e)}{=} \sum_{i=1}^T[h(S_{2G,i}X_{top1G,i},Y_{F_2i}|X_{2i},S_{2G}^{i-1},X_{top1G}^{i-1},Y_{F_2}^{i-1},W_2)]-h(Z_1^T)-h(Z_2^T)-h(\tilde{Z}_2^T)-h(\tilde{Z}_1^T)+h(Y_{F_1}^T|W_1)\\ &\le \sum_{i=1}^T[h(S_{2G,i},X_{top1G,i},Y_{F_2i}|X_{2i})-h(Z_{1i})-h(Z_{2i})-h(\tilde{Z}_{2i})-h(\tilde{Z}_{1i})]+h(Y_{F_1}^T|W_1)\\ &\le \sum_{i=1}^T[h(S_{2G,i}|X_{2i})+h(X_{top1G,i}|X_{2i},S_{2G,i})+h(Y_{F_2i}|X_{2i},S_{2G,i},X_{top1G,i})-h(Z_{1i})-h(Z_{2i})-h(\tilde{Z}_{2i})\\ &\quad -h(\tilde{Z}_{1i})]+h(Y_{F_1}^T|W_1)\\ &= \sum_{i=1}^T[h(Z_{1i}|X_{2i})+h(X_{top1G,i}|X_{2i},S_{2Gi})+h(Y_{F_2i}|X_{2i},S_{2G,i},X_{top1G,i})-h(Z_{1i})-h(Z_{2i})-h(\tilde{Z}_{2i})\\ &\quad -h(\tilde{Z}_{1i})]+h(Y_{F_1}^T|W_1)\end{aligned}$$

$$\overset{(f)}{\leq} \sum_{i=1}^{T}[h(X_{top1G,i}|S_{2G,i}) + h(Y_{F_2 i}|X_{2i}, X_{top1G,i}) - h(Z_{2i}) - h(\tilde{Z}_{1i}) - h(\tilde{Z}_{2i})] + h(Y_{F_1}^T|W_1)$$

where

(a) follows from the facts that W_1 and W_2 are independent, and that X_1^T is a function of $Y_{F_1}^{T-1}$ and W_1;
(b) follows from the fact we can remove the known information X_1^T from X_{top1G}^T;
(c) follows from the fact that more conditioning reduces the entropy;
(d) follows from the fact that more conditioning reduces the entropy;
(e) follows from the fact that X_{ji} is a function of $(Y_{F_j}^{i-1}, W_j)$ for $j = 1, 2$; and
(f) in this step, we try to obtain a bound that is equivalent to that in the linear deterministic model.

