The density of discriminants of quintic rings and fields

Manjul Bhargava

June 1, 2010

1 Introduction

Let $N_n(X)$ denote the number of isomorphism classes of number fields of degree n having absolute discriminant at most X. Then it is an old folk conjecture that the limit

$$c_n = \lim_{X\to\infty} \frac{N_n(X)}{X} \tag{1}$$

exists and is positive for $n > 1$. The conjecture is trivial for $n \leq 2$, while for $n = 3$ and $n = 4$ it is a theorem of Davenport and Heilbronn [13] and of the author [4], respectively. In degrees $n \geq 5$, where number fields tend to be predominantly nonsolvable, the conjecture has not previously been known to be true for any value of n.

The primary purpose of this article is to prove the above conjecture for $n = 5$. In particular, we are able to determine the constant c_5 explicitly. More precisely, we prove:

Theorem 1 *Let $N_5^{(i)}(\xi,\eta)$ denote the number of quintic fields K, up to isomorphism, having $5-2i$ real embeddings and satisfying $\xi < \mathrm{Disc}(K) < \eta$. Then*

$$\begin{aligned}
&\text{(a)}\quad \lim_{X\to\infty}\frac{N_5^{(0)}(0,X)}{X} &=& \frac{1}{240}\prod_p (1+p^{-2}-p^{-4}-p^{-5});\\
&\text{(b)}\quad \lim_{X\to\infty}\frac{N_5^{(1)}(-X,0)}{X} &=& \frac{1}{24}\prod_p (1+p^{-2}-p^{-4}-p^{-5});\\
&\text{(c)}\quad \lim_{X\to\infty}\frac{N_5^{(2)}(0,X)}{X} &=& \frac{1}{16}\prod_p (1+p^{-2}-p^{-4}-p^{-5}).
\end{aligned}$$

The constants appearing in Theorem 1 (and thus their sum, $c_5 = \frac{13}{120}\prod_p(1+p^{-2}-p^{-4}-p^{-5})$) turn out to have very natural interpretations. Indeed, the constant c_5 takes the form of an Euler product, where the Euler factor at a place ν "counts" the total number of local étale quintic extensions of $\mathbb{Q}_\nu$, where each isomorphism class of local extension K_ν is counted with a certain natural weight to reflect the probability that a quintic number field K has localization $K \otimes \mathbb{Q}_\nu$ isomorphic to K_ν at ν. More precisely, let

$$\beta_\infty = \frac{1}{2}\sum_{[K_\infty:\mathbb{R}]=5 \text{ étale}} \frac{1}{|\mathrm{Aut}_{\mathbb{R}}(K_\infty)|}, \tag{2}$$

where the sum is over all isomorphism classes K_∞ of étale extensions of $\mathbb{R}$ of degree 5. Since $\mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^5) = 120$, $\mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^3\oplus\mathbb{C}) = 12$, and $\mathrm{Aut}_{\mathbb{R}}(\mathbb{R}\oplus\mathbb{C}^2) = 8$, we have $\beta_\infty = \frac{1}{240}+\frac{1}{24}+\frac{1}{16} = \frac{13}{120}$.

Similarly, for each prime p, let

$$\beta_p = \frac{p-1}{p} \sum_{[K_p:\mathbb{Q}_p]=5 \text{ étale}} \frac{1}{|\mathrm{Aut}_{\mathbb{Q}_p}(K_p)|} \cdot \frac{1}{\mathrm{Disc}_p(K_p)}, \tag{3}$$

where the sum is over all isomorphism classes K_p of étale extensions of $\mathbb{Q}_p$ of degree 5, and $\mathrm{Disc}_p(K_p)$ denotes the discriminant of K_p viewed as a power of p. Then

$$c_5 = \beta_\infty \cdot \prod_p \beta_p, \tag{4}$$

since we will show that

$$\beta_p = 1 + p^{-2} - p^{-4} - p^{-5}. \tag{5}$$

Thus we obtain a natural interpretation of c_5 as a product of counts of local field extensions. For more details on the evaluation of local sums of the form (3), and for global heuristics on the expected values of the asymptotic constants associated to general degree n S_n-number fields, see [5].

We obtain several additional results as by-products. First, our methods enable us to analogously count all *orders* in quintic fields:

Theorem 2 *Let $M_5^{(i)}(\xi,\eta)$ denote the number of isomorphism classes of orders $\mathcal{O}$ in quintic fields having $5-2i$ real embeddings and satisfying $\xi < \mathrm{Disc}(\mathcal{O}) < \eta$. Then there exists a positive constant α such that*

$$\begin{array}{lll}
\text{(a)} & \displaystyle\lim_{X\to\infty} \frac{M_5^{(0)}(0,X)}{X} & = \displaystyle\frac{\alpha}{240}; \\
\text{(b)} & \displaystyle\lim_{X\to\infty} \frac{M_5^{(1)}(-X,0)}{X} & = \displaystyle\frac{\alpha}{24}; \\
\text{(c)} & \displaystyle\lim_{X\to\infty} \frac{M_5^{(2)}(0,X)}{X} & = \displaystyle\frac{\alpha}{16}.
\end{array}$$

The constant α in Theorem 2 has an analogous interpretation. Let α_p denote the analogue of the sum (3) for orders, i.e.,

$$\alpha_p = \frac{p-1}{p} \sum_{[R_p:\mathbb{Z}_p]=5} \frac{1}{|\mathrm{Aut}_{\mathbb{Z}_p}(R_p)|} \cdot \frac{1}{\mathrm{Disc}_p(R_p)}, \tag{6}$$

where the sum is over all isomorphism classes of $\mathbb{Z}_p$-algebras R_p of rank 5 over $\mathbb{Z}_p$ with nonzero discriminant. Then we will show that the constant α appearing in Theorem 2 is given by

$$\alpha = \prod_p \alpha_p, \tag{7}$$

thus expressing α as a product of counts of local ring extensions. It is an interesting combinatorial problem to explicitly evaluate α_p in "closed form", analogous to the formula (5) that we obtain for β_p; see [6] for some further discussion on the evaluation of such sums.

Second, we note that the proof of Theorem 1 contains a determination of the densities of the various splitting types of primes in S_5-quintic fields. If K is an S_5-quintic field and K_{120} denotes the Galois closure of K, then the Artin symbol (K_{120}/p) is defined as a conjugacy class in S_5, its values being $\langle e\rangle$, $\langle(12)\rangle$, $\langle(123)\rangle$, $\langle(1234)\rangle$, $\langle(12345)\rangle$, $\langle(12)(34)\rangle$, or $\langle(12)(345)\rangle$, where $\langle x\rangle$ denotes the conjugacy class of x in S_5. It follows from the Cebotarev density theorem that for fixed K and varying p (unramified in K), the values $\langle e\rangle$, $\langle(12)\rangle$, $\langle(123)\rangle$, $\langle(1234)\rangle$, $\langle(12345)\rangle$, $\langle(12)(34)\rangle$, or $\langle(12)(345)\rangle$ occur with relative frequency $1:10:20:30:24:15:20$ (i.e., proportional to the size of the respective conjugacy class). We prove the following complement to Cebatorev density:

Theorem 3 *Let p be a fixed prime, and let K run through all S_5-quintic fields in which p does not ramify, the fields being ordered by the size of the discriminants. Then the Artin symbol (K_{120}/p) takes the values $\langle e\rangle$, $\langle(12)\rangle$, $\langle(123)\rangle$, $\langle(1234)\rangle$, $\langle(12345)\rangle$, $\langle(12)(34)\rangle$, or $\langle(12)(345)\rangle$ with relative frequency* $1:10:20:30:24:15:20$.

Actually, we do a little more: we determine for each prime p the density of S_5-quintic fields K in which p has the various possible ramification types. For example, it follows from our methods that a proportion of precisely $\frac{(p+1)(p^2+p+1)}{p^4+p^3+2p^2+2p+1}$ of S_5-quintic fields are ramified at p.

Lastly, our proof of Theorem 1 implies that nearly all—i.e., a density of 100% of—quintic fields have full Galois group S_5. This is in stark contrast to the quartic case [4, Theorem 3], where we showed that only about 91% of quartic fields have associated Galois group S_4:

Theorem 4 *When ordered by absolute discriminant, a density of* 100% *of quintic fields have associated Galois group S_5.*

In particular, it follows that 100% of quintic fields are nonsolvable.

Note that, rather than counting quintic fields and orders up to isomorphism, we could instead count these objects within a fixed algebraic closure of $\mathbb{Q}$. This would simply multiply all constants appearing in Theorems 1 and 2 by five. Meanwhile, Theorems 3 and 4 of course remain true regardless of whether one counts quintic extensions up to isomorphism or within an algebraic closure of $\mathbb{Q}$.

The key ingredient that allows us to prove the above results for quintic (and thus predominantly nonsolvable) fields is a parametrization of isomorphism classes of quintic orders by means of four integral alternating bilinear forms in five variables, up to the action of $\mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$, which we established in [3]. The proofs of Theorems 1–4 can then be reduced to counting appropriate integer points in certain fundamental regions, as in [4]. However, the current case is considerably more involved than the quartic case, since the relevant space is now 40-dimensional rather than 12-dimensional! The primary difficulty lies in counting points in the rather complicated cusps of these 40-dimensional fundamental regions (see Lemmas 8–11).

The necessary point-counting is accomplished in Section 2, by carefully dissecting the "irreducible" portions of the fundamental regions into 152 pieces, and then applying a new adaptation of the averaging methods of [4] to each piece (see Lemma 11). The resulting counting theorem (see Theorem 6), in conjunction with the results of [3], then yields the asymptotic density of discriminants of pairs (R, R'), where R is an order in a quintic field and R' is a *sextic resolvent ring* of R. Obtaining Theorems 1–4 from this general density result then requires a sieve, which in turn uses certain counting results on resolvent rings and subrings obtained in [3] and in the recent work of Brakenhoff [10], respectively. This sieve is carried out in the final Section 3.

We note that the space of binary cubic forms that was used in the work of Davenport-Heilbronn to count cubic fields, the space of pairs of ternary quadratic forms that we used in [4] to count quartic fields, and the space of quadruples of alternating 2-forms in five variables that we use in this article, are all examples of what are known as prehomogeneous vector spaces. A *prehomogeneous vector space* is a pair (G, V), where G is a reductive group and V is a linear representation of G such that $G_{\mathbb{C}}$ has a Zariski open orbit on $V_{\mathbb{C}}$. The concept was introduced by Sato in the 1960's and a classification of all irreducible prehomogeneous vector spaces was given in the work of Sato-Kimura [16], while Sato-Shintani [17] and Shintani [18] developed a theory of zeta functions associated to these spaces.

The connection between prehomogeneous vector spaces and field extensions was first studied systematically in the beautiful 1992 paper of Wright-Yukie [19]. In this work, Wright and Yukie determined the rational orbits and stabilizers in a number of prehomogeneous vector spaces, and showed that these orbits correspond to field extensions of degree 2, 3, 4, or 5. In their paper, they laid out a program to determine the density of discriminants of number fields of degree up to five, by considering adelic versions of Sato-Shintani's zeta functions as developed by Datskovsky and Wright [14] in their extensive work on cubic extensions.

However, despite looking very promising, the program via adelic Shintani zeta functions encountered some difficulties and has not succeeded to date beyond the cubic case. The primary difficulties have been: (a) establishing cancellations among various divergent zeta integrals, in order to establish a "principal part formula" for the associated adelic Shintani zeta function; and (b) "filtering" out the correct count of extensions from the overcount of extensions that is inherent in the definition of the zeta function. In the quartic case, difficulty (a) was overcome in the impressive 1995 treatise of Yukie [20], while (b) remained an obstacle. In the quintic case, both (a) and (b) have remained impediments to obtaining a correct count of quintic field extensions by discriminant. (For more on the Shintani adelic zeta function approach and these related difficulties, see [4, §1] and [20].)

In [4] and in the current article, we overcome the problems (a) and (b) above, for quartic and quintic fields respectively, by introducing a different counting method that relies more on geometry-of-numbers arguments. Thus, although our methods are different, this article may be viewed as completing the program first laid out by Wright and Yukie [19] to count field extensions in degrees up to 5 via the use of appropriate prehomogeneous vector spaces.

We now describe in more detail the methods of this paper, and give a comparison with previous methods. At least initially, our approach to counting quintic extensions using the prehomogeneous vector space $\mathbb{C}^4 \otimes \wedge^2\mathbb{C}^5$ is quite similar in spirit to Davenport-Heilbronn's original method in the cubic case [13] and its refinements developed in the quartic case [4]. Namely, we begin by giving an algebraic interpretation of the *integer* orbits on the associated prehomogeneous vector space which, in the quintic case, are the orbits of the group $G_{\mathbb{Z}} = \mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$ on the 40-dimensional lattice $V_{\mathbb{Z}} = \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$. As we showed in [3], these integer orbits have an extremely rich algebraic interpretation and structure (see Theorem 5 for a precise statement), enabling us to consider not only quintic fields, but also more refined data such as all *orders* in quintic fields, the local behaviors of these orders, and their sextic resolvent rings. This interpretation of the integer orbits then allows us to reduce our problem of counting orders and fields to that of enumerating appropriate lattice points in a fundamental domain for the action of the discrete group $G_{\mathbb{Z}}$ on the real vector space $V_{\mathbb{R}} = V_{\mathbb{Z}} \otimes \mathbb{R}$.

Just as in [13] and [4], the main difficulty in counting lattice points in such a fundamental region is that this region is not compact, but instead has cusps (or "tentacles") going off to infinity. To make matters even more interesting, unlike the case of binary cubic forms in Davenport-Heilbronn's work—where there is one relatively simple cusp defined by small degree inequalities in four variables—in the case of quadruples of quinary alternating 2-forms, the cusps are numerous in number and are defined by polynomial inequalities of extremely high degree in 40 variables! These difficulties are further exacerbated by the fact that—contrary to the cubic case—in the quartic and quintic cases the number of nondegenerate lattice points in the cuspidal regions is of strictly *greater* order than the number of points in the noncuspidal part ("main body") of the corresponding fundamental domains. The latter issue is indeed what lies behind the problems (a) and (b) above in the adelic zeta function method.

