# EIGENFUNCTIONS WITH INFINITELY MANY ISOLATED CRITICAL POINTS 

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#### Abstract

We construct a Riemannian metric on the 2-dimensional torus, such that for infinitely many eigenvalues of the Laplace-Beltrami operator, a corresponding eigenfunction has infinitely many isolated critical points. A minor modification of our construction implies that each of these eigenfunctions has a level set with infinitely many connected components (i.e., a linear combination of two eigenfunctions may have infinitely many nodal domains).


## 1. Introduction and the main result

Let $(X, g)$ be a compact connected Riemannian manifold without boundary. The Riemannian structure $g$ on $X$ defines the Laplace-Beltrami operator $\Delta_{g}$ on the space of smooth functions on $X$. The operator $\Delta_{g}$ admits a spectral decomposition via orthonormal basis of $L^{2}\left(X, \Omega_{g}\right)$ (where $\Omega_{g}$ is the volume density on $X$ induced by $g$ ), consisting of smooth real-valued functions, and with corresponding real and non-negative eigenvalues: $\Delta_{g} \varphi_{j}+\lambda_{j} \varphi_{j}=0$, where $\lambda_{0}=0<\lambda_{1} \leqslant \lambda_{2} \leqslant \ldots$, and $\varphi_{0} \equiv 1$.

The number of connected components of the set $\operatorname{Crit}\left(\varphi_{k}\right)$ of the critical points of $\varphi_{k}$ is an important geometric characteristic of the eigenfunction $\varphi_{k}$. We denote it by $N_{\text {crit }}(k)$. In [11], Yau asked whether there exists a non-trivial asymptotic lower bound on $N_{\text {crit }}(k)$. The following theorem proven by Jakobson and Nadirashvili in [7] shows the negative answer to this question:

Theorem (Jakobson-Nadirashvili). There exists a Riemannian metric on $\mathbb{T}^{2}$ and a sequence of eigenfunctions of the corresponding Laplace-Beltrami operator, such that the corresponding eigenvalues converge to infinity, but the number of critical points of the eigenfunctions from the sequence remains bounded.

Our result states that generally one cannot hope for an asymptotic upper bound on $N_{\text {crit }}(k)$ :

Theorem 1. On $\mathbb{T}^{2}$ there exists a Riemannian metric and a sequence of eigenfunctions of the corresponding Laplace-Beltrami operator, with eigenvalues converging to infinity, such
that each one of the eigenfunctions from the sequence has an infinite number of isolated critical points.

Similarly to the construction of Jakobson and Nadirashvili, we also will construct a Riemannian metric on $\mathbb{T}^{2}$ of the Liouville type. The "punch-line argument" in our proof of Theorem 1 invokes the Brower's fixed point theorem.

Let us mention that Enciso and Peralta-Salas [6] constructed a smooth metric such that the first eigenfunction has arbitrarily large number of isolated critical points. Results of opposite spirit were proven by Polterovich and Sodin [10] and Polterovich-PolterovichStojisavljević 9 .

It is worth to point out that, at present, we do not know what happens in the case when the Riemannian metric $g$ is real-analytic. In that case, an eigenfunction must have a finite number of isolated critical points, however, we do not know whether there exists an asymptotic upper bound for the number of critical points in terms of the corresponding eigenvalue.

At last, we mention that minor modification of our construction implies that each of the eigenfunctions in the statement of Theorem 1 has a level set with infinitely many connected components. This feature is interesting in view of the failure of the Courant nodal domains theorem for linear combinations of eigenfunctions, see Gladwell-Zhou [4], Arnold [1], Bérard-Helffer [3], Bérard-Charron-Helffer [2] and references therein. We will outline the needed modification at the end of this note.