■

Now, we are going to prove the bound (176). We have

$$\begin{aligned}
&T(2R_1 + R_2 - p_e^T)\\
&\leq 2I(W_1; Y_1^T) + I(W_2; Y_2^T)\\
&\leq I(W_1; Y_1^T, Y_{F_1}^T) + I(W_1; Y_1^T, Y_{F_2}^T|W_2) + I(W_2; Y_2^T, Y_{F_2}^T)\\
&= h(Y_1^T) + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_1^T|Y_{F_1}^T, W_1) + h(Y_{F_2}^T|W_2) - h(Y_{F_2}^T|W_2, W_1) + I(W_1; Y_1^T|Y_{F_2}^T, W_2)\\
&\quad + h(Y_2^T) + h(Y_{F_2}^T|Y_2^T) - h(Y_{F_2}^T|W_2) - h(Y_2^T|Y_{F_2}^T, W_2)\\
&= h(Y_1^T) - h(Y_1^T|Y_{F_1}^T, W_1) + I(W_1; Y_1^T|Y_{F_2}^T, W_2) + h(Y_2^T) - h(Y_2^T|Y_{F_2}^T, W_2)\\
&\quad + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&\overset{(a)}{=} h(Y_1^T) - h(S_{2G}^T|Y_{F_1}^T, W_1) + I(W_1; Y_1^T|Y_{F_2}^T, W_2) + h(Y_2^T) - h(S_{1G}^T|Y_{F_2}^T, W_2)\\
&\quad + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&\overset{(b)}{=} h(Y_1^T) - h(S_{2G}^T, Z_2^T|Y_{F_1}^T, W_1, X_1^T) + h(Z_2^T|Y_{F_1}^T, W_1, X_1^T S_{2G}^T) + I(W_1; Y_1^T|Y_{F_2}^T, W_2) + h(Y_2^T) - h(S_{1G}^T|Y_{F_2}^T, W_2)\\
&\quad + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&\overset{(c)}{\leq} h(Y_1^T) - h(S_{2G}^T, X_{top1G}^T|Y_{F_1}^T, W_1) + I(W_1; Y_1^T|Y_{F_2}^T, W_2) + h(Y_2^T) - h(S_{1G}^T|Y_{F_2}^T, W_2)\\
&\quad + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&\overset{(d)}{\leq} h(Y_1^T) - h(S_{2G}^T, X_{top1G}^T|Y_{F_1}^T, W_1) + I(W_1; Y_1^T, S_{1G}^T|Y_{F_2}^T, W_2) + h(Y_2^T) - h(S_{1G}^T|Y_{F_2}^T, W_2)\\
&\quad + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T, |Y_2^T)\\
&= h(Y_1^T) - h(S_{2G}^T, X_{top1G}^T|Y_{F_1}^T, W_1) + h(Y_1^T, S_{1G}^T|Y_{F_2}^T, W_2) - h(Y_1^T, S_{1G}^T|Y_{F_2}^T, W_2, W_1) + h(Y_2^T)\\
&\quad - h(S_{1G}^T|Y_{F_2}^T, W_2) + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&\overset{(e)}{\leq} h(Y_2^T) - h(S_{2G}^T, X_{top1G}^T|Y_{F_1}^T, W_1) + h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - h(Z_1^T, Z_2^T)\\
&\quad + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&\overset{(f)}{\leq} h(Y_2^T) + [I(S_{2G}^T, X_{top1G}^T; Y_{F_1}^T, W_1) - h(S_{2G}^T, X_{top1G}^T)] + [h(S_{2G}^T, X_{top1G}^T|Y_2^T) - h(S_{2G}^T, X_{top1G}^T|Y_2^T, X_2^T, X_1^T)]\\
&\quad + h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - h(Z_1^T) - h(Z_2^T) + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&= I(S_{2G}^T, X_{top1G}^T; Y_{F_1}^T, W_1) + h(Y_2^T|S_{2G}^T, X_{top1G}^T) - h(Z_1^T, Z_2^T|Y_2^T, X_2^T, X_1^T) + h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2)\\
&\quad + h(Y_1^T) - h(Z_1^T) - h(Z_2^T) + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&\overset{(g)}{\leq} I(S_{2G}^T, X_{top1G}^T, Y_{F_2}^T, W_2; Y_{F_1}^T, W_1) + h(Y_2^T|S_{2G}^T, X_{top1G}^T) + h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - 2h(Z_1^T) - h(Z_2^T)\\
&\quad + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&\overset{(h)}{\leq} \sum_{i=1}^{T}[h(X_{top1G,i}|S_{2G,i}) + h(Y_{F_2 i}|X_{2i}, X_{top1G,i}) - h(Z_{2i}) - h(\tilde{Z}_{1i}) - h(\tilde{Z}_{2i})] + h(Y_{F_1}^T|W_1) + h(Y_2^T|S_{2G}^T, X_{top1G}^T)\\
&\quad + h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - 2h(Z_1^T) - h(Z_2^T) + h(Y_{F_1}^T|Y_1^T) - h(Y_{F_1}^T|W_1) - h(Y_{F_2}^T|W_2, W_1) + h(Y_{F_2}^T|Y_2^T)\\
&\overset{(i)}{\leq} \sum_{i=1}^{T}[h(X_{top1G,i}|S_{2Gi}) + h(Y_{F_2 i}|X_{2i}, X_{top1G,i}) - h(Z_{2i}) - h(\tilde{Z}_{1i}) - h(\tilde{Z}_{2i})] + h(Y_2^T|S_{2G}^T, X_{top1G}^T)
\end{aligned}$$

$$+ h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2) + h(Y_1^T) - 2h(Z_1^T) - h(Z_2^T) + h(\tilde{Z}_1^T|Y_1^T) - h(Y_{F_2}^T|W_2, W_1, Y_2^T) + h(\tilde{Z}_2^T|Y_2^T)$$

$$= \sum_{i=1}^{T}[h(X_{top1G,i}|S_{2Gi}) + h(Y_{F_2 i}|X_{2,i}, X_{top1G,i}) - 2h(Z_{1i}) - 2h(Z_{2i})] + h(Y_2^T|S_{2G}^T, X_{top1G}^T) + h(Y_1^T|S_{1G}^T, Y_{F_2}^T, W_2, X_2^T)$$

$$+ h(Y_1^T) - h(\tilde{Z}_2^T|W_2, W_1, Y_2^T)$$

$$\leq \sum_{i=1}^{T}[h(X_{top1G,i}|S_{2G,i}) + h(Y_{F_2 i}|X_{2i}, X_{top1G,i}) + h(Y_{2i}|S_{2G,i}, X_{top1G,i}) + h(Y_{1i}|S_{1G,i}, X_{2i}) + h(Y_{1i})$$

$$- 2h(Z_{1i}) - 2h(Z_{2i}) - h(\tilde{Z}_{2i})]$$

where

(a) follows from Lemma 4;
(b) follows from the fact that X_1^T is a function of $(Y_{F_1}^T, W_1)$;
(c) follows from the fact X_{top1G}^T is a function of Z_2^T and X_1^T, which is in turn a function of $(Y_{F_1}^T, W_1)$;
(d) follows from the fact, more side information increases the mutual information;
(e) follows from $h(Y_1^T, S_{1G}^T|Y_{F_2}^T, W_2, W_1) \geq h(Y_1^T, S_{1G}^T|Y_{F_2}^T, W_2, W_1, X_1^T) = h(Z_1^T, Z_2^T)$;
(f) follows from the fact more conditioning reduces the entropy;
(g) follows from the fact, more side information increases the mutual information;
(h) follows from utilization of Lemma 5; and
(i) follows from the fact more conditioning reduces the entropy.

E. Proof of Theorem 3

Choose (U, U_i, V_i, X_{ip}), for $i \in \{1, 2\}$ as jointly Gaussian, independent random variables which satisfy

$$U \sim \mathcal{N}(0, 0), \tag{209}$$

$$U_i \sim \mathcal{N}(0, P_{cc}), \tag{210}$$

$$V_i \sim \mathcal{N}(0, P_{nc}), \tag{211}$$

$$X_{ip} \sim \mathcal{N}(0, P_p), \tag{212}$$

$$P_{cc} + P_{nc} + P_p = 1. \tag{213}$$

Set $X_i = U + U_i + V_i + X_{ip}$. With this choice of random variables, Theorem 3 is a direct corollary of Lemma 2.