Following our work in the quartic case [4], we overcome these problems that arise from the cuspidal regions by counting lattice points not in a single fundamental domain, but over a continuous, compact set of fundamental domains. This allows one to "thicken" the cusps, thereby gaining a good deal of control on the integer points in these cuspidal regions. A basic version of this "averaging" method was introduced and used in [4] in the quartic case to handle points in these cusps, and thus enumerate quartic extensions by discriminant (see [4, §1] for more details). However, since the number, complexity and dimensions of the cuspidal regions are so much greater in the quintic case than in the quartic case, a number of new ideas and modifications are needed to successfully carry out the same averaging method in the quintic case.

The primary technical contribution of this article is the introduction of a method that allows one to systematically and canonically dissect the cuspidal regions into certain "nice" subregions on which a slightly refined averaging technique (see Sections 2.1–2.2) can then be applied in a uniform manner. Using this method, we divide up the fundamental region into 159 pieces. The first piece is the main body of the region, where we show using geometry-of-numbers arguments that the number of lattice points in the region is essentially its volume. For each of the remaining 158 cuspidal pieces, we show, by a uniform argument, that either the number of lattice points in that region is negligible (see Table 1, Lemma 11), *or* that the lattice points in that cuspidal piece are all *reducible*, i.e., they correspond to quintic rings that are not integral domains (see Lemma 10). An asymptotic formula for the number of irreducible integer points in the entire fundamental domain is then attained. The interesting interaction between the algebraic properties of the lattice points (via the correspondence in [3]) and their geometric locations within the fundamental domain is therefore what allows us to overcome the problems (a) and (b) arising in the adelic Shintani zeta function method. As explained earlier, a sieving method can then be used to prove Theorems 1–4.

Our counting method in this article is quite robust and systematic, and should be applicable in many other situations. First, it can be used to reprove the density of discriminants of cubic and quartic fields, with much stronger error terms than have previously been known (in fact, in the cubic case it can be used, in conjunction with a sieve, to obtain an exact second order term; see [9]). Second, the method can be suitably adapted to count cubic, quartic, and quintic field extensions of any base number field (see [8]). Third, the method can be used on prehomogeneous vector spaces having *infinite* stabilizer groups, which would also have a number of interesting applications (see, e.g., [7]). Finally, we expect that the methods should also be adaptable to representations of algebraic groups that are not necessarily prehomogeneous. We hope that these directions will be pursued further in future work.

2 On the class numbers of quadruples of 5×5 skew-symmetric matrices

Let $V = V_{\mathbb{R}}$ denote the space of quadruples of 5×5 skew-symmetric matrices over the real numbers. We write an element of $V_{\mathbb{R}}$ as an ordered quadruple (A, B, C, D), where the 5×5 matrices A, B, C, D have entries a_{ij}, b_{ij}, c_{ij}, d_{ij} respectively. Such a quadruple (A, B, C, D) is said to be *integral* if all entries of the matrices A, B, C, D are integral.

The group $G_{\mathbb{Z}} = \mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$ acts naturally on the space $V_{\mathbb{R}}$. Namely, an element $g_4 \in \mathrm{GL}_4(\mathbb{Z})$ acts by changing the basis of the $\mathbb{Z}$-module of matrices spanned by A, B, C, D; in terms of matrix multiplication, we have $(A\ B\ C\ D)^t \mapsto g_4\,(A\ B\ C\ D)^t$. Similarly, an element $g_5 \in \mathrm{SL}_5(\mathbb{Z})$ changes the basis of the five-dimensional space on which the skew-symmetric forms A, B, C, D take values, i.e., $g_5 \cdot (A, B, C, D) = (g_5 A g_5^t, g_5 B g_5^t, g_5 C g_5^t, g_5 D g_5^t)$. It is clear that the actions of g_4 and g_5 commute, and that this action of $G_{\mathbb{Z}}$ preserves the lattice $V_{\mathbb{Z}}$ consisting of the integral elements of $V_{\mathbb{R}}$.

The action of $G_{\mathbb{Z}}$ on $V_{\mathbb{R}}$ (or $V_{\mathbb{Z}}$) has a unique polynomial invariant, which we call the *discriminant*. It is a degree 40 polynomial in 40 variables, and is much too large to write down. An easy method to compute it for any given element in V was described in [3].

The integer orbits of $G_{\mathbb{Z}}$ on $V_{\mathbb{Z}}$ have an important arithmetic significance. Recall that a *quintic ring* is any ring with unit that is isomorphic to $\mathbb{Z}^5$ as a $\mathbb{Z}$-module; for example, an order in a quintic number field is a quintic ring. In [3] we showed how quintic rings may be parametrized in terms of the $G_{\mathbb{Z}}$-orbits on $V_{\mathbb{Z}}$:

Theorem 5 *There is a canonical bijection between the set of $G_{\mathbb{Z}}$-equivalence classes of elements $(A,B,C,D) \in V_{\mathbb{Z}}$, and the set of isomorphism classes of pairs (R,R'), where R is a quintic ring and R' is a sextic resolvent ring of R. Under this bijection, we have* $\mathrm{Disc}(A,B,C,D) = \mathrm{Disc}(R) = \frac{1}{16}\cdot \mathrm{Disc}(R')^{1/3}$.

A *sextic resolvent* of a quintic ring R is a sextic ring R' equipped with a certain *resolvent mapping* $R \to \wedge^2 R'$ whose precise definition will not be needed here (see [3] for details). In view of Theorem 5, we wish to try and understand the number of $G_{\mathbb{Z}}$-orbits on $V_{\mathbb{Z}}$ having absolute discriminant at most X, as $X \to \infty$. The number of integral orbits on $V_{\mathbb{Z}}$ having a fixed discriminant Δ is called a "class number", and we wish to understand the behavior of this class number on average.

From the point of view of Theorem 5, we would like to restrict the elements of $V_{\mathbb{Z}}$ under consideration to those that are "irreducible" in an appropriate sense. More precisely, we call an element $(A,B,C,D) \in V_{\mathbb{Z}}$ *irreducible* if, in the corresponding pair of rings (R,R') in Theorem 5, the ring R is an integral domain. The quotient field of R is thus a quintic field in that case. We say (A,B,C,D) is *reducible* otherwise.

One may also describe reducibility and irreducibility in more geometric terms. If $(A,B,C,D) \in V_{\mathbb{Z}}$, then one may consider the 4×4 sub-Pfaffians $Q_1(t_1,t_2,t_3,t_4),\ldots,Q_5(t_1,t_2,t_3,t_4)$ of the single 5×5 skew-symmetric matrix $At_1+Bt_2+Ct_3+Dt_4$ whose entries are linear forms in t_1,t_2,t_3,t_4. In other words, $Q_i = Q_i(w,x,y,z)$ is defined as a canonical squareroot of the determinant of the 4×4 matrix obtained from $t_1A+t_2B+t_3C+t_4D$ by removing its ith row and column. Thus these 4×4 Pfaffians $Q_1,\ldots,Q_5$ are quaternary quadratic forms and so define five quadrics in $\mathbb{P}^3$. If the element $(A,B,C,D) \in V_{\mathbb{Z}}$ has nonzero discriminant, then it is known that these five quadrics intersect in exactly five points in $\mathbb{P}^3$ (counting multiplicities); see e.g., [19], [3]. We refer to these five points as the *zeroes of* (A,B,C,D) in $\mathbb{P}^3$. In [3] we showed that if (A,B,C,D) corresponds to (R,R'), where R is isomorphic to an order in a quintic field K, then there exists a zero of (A,B,C,D) in $\mathbb{P}^3$ whose field of definition is K. (The other zeroes of $(A,B,C,D) \in V_{\mathbb{Z}}$ are thus defined over the conjugates of K.) Therefore, geometrically, we may say that (A,B,C,D) is irreducible if and only if it possesses a zero in $\mathbb{P}^3$ having field of definition K, where K is a quintic field extension of $\mathbb{Q}$. On the other hand, (A,B,C,D) is reducible if and only if (A,B,C,D) possesses a zero in $\mathbb{P}^3$ defined over a number field of degree smaller than five.

The main result of this section is the following theorem:

Theorem 6 *Let $N(V_{\mathbb{Z}}^{(i)};X)$ denote the number of $G_{\mathbb{Z}}$-equivalence classes of irreducible elements $(A,B,C,D) \in V_{\mathbb{Z}}$ having $5-2i$ real zeroes in $\mathbb{P}^3$ and satisfying $|\mathrm{Disc}(A,B,C,D)| < X$. Then*

$$\begin{aligned}
\text{(a)}\quad \lim_{X\to\infty}\frac{N(V_{\mathbb{Z}}^{(0)};X)}{X} &= \frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{240};\\
\text{(b)}\quad \lim_{X\to\infty}\frac{N(V_{\mathbb{Z}}^{(1)};X)}{X} &= \frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{24};\\
\text{(c)}\quad \lim_{X\to\infty}\frac{N(V_{\mathbb{Z}}^{(2)};X)}{X} &= \frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{16}.
\end{aligned}$$

Theorem 6 is proven in several steps. In Subsection 2.1, we outline the necessary reduction theory needed to establish some particularly useful fundamental domains for the action of $G_{\mathbb{Z}}$ on $V_{\mathbb{R}}$. In Subsections 2.2 and 2.3, we describe a refinement of the "averaging" method from [4] that allows us to efficiently count integer points in various components of these fundamental domains in terms of their volumes. In Subsections 2.4 and 2.5, we investigate the distribution of reducible and irreducible integral points within these fundamental domains. The volumes of the resulting "irreducible" components of these fundamental domains are then computed in Subsection 2.6, proving Theorem 6. A version of Theorem 6 for elements in $V_{\mathbb{Z}}$ satisfying any specified set of congruence conditions is then obtained in Subsection 2.7.

In Section 3, we will show how these counting methods—together with a sieving argument—can be used to prove Theorems 1–4.

2.1 Reduction theory

The action of $G_{\mathbb{R}} = \mathrm{GL}_4(\mathbb{R}) \times \mathrm{SL}_5(\mathbb{R})$ on $V_{\mathbb{R}}$ has three nondegenerate orbits $V_{\mathbb{R}}^{(0)}, V_{\mathbb{R}}^{(1)}, V_{\mathbb{R}}^{(2)}$, where $V_{\mathbb{R}}^{(i)}$ consists of those elements (A,B,C,D) in $V_{\mathbb{R}}$ having nonzero discriminant and $5-2i$ real zeroes in $\mathbb{P}^3$. We wish to understand the number of irreducible $G_{\mathbb{Z}}$-orbits on $V_{\mathbb{Z}}^{(i)} = V_{\mathbb{R}}^{(i)} \cap V_{\mathbb{Z}}$ having absolute discriminant at most X $(i = 0,1,2)$. We accomplish this by counting the number of integer points of absolute discriminant at most X in suitable fundamental domains for the action of $G_{\mathbb{Z}}$ on $V_{\mathbb{R}}$.

These fundamental regions are constructed as follows. First, let $\mathcal{F}$ denote a fundamental domain in $G_{\mathbb{R}}$ for $G_{\mathbb{Z}}\backslash G_{\mathbb{R}}$. We may assume that $\mathcal{F}$ is contained in a standard Siegel set, i.e., we may assume $\mathcal{F}$ is of the form $\mathcal{F} = \{nak\lambda : n \in N'(a), a \in A', k \in K, \lambda \in \Lambda\}$, where

$$
\begin{aligned}
K &= \{\text{special orthogonal transformations in } G_{\mathbb{R}}\};\\
A' &= \{a(s_1,s_2,\ldots,s_7) : s_1,s_2,\ldots,s_7 \geq c\}, \text{ where}\\
& \quad a(s) = \left(\begin{pmatrix} s_1^{-3}s_2^{-1}s_3^{-1} & & & \\ & s_1s_2^{-1}s_3^{-1} & & \\ & & s_1s_2s_3^{-1} & \\ & & & s_1s_2s_3^{3} \end{pmatrix}, \begin{pmatrix} s_4^{-4}s_5^{-3}s_6^{-2}s_7^{-1} & & & & \\ & s_4s_5^{-3}s_6^{-2}s_7^{-1} & & & \\ & & s_4s_5^{2}s_6^{-2}s_7^{-1} & & \\ & & & s_4s_5^{2}s_6^{3}s_7^{-1} & \\ & & & & s_4s_5^{2}s_6^{3}s_7^{4} \end{pmatrix} \right);\\
\bar{N}' &= \{n(u_1,u_2,\ldots,u_{16}) : u = (u_1,u_2,\ldots,u_{16}) \in \nu(a)\}, \text{ where}\\
& \quad n(u) = \left(\begin{pmatrix} 1 & & & \\ u_1 & 1 & & \\ u_2 & u_3 & 1 & \\ u_4 & u_5 & u_6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ u_7 & 1 & & & \\ u_8 & u_9 & 1 & & \\ u_{10} & u_{11} & u_{12} & 1 & \\ u_{13} & u_{14} & u_{15} & u_{16} & 1 \end{pmatrix} \right);\\
\Lambda &= \{\{\lambda : \lambda > 0\}, \text{ where}\\
& \quad \lambda \text{ acts by } \left(\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right);
\end{aligned}
$$

here $c > 0$ is an absolute constant and $\nu(a)$ is an absolutely bounded measurable subset of $\mathbb{R}^{16}$ dependent only on the value of $a \in A'$.