F. Proof of Theorem 4

The strategy to prove this theorem is that we need to carefully choose the right power allocations P_p, P_{nc}, P_{cc} such that the achievable rate region approximates the capacity region within a constant gap.

When $\text{INR} < 1$, by treating interference as noise and not using any feedback, each receiver can achieve a rate of

$$\frac{1}{2}\log\left(1 + \frac{\text{SNR}}{\text{INR}+1}\right). \tag{214}$$

We have

$$\psi_1 - \frac{1}{2}\log\left(1 + \frac{\text{SNR}}{\text{INR}+1}\right) = \frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) - \frac{1}{2}\log\left(1 + \frac{\text{SNR}}{\text{INR}+1}\right) \tag{215}$$

$$\leq \frac{1}{2}\log(3\text{SNR} + 3) - \frac{1}{2}\log\left(\frac{1}{2} + \frac{\text{SNR}}{2}\right) \tag{216}$$

$$= \frac{1}{2}\log 3 + \frac{1}{2} = 1.3 \text{ bits} \tag{217}$$

Thus, $\delta_R \leq 1.3$.

Next, we have

$$\psi_3 - 2\cdot\frac{1}{2}\log\left(1 + \frac{\text{SNR}}{\text{INR}+1}\right) = \psi_1 - \frac{1}{2}\log\left(1 + \frac{\text{SNR}}{\text{INR}+1}\right) \tag{218}$$

$$\leq \frac{1}{2}\log 3 + \frac{1}{2} = 1.3 \text{ bits}. \tag{219}$$

Thus, $\delta_{2R} \leq 1.3$.

Subsequently,

$$\psi_3 + \psi_1 - 3 \cdot \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{\mathrm{INR}+1}\right) \leq 2(\frac{1}{2}\log 3 + \frac{1}{2}) = 2.6 \text{ bits}. \tag{220}$$

Thus, $\delta_{3R} \leq 2.6$.

Therefore, the outer bounds in Theorem 2 is within 2.6 bits/s/Hz away from the achievable rate region. Thus, our main focus in this subsection now is to quantify the gaps for $\mathrm{INR} \geq 1$.

- **Case 1**: $1 \leq \alpha_G$
  - Sub-case 1.1: $\mathrm{SNR}_F < \mathrm{SNR}$
    Choose $P_p = 0, P_{cc} = 0, P_{nc} = 1$. With this power allocation, from Theorem 3 we have

$$\tau_6 = \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) \tag{221}$$

$$\tau_5 = \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 1) \geq \frac{1}{2}\log\mathrm{INR} \tag{222}$$

$$\tau_4 = \frac{1}{2}\log(\mathrm{SNR} + 1) \tag{223}$$

$$\tau_3 = \frac{1}{2}\log(\mathrm{INR} + 1) \geq \frac{1}{2}\log\mathrm{INR} \tag{224}$$

$$\tau_2 = \tau_1 = 0 \tag{225}$$

Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\mathrm{SNR}+1) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+1} + 1\right) \tag{226}$$

$$\leq \frac{1}{2}\log(\mathrm{SNR}+1) + \frac{1}{2}\log(2). \tag{227}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1} + 1\right) + \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right) \tag{228}$$

$$\leq \frac{1}{2}\log(2) + \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right). \tag{229}$$

$$\psi_5 = \frac{1}{2}\log\left(\frac{\mathrm{INR}^2 + \mathrm{SNR} + 2\mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1}{\mathrm{INR}+1}\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1} + 1\right) \tag{230}$$

$$+ \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1} + 1\right) + \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right) \tag{231}$$

$$\leq \frac{1}{2}\log\left(\frac{\mathrm{INR}^2 + 5\mathrm{INR} + 4}{\mathrm{INR}+1}\right) + \frac{1}{2}\log\left(\mathrm{SNR}_F + 1\right) \tag{232}$$

$$+ \frac{1}{2}\log(2) + \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right) \tag{233}$$

$$\leq \frac{1}{2}\log\left(\mathrm{INR}+4\right) + \frac{1}{2}\log\left(\mathrm{SNR}_F + 1\right) + \frac{1}{2} + \frac{1}{2}\log\left(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1\right). \tag{234}$$

Now, the gap can be quantified easily.

$$\psi_1 - \tau_6 = 0 \tag{235}$$

$$\psi_2 - (\tau_4 + \tau_1) \leq \frac{1}{2} \tag{236}$$

$$\psi_1 - (\tau_1 + \tau_2 + \tau_3) \leq \frac{1}{2}\log(5\mathrm{INR}) - \frac{1}{2}\log(\mathrm{INR}) \tag{237}$$

$$= \frac{1}{2}\log(5) = 1.2. \tag{238}$$

Thus, $\delta_R \leq 1.2$.
Next, we have

$$\psi_3 - (\tau_2 + \tau_6) \leq \frac{1}{2}, \tag{239}$$

$$\psi_3 - (2\tau_1 + \tau_2 + \tau_5) \leq \frac{1}{2} + \frac{1}{2}\log(5\text{INR}) - \frac{1}{2}\log(\text{INR}) \tag{240}$$

$$= \frac{1}{2} + \frac{1}{2}\log(5) = 1.7 \tag{241}$$

$$\psi_3 - (2\tau_1 + 2\tau_3) \leq \frac{1}{2} + \frac{1}{2}\log(5\text{INR}) - \frac{1}{2}\log(\text{INR}^2) \tag{242}$$

$$= \frac{1}{2} + \frac{1}{2}\log(5) = 1.7 \tag{243}$$

Thus, $\delta_{2R} \leq 1.7$.
Next, we have

$$(\psi_1 + \psi_3) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) = (\psi_1 - \tau_6) + (\psi_3 - \tau_3) \tag{244}$$

$$\leq \frac{1}{2} + \frac{1}{2}\log(5\text{INR}) - \frac{1}{2}\log(\text{INR}) \tag{245}$$

$$= \frac{1}{2} + \frac{1}{2}\log(5) = 1.7 \tag{246}$$

$$(\psi_1 + \psi_3) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5) \tag{247}$$

$$\leq [\frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) + \frac{1}{2}\log(2) + \frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1)] \tag{248}$$

$$-[\frac{1}{2}\log(\text{INR}) + \frac{1}{2}\log(\text{SNR} + \text{INR} + 1)] \tag{249}$$

$$\leq [\log 3(\text{SNR} + \text{INR} + 1) + \frac{1}{2}\log(2) + \log(5\text{INR})] - [\frac{1}{2}\log(\text{INR}) + \frac{1}{2}\log(\text{SNR} + \text{INR} + 1)] \tag{250}$$

$$= \frac{1}{2}\log(30) = 2.5 \tag{251}$$

Thus, $\delta_{3R} \leq 2.5$.
Therefore, with current power allocation, in this sub-case, the achievable region is at most 2.5 bits/s/Hz from the outer bounds.

– Sub-case 1.2: $\text{SNR} \leq \text{SNR}_F \leq \text{INR}$
Choose $P_p = 0, P_{cc} = 1, P_{nc} = 0$. With this power allocation, from Theorem 3 we have

$$\tau_6 = \frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) \tag{252}$$

$$\tau_5 = \tau_4 = \tau_3 = \tau_2 = 0 \tag{253}$$

$$\tau_1 = \frac{1}{2}\log\frac{\frac{\text{SNR}_F}{\text{SNR}+\text{INR}+2\rho\sqrt{\text{SNR.INR}}+1}\cdot(\text{INR}+1)+1}{\frac{\text{SNR}_F}{\text{SNR}+\text{INR}+2\rho\sqrt{\text{SNR}\cdot\text{INR}}+1}+1} \tag{254}$$

$$\geq \frac{1}{2}\log\frac{\frac{\text{SNR}_F}{5\text{INR}}\cdot\text{INR}}{2} \tag{255}$$

$$= \frac{1}{2}\log\text{SNR}_F - \frac{1}{2}\log 10 \tag{256}$$

Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\text{SNR}+1) + \frac{1}{2}\log\left(\frac{\text{SNR}_F}{\text{SNR}+1} + 1\right) \tag{257}$$

$$\leq \frac{1}{2}\log(3\text{SNR}_F). \tag{258}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}+1} + 1\right) + \frac{1}{2}\log\left(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1\right) \tag{259}$$

$$\leq \frac{1}{2}\log(2) + \frac{1}{2}\log\left(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1\right). \tag{260}$$

Now, the gap can be quantified easily.

$$\psi_1 - \tau_6 = 0 \tag{261}$$

$$\psi_2 - (\tau_4 + \tau_1) \leq \frac{1}{2}\log 30 = 2.5 \tag{262}$$

$$\psi_2 - (\tau_1 + \tau_2 + \tau_3) \leq \frac{1}{2}\log 30 = 2.5. \tag{263}$$

Thus, $\delta_R \leq 2.5$.
Next, we have

$$\psi_3 - (\tau_2 + \tau_6) \leq \frac{1}{2}, \tag{264}$$

$$2\psi_2 - (2\tau_1 + \tau_2 + \tau_5) \leq \log(30) = 4.9 \tag{265}$$

$$2\psi_2 - (2\tau_1 + 2\tau_3) \leq \log(30) = 4.9 \tag{266}$$

Thus, $\delta_{2R} \leq 4.9$.
Next, we have

$$(\psi_2 + \psi_3) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) \leq \frac{1}{2}\log(60) = 3.0 \tag{267}$$

$$(3\psi_2) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5) \leq \frac{1}{2}\log(27000) = 7.4 \tag{268}$$

Thus, $\delta_{3R} \leq 7.4$.
Therefore, with current power allocation, in this sub-case, the achievable region is at most 7.4 bits/s/Hz from the outer bounds.