For $i = 0,1,2$, let n_i denote the cardinality of the stabilizer in $G_{\mathbb{R}}$ of any element $v \in V_{\mathbb{R}}^{(i)}$ (it follows from Proposition 15 below that $n_1 = 120$, $n_2 = 12$, and $n_3 = 8$). Then for any $v \in V_{\mathbb{R}}^{(i)}$, $\mathcal{F}v$ will be the union of n_i fundamental domains for the action of $G_{\mathbb{Z}}$ on $V_{\mathbb{R}}^{(i)}$. Since this union is not necessarily disjoint, $\mathcal{F}v$ is best viewed as a multiset, where the multiplicity of a point x in $\mathcal{F}v$ is given by the cardinality of the set $\{g \in \mathcal{F} \mid gv = x\}$. Evidently, this multiplicity is a number between 1 and n_i.

Even though the multiset $\mathcal{F}v$ is the union of n_i fundamental domains for the action of $G_{\mathbb{Z}}$ on $V_{\mathbb{R}}^{(i)}$, not all elements in $G_{\mathbb{Z}}\backslash V_{\mathbb{Z}}$ will be represented in $\mathcal{F}v$ exactly n_i times. In general, the number of times the $G_{\mathbb{Z}}$-equivalence class of an element $x \in V_{\mathbb{Z}}$ will occur in $\mathcal{F}v$ is given by $n_i/m(x)$, where $m(x)$ denotes the size of the stabilizer of x in $G_{\mathbb{Z}}$. We define $N(V_{\mathbb{Z}}^{(i)};X)$ to be the (weighted) number of irreducible $G_{\mathbb{Z}}$-orbits on $V_{\mathbb{Z}}^{(i)}$ having absolute discriminant at most X, where each orbit is counted by a weight of $1/m(x)$ for any point x in that orbit. Thus $n_i \cdot N(V_{\mathbb{Z}}^{(i)};X)$ is the (weighted) number of points in $\mathcal{F}v$ having absolute discriminant at most X, where each point x in the multiset $\mathcal{F}v$ is counted with a weight of $1/m(x)$.

We note that the $G_{\mathbb{Z}}$-orbits in $V_{\mathbb{Z}}$ corresponding to orders in non-Galois quintic fields will then each be counted simply with a weight of 1, since such orders can have no automorphisms. We will show (see Lemma 14) that orbits having weight < 1 are negligible in number in comparison to those having weight 1, and so points of weight < 1 will not be important as they will not affect the main term of the asymptotics of $N(V_{\mathbb{Z}}^{(i)};X)$ as $X \to \infty$.

Now the number of integer points can be difficult to count in a single fundamental region $\mathcal{F}v$. The main technical obstacle is that the fundamental region $\mathcal{F}v$ is not compact, but rather has a system of cusps going off to infinity which in fact contains infinitely many points, including many irreducible points. We simplify the counting of such points by "thickening" the cusp; more precisely, we compute the number of points in the fundamental region $\mathcal{F}v$ by averaging over lots of such fundamental domains, i.e., by averaging over a continuous range of points v lying in a certain special compact subset H of V.

2.2 Averaging over fundamental domains

Let $H = H(J) = \{w \in V : \|w\| \leq J,\ |\mathrm{Disc}(w)| \geq 1\}$, where $\|w\|$ denotes a Euclidean norm on V fixed under the action of K, and J is sufficiently large so that H is nonempty and of nonzero volume. We write $V^{(i)} := V_{\mathbb{R}}^{(i)}$. Then we have

$$N(V_{\mathbb{Z}}^{(i)};X) = \frac{\int_{v\in H\cap V^{(i)}} \#\{x \in \mathcal{F}v \cap V_{\mathbb{Z}}^{\mathrm{irr}} : |\mathrm{Disc}(x)| < X\}\ |\mathrm{Disc}(v)|^{-1} dv}{n_i \cdot \int_{v\in H\cap V^{(i)}}\ |\mathrm{Disc}(v)|^{-1} dv}, \tag{8}$$

where $V_{\mathbb{Z}}^{\mathrm{irr}} \subset V_{\mathbb{Z}}$ denotes the subset of irreducible points in $V_{\mathbb{Z}}$. The denominator of the latter expression is, by construction, a finite absolute constant $M_i = M_i(J)$ greater than zero. We have chosen the measure $|\mathrm{Disc}(v)|^{-1}\,dv$ because it is a $G_{\mathbb{R}}$-invariant measure.

More generally, for any $G_{\mathbb{Z}}$-invariant subset $S \subset V_{\mathbb{Z}}^{(i)}$, let $N(S;X)$ denote the number of irreducible $G_{\mathbb{Z}}$-orbits on S having discriminant less than X. Then $N(S;X)$ can be expressed as

$$N(S;X) = \frac{\int_{v\in H\cap V^{(i)}} \#\{x \in \mathcal{F}v \cap S^{\mathrm{irr}} : |\mathrm{Disc}(x)| < X\}\ |\mathrm{Disc}(v)|^{-1} dv}{n_i \cdot \int_{v\in H\cap V^{(i)}}\ |\mathrm{Disc}(v)|^{-1} dv}, \tag{9}$$

where $S^{\mathrm{irr}} \subset S$ denotes the subset of irreducible points in S. We shall use this definition of $N(S;X)$ for any $S \subset V_{\mathbb{Z}}$, even if S is not $G_{\mathbb{Z}}$-invariant. Note that for disjoint $S_1, S_2 \subset V_{\mathbb{Z}}$, we have $N(S_1 \cup S_2) = N(S_1) + N(S_2)$.

Now since $|\mathrm{Disc}(v)|^{-1}\,dv$ is a $G_{\mathbb{R}}$-invariant measure, we have for any $f \in C_0(V^{(i)})$, with $v, x \in V_{\mathbb{R}}^{(i)}$ and $g \in G_{\mathbb{R}}$ satisfying $v = gx$, that $f(v)|\mathrm{Disc}(v)|^{-1}dv = r_i\, f(gx)\, dg$ for some constant r_i

dependent only on whether $i = 0$, 1 or 2; here dg denotes a left-invariant Haar measure on $G_{\mathbb{R}}$. We may thus express the above formula for $N(S;X)$ as an integral over $\mathcal{F} \subset G_{\mathbb{R}}$:

$$N(S;X) = \frac{r_i}{M_i}\int_{g\in\mathcal{F}} \#\{x \in S^{\rm irr} \cap gH : |{\rm Disc}(x)| < X\}\, dg \tag{10}$$

$$= \frac{r_i}{M_i}\int_{g\in N'(a)A'\Lambda K} \#\{x \in S \cap \bar{n}(u)a(s)\lambda k H : |{\rm Disc}(x)| < X\}\, dg\,. \tag{11}$$

Let us write $H(u,s,\lambda,X) = \bar{n}(u)a(s)\lambda H \cap \{v \in V^{(i)} : |{\rm Disc}(v)| < X\}$. Noting that $KH = H$, $\int_K dk = 1$ (by convention), and $dg = s_1^{-12}s_2^{-8}s_3^{-12}s_4^{-20}s_5^{-30}s_6^{-30}s_7^{-20}du\, d^\times s\, d^\times\lambda\, dk$ (up to scaling), we have

$$N(S;X) = \frac{r_i}{M_i}\int_{g\in N'(a)A'\Lambda} \#\{x \in S^{\rm irr} \cap H(u,s,\lambda,X)\}\, s_1^{-12}s_2^{-8}s_3^{-12}s_4^{-20}s_5^{-30}s_6^{-30}s_7^{-20}\, du\, d^\times t\, d^\times\lambda\,. \tag{12}$$

We note that the same counting method may be used even if we are interested in counting both reducible and irreducible orbits in $V_{\mathbb{Z}}$. For any set $S \subset V_{\mathbb{Z}}^{(i)}$, let $N^*(S;X)$ be defined by (9), but where the superscript "irr" is removed. Thus for a $G_{\mathbb{Z}}$-invariant set $S \subset V_{\mathbb{Z}}^{(i)}$, $n_i \cdot N^*(S;X)$ counts the total (weighted) number of $G_{\mathbb{Z}}$-orbits in S having absolute discriminant nonzero and less than X (not just the irreducible ones). By the same reasoning, we have

$$N^*(S;X) = \frac{r_i}{M_i}\int_{g\in N'(a)A'\Lambda} \#\{x \in S \cap H(u,s,\lambda,X)\}\, s_1^{-12}s_2^{-8}s_3^{-12}s_4^{-20}s_5^{-30}s_6^{-30}s_7^{-20}\, du\, d^\times t\, d^\times\lambda\,. \tag{13}$$

The expression (12) for $N(S;X)$, and its analogue (13) for $N^*(S,X)$, will be useful in the sections that follow.

2.3 A lemma from geometry of numbers

To estimate the number of lattice points in $H(u,s,\lambda,X)$, we have the following elementary proposition from the geometry-of-numbers, which is essentially due to Davenport [11]. To state the proposition, we require the following simple definitions. A multiset $\mathcal{R} \subset \mathbb{R}^n$ is said to be *measurable* if $\mathcal{R}_k$ is measurable for all k, where $\mathcal{R}_k$ denotes the set of those points in $\mathcal{R}$ having a fixed multiplicity k. Given a measurable multiset $\mathcal{R} \subset \mathbb{R}^n$, we define its volume in the natural way, that is, ${\rm Vol}(\mathcal{R}) = \sum_k k \cdot {\rm Vol}(\mathcal{R}_k)$, where ${\rm Vol}(\mathcal{R}_k)$ denotes the usual Euclidean volume of $\mathcal{R}_k$.

Lemma 7 *Let $\mathcal{R}$ be a bounded, semi-algebraic multiset in $\mathbb{R}^n$ having maximum multiplicity m, and which is defined by at most k polynomial inequalities each having degree at most ℓ. Let $\mathcal{R}'$ denote the image of $\mathcal{R}$ under any (upper or lower) triangular, unipotent transformation of $\mathbb{R}^n$. Then the number of integer lattice points (counted with multiplicity) contained in the region $\mathcal{R}'$ is*

$${\rm Vol}(\mathcal{R}) + O(\max\{{\rm Vol}(\bar{\mathcal{R}}), 1\}),$$

where ${\rm Vol}(\bar{\mathcal{R}})$ denotes the greatest d-dimensional volume of any projection of $\mathcal{R}$ onto a coordinate subspace obtained by equating $n-d$ coordinates to zero, where d takes all values from 1 to $n-1$. The implied constant in the second summand depends only on n, m, k, and ℓ.

Although Davenport states the above lemma only for compact semi-algebraic sets $\mathcal{R} \subset \mathbb{R}^n$, his proof adapts without essential change to the more general case of a bounded semi-algebraic multiset $\mathcal{R} \subset \mathbb{R}^n$, with the same estimate applying also to any image $\mathcal{R}'$ of $\mathcal{R}$ under a unipotent triangular transformation.

2.4 Estimates on reducible quadruples (A, B, C, D)

In this section we describe the relative frequencies with which reducible and irreducible elements sit inside various parts of the fundamental domain $\mathcal{F}v$, as v varies over the compact region H.

We begin by describing some sufficient conditions that guarantee that a point in $V_{\mathbb{Z}}$ is reducible.

Lemma 8 *Let $(A, B, C, D) \in V_{\mathbb{Z}}$ be an element such that some non-trivial $\mathbb{Q}$-linear combination of A, B, C, D has rank ≤ 2. Then (A, B, C, D) is reducible.*

Proof: Suppose $E = rA+sB+tC+uD$, where $r, s, t, u \in \mathbb{Q}$ are not all zero. Let $Q_1, \ldots, Q_5$ denote the five 4×4 sub-Pfaffians of (A, B, C, D). Then we have proven in [3] that if $(A, B, C, D) \in V_{\mathbb{Z}}$ is irreducible, then the quadrics $Q_1 = 0, \ldots, Q_5 = 0$ intersect in five points in $\mathbb{P}^3(\bar{\mathbb{Q}})$, and moreover, these five points are defined over conjugate quintic extensions of $\mathbb{Q}$. However, if $\operatorname{rank}(E) \leq 2$, then $[r, s, t, u] \in \mathbb{P}^3(\mathbb{Q})$ is a common zero of $Q_1, \ldots, Q_5$ and it is defined over $\mathbb{Q}$, contradicting the irreducibility of (A, B, C, D). $\Box$

Lemma 9 *Let $(A, B, C, D) \in V_{\mathbb{Z}}$ be an element such that some non-trivial $\mathbb{Q}$-linear combination of $Q_1, \ldots, Q_5$ factors over $\mathbb{Q}$ into two linear factors, where $Q_1, \ldots, Q_5$ denote the five 4×4 sub-Pfaffians of (A, B, C, D). Then (A, B, C, D) is reducible.*

Proof: As noted in the proof of Lemma 8, the five associated quadratic forms $Q_1, \ldots, Q_5$ of an irreducible element $(A, B, C, D) \in V_{\mathbb{Z}}$ possess five common zeroes that are defined over conjugate quintic fields, and these zeroes are conjugate to each other over $\mathbb{Q}$. It follows that each of the $\binom{5}{3} = 10$ planes, going through subsets of three of those five points, cannot be defined over $\mathbb{Q}$, as these planes will each be part of a $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-orbit of size at least 5. However, if some rational quaternary quadratic form Q factors over $\mathbb{Q}$ into linear factors, then (by the pigeonhole principle) at least one of these two rational factors must vanish at three of the five common points of intersection, a contradiction. $\Box$

Lemma 10 *Let $(A, B, C, D) \in V_{\mathbb{Z}}$ be an element such that all the variables in at least one of the following sets vanish:*

(i) $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}\}$

(ii) $\{a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}\}$

(iii) $\{a_{12}, a_{13}, a_{14}, a_{15}\} \cup \{b_{12}, b_{13}, b_{14}, b_{15}\}$

(iv) $\{a_{12}, a_{13}, a_{14}, a_{23}, a_{24}\} \cup \{b_{12}, b_{13}, b_{14}, b_{23}, b_{24}\}$

(v) $\{a_{12}, a_{13}, a_{14}\} \cup \{b_{12}, b_{13}, b_{14}\} \cup \{c_{12}, c_{13}, c_{14}\}$

(vi) $\{a_{12}, a_{13}, a_{23}\} \cup \{b_{12}, b_{13}, b_{23}\} \cup \{c_{12}, c_{13}, c_{23}\}$

(vii) $\{a_{12}, a_{13}\} \cup \{b_{12}, b_{13}\} \cup \{c_{12}, c_{13}\} \cup \{d_{12}, d_{13}\}$

Then (A, B, C, D) is reducible.