- **Case 2**: $\frac{1}{2} \leq \alpha_G \leq 1$

  – Sub-case 2.1: $\mathrm{SNR}_F \leq \mathrm{INR}$
  Choose $P_p = \frac{1}{\mathrm{INR}}, P_{nc} = 1 - P_p, P_{cc} = 0$. With this power allocation, from Theorem 3 we have

$$\tau_6 = \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) - \frac{1}{2} \tag{269}$$

$$\tau_5 = \frac{1}{2}\log(\mathrm{SNR} + \mathrm{INR} + 1) - \frac{1}{2} \tag{270}$$

$$\tau_4 = \frac{1}{2}\log(\mathrm{SNR} + 2) - \frac{1}{2} \geq \frac{1}{2}\log(\mathrm{SNR} + 1) - \frac{1}{2} \tag{271}$$

$$\tau_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}} + \mathrm{INR} + 1\right) - \frac{1}{2} \geq \frac{1}{2}\log(\mathrm{INR}) - \frac{1}{2} \tag{272}$$

$$\tau_2 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}} + 2\right) - \frac{1}{2} \geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) - \frac{1}{2} \tag{273}$$

$$\tau_1 = 0. \tag{274}$$

Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\mathrm{SNR} + 1) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR} + 1} + 1\right) \tag{275}$$

$$\leq \frac{1}{2}\log(\mathrm{SNR} + 1) + \frac{1}{2}\log(2). \tag{276}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR} + 1} + 1\right) + \log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1) \tag{277}$$

$$\leq \frac{1}{2}\log\left(2\frac{\mathrm{SNR}}{\mathrm{INR}}\right) + \log(\mathrm{SNR} + \mathrm{INR} + 2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}} + 1). \tag{278}$$

$$\psi_4 = \log\left(\frac{\mathrm{INR}^3+\mathrm{INR}^2}{\mathrm{SNR}(\mathrm{INR}+1)}+1\right)+\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right)+\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\log 3 \tag{279}$$
$$\leq \log\left(\frac{3\mathrm{INR}^2}{\mathrm{SNR}}\right)+\log(2)+\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\log 3 \tag{280}$$
$$= \log 18+\log \mathrm{INR} \tag{281}$$

$$\psi_5 = \frac{1}{2}\log\left(\frac{\mathrm{INR}^3+\mathrm{INR}^2}{\mathrm{SNR}(\mathrm{INR}+1)}+1\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log 3 \tag{282}$$
$$+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{283}$$
$$\leq \frac{1}{2}\log\left(\frac{3\mathrm{INR}^2}{\mathrm{SNR}}\right)+\frac{1}{2}\log(2)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log 3 \tag{284}$$
$$+\frac{1}{2}\log\left(\frac{2\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{285}$$
$$=^{(a)} \frac{1}{2}\log 36+\frac{1}{2}\log \mathrm{SNR}+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{286}$$
$$\leq^{(b)} \frac{1}{2}\log 36+\frac{1}{2}\log \mathrm{SNR}+\frac{1}{2}\log 3(\mathrm{SNR}+\mathrm{INR}+1) \tag{287}$$

where, depending on our need, we use either the form (a) or the form (b).
Now, the gap can be quantified easily.

$$\psi_1-\tau_6 = \frac{1}{2} \tag{288}$$
$$\psi_2-(\tau_4+\tau_1) \leq 1 \tag{289}$$
$$\psi_2-(\tau_1+\tau_2+\tau_3) \leq \left(\frac{1}{2}\log(2\mathrm{SNR})+\frac{1}{2}\right)-\left(\frac{1}{2}\log(\mathrm{INR})-\frac{1}{2}+\frac{1}{2}\log(\frac{\mathrm{SNR}}{\mathrm{INR}})-\frac{1}{2}\right) \tag{290}$$
$$= \frac{1}{2}\log(16) = 2. \tag{291}$$

Thus, $\delta_R \leq 2$.
Next, we have

$$\psi_3-(\tau_2+\tau_6) \leq \frac{3}{2}, \tag{292}$$
$$\psi_3-(2\tau_1+\tau_2+\tau_5) \leq \frac{1}{2}\log\left(2\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{293}$$
$$-\left(\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)-\frac{1}{2}+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+1)-\frac{1}{2}\right) \tag{294}$$
$$\leq \frac{1}{2}\log(2)+\log 3(\mathrm{SNR}+\mathrm{INR}+1) \tag{295}$$
$$-\left(-\frac{1}{2}+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+1)-\frac{1}{2}\right) \tag{296}$$
$$= \frac{1}{2}\log(24) = 2.3 \tag{297}$$
$$\psi_4-(2\tau_1+2\tau_3) \leq [\log 18+\log(\mathrm{INR})]-[\log(\mathrm{INR})-1] \tag{298}$$
$$= \log(36) = 5.2 \tag{299}$$

Thus, $\delta_{2R} \leq 5.2$.
Next, we have

$$(\psi_5)-(\tau_1+\tau_2+\tau_3+\tau_6) \leq \frac{1}{2}\log 36+\frac{3}{2} = 4.1 \tag{300}$$
$$(\psi_5)-(3\tau_1+\tau_2+\tau_3+\tau_5) \leq \frac{1}{2}\log 108+\frac{3}{2} = 4.9 \tag{301}$$

Thus, $\delta_{3R} \leq 4.9$.
Therefore, with current power allocation, in this sub-case, the achievable region is at most 5.2 bits/s/Hz from the outer bounds.

– Sub-case 2.2: $\mathrm{INR} \leq \mathrm{SNR}_F \leq \mathrm{SNR}$
Choose $P_p = \frac{1}{\mathrm{INR}}, P_{nc} = \frac{\mathrm{SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}} - P_p, P_{cc} = \frac{\mathrm{INR}\cdot\mathrm{SNR}_F}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}}$. With this power allocation, from Theorem 3 we have

$$\tau_6 = \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR.INR}}+1) - \frac{1}{2} \tag{302}$$

$$\tau_5 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}} + \frac{\mathrm{INR.SNR}}{\mathrm{INR.SNR}_F+\mathrm{SNR}}+1\right) - \frac{1}{2} \tag{303}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{2\mathrm{INR}\cdot\mathrm{SNR}_F}\right) - \frac{1}{2} \tag{304}$$

$$\tau_4 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}}+2\right) - \frac{1}{2} \tag{305}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{2\mathrm{INR}\cdot\mathrm{SNR}_F}\right) - \frac{1}{2} \tag{306}$$

$$\tau_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}} + \frac{\mathrm{INR.SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}}+1\right) - \frac{1}{2} \tag{307}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) - \frac{1}{2} \tag{308}$$

$$\tau_2 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}+2\right) - \frac{1}{2} \tag{309}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) - \frac{1}{2} \tag{310}$$

$$\tau_1 = \frac{1}{2}\log\frac{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}(\mathrm{INR}+1)+1}{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}(\frac{\mathrm{INR}\cdot\mathrm{SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}}+1)+1} \tag{311}$$

$$\geq \frac{1}{2}\log\frac{\frac{\mathrm{SNR}_F}{5\mathrm{SNR}}(\mathrm{INR})}{\frac{\mathrm{SNR}_F}{\mathrm{SNR}}(\frac{\mathrm{INR}\cdot\mathrm{SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F}+1)+1} \tag{312}$$

$$\geq \frac{1}{2}\log\frac{\mathrm{SNR}_F}{\mathrm{SNR}}(\mathrm{INR}) - \frac{1}{2}\log 15 \tag{313}$$

Next, we simplify some outer bounds first.

$$\psi_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right) + \log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{314}$$

$$\leq \frac{1}{2}\log\left(2\frac{\mathrm{SNR}}{\mathrm{INR}}\right) + \log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1). \tag{315}$$

$$\psi_4 = \log\left(\frac{\mathrm{INR}^3+\mathrm{INR}^2}{\mathrm{SNR}(\mathrm{INR}+1)}+1\right) + \log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right) + \log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) + \log 3 \tag{316}$$

$$\leq \log\left(\frac{3\mathrm{INR}^2}{\mathrm{SNR}}\right) + \log\left(\frac{2\mathrm{SNR}_F}{\mathrm{INR}}\right) + \log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) + \log 3 \tag{317}$$

$$= \log 18 + \log\mathrm{SNR}_F \tag{318}$$

$$\psi_5 = \frac{1}{2}\log\left(\frac{\mathrm{INR}^3+\mathrm{INR}^2}{\mathrm{SNR}(\mathrm{INR}+1)}+1\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{INR}}+1\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) + \frac{1}{2}\log 3 \tag{319}$$

$$+ \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right) + \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{320}$$

$$\leq \frac{1}{2}\log\left(\frac{3\text{INR}^2}{\text{SNR}}\right) + \frac{1}{2}\log\left(\frac{2\text{SNR}_F}{\text{INR}}\right) + \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}}\right) + \frac{1}{2}\log 3 \tag{321}$$

$$+ \frac{1}{2}\log\left(\frac{2\text{SNR}}{\text{INR}}\right) + \frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) \tag{322}$$

$$= \frac{1}{2}\log 36 + \frac{1}{2}\log\frac{\text{SNR}\cdot\text{SNR}_F}{\text{INR}} + \frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) \tag{323}$$

Now, the gap can be quantified easily.

$$\psi_1 - \tau_6 = \frac{1}{2} \tag{324}$$

$$\psi_1 - (\tau_4 + \tau_1) \leq \frac{1}{2}\log(5\text{SNR}) - \left[\frac{1}{2}\log\frac{\text{SNR}_F}{\text{SNR}}(\text{INR}) - \frac{1}{2}\log 15 + \frac{1}{2}\log\left(\frac{\text{SNR}^2}{2\text{INR}\cdot\text{SNR}_F}\right) - \frac{1}{2}\right] \tag{325}$$

$$= \frac{1}{2}\log(150) + \frac{1}{2} = 4.1 \tag{326}$$

$$\psi_1 - (\tau_1 + \tau_2 + \tau_3) \leq \frac{1}{2}\log(75) + 1 = 4.1 \tag{327}$$

Thus, $\delta_R \leq 4.1$.
Next, we have

$$\psi_3 - (\tau_2 + \tau_6) \leq \frac{3}{2} \tag{328}$$

$$\psi_4 - (2\tau_1 + \tau_2 + \tau_5) = \log(270) + 1 = 9.1 \tag{329}$$

$$\psi_4 - (2\tau_1 + 2\tau_3) \leq \log(270) + 1 = 9.1 \tag{330}$$

Thus, $\delta_{2R} \leq 9.1$.
Next, we have

$$(\psi_5) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) \leq \frac{1}{2}\log 540 + \frac{3}{2} = 6.0 \tag{331}$$

$$(\psi_5) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5) \leq \frac{1}{2}\log 607500 + \frac{3}{2} = 11.1 \tag{332}$$

Thus, $\delta_{3R} \leq 11.1$.
Therefore, with current power allocation, in this sub-case, the achievable region is at most 11.1 bits/s/Hz from the outer bounds.