Proof: In cases (i) and (ii), one sees that A has rank ≤ 2, and thus (A, B, C, D) is reducible by Lemma 8. In the remaining cases (iii)–(vii), one finds that Q_5 factors into rational linear factors, and thus the result in these cases follows from Lemma 9. $\Box$

We are now ready to give an estimate on the number of irreducible elements in $\mathcal{F}v$, on average, satisfying $a_{12} = 0$:

Lemma 11 *Let v take a random value in H uniformly with respect to the measure $|\mathrm{Disc}(v)|^{-1}\,dv$. Then the expected number of irreducible elements $(A,B,C,D)\in\mathcal{F}v$ such that $|\mathrm{Disc}(A,B,C,D)|<X$ and $a_{12}=0$ is $O(X^{39/40})$.*

Proof: As in [4], we divide the set of all $(A,B,C,D)\in V_{\mathbb{Z}}$ into a number of cases depending on which initial coordinates are zero and which are nonzero. These cases are described in the second column of Table 1. The vanishing conditions in the various subcases of Case $n+1$ are obtained by setting equal to 0—one at a time—each variable that was assumed to be nonzero in Case n. If such a resulting subcase satisfies the reducibility conditions of Lemma 10, it is not listed. In this way, it becomes clear that any irreducible element in $V_{\mathbb{Z}}$ must satisfy precisely one of the conditions enumerated in the second column of Table 1. In particular, there is no Case 14, because assuming any nonzero variables in Case 13 to be zero immediately results in reducibility by Lemma 10.

Let T denote the set of all forty variables $a_{ij},b_{ij},c_{ij},d_{ij}$. For a subcase $\mathcal{C}$ of Table 1, we use $T_0=T_0(\mathcal{C})$ to denote the set of variables in T assumed to be 0 in Subcase $\mathcal{C}$, and T_1 to denote the set of variables in T assumed to be nonzero.

Each variable $t\in T$ has a *weight*, defined as follows. The action of $a(s_1,s_2,\ldots,s_7)\cdot\lambda$ on $(A,B,C,D)\in V$ causes each variable t to multiply by a certain weight which we denote by $w(t)$. These weights $w(t)$ are evidently rational functions in $\lambda,s_1,\ldots,s_7$.

Let $V(\mathcal{C})$ denote the set of $(A,B,C,D)\in V_{\mathbb{R}}$ such that (A,B,C,D) satisfies the vanishing and nonvanishing conditions of Subcase $\mathcal{C}$. For example, in Subcase 2a we have $T_0(2\mathrm{a})=\{a_{12},a_{13}\}$ and $T_1(2\mathrm{a})=\{a_{14},a_{23},b_{12}\}$; thus $V(2\mathrm{a})$ denotes the set of all $(A,B,C,D)\in V_{\mathbb{Z}}$ such that $a_{12}=a_{13}=0$ but $a_{14},a_{23},b_{12}\neq 0$.

For each subcase $\mathcal{C}$ of Case n ($n>0$), we wish to show that $N(V(\mathcal{C});X)$, as defined by (9), is $O(X^{39/40})$. Since $N'(a)$ is absolutely bounded, the equality (13) implies that

$$N^*(V(\mathcal{C});X)\ll\int_{\lambda=c'}^{X^{1/40}}\int_{s_1,s_2,\ldots,s_7=c}^{\infty}\sigma(V(\mathcal{C}))\,s_1^{-12}s_2^{-8}s_3^{-12}s_4^{-20}s_5^{-30}s_6^{-30}s_7^{-20}\,d^\times s\,d^\times\lambda,\tag{14}$$

where $\sigma(V(\mathcal{C}))$ denotes the number of integer points in the region $H(u,s,\lambda,X)$ that also satisfy the conditions

$$t=0\text{ for }t\in T_0\text{ and }|t|\geq 1\text{ for }t\in T_1.\tag{15}$$

Now for an element $(A,B,C,D)\in H(u,s,\lambda,X)$, we evidently have

$$|t|\leq Jw(t)\tag{16}$$

and therefore the number of integer points in $H(u,s,\lambda,X)$ satisfying (15) will be nonzero only if we have

$$Jw(t)\geq 1\tag{17}$$

for all weights $w(t)$ such that $t\in T_1$. Now the sets T_1 in each subcase of Table 1 have been chosen to be precisely the set of variables having the minimal weights $w(t)$ among the variables $t\in T\setminus T_0$ (by "minimal weight" in $T\setminus T_0$, we mean there is no other variable $t\in T\setminus T_0$ with weight having smaller exponents for all parameters $\lambda,s_1,s_2,\ldots,s_7$). Thus if the condition (17) holds for all weights $w(t)$ corresponding to $t\in T_1$, then—by the very choice of T_1—we will also have $Jw(t)\gg 1$ for all weights $w(t)$ such that $t\in T\setminus T_0$.

Therefore, if the region $\mathcal{H}=\{(A,B,C,D)\in H(u,s,\lambda,X):t=0\ \ \forall t\in T_0;\ \ |t|\geq 1\ \ \forall t\in T_1\}$ contains an integer point, then (17) and Lemma 7 together imply that the number of integer points in $\mathcal{H}$ is $O(\mathrm{Vol}(\mathcal{H}))$, since the volumes of all the projections of $u^{-1}\mathcal{H}$ will in that case also be $O(\mathrm{Vol}(\mathcal{H}))$. Now clearly

$$\mathrm{Vol}(\mathcal{H}) = O\Big(J^{40-|T_0|}\prod_{t\in T\setminus T_0} w(t)\Big),$$

so we obtain

$$N(V(\mathcal{C});X) \ll \int_{\lambda=c'}^{X^{1/40}} \int_{s_1,s_2,\ldots,s_7=c}^{\infty} \prod_{t\in T\setminus T_0} w(t)\; s_1^{-12}s_2^{-8}s_3^{-12}s_4^{-20}s_5^{-30}s_6^{-30}s_7^{-20}\;\; d^\times s\, d^\times\lambda. \qquad (18)$$

The latter integral can be explicitly carried out for each of the subcases in Table 1. It will suffice, however, to have a simple estimate of the form $O(X^r)$, with $r<1$, for the integral corresponding to each subcase. For example, if the total exponent of s_i in (18) is negative for all i in $\{1,\ldots,7\}$, then it is clear that the resulting integral will be at most $O(X^{(40-|T_0|)/40})$ in value. This condition holds for many of the subcases in Table 1 (indicated in the fourth column by "-"), immediately yielding the estimates given in the third column.

For cases where this negative exponent condition does not hold, the estimate given in the third column can be obtained as follows. The factor π given in the fourth column is a product of variables in T_1, and so it is at least one in absolute value. The integrand in (18) may thus be multiplied by π without harm, and the estimate (18) will remain true; we may then apply the inequalities (16) to each of the variables in π, yielding

$$N(V(\mathcal{C});X) \ll \int_{\lambda=c'}^{X^{1/40}} \int_{s_1,s_2,\ldots,s_7=c}^{\infty} \prod_{t\in T\setminus T_0} w(t)\; w(\pi)\; s_1^{-12}s_2^{-8}s_3^{-12}s_4^{-20}s_5^{-30}s_6^{-30}s_7^{-20}\;\; d^\times s\, d^\times\lambda. \qquad (19)$$

where we extend the notation w multiplicatively, i.e., $w(ab)=w(a)w(b)$. In each subcase of Table 1, we have chosen the factor π so that the total exponent of each s_i in (19) is negative. Thus we obtain from (19) that $N(V(\mathcal{C});X)=O(X^{(40-\#T_0(\mathcal{C})+\#\pi)/40}$, where $\#\pi$ denotes the total number of variables of T appearing in π (counted with multiplicity), and this is precisely the estimate given in the third column of Table 1. In every subcase, aside from Case 0, we see that $40-\#T_0+\#\pi<40$, as desired. $\Box$

Case	The set $S \subset V_{\mathbb{Z}}$ defined by	$N(S;X) \ll$	Use factor
0.	$a_{12} \neq 0$	$X^{40/40}$	-
1.	$a_{12} = 0$; $a_{13}, b_{12} \neq 0$	$X^{39/40}$	-
2a.	$a_{12}, a_{13} = 0$; $a_{14}, a_{23}, b_{12} \neq 0$	$X^{38/40}$	-
2b.	$a_{12}, b_{12} = 0$; $a_{13}, c_{12} \neq 0$	$X^{38/40}$	-
3a.	$a_{12}, a_{13}, a_{14} = 0$; $a_{15}, a_{23}, b_{12} \neq 0$	$X^{37/40}$	-
3b.	$a_{12}, a_{13}, a_{23} = 0$; $a_{14}, b_{12} \neq 0$	$X^{37/40}$	-
3c.	$a_{12}, a_{13}, b_{12} = 0$; $a_{14}, a_{23}, b_{13}, c_{12} \neq 0$	$X^{37/40}$	-
3d.	$a_{12}, b_{12}, c_{12} = 0$; $a_{13}, d_{12} \neq 0$	$X^{37/40}$	-
4a.	$a_{12}, a_{13}, a_{14}, a_{15} = 0$; $a_{23}, b_{12} \neq 0$	$X^{37/40}$	a_{23}
4b.	$a_{12}, a_{13}, a_{14}, a_{23} = 0$; $a_{15}, a_{24}, b_{12} \neq 0$	$X^{37/40}$	a_{24}
4c.	$a_{12}, a_{13}, a_{14}, b_{12} = 0$; $a_{15}, a_{23}, b_{13}, c_{12} \neq 0$	$X^{36/40}$	-
4d.	$a_{12}, a_{13}, a_{23}, b_{12} = 0$; $a_{14}, b_{13}, c_{12} \neq 0$	$X^{36/40}$	-
4e.	$a_{12}, a_{13}, b_{12}, b_{13} = 0$; $a_{14}, a_{23}, c_{12} \neq 0$	$X^{36/40}$	-
4f.	$a_{12}, a_{13}, b_{12}, c_{12} = 0$; $a_{14}, a_{23}, b_{13}, d_{12} \neq 0$	$X^{36/40}$	-
4g.	$a_{12}, b_{12}, c_{12}, d_{12} = 0$; $a_{13} \neq 0$	$X^{36/40}$	-
5a.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23} = 0$; $a_{24}, b_{12} \neq 0$	$X^{37/40}$	a_{24}^2
5b.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12} = 0$; $a_{23}, b_{13}, c_{12} \neq 0$	$X^{35/40}$	-
5c.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24} = 0$; $a_{15}, a_{34}, b_{12} \neq 0$	$X^{37/40}$	a_{34}^2
5d.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12} = 0$; $a_{15}, a_{24}, b_{13}, c_{12} \neq 0$	$X^{35/40}$	-

Table 1. Subcases 0–5d.

Case	The set $S \subset V_{\mathbb{Z}}$ defined by	$N(S;X) \ll$	Use factor
5e.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13} = 0\,;$ $a_{15}, a_{23}, b_{14}, c_{12} \neq 0$	$X^{35/40}$	-
5f.	$a_{12}, a_{13}, a_{14}, b_{12}, c_{12} = 0\,;$ $a_{15}, a_{23}, b_{13}, d_{12} \neq 0$	$X^{35/40}$	-
5g.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13} = 0\,;$ $a_{14}, b_{23}, c_{12} \neq 0$	$X^{35/40}$	-
5h.	$a_{12}, a_{13}, a_{23}, b_{12}, c_{12} = 0\,;$ $a_{14}, b_{13}, d_{12} \neq 0$	$X^{35/40}$	-
5i.	$a_{12}, a_{13}, b_{12}, b_{13}, c_{12} = 0\,;$ $a_{14}, a_{23}, c_{13}, d_{12} \neq 0$	$X^{35/40}$	-
5j.	$a_{12}, a_{13}, b_{12}, c_{12}, d_{12} = 0\,;$ $a_{14}, a_{23}, b_{13} \neq 0$	$X^{35/40}$	-
6a.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24} = 0\,;$ $a_{25}, a_{34}, b_{12} \neq 0$	$X^{37/40}$	a_{34}^3
6b.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12} = 0\,;$ $a_{24}, b_{13}, c_{12} \neq 0$	$X^{35/40}$	a_{24}
6c.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13} = 0\,;$ $a_{23}, b_{14}, c_{12} \neq 0$	$X^{34/40}$	-
6d.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, c_{12} = 0\,;$ $a_{23}, b_{13}, d_{12} \neq 0$	$X^{34/40}$	-
6e.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12} = 0\,;$ $a_{15}, a_{34}, b_{13}, c_{12} \neq 0$	$X^{35/40}$	a_{34}
6f.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13} = 0\,;$ $a_{15}, a_{24}, b_{14}, b_{23}, c_{12} \neq 0$	$X^{34/40}$	-
6g.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, c_{12} = 0\,;$ $a_{15}, a_{24}, b_{13}, d_{12} \neq 0$	$X^{34/40}$	-
6h.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13}, b_{14} = 0\,;$ $a_{15}, a_{23}, c_{12} \neq 0$	$X^{34/40}$	-
6i.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13}, c_{12} = 0\,;$ $a_{15}, a_{23}, b_{14}, c_{13}, d_{12} \neq 0$	$X^{34/40}$	-
6j.	$a_{12}, a_{13}, a_{14}, b_{12}, c_{12}, d_{12} = 0\,;$ $a_{15}, a_{23}, b_{13} \neq 0$	$X^{34/40}$	-
6k.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, b_{23} = 0\,;$ $a_{14}, c_{12} \neq 0$	$X^{34/40}$	-
6l.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, c_{12} = 0\,;$ $a_{14}, b_{23}, c_{13}, d_{12} \neq 0$	$X^{34/40}$	-
6m.	$a_{12}, a_{13}, a_{23}, b_{12}, c_{12}, d_{12} = 0\,;$ $a_{14}, b_{13} \neq 0$	$X^{34/40}$	-

Table 1. Subcases 5e–6m.