- **Case 3**: $0 \leq \alpha_G \leq \frac{1}{2}$

  – Sub-case 3.1: $\text{SNR}_F \leq \frac{\text{SNR}}{\text{INR}}$
  Choose $P_p = \frac{1}{\text{INR}}, P_{nc} = 1 - P_p, P_{cc} = 0$. With this power allocation, from Theorem 3 we have

$$\tau_6 = \frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) - \frac{1}{2} \tag{333}$$

$$\tau_5 = \frac{1}{2}\log(\text{SNR} + \text{INR} + 1) - \frac{1}{2} \tag{334}$$

$$\tau_4 = \frac{1}{2}\log(\text{SNR} + 2) - \frac{1}{2} \geq \frac{1}{2}\log(\text{SNR} + 1) - \frac{1}{2} \tag{335}$$

$$\tau_3 = \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}} + \text{INR} + 1\right) - \frac{1}{2} \geq \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}}\right) - \frac{1}{2} \tag{336}$$

$$\tau_2 = \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}} + 2\right) - \frac{1}{2} \geq \frac{1}{2}\log\left(\frac{\text{SNR}}{\text{INR}}\right) - \frac{1}{2} \tag{337}$$

$$\tau_1 = 0. \tag{338}$$

  Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\text{SNR} + 1) + \frac{1}{2}\log\left(\frac{\text{SNR}_F}{\text{SNR} + 1} + 1\right) \tag{339}$$

$$\leq \frac{1}{2}\log(\text{SNR} + 1) + \frac{1}{2}\log(2). \tag{340}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right) + \log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{341}$$

$$\leq \frac{1}{2}\log\left(2\frac{\mathrm{SNR}}{\mathrm{INR}}\right) + \log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1). \tag{342}$$

$$\psi_4 = \log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}{\mathrm{INR}+1}\right) + \log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right) \tag{343}$$

$$\leq \log\left(\frac{7\mathrm{SNR}}{\mathrm{INR}}\right) + \log(3) = \log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right) + \log(21) \tag{344}$$

$$\psi_5 = \frac{1}{2}\log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}{\mathrm{INR}+1}\right) + \frac{1}{2}\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right) \tag{345}$$

$$+ \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right) + \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{346}$$

$$\leq \frac{1}{2}\log\left(\frac{7\mathrm{SNR}}{\mathrm{INR}}\right) + \frac{1}{2}\log(3) + \frac{1}{2}\log\left(\frac{2\mathrm{SNR}}{\mathrm{INR}}\right) + \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{347}$$

$$\leq \frac{1}{2}\log 42 + \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}^2}\right) + \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{348}$$

$$\leq \frac{1}{2}\log 42 + \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}^2}\right) + \frac{1}{2}\log 3(\mathrm{SNR}+\mathrm{INR}+1) \tag{349}$$

Now, the gap can be quantified easily.

$$\psi_1 - \tau_6 = \frac{1}{2} \tag{350}$$

$$\psi_2 - (\tau_4+\tau_1) \leq 1 \tag{351}$$

$$\psi_2 - (\tau_1+\tau_2+\tau_3) \leq \frac{1}{2}\log(16) = 2. \tag{352}$$

Thus, $\delta_R \leq 2$.
Next, we have

$$\psi_3 - (\tau_2+\tau_6) \leq \frac{3}{2}, \tag{353}$$

$$\psi_3 - (2\tau_1+\tau_2+\tau_5) \leq \frac{1}{2}\log(24) = 2.3 \tag{354}$$

$$\psi_4 - (2\tau_1+2\tau_3) \leq \log(21)+1 = 5.4 \tag{355}$$

Thus, $\delta_{2R} \leq 5.4$.
Next, we have

$$(\psi_5) - (\tau_1+\tau_2+\tau_3+\tau_6) \leq \frac{1}{2}\log 42 + \frac{3}{2} = 4.2 \tag{356}$$

$$(\psi_5) - (3\tau_1+\tau_2+\tau_3+\tau_5) \leq \frac{1}{2}\log 126 + \frac{3}{2} = 5.0 \tag{357}$$

$$\tag{358}$$

Thus, $\delta_{3R} \leq 5.0$.
Therefore, with current power allocation, in this sub-case, the achievable region is at most 5.4 bits/s/Hz from the outer bounds.

- Sub-case 3.2: $\frac{\mathrm{SNR}}{\mathrm{INR}} \leq \mathrm{SNR}_F \leq \mathrm{SNR}$
  Choose $P_p = \frac{1}{\mathrm{INR}}, P_{nc} = \frac{\mathrm{SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}} - P_p, P_{cc} = \frac{\mathrm{INR}\cdot\mathrm{SNR}_F}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}}$. With this power allocation, from Theorem

3 we have

$$\tau_6 = \frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1)-\frac{1}{2} \tag{359}$$

$$\tau_5 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}.\mathrm{SNR}_F+\mathrm{SNR}}+\frac{\mathrm{INR}\cdot\mathrm{SNR}}{\mathrm{INR}.\mathrm{SNR}_F+\mathrm{SNR}}+1\right)-\frac{1}{2} \tag{360}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{2\mathrm{INR}\cdot\mathrm{SNR}_F}\right)-\frac{1}{2} \tag{361}$$

$$\tau_4 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}}+2\right)-\frac{1}{2} \tag{362}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}^2}{2\mathrm{INR}\cdot\mathrm{SNR}_F}\right)-\frac{1}{2} \tag{363}$$