Case	The set $S \subset V_{\mathbb{Z}}$ defined by	$N(S;X) \ll$	Use factor
6n.	$a_{12}, a_{13}, b_{12}, b_{13}, c_{12}, c_{13} = 0\,;$ $a_{14}, a_{23}, d_{12} \neq 0$	$X^{34/40}$	-
6o.	$a_{12}, a_{13}, b_{12}, b_{13}, c_{12}, d_{12} = 0\,;$ $a_{14}, a_{23}, c_{13} \neq 0$	$X^{34/40}$	-
7a.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12} = 0\,;$ $a_{25}, a_{34}, b_{13}, c_{12} \neq 0$	$X^{35/40}$	a_{34}^2
7b.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13} = 0\,;$ $a_{24}, b_{14}, b_{23}, c_{12} \neq 0$	$X^{34/40}$	a_{24}
7c.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, c_{12} = 0\,;$ $a_{24}, b_{13}, d_{12} \neq 0$	$X^{34/40}$	a_{24}
7d.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13}, b_{14} = 0\,;$ $a_{23}, b_{15}, c_{12} \neq 0$	$X^{34/40}$	b_{15}
7e.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13}, c_{12} = 0\,;$ $a_{23}, b_{14}, c_{13}, d_{12} \neq 0$	$X^{34/40}$	d_{12}
7f.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, c_{12}, d_{12} = 0\,;$ $a_{23}, b_{13} \neq 0$	$X^{34/40}$	b_{13}
7g.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13} = 0\,;$ $a_{15}, a_{34}, b_{14}, b_{23}, c_{12} \neq 0$	$X^{34/40}$	a_{34}
7h.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, c_{12} = 0\,;$ $a_{15}, a_{34}, b_{13}, d_{12} \neq 0$	$X^{34/40}$	a_{34}
7i.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{14} = 0\,;$ $a_{15}, a_{24}, b_{23}, c_{12} \neq 0$	$X^{33/40}$	-
7j.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{23} = 0\,;$ $a_{15}, a_{24}, b_{14}, c_{12} \neq 0$	$X^{33/40}$	-
7k.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, c_{12} = 0\,;$ $a_{15}, a_{24}, b_{14}, b_{23}, c_{13}, d_{12} \neq 0$	$X^{33/40}$	-
7l.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, c_{12}, d_{12} = 0\,;$ $a_{15}, a_{24}, b_{13} \neq 0$	$X^{33/40}$	-
7m.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13}, b_{14}, c_{12} = 0\,;$ $a_{15}, a_{23}, c_{13}, d_{12} \neq 0$	$X^{34/40}$	d_{12}
7n.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13}, c_{12}, c_{13} = 0\,;$ $a_{15}, a_{23}, b_{14}, d_{12} \neq 0$	$X^{34/40}$	d_{12}
7o.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13}, c_{12}, d_{12} = 0\,;$ $a_{15}, a_{23}, b_{14}, c_{13} \neq 0$	$X^{34/40}$	c_{13}
7p.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12} = 0\,;$ $a_{14}, c_{13}, d_{12} \neq 0$	$X^{33/40}$	-
7q.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, c_{12}, c_{13} = 0\,;$ $a_{14}, b_{23}, d_{12} \neq 0$	$X^{33/40}$	-

Table 1. Subcases 6n–7q.

Case	The set $S \subset V_{\mathbb{Z}}$ defined by	$N(S;X) \ll$	Use factor
7r.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, c_{12}, d_{12} = 0$; $a_{14}, b_{23}, c_{13} \neq 0$	$X^{33/40}$	-
7s.	$a_{12}, a_{13}, b_{12}, b_{13}, c_{12}, c_{13}, d_{12} = 0$; $a_{14}, a_{23}, d_{13} \neq 0$	$X^{34/40}$	d_{13}
8a.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13} = 0$; $a_{25}, a_{34}, b_{14}, b_{23}, c_{12} \neq 0$	$X^{34/40}$	a_{34}^2
8b.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, c_{12} = 0$; $a_{25}, a_{34}, b_{13}, d_{12} \neq 0$	$X^{34/40}$	$a_{25}a_{34}$
8c.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{14} = 0$; $a_{24}, b_{15}, b_{23}, c_{12} \neq 0$	$X^{34/40}$	$a_{24}b_{15}$
8d.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{23} = 0$; $a_{24}, b_{14}, c_{12} \neq 0$	$X^{33/40}$	a_{24}
8e.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, c_{12} = 0$; $a_{24}, b_{14}, b_{23}, c_{13}, d_{12} \neq 0$	$X^{34/40}$	$a_{24}d_{12}$
8f.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, c_{12}, d_{12} = 0$; $a_{24}, b_{13} \neq 0$	$X^{34/40}$	$a_{24}b_{13}$
8g.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13}, b_{14}, c_{12} = 0$; $a_{23}, b_{15}, c_{13}, d_{12} \neq 0$	$X^{34/40}$	$b_{15}d_{12}$
8h.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13}, c_{12}, c_{13} = 0$; $a_{23}, b_{14}, d_{12} \neq 0$	$X^{34/40}$	$b_{14}d_{12}$
8i.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13}, c_{12}, d_{12} = 0$; $a_{23}, b_{14}, c_{13} \neq 0$	$X^{34/40}$	c_{13}^2
8j.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14} = 0$; $a_{15}, a_{34}, b_{23}, c_{12} \neq 0$	$X^{33/40}$	a_{34}
8k.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{23} = 0$; $a_{15}, a_{34}, b_{14}, c_{12} \neq 0$	$X^{33/40}$	a_{34}
8l.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, c_{12} = 0$; $a_{15}, a_{34}, b_{14}, b_{23}, c_{13}, d_{12} \neq 0$	$X^{33/40}$	a_{34}
8m.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, c_{12}, d_{12} = 0$; $a_{15}, a_{34}, b_{13} \neq 0$	$X^{33/40}$	a_{15}
8n.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{14}, b_{23} = 0$; $a_{15}, a_{24}, c_{12} \neq 0$	$X^{33/40}$	a_{24}
8o.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{14}, c_{12} = 0$; $a_{15}, a_{24}, b_{23}, c_{13}, d_{12} \neq 0$	$X^{32/40}$	-
8p.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12} = 0$; $a_{15}, a_{24}, b_{14}, c_{13}, d_{12} \neq 0$	$X^{32/40}$	-
8q.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, c_{12}, c_{13} = 0$; $a_{15}, a_{24}, b_{14}, b_{23}, d_{12} \neq 0$	$X^{32/40}$	-

Table 1. Subcases 7r–8q.

Case	The set $S \subset V_{\mathbb{Z}}$ defined by	$N(S;X) \ll$	Use factor
8r.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, c_{12}, d_{12} = 0\,;$ $a_{15}, a_{24}, b_{14}, b_{23}, c_{13} \neq 0$	$X^{32/40}$	-
8s.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13}, b_{14}, c_{12}, c_{13} = 0\,;$ $a_{15}, a_{23}, c_{14}, d_{12} \neq 0$	$X^{34/40}$	$c_{14}d_{12}$
8t.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13}, b_{14}, c_{12}, d_{12} = 0\,;$ $a_{15}, a_{23}, c_{13} \neq 0$	$X^{34/40}$	c_{13}^2
8u.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13}, c_{12}, c_{13}, d_{12} = 0\,;$ $a_{15}, a_{23}, b_{14}, d_{13} \neq 0$	$X^{34/40}$	d_{13}^2
8v.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12}, c_{13} = 0\,;$ $a_{14}, c_{23}, d_{12} \neq 0$	$X^{33/40}$	d_{12}
8w.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12}, d_{12} = 0\,;$ $a_{14}, c_{13} \neq 0$	$X^{33/40}$	c_{13}
8x.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, c_{12}, c_{13}, d_{12} = 0\,;$ $a_{14}, b_{23}, d_{13} \neq 0$	$X^{33/40}$	d_{13}
9a.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14} = 0\,;$ $a_{25}, a_{34}, b_{15}, b_{23}, c_{12} \neq 0$	$X^{34/40}$	$a_{34}^2 b_{15}$
9b.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, b_{23} = 0\,;$ $a_{25}, a_{34}, b_{14}, c_{12} \neq 0$	$X^{33/40}$	a_{34}^2
9c.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, c_{12} = 0\,;$ $a_{25}, a_{34}, b_{14}, b_{23}, c_{13}, d_{12} \neq 0$	$X^{34/40}$	$a_{34}^2 d_{12}$
9d.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, c_{12}, d_{12} = 0\,;$ $a_{25}, a_{34}, b_{13} \neq 0$	$X^{34/40}$	$a_{25}^2 b_{13}$
9e.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{14}, b_{23} = 0\,;$ $a_{24}, b_{15}, c_{12} \neq 0$	$X^{33/40}$	$a_{24}b_{15}$
9f.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{14}, c_{12} = 0\,;$ $a_{24}, b_{15}, b_{23}, c_{13}, d_{12} \neq 0$	$X^{34/40}$	$a_{24}b_{15}d_{12}$
9g.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12} = 0\,;$ $a_{24}, b_{14}, c_{13}, d_{12} \neq 0$	$X^{31/40}$	-
9h.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, c_{12}, c_{13} = 0\,;$ $a_{24}, b_{14}, b_{23}, d_{12} \neq 0$	$X^{34/40}$	$a_{24}b_{14}d_{12}$
9i.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, c_{12}, d_{12} = 0\,;$ $a_{24}, b_{14}, b_{23}, c_{13} \neq 0$	$X^{34/40}$	$a_{24}c_{13}^2$
9j.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13}, b_{14}, c_{12}, c_{13} = 0\,;$ $a_{23}, b_{15}, c_{14}, d_{12} \neq 0$	$X^{34/40}$	$b_{15}c_{14}d_{12}$
9k.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13}, b_{14}, c_{12}, d_{12} = 0\,;$ $a_{23}, b_{15}, c_{13} \neq 0$	$X^{34/40}$	$b_{15}c_{13}^2$
9l.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13}, c_{12}, c_{13}, d_{12} = 0\,;$ $a_{23}, b_{14}, d_{13} \neq 0$	$X^{34/40}$	$b_{14}d_{13}^2$

Table 1. Subcases 8r–9l.

Case	The set $S \subset V_{\mathbb{Z}}$ defined by	$N(S;X) \ll$	Use factor
9m.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, b_{23} = 0\,;$ $a_{15}, a_{34}, b_{24}, c_{12} \neq 0$	$X^{33/40}$	$a_{34}b_{24}$
9n.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, c_{12} = 0\,;$ $a_{15}, a_{34}, b_{23}, c_{13}, d_{12} \neq 0$	$X^{31/40}$	-
9o.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{23}, c_{12} = 0\,;$ $a_{15}, a_{34}, b_{14}, c_{13}, d_{12} \neq 0$	$X^{31/40}$	-
9p.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, c_{12}, c_{13} = 0\,;$ $a_{15}, a_{34}, b_{14}, b_{23}, d_{12} \neq 0$	$X^{31/40}$	-
9q.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, c_{12}, d_{12} = 0\,;$ $a_{15}, a_{34}, b_{14}, b_{23}, c_{13} \neq 0$	$X^{31/40}$	-
9r.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{14}, b_{23}, c_{12} = 0\,;$ $a_{15}, a_{24}, c_{13}, d_{12} \neq 0$	$X^{31/40}$	-
9s.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{14}, c_{12}, c_{13} = 0\,;$ $a_{15}, a_{24}, b_{23}, c_{14}, d_{12} \neq 0$	$X^{32/40}$	c_{14}
9t.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{14}, c_{12}, d_{12} = 0\,;$ $a_{15}, a_{24}, b_{23}, c_{13} \neq 0$	$X^{32/40}$	c_{13}
9u.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12}, c_{13} = 0\,;$ $a_{15}, a_{24}, b_{14}, c_{23}, d_{12} \neq 0$	$X^{32/40}$	c_{23}
9v.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12}, d_{12} = 0\,;$ $a_{15}, a_{24}, b_{14}, c_{13} \neq 0$	$X^{32/40}$	c_{13}
9w.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, c_{12}, c_{13}, d_{12} = 0\,;$ $a_{15}, a_{24}, b_{14}, b_{23}, d_{13} \neq 0$	$X^{32/40}$	d_{13}
9x.	$a_{12}, a_{13}, a_{14}, b_{12}, b_{13}, b_{14}, c_{12}, c_{13}, d_{12} = 0\,;$ $a_{15}, a_{23}, c_{14}, d_{13} \neq 0$	$X^{34/40}$	$c_{14}d_{13}^2$
9y.	$a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12}, c_{13}, d_{12} = 0\,;$ $a_{14}, c_{23}, d_{13} \neq 0$	$X^{33/40}$	d_{13}^2
10a.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, b_{23} = 0\,;$ $a_{25}, a_{34}, b_{15}, b_{24}, c_{12} \neq 0$	$X^{33/40}$	$a_{34}^2b_{15}$
10b.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, c_{12} = 0\,;$ $a_{25}, a_{34}, b_{15}, b_{23}, c_{13}, d_{12} \neq 0$	$X^{34/40}$	$a_{34}^2b_{15}d_{12}$
10c.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, b_{23}, c_{12} = 0\,;$ $a_{25}, a_{34}, b_{14}, c_{13}, d_{12} \neq 0$	$X^{31/40}$	a_{34}
10d.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, c_{12}, c_{13} = 0\,;$ $a_{25}, a_{34}, b_{14}, b_{23}, d_{12} \neq 0$	$X^{34/40}$	$a_{34}^2b_{14}d_{12}$
10e.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, c_{12}, d_{12} = 0\,;$ $a_{25}, a_{34}, b_{14}, b_{23}, c_{13} \neq 0$	$X^{34/40}$	$a_{25}^2c_{13}^2$
10f.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{14}, b_{23}, c_{12} = 0\,;$ $a_{24}, b_{15}, c_{13}, d_{12} \neq 0$	$X^{31/40}$	b_{15}

Table 1. Subcases 9m–10f.

Case	The set $S \subset V_{\mathbb{Z}}$ defined by	$N(S;X) \ll$	Use factor
10g.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{14}, c_{12}, c_{13} = 0$; $a_{24}, b_{15}, b_{23}, c_{14}, d_{12} \neq 0$	$X^{34/40}$	$a_{24}b_{15}c_{14}d_{12}$
10h.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{14}, c_{12}, d_{12} = 0$; $a_{24}, b_{15}, b_{23}, c_{13} \neq 0$	$X^{34/40}$	$a_{24}b_{15}c_{13}^2$
10i.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12}, c_{13} = 0$; $a_{24}, b_{14}, c_{23}, d_{12} \neq 0$	$X^{33/40}$	$a_{24}b_{14}d_{12}$
10j.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12}, d_{12} = 0$; $a_{24}, b_{14}, c_{13} \neq 0$	$X^{31/40}$	c_{13}
10k.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, b_{12}, b_{13}, c_{12}, c_{13}, d_{12} = 0$; $a_{24}, b_{14}, b_{23}, d_{13} \neq 0$	$X^{34/40}$	$a_{24}b_{14}d_{13}^2$
10l.	$a_{12}, a_{13}, a_{14}, a_{15}, b_{12}, b_{13}, b_{14}, c_{12}, c_{13}, d_{12} = 0$; $a_{23}, b_{15}, c_{14}, d_{13} \neq 0$	$X^{34/40}$	$b_{15}c_{14}d_{13}^2$
10m.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, b_{23}, c_{12} = 0$; $a_{15}, a_{34}, b_{24}, c_{13}, d_{12} \neq 0$	$X^{31/40}$	b_{24}
10n.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, c_{12}, c_{13} = 0$; $a_{15}, a_{34}, b_{23}, c_{14}, d_{12} \neq 0$	$X^{31/40}$	c_{14}
10o.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, c_{12}, d_{12} = 0$; $a_{15}, a_{34}, b_{23}, c_{13} \neq 0$	$X^{31/40}$	c_{13}
10p.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{23}, c_{12}, c_{13} = 0$; $a_{15}, a_{34}, b_{14}, c_{23}, d_{12} \neq 0$	$X^{31/40}$	c_{23}
10q.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, b_{23}, c_{12}, d_{12} = 0$; $a_{15}, a_{34}, b_{14}, c_{13} \neq 0$	$X^{31/40}$	c_{13}
10r.	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, b_{12}, b_{13}, c_{12}, c_{13}, d_{12} = 0$; $a_{15}, a_{34}, b_{14}, b_{23}, d_{13} \neq 0$	$X^{31/40}$	d_{13}
10s.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{14}, b_{23}, c_{12}, c_{13} = 0$; $a_{15}, a_{24}, c_{14}, c_{23}, d_{12} \neq 0$	$X^{33/40}$	$a_{24}c_{14}d_{12}$
10t.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{14}, b_{23}, c_{12}, d_{12} = 0$; $a_{15}, a_{24}, c_{13} \neq 0$	$X^{31/40}$	c_{13}
10u.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{14}, c_{12}, c_{13}, d_{12} = 0$; $a_{15}, a_{24}, b_{23}, c_{14}, d_{13} \neq 0$	$X^{32/40}$	$c_{14}d_{13}$
10v.	$a_{12}, a_{13}, a_{14}, a_{23}, b_{12}, b_{13}, b_{23}, c_{12}, c_{13}, d_{12} = 0$; $a_{15}, a_{24}, b_{14}, c_{23}, d_{13} \neq 0$	$X^{32/40}$	$c_{23}d_{13}$
11a.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, b_{23}, c_{12} = 0$; $a_{25}, a_{34}, b_{15}, b_{24}, c_{13}, d_{12} \neq 0$	$X^{31/40}$	$a_{34}b_{15}$
11b.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, c_{12}, c_{13} = 0$; $a_{25}, a_{34}, b_{15}, b_{23}, c_{14}, d_{12} \neq 0$	$X^{34/40}$	$a_{34}^2b_{15}c_{14}d_{12}$
11c.	$a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, b_{12}, b_{13}, b_{14}, c_{12}, d_{12} = 0$; $a_{25}, a_{34}, b_{15}, b_{23}, c_{13} \neq 0$	$X^{36/40}$	$a_{25}^2a_{34}b_{15}c_{13}^3$

Table 1. Subcases 10g–11c.

Case	The set $S \subset V_{\mathbb{Z}}$ defined by	$N(S;X) \ll$	Use factor
11d.	$a_{12},a_{13},a_{14},a_{15},a_{23},a_{24},b_{12},b_{13},b_{23},c_{12},c_{13}=0$; $a_{25},a_{34},b_{14},c_{23},d_{12}\neq 0$	$X^{33/40}$	$a_{34}^2b_{14}d_{12}$
11e.	$a_{12},a_{13},a_{14},a_{15},a_{23},a_{24},b_{12},b_{13},b_{23},c_{12},d_{12}=0$; $a_{25},a_{34},b_{14},c_{13}\neq 0$	$X^{31/40}$	$a_{25}c_{13}$
11f.	$a_{12},a_{13},a_{14},a_{15},a_{23},a_{24},b_{12},b_{13},c_{12},c_{13},d_{12}=0$; $a_{25},a_{34},b_{14},b_{23},d_{13}\neq 0$	$X^{34/40}$	$a_{25}^2b_{14}d_{13}^2$
11g.	$a_{12},a_{13},a_{14},a_{15},a_{23},b_{12},b_{13},b_{14},b_{23},c_{12},c_{13}=0$; $a_{24},b_{15},c_{14},c_{23},d_{12}\neq 0$	$X^{33/40}$	$a_{24}b_{15}c_{14}d_{12}$
11h.	$a_{12},a_{13},a_{14},a_{15},a_{23},b_{12},b_{13},b_{14},b_{23},c_{12},d_{12}=0$; $a_{24},b_{15},c_{13}\neq 0$	$X^{31/40}$	$b_{15}c_{13}$
11i.	$a_{12},a_{13},a_{14},a_{15},a_{23},b_{12},b_{13},b_{14},c_{12},c_{13},d_{12}=0$; $a_{24},b_{15},b_{23},c_{14},d_{13}\neq 0$	$X^{34/40}$	$a_{24}b_{15}c_{14}d_{13}^2$
11j.	$a_{12},a_{13},a_{14},a_{15},a_{23},b_{12},b_{13},b_{23},c_{12},c_{13},d_{12}=0$; $a_{24},b_{14},c_{23},d_{13}\neq 0$	$X^{33/40}$	$a_{24}b_{14}d_{13}^2$
11k.	$a_{12},a_{13},a_{14},a_{23},a_{24},b_{12},b_{13},b_{14},b_{23},c_{12},c_{13}=0$; $a_{15},a_{34},b_{24},c_{14},c_{23},d_{12}\neq 0$	$X^{33/40}$	$a_{34}b_{24}c_{14}d_{12}$
11l.	$a_{12},a_{13},a_{14},a_{23},a_{24},b_{12},b_{13},b_{14},b_{23},c_{12},d_{12}=0$; $a_{15},a_{34},b_{24},c_{13}\neq 0$	$X^{31/40}$	$b_{24}c_{13}$
11m.	$a_{12},a_{13},a_{14},a_{23},a_{24},b_{12},b_{13},b_{14},c_{12},c_{13},d_{12}=0$; $a_{15},a_{34},b_{23},c_{14},d_{13}\neq 0$	$X^{31/40}$	$c_{14}d_{13}$
11n.	$a_{12},a_{13},a_{14},a_{23},a_{24},b_{12},b_{13},b_{23},c_{12},c_{13},d_{12}=0$; $a_{15},a_{34},b_{14},c_{23},d_{13}\neq 0$	$X^{31/40}$	$c_{23}d_{13}$
11o.	$a_{12},a_{13},a_{14},a_{23},b_{12},b_{13},b_{14},b_{23},c_{12},c_{13},d_{12}=0$; $a_{15},a_{24},c_{14},c_{23},d_{13}\neq 0$	$X^{33/40}$	$a_{24}c_{14}d_{13}^2$
12a.	$a_{12},a_{13},a_{14},a_{15},a_{23},a_{24},b_{12},b_{13},b_{14},b_{23},c_{12},c_{13}=0$; $a_{25},a_{34},b_{15},b_{24},c_{14},c_{23},d_{12}\neq 0$	$X^{33/40}$	$a_{34}^2b_{15}c_{14}d_{12}$
12b.	$a_{12},a_{13},a_{14},a_{15},a_{23},a_{24},b_{12},b_{13},b_{14},b_{23},c_{12},d_{12}=0$; $a_{25},a_{34},b_{15},b_{24},c_{13}\neq 0$	$X^{36/40}$	$a_{25}^2a_{34}b_{15}b_{24}c_{13}^3$
12c.	$a_{12},a_{13},a_{14},a_{15},a_{23},a_{24},b_{12},b_{13},b_{14},c_{12},c_{13},d_{12}=0$; $a_{25},a_{34},b_{15},b_{23},c_{14},d_{13}\neq 0$	$X^{36/40}$	$a_{25}^2a_{34}b_{15}c_{14}^2d_{13}^2$
12d.	$a_{12},a_{13},a_{14},a_{15},a_{23},a_{24},b_{12},b_{13},b_{23},c_{12},c_{13},d_{12}=0$; $a_{25},a_{34},b_{14},c_{23},d_{13}\neq 0$	$X^{33/40}$	$a_{25}^2b_{14}d_{13}^2$
12e.	$a_{12},a_{13},a_{14},a_{15},a_{23},b_{12},b_{13},b_{14},b_{23},c_{12},c_{13},d_{12}=0$; $a_{24},b_{15},c_{14},c_{23},d_{13}\neq 0$	$X^{33/40}$	$a_{24}b_{15}c_{14}d_{13}^2$
12f.	$a_{12},a_{13},a_{14},a_{23},a_{24},b_{12},b_{13},b_{14},b_{23},c_{12},c_{13},d_{12}=0$; $a_{15},a_{34},b_{24},c_{14},c_{23},d_{13}\neq 0$	$X^{33/40}$	$a_{15}b_{24}c_{23}d_{13}^2$
13.	$a_{12},a_{13},a_{14},a_{15},a_{23},a_{24},b_{12},b_{13},b_{14},b_{23},c_{12},c_{13},d_{12}=0$; $a_{25},a_{34},b_{15},b_{24},c_{14},c_{23},d_{13}\neq 0$	$X^{37/40}$	$a_{25}^2a_{34}b_{24}^2c_{14}^2d_{13}^3$

Table 1. Subcases 11d–13.

Therefore, for the purposes of proving Theorem 6, we may assume that $a_{12} \neq 0$.

2.5 The main term

Let $\mathcal{R}_X(v)$ denote the multiset $\{x \in \mathcal{F}v : |\mathrm{Disc}(x)| < X\}$. Then we have the following result counting the number of integral points in $\mathcal{R}_X(v)$, on average, satisfying $a_{12} \neq 0$:

Proposition 12 *Let v take a random value in $H \cap V^{(i)}$ uniformly with respect to the measure $|\mathrm{Disc}(v)|^{-1}\,dv$. Then the expected number of integral elements $(A,B,C,D) \in \mathcal{F}v$ such that $|\mathrm{Disc}(A,B,C,D)| < X$ and $a_{12} \neq 0$ is $\mathrm{Vol}(\mathcal{R}_X(v_i)) + O(X^{39/40})$, where v_i is any vector in $V^{(i)}$.*

Proof: Following the proof of Lemma 11, let $V^{(i)}(0)$ denote the subset of $V_{\mathbb{R}}$ such that $a_{12} \neq 0$. We wish to show that

$$N^*(V^{(i)}(0); X) = \frac{1}{n_i} \cdot \mathrm{Vol}(\mathcal{R}_X(v_i)) + O(X^{39/40}). \tag{20}$$

We have

$$N^*(V^{(i)}(0); X) = \frac{r_i}{M_i} \int_{\lambda=c'}^{X^{1/40}} \int_{s_1,s_2,\ldots,s_7=c}^{\infty} \int_{u \in N'(a(s))} \sigma(V(0))\, s_1^{-12} s_2^{-8} s_3^{-12} s_4^{-20} s_5^{-30} s_6^{-30} s_7^{-20}\, du\, d^\times s\, d^\times \lambda, \tag{21}$$

where $\sigma(V(0))$ denotes the number of integer points in the region $H(u,s,\lambda,X)$ satisfying $|a_{12}| \geq 1$. Evidently, the number of integer points in $H(u,s,\lambda,X)$ with $|a_{12}| \geq 1$ can be nonzero only if we have

$$Jw(a_{12}) = J \cdot \frac{\lambda}{s_1^3 s_2 s_3 s_4^3 s_5^6 s_6^4 s_7^2} \geq 1. \tag{22}$$

Therefore, if the region $\mathcal{H} = \{(A,B,C,D) \in H(u,s,\lambda,X) : |a_{12}| \geq 1\}$ contains an integer point, then (22) and Lemma 7 imply that the number of integer points in $\mathcal{H}$ is $\mathrm{Vol}(\mathcal{H}) + O(J^{-1}\mathrm{Vol}(\mathcal{H})/w(a_{12}))$, since all smaller-dimensional projections of $u^{-1}\mathcal{H}$ are clearly bounded by a constant times the projection of $\mathcal{H}$ onto the hyperplane $a_{12} = 0$ (since a_{12} has minimal weight).