$$\tau_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}+\frac{\mathrm{INR}\cdot\mathrm{SNR}}{\mathrm{INR}.\mathrm{SNR}_F+\mathrm{SNR}}+1\right)-\frac{1}{2} \tag{364}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)-\frac{1}{2} \tag{365}$$

$$\tau_2 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}+2\right)-\frac{1}{2} \tag{366}$$

$$\geq \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}}\right)-\frac{1}{2} \tag{367}$$

$$\tau_1 = \frac{1}{2}\log\frac{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}(\mathrm{INR}+1)+1}{\frac{\mathrm{SNR}_F}{\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}(\frac{\mathrm{INR}\cdot\mathrm{SNR}}{\mathrm{INR}\cdot\mathrm{SNR}_F+\mathrm{SNR}}+1)+1} \tag{368}$$

$$\geq \frac{1}{2}\log\frac{\frac{\mathrm{SNR}_F}{5\mathrm{SNR}}(\mathrm{INR})}{\frac{\mathrm{SNR}_F}{\mathrm{SNR}}(\frac{\mathrm{INR}\cdot\mathrm{SNR}}{\mathrm{INR}.\mathrm{SNR}_F}+1)+1} \tag{369}$$

$$\geq \frac{1}{2}\log\frac{\mathrm{SNR}_F}{\mathrm{SNR}}(\mathrm{INR})-\frac{1}{2}\log 15 \tag{370}$$

Next, we simplify some outer bounds first.

$$\psi_2 = \frac{1}{2}\log(\mathrm{SNR}+1)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F}{\mathrm{SNR}+1}+1\right) \tag{371}$$

$$\leq \frac{1}{2}\log(2\mathrm{SNR})+\frac{1}{2}\log(2). \tag{372}$$

$$\psi_3 = \frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)+\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{373}$$

$$\leq \frac{1}{2}\log\left(2\frac{\mathrm{SNR}}{\mathrm{INR}}\right)+\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1). \tag{374}$$

$$\psi_4 = \log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}{\mathrm{INR}+1}\right)+\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right) \tag{375}$$

$$\leq \log\left(\frac{7\mathrm{SNR}}{\mathrm{INR}}\right)+\log\left(\frac{3\mathrm{SNR}_F\mathrm{INR}}{\mathrm{SNR}}\right) \tag{376}$$

$$\leq \log(\mathrm{SNR}_F)+\log(21) \tag{377}$$

$$\psi_5 = \frac{1}{2}\log\left(\frac{\mathrm{INR}^2+\mathrm{SNR}+2\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1}{\mathrm{INR}+1}\right)+\frac{1}{2}\log\left(\frac{\mathrm{SNR}_F(\mathrm{INR}+1)}{\mathrm{SNR}+\mathrm{INR}+1}+1\right) \tag{378}$$

$$+\frac{1}{2}\log\left(\frac{\mathrm{SNR}}{\mathrm{INR}+1}+1\right)+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{379}$$

$$\leq \frac{1}{2}\log\left(\frac{7\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log\left(\frac{3\mathrm{SNR}_F\mathrm{INR}}{\mathrm{SNR}}\right)+\frac{1}{2}\log\left(\frac{2\mathrm{SNR}}{\mathrm{INR}}\right)+\frac{1}{2}\log(\mathrm{SNR}+\mathrm{INR}+2\rho\sqrt{\mathrm{SNR}\cdot\mathrm{INR}}+1) \tag{380}$$

$$\leq \frac{1}{2}\log 42 + \frac{1}{2}\log\left(\frac{\text{SNR}_F\text{SNR}}{\text{INR}}\right) + \frac{1}{2}\log(\text{SNR} + \text{INR} + 2\rho\sqrt{\text{SNR}\cdot\text{INR}} + 1) \tag{381}$$

Now, the gap can be quantified easily.

$$\psi_1 - \tau_6 = \frac{1}{2} \tag{382}$$

$$\psi_2 - (\tau_4 + \tau_1) \leq \frac{1}{2}\log(60) + 1 = 4.0 \tag{383}$$

$$\psi_2 - (\tau_1 + \tau_2 + \tau_3) \leq \frac{1}{2}\log(60) + 1 = 4.0 \tag{384}$$

Thus, $\delta_R \leq 4.0$.
Next, we have

$$\psi_3 - (\tau_2 + \tau_6) \leq \frac{3}{2} \tag{385}$$

$$\psi_3 - (2\tau_1 + \tau_2 + \tau_5) = \frac{1}{2}\log(2250) + \frac{3}{2} = 7.1 \tag{386}$$

$$\psi_4 - (2\tau_1 + 2\tau_3) \leq \log(315) + 1 = 9.3 \tag{387}$$

Thus, $\delta_{2R} \leq 9.3$.
Next, we have

$$(\psi_5) - (\tau_1 + \tau_2 + \tau_3 + \tau_6) \leq \frac{1}{2}\log 630 + \frac{3}{2} = 6.2 \tag{388}$$

$$(\psi_5) - (3\tau_1 + \tau_2 + \tau_3 + \tau_5) \leq \frac{1}{2}\log 708750 + 2 = 11.7 \tag{389}$$

$$\tag{390}$$

Thus, $\delta_{3R} \leq 11.7$.
Therefore, with current power allocation, in this sub-case, the achievable region is at most 11.7 bits/s/Hz from the outer bounds.

In conclusion, we have proved that the outer bounds are at most 11.7 bits/s/Hz from the achievable rate region.

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