Therefore, since $\mathcal{H} = H(u,s,\lambda,X) - \big(H(u,s,\lambda,X) - \mathcal{H}\big)$, we may write

$$\begin{aligned} N^*(V^{(i)}(0); X) \;=\; \frac{r_i}{M_i} \int_{\lambda=c'}^{X^{1/40}} \int_{s_1,\ldots,s_7=c}^{\infty} \int_{u \in N'(a(s))} \Big(\mathrm{Vol}\big(H(u,s,\lambda,X)\big) - \mathrm{Vol}\big(H(u,s,\lambda,X) - \mathcal{H}\big) + \\ O(\max\{J^{39}\lambda^{39} s_1^3 s_2 s_3 s_4^3 s_5^6 s_6^4 s_7^2, 1\})\Big)\, s_1^{-12} s_2^{-8} s_3^{-12} s_4^{-20} s_5^{-30} s_6^{-30} s_7^{-20}\, du\, d^\times s\, d^\times \lambda. \end{aligned} \tag{23}$$

The integral of the first term in (23) is $(1/r_i) \cdot \int_{v \in H \cap V^{(i)}} \mathrm{Vol}(\mathcal{R}_X(v)) |\mathrm{Disc}(v)|^{-1} dv$. Since $\mathrm{Vol}(\mathcal{R}_X(v))$ does not depend on the choice of $v \in V^{(i)}$ (see Section 2.6), the latter integral is simply $[M_i/(n_i\, r_i)] \cdot \mathrm{Vol}(\mathcal{R}_X(v))$.

To estimate the integral of the second term in (23), let $\mathcal{H}' = H(u,s,t,X) - \mathcal{H}$, and for each $|a_{12}| \leq 1$, let $\mathcal{H}'(a_{12})$ be the subset of all elements $(A,B,C,D) \in \mathcal{H}'$ with the given value of a_{12}. Then the 39-dimensional volume of $\mathcal{H}'(a_{12})$ is at most $O\Big(J^{39} \prod_{t \in T \setminus \{a_{12}\}} w(t)\Big)$, and so we have the estimate

$$\mathrm{Vol}(\mathcal{H}') \ll \int_{-1}^{1} J^{39} \prod_{t \in T \setminus \{a_{12}\}} w(t)\; da_{12} = O\Big(J^{39} \prod_{t \in T \setminus \{a_{12}\}} w(t)\Big).$$

The second term of the integrand in (23) can thus be absorbed into the third term.

Finally, one easily computes the integral of the third term in (23) to be $O(J^{39}X^{39/40})$. We thus obtain, for any $v \in V^{(i)}$, that

$$N^*(V^{(i)};X) = \frac{1}{n_i} \cdot \mathrm{Vol}(\mathcal{R}_X(v)) + O(J^{39}X^{39/40}/M_i(J)). \tag{24}$$

□

Note that the above proposition counts all integer points in $\mathcal{R}_X(v)$ satisfying $a_{12} \neq 0$, not just the irreducible ones. However, in this regard we have the following lemma:

Lemma 13 *Let $v \in H \cap V^{(i)}$. Then the number of $(A,B,C,D) \in \mathcal{F}v$ such that $a_{12} \neq 0$, $|\mathrm{Disc}(A,B,C,D)| < X$, and (A,B,C,D) is not irreducible is $o(X)$.*

Lemma 13 will in fact follow from a stronger lemma. We say that an element $(A,B,C,D) \in V_{\mathbb{Z}}$ is *absolutely irreducible* if it is irreducible and the fraction field of its associated quintic ring is an S_5-quintic field (equivalently, if the fields of definition of its common zeroes in $\mathbb{P}^3$ are S_5-quintic fields). Then we have the following lemma, whose proof is postponed to Section 3:

Lemma 14 *Let $v \in H \cap V^{(i)}$. Then the number of $(A,B,C,D) \in \mathcal{F}v$ such that $a_{12} \neq 0$, $|\mathrm{Disc}(A,B,C,D)| < X$, and (A,B,C,D) is not absolutely irreducible is $o(X)$.*

Therefore, to prove Theorem 6, it remains only to compute the fundamental volume $\mathrm{Vol}(\mathcal{R}_X(v))$ for $v \in V^{(i)}$. This is handled in the next subsection.

2.6 Computation of the fundamental volume

In this subsection, we compute $\mathrm{Vol}(\mathcal{R}_X(v))$, where $\mathcal{R}_X(v)$ is defined as in Section 2.5. We will see that this volume depends only on whether v lies in $V^{(0)}$, $V^{(1)}$, or $V^{(2)}$; here $V^{(i)}$ again denotes the $G_{\mathbb{R}}$-orbit in $V_{\mathbb{R}}$ consisting of those elements (A,B,C,D) having nonzero discriminant and possessing $5-2i$ real zeros in $\mathbb{P}^3$.

Before performing this computation, we first state two propositions regarding the group $G = \mathrm{GL}_4 \times \mathrm{SL}_5$ and its 40-dimensional representation V.

Proposition 15 *The group $G_{\mathbb{R}}$ acts transitively on $V^{(i)}$, and the isotropy groups for $v \in V^{(i)}$ are given as follows:*

(i) *S_5, if $v \in V^{(0)}$;*

(ii) *$S_3 \times C_2$, if $v \in V^{(1)}$; and*

(iii) *D_4, if $v \in V^{(2)}$.*

In view of Proposition 15, it will be convenient to use the notation n_i to denote the order of the stabilizer of any vector $v \in V^{(i)}$. Proposition 15 implies that we have $n_0 = 120$, $n_1 = 12$, and $n_2 = 8$.

Now define the usual subgroups N, $\bar{N}$, A, and Λ of $G_{\mathbb{R}}$ as follows:

$$\begin{aligned}
N &= \{n(x_1,x_2,\ldots,x_{16}) : x_i \in \mathbb{R}\}, \text{ where} \\
& \quad n(x) = \left(\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & x_4 & x_5 \\ & & 1 & x_6 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_7 & x_8 & x_9 & x_{10} \\ & 1 & x_{11} & x_{12} & x_{13} \\ & & 1 & x_{14} & x_{15} \\ & & & 1 & x_{16} \\ & & & & 1 \end{pmatrix} \right); \\
\bar{N} &= \{\bar{n}(u_1,u_2,\ldots,u_{16}) : u_i \in \mathbb{R}\}, \text{ where} \\
& \quad \bar{n}(u) = \left(\begin{pmatrix} 1 & & & \\ u_1 & 1 & & \\ u_2 & u_3 & 1 & \\ u_4 & u_5 & u_6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ u_7 & 1 & & & \\ u_8 & u_9 & 1 & & \\ u_{10} & u_{11} & u_{12} & 1 & \\ u_{13} & u_{14} & u_{15} & u_{16} & 1 \end{pmatrix} \right); \\
A &= \{a(t_1,t_2,\ldots,t_7) : t_1,t_2,\ldots,t_7 \in \mathbb{R}_+\}, \text{ where} \\
& \quad a(\lambda,t) = \left(\begin{pmatrix} t_1 & & & \\ & t_2/t_1 & & \\ & & t_3/t_2 & \\ & & & 1/t_3 \end{pmatrix}, \begin{pmatrix} t_4 & & & & \\ & t_5/t_4 & & & \\ & & t_6/t_5 & & \\ & & & t_7/t_6 & \\ & & & & 1/t_7 \end{pmatrix} \right); \\
\Lambda &= \{\{\lambda : \lambda > 0\}, \text{ where} \\
& \quad \lambda \text{ acts by } \left(\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right).
\end{aligned}$$

We define an invariant measure dg on $G_{\mathbb{R}}$ by

$$\int_G f(g)dg = \int_{\mathbb{R}_+^\times} \int_{\mathbb{R}_+^{\times 7}} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} f(n(x)\bar{n}(u)a(t)\lambda)\, dx\, du\, d^\times t\, d^\times \lambda. \tag{25}$$

With this choice of Haar measure on $G_{\mathbb{R}}$, it is known that

$$\int_{G_{\mathbb{Z}}\backslash G_{\mathbb{R}}^{\pm 1}} dg = [\zeta(2)\zeta(3)\zeta(4)] \cdot [\zeta(2)\zeta(3)\zeta(4)\zeta(5)],$$

where $G_{\mathbb{R}}^{\pm 1} \subset G_{\mathbb{R}}$ denotes the subgroup $\{(g_4,g_5) \in G_{\mathbb{R}} : \det(g_4) = \pm 1\}$ (see, e.g., [15]).

Now let $dy = dy_1\, dy_2 \cdots dy_{40}$ be the standard Euclidean measure on $V_{\mathbb{R}}$. Then we have:

Proposition 16 *For $i = 0$, 1, or 2, let $f \in C_0(V^{(i)})$, and let y denote any element of $V^{(i)}$. Then*

$$\int_{g \in G_{\mathbb{R}}} f(g \cdot y)dg \;=\; \frac{n_i}{20} \cdot \int_{v \in V^{(i)}} |\mathrm{Disc}(v)|^{-1} f(v)\, dv. \tag{26}$$

Proof: Put

$$(z_1, \ldots, z_{40}) = n(x)\bar{n}(u)a(t) \cdot y.$$

Then the form $\mathrm{Disc}(z)^{-1} dz_1 \wedge \cdots \wedge dz_{12}$ is a $G_{\mathbb{R}}$-invariant measure, and so we must have

$$\mathrm{Disc}(z)^{-1} dz_1 \wedge \cdots \wedge dz_{40} = c\ dx \wedge du \wedge d^\times t \wedge d^\times \lambda$$

for some constant factor c. An explicit Jacobian calculation shows that $c = -20$. (To make easier the calculation, we note that it suffices to check this on any fixed representative y in $V^{(0)}$, $V^{(1)}$, or $V^{(2)}$.) By Proposition 15, the group $G_{\mathbb{R}}$ is an n_i-fold covering of $V^{(i)}$ via the map $g \to g \cdot y$. Hence

$$\int_{G_{\mathbb{R}}} f(g \cdot y) dg = \frac{n_i}{20} \cdot \int_{V^{(i)}} |\mathrm{Disc}(v)|^{-1} f(v) dv.$$

as desired. $\square$

Finally, for any vector $y \in V^{(i)}$ of absolute discriminant 1, we obtain using Proposition 16 that

$$\frac{1}{n_i} \cdot \mathrm{Vol}(\mathcal{R}_X(y)) = \frac{20}{n_i} \int_1^{X^{1/40}} \lambda^{40} d^\times \lambda \int_{G_{\mathbb{Z}} \backslash G_{\mathbb{R}}^{\pm 1}} dg = \frac{\zeta(2)^2 \zeta(3)^2 \zeta(4)^2 \zeta(5)}{2n_i} X,$$

proving Theorem 6.

2.7 Congruence conditions

We may prove a version of Theorem 6 for a set in $V^{(i)}$ defined by a finite number of congruence conditions:

Theorem 17 *Suppose S is a subset of $V_{\mathbb{Z}}^{(i)}$ defined by finitely many congruence conditions. Then we have*

$$\lim_{X \to \infty} \frac{N(S \cap V^{(i)}; X)}{X} = \frac{\zeta(2)^2 \zeta(3)^2 \zeta(4)^2 \zeta(5)}{2n_i} \prod_p \mu_p(S), \tag{27}$$

where $\mu_p(S)$ denotes the p-adic density of S in $V_{\mathbb{Z}}$, and $n_i = 120$, 12, or 8 for $i = 0$, 1, or 2 respectively.

To obtain Theorem 17, suppose S is defined by congruence conditions modulo some integer m. Then S may be viewed as the union of (say) k translates $L_1, \ldots, L_k$ of the lattice $m \cdot V_{\mathbb{Z}}$. For each such lattice translate L_j, we may use formula (12) and the discussion following that formula to compute $N(S; X)$, but where each d-dimensional volume is scaled by a factor of $1/m^d$ to reflect the fact that our new lattice has been scaled by a factor of m. For a fixed value of m, we thus obtain

$$N(L_j; X) = m^{-40}\, \mathrm{Vol}(\mathcal{R}_X(v)) + O(m^{-39} J^{39} X^{39/40} / M_i(J)) \tag{28}$$

for $v \in V^{(i)}$, where the implied constant is also independent of m provided $m = O(X^{1/40})$. Summing (28) over j, and noting that $km^{-40} = \prod_p \mu_p(S)$, yields (27).

3 Quadruples of 5×5 skew-symmetric matrices and Theorems 1–4

Theorems 5 and 6 of the previous section now immediately imply the following.

Theorem 18 *Let $M_5^{*(i)}(\xi,\eta)$ denote the number of isomorphism classes of pairs (R,R') such that R is an order in an S_5-quintic field with $5-2i$ real embeddings, R' is a sextic resolvent ring of R, and $\xi < \mathrm{Disc}(R) < \eta$. Then*

$$\begin{array}{llcl} \text{(a)} & \displaystyle\lim_{X\to\infty} \frac{M_5^{*(0)}(0,X)}{X} & = & \displaystyle\frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{240}; \\ \text{(b)} & \displaystyle\lim_{X\to\infty} \frac{M_5^{*(1)}(-X,0)}{X} & = & \displaystyle\frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{24}; \\ \text{(c)} & \displaystyle\lim_{X\to\infty} \frac{M_5^{*(2)}(0,X)}{X} & = & \displaystyle\frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{16}. \end{array}$$

To obtain finer asymptotic information on the distribution of quintic rings (in particular, without the weighting by the number of sextic resolvents), we need to be able to count irreducible equivalence classes in $V_{\mathbb{Z}}$ lying in certain subsets $S \subset V_{\mathbb{Z}}$. If S is defined, say, by *finitely many* congruence conditions, then Theorem 17 applies in that case.

However, the set S of elements $(A,B,C,D) \in V_{\mathbb{Z}}$ corresponding to maximal quintic orders is defined by infinitely many congruence conditions (see [3, §12]). To prove that (27) still holds for such a set, we require a uniform estimate on the error term when only finitely many factors are taken in (27). This estimate is provided in Section 3.1. In Section 3.2, we prove Lemma 14. Finally, in Section 3.3, we complete the proofs of Theorems 1–4.

3.1 A uniformity estimate

As in [3], for a prime number p let us denote by $\mathcal{U}_p$ the set of all $(A,B,C,D) \in V_{\mathbb{Z}}$ corresponding to quintic orders R that are maximal at p. Let $\mathcal{W}_p = V_{\mathbb{Z}} - \mathcal{U}_p$. In order to apply a sieve to obtain Theorems 1–4, we require the following proposition, analogous to Proposition 1 in [13] and Proposition 23 in [4].

Proposition 19 $N(\mathcal{W}_p;X) = O(X/p^2)$, *where the implied constant is independent of p.*

Proof: We begin with the following lemma.

Lemma 20 *The number of maximal orders in quintic fields, up to isomorphism, having absolute discriminant less than X is $O(X)$.*

Lemma 20 follows immediately from Theorem 18, since we have shown that every quintic ring has a sextic resolvent ring ([3, Corollary 4]).

To estimate $N(\mathcal{W}_p;X)$ using Lemma 20, we only need to know that (a) the number of subrings of index p^k ($k \geq 1$) in a maximal quintic ring R does not grow too rapidly with k; and (b) the number of sextic resolvents that such a subring possesses is also not too large relative to p^k. For (a), an even stronger result than we need here has recently been proven in the Ph.D. thesis [10] of Jos Brakenhoff, who shows that the number of orders having index p^k in a maximal quintic ring R is at most $O(p^{\min\{2k-2,\frac{20}{11}k\}})$ for $k \geq 1$, where the implied constant is independent of p, k, and R. Any such order will of course have discriminant $p^{2k}\mathrm{Disc}(R)$. As for (b), it follows from [3, Proof of Corollary 4] that the number of sextic resolvents of a quintic ring having content n is $O(n^6)$;

moreover, the number of sextic resolvents of a maximal quintic ring is 1. (Recall that the *content* of a quintic ring R is the largest integer n such that $R = \mathbb{Z} + nR'$ for some quintic ring R'.)

Since every content n quintic ring R arises as $\mathbb{Z} + nR'$ for a unique content 1 quintic ring R', and $\mathrm{Disc}(R) = n^8\,\mathrm{Disc}(R')$, we have

$$N(\mathcal{W}_p; X) = \sum_{n=1}^{\infty} \frac{O(n^6)}{n^8} \sum_{k=1}^{\infty} \frac{O(p^{\min\{2k-2,\frac{20}{11}k\}})}{p^{2k}} O(X) = O(X/p^2),$$

as desired. □

3.2 Proof of Lemma 14

We say a quintic ring is an S_5-*quintic ring* if it is an order in an S_5-quintic field. To prove Lemma 14, we wish to show that the expected number of integral elements $(A, B, C, D) \in \mathcal{F}v$ ($v \in V^{(i)}$) that correspond to quintic rings that are not S_5-quintic rings, and such that $|\mathrm{Disc}(A, B, C, D)| < X$ and $a_{12} \neq 0$, is $o(X)$.

Now if a quintic ring $R = R(A, B, C, D)$ is not an S_5-quintic ring, then we claim that either the splitting type (1112) or (5) does not occur in R. Indeed, if both of these splitting types occur in R, then R is clearly a domain (since $R/pR \cong \mathbb{F}_{p^5}$ for some prime p) and the Galois group associated with the quotient field of R then must contain a 5-cycle and a transposition, implying that the Galois group is in fact S_5.

Therefore, to obtain an upper bound on the expected number of integral elements $(A, B, C, D) \in \mathcal{F}v$ such that $R(A, B, C, D)$ is not an S_5-quintic ring, $|\mathrm{Disc}(A, B, C, D)| < X$, and $a_{12} \neq 0$, we may simply count those quintic rings in which p does not split as (1112) in R for any prime $p < N$ and those quintic rings for which p does not have splitting type (5) for any prime $p < N$ (for some sufficiently large N). Now the p-adic density $\mu_p(T_p(1112))$ in $V_{\mathbb{Z}}$ of the set of those $(A, B, C, D) \in T_p(1112)$ approaches 1/12 as $p \to \infty$ while the p-adic density $\mu_p(T_p(5))$ of those $(A, B, C, D) \in T_p(5)$ approaches 1/5 as $p \to \infty$ (by [3, Lemma 20]). We conclude from (27) that the total number of such $(A, B, C, D) \in \mathcal{F}v$ that do not lie in $T_p(1112)$ for any $p < N$ or do not lie in $T_p(5)$ for any $p < N$, and satisfy $|\mathrm{Disc}(A, B, C, D)| < X$ for sufficiently large $X = X(N)$, is at most

$$\frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{2n_i}\Big(\prod_{p<N}\big(1 - \mu_p(T_p(1112))\big) + \prod_{p<N}\big(1 - \mu_p(T_p(5))\big)\Big)X + o(X).$$

Letting $N \to \infty$, we see that asymptotically the above count of (A, B, C, D) is less than cX for any fixed positive constant c, and this completes the proof.

3.3 Proofs of Theorems 1–4

Proof of Theorem 1: Again, let $\mathcal{U}_p$ denote the set of all $(A, B, C, D) \in V_{\mathbb{Z}}$ that correspond to pairs (R, R') where R is maximal at p, and let $\mathcal{U} = \cap_p \mathcal{U}_p$. Then $\mathcal{U}$ is the set of $(A, B, C, D) \in V_{\mathbb{Z}}$ corresponding to maximal quintic rings R. In [3, Theorem 21], we determined the p-adic density $\mu(\mathcal{U}_p)$ of $\mathcal{U}_p$:

$$\mu(\mathcal{U}_p) = (p-1)^8 p^{12}(p+1)^4(p^2+1)^2(p^2+p+1)^2(p^4+p^3+p^2+p+1)(p^4+p^3+2p^2+2p+1)\,/\,p^{40}\,. \quad (29)$$

Suppose Y is any positive integer. It follows from (27) and (29) that

$$\lim_{X\to\infty} \frac{N(\cap_{p<Y}\mathcal{U}_p \cap V^{(i)};X)}{X}$$
$$= \frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{2n_i} \prod_{p<Y} [p^{-28}\,(p^2-1)^2(p^3-1)^2(p^4-1)^2(p^5-1)(p^5+p^3-p-1)].$$

Letting Y tend to ∞, we obtain immediately that

$$\limsup_{X\to\infty} \frac{N(\mathcal{U}\cap V^{(i)};X)}{X}$$
$$\leq \frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{2n_i} \prod_p [p^{-28}(p^2-1)^2(p^3-1)^2(p^4-1)^2(p^5-1)(p^5+p^3-p-1)]$$
$$= \frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{2n_i} \prod_p [(1-p^{-2})^2(1-p^{-3})^2(1-p^{-4})^2(1-p^{-5})(1+p^{-2}-p^{-4}-p^{-5})].$$
$$= \frac{1}{2n_i}\prod_p (1+p^{-2}-p^{-4}-p^{-5}).$$

To obtain a lower bound for $N(\mathcal{U}\cap V^{(i)};X)$, we note that

$$\bigcap_{p<Y} \mathcal{U}_p \subset (\mathcal{U} \cup \bigcup_{p\geq Y} \mathcal{W}_p).$$

Hence by Proposition 19,

$$\lim_{X\to\infty} \frac{N(\mathcal{U}\cap V^{(i)};X)}{X}$$
$$\geq \frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{2n_i} \prod_{p<Y} [p^{-28}(p^2-1)^2(p^3-1)^2(p^4-1)^2(p^5-1)(p^5+p^3-p-1)] - O(\sum_{p\geq Y} p^{-2}).$$

Letting Y tend to infinity completes the proof of Theorem 1. $\square$

Proof of Theorem 2: For each (isomorphism class of) quintic ring R, we make a choice of sextic resolvent ring R', and let $S \subset V_{\mathbb{Z}}$ denote the set of all elements in $V_{\mathbb{Z}}$ that yield the pair (R, R') (under the bijection of Theorem 5) for some R. Then we wish to determine $N(S \cap V^{(i)}; X)$ for $i = 0, 1, 2$; by equation (27), this amounts to determining the p-adic density $\mu_p(S)$ of S for each prime p for our choice of S. In this regard we have the following formula, which follows easily from the arguments in [3, Proof of Lemma 20]:

$$\mu_p(S) = \frac{|G(\mathbb{F}_p)|}{\mathrm{Disc}_p(R)\cdot|\mathrm{Aut}_{\mathbb{Z}_p}(R)|}. \tag{30}$$

Combining (27) and (30) together with the fact that

$$|G(\mathbb{F}_p)| = (p-1)^8\,p^{16}\,(p+1)^4\,(p^2+1)^2\,(p^2+p+1)^2\,(p^4+p^3+p^2+p+1),$$

and proceeding as in Theorem 1, now yields Theorem 2. $\square$

Proof of Theorem 3: Let K_5 be an S_5-quintic field, and K_{120} its Galois closure. It is known that the Artin symbol (K_{120}/p) equals $\langle e\rangle$, $\langle(12)\rangle$, $\langle(123)\rangle$, $\langle(1234)\rangle$, $\langle(12345)\rangle$, $\langle(12)(34)\rangle$, or $\langle(12)(345)\rangle$

precisely when the splitting type of p in R is (11111), (1112), (113), (14), (5), (122), or (23) respectively, where R denotes the ring of integers in K_5. As in [3], let $U_p(\sigma)$ denote the set of all $(A,B,C,D) \in V_{\mathbb{Z}}$ that correspond to maximal quintic rings R having a specified splitting type σ at p. Then by the same argument as in the proof of Theorem 1, we have

$$\lim_{X\to\infty} \frac{N(U_p(\sigma)\cap V^{(i)};X)}{X} = \frac{\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)}{2n_i}\mu_p(U_p(\sigma))\prod_{q\neq p}\mu_q(\mathcal{U}_q).$$

On the other hand, Lemma 20 of [3] gives the p-adic densities of $U_p(\sigma)$ for all splitting and ramification types σ; in particular, the values of $\mu_p(U_p(\sigma))$ for $\sigma =$ (11111), (1112), (113), (14), (5), (122), or (23) are seen to occur in the ratio $1:10:20:30:24:15:20$ for any value of p; this is the desired result. $\Box$

Proof of Theorem 4: This follows immediately from Theorem 1, Lemma 11, and Lemma 14.

Acknowledgments

I am very grateful to B. Gross, H. W. Lenstra, P. Sarnak, A. Shankar, A. Wiles, and M. Wood for many helpful discussions during this work. I am also very thankful to the Packard Foundation for their kind support of this project.

References

[1] M. Bhargava, *Higher Composition Laws*, Ph.D. Thesis, Princeton University, June 2001.

[2] M. Bhargava, Higher composition laws III: The parametrization of quartic rings, *Annals of Math.* **159** (2004), no. 3, 1329–1360.

[3] M. Bhargava, Higher composition laws IV: The parametrization of quintic rings, *Annals of Math.* **167** (2008), no. 1, 53–94.

[4] M. Bhargava, The density of discriminants of quartic rings and fields, *Annals of Math.* **162** (2005), no. 2, 1031–1063.

[5] M. Bhargava, Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants, *Int. Math. Res. Not.* (2007), Vol. 2007: article ID rnm052, 20 pages.

[6] M. Bhargava, On mass formulae for algebras over $\mathbb{F}_p$ and $\mathbb{Z}_p$, in progress.

[7] M. Bhargava, On Gauss-Siegel class number–regulator summation formulae for cubic fields, in progress.

[8] M. Bhargava and A. Shankar, The density of discriminants of cubic, quartic, and quintic extensions of a number field, preprint.

[9] M. Bhargava, A. Shankar, and J. Tsimerman, On the Davenport-Heilbronn theorems, and second order terms, preprint.

[10] J. Brakenhoff, *Counting problems for number rings*, Ph.D. Thesis, Leiden University, 2009.

[11] H. Davenport, On a principle of Lipshitz, *J. London Math. Soc.* **26** (1951), 179–183. Corrigendum: "On a principle of Lipschitz", *J. London Math. Soc.* **39** (1964), 580.

[12] H. Davenport, On the class-number of binary cubic forms I and II, *J. London Math. Soc.* **26** (1951), 183–198.

[13] H. Davenport and H. Heilbronn, On the density of discriminants of cubic fields II, *Proc. Roy. Soc. London Ser. A* **322** (1971), no. 1551, 405–420.

[14] B. Datskovsky and D. J. Wright, The adelic zeta function associated to the space of binary cubic forms II: Local theory, *J. Reine Angew. Math.* **367** (1986), 27–75.

[15] R. Langlands, The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups, in *Algebraic Subgroups and Discontinuous Subgroups*, Proceedings of Symposia in Pure Mathematics IX, Amer. Math. Soc. (1966), pp. 143–148.

[16] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, *Nagoya Math. J.* **65** (1977), 1–155.

[17] M. Sato and T. Shintani, On zeta functions associated with prehomogeneous vector spaces, *Annals of Math.* (2) **100** (1974), 131–170.

[18] T. Shintani, On Dirichlet series whose coefficients are class numbers of integral binary cubic forms. *J. Math. Soc. Japan* **24** (1972), 132–188.

[19] D. J. Wright and A. Yukie, Prehomogeneous vector spaces and field extensions, *Invent. Math.* **110** (1992), 283–314.

[20] A. Yukie, *Shintani zeta functions*, London Mathematical Society Lecture Note Series **183**, Cambridge University Press, Cambridge, 1993.