## 2. Proof of Theorem 1

2.1. The metric and 1-dim reduction. Consider the coordinates $(x, y) \in \mathbb{T}^{2}=(\mathbb{R} / 2 \pi \mathbb{Z})^{2}$ on the torus. Our Riemannian metric $g$ on $\mathbb{T}^{2}$ will be of the form $d s^{2}=Q(x)\left(d x^{2}+d y^{2}\right)$, where $Q$ is a $C^{\infty}$ smooth, positive, $2 \pi$ periodic function. Then the Laplace-Beltrami operator is $\Delta_{g}=\frac{1}{Q(x)}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right)$. Searching eigenfunctions $\varphi$ of the form $\varphi(x, y)=F(x) G(y)$ with $2 \pi$-periodic functions $F$ and $G$, the equation $\Delta_{g} \varphi+\lambda \varphi=0$ becomes equivalent to the system

$$
\left\{\begin{array}{l}
F^{\prime \prime}(x)+(\lambda Q(x)-\mu) F(x)=0  \tag{1}\\
G^{\prime \prime}(y)+\mu G(y)=0
\end{array}\right.
$$

for some $\mu \in \mathbb{R}$. Thus we get a sequence of solutions $G_{m}(y)=e^{i m y}$ with $\mu_{m}=m^{2}$, where $m \in \mathbb{Z}$, and for each such $m$, the first equation in the system (1) becomes

$$
\begin{equation*}
F^{\prime \prime}(x)+\left(\lambda Q(x)-m^{2}\right) F(x)=0 \tag{2}
\end{equation*}
$$

For each given $m$, the eigenvalue problem (2) admits a spectral decomposition, where we have a sequence of smooth solutions $F=F_{m, 1}, F_{m, 2}, \ldots$ with corresponding eigenvalues $\lambda_{m, 1} \leqslant \lambda_{m, 2} \leqslant \ldots$ Going back to our original eigenvalue problem on the torus, $\Delta_{g} \varphi+$ $\lambda \varphi=0$, we have found a sequence of solutions $F_{m, k}(x) \cos m y, F_{m, k}(x) \sin m y$. It is easy to see that this sequence of solutions is complete in $L^{2}\left(\mathbb{T}^{2}, Q \mathrm{~d} x \mathrm{~d} y\right)$, and hence gives a complete spectral decomposition of our original eigenvalue problem on the torus we will not use this fact).

Our goal now is to find a smooth positive function $Q \in C^{\infty}(\mathbb{R} / 2 \pi \mathbb{Z})$, such that for infinitely many integers $m$, there will exist a solution $F_{m}$ of the equation (2) having infinitely many isolated critical points in $\mathbb{R} / 2 \pi \mathbb{Z}$. For such $Q$ and $F_{m}$, the eigenfunction $F_{m}(x) \cos m y$ (as well as the eigenfunction $F_{m}(x) \sin m y$ ) will have infinitely many isolated critical points.

We have reduced 2-dimensional eigenvalue problem to a 1-dimensional problem, and for convenience, we will consider periodic functions on $\mathbb{R}$, instead of functions on $\mathbb{R} / 2 \pi \mathbb{Z}$. Moreover, by rescaling, we may assume that the functions are 8-periodic rather than $2 \pi$-periodic.
2.2. The main observation. Assume that $u: \mathbb{R} \rightarrow \mathbb{R}$ is a non-trivial solution of the differential equation $u^{\prime \prime}(x)+K(x) u(x)=0$, where $K: \mathbb{R} \rightarrow \mathbb{R}$ is some smooth function. Also assume that for some point $x_{*} \in \mathbb{R}$, we have $K\left(x_{*}\right)=0$, and moreover, $K$ makes a large number of oscillations near $x_{*}$, such that these oscillations decay very fast when we approach $x_{*}$. To be more precise, we suppose that for some

$$
x_{*}=x_{N}<x_{N-1}<\ldots<x_{1}<x_{*}+\varepsilon
$$

we have $\left.(-1)^{i} K\right|_{\left(x_{i+1}, x_{i}\right)}<0$, and moreover, for every $1 \leqslant i \leqslant N-2$,

$$
\left|\int_{x_{i+1}}^{x_{i}} K(x) d x\right| \text { is much larger than } \sum_{j=i+1}^{N-1}\left|\int_{x_{j+1}}^{x_{j}} K(x) d x\right| \text {. }
$$

Furthermore, assume that $u^{\prime}\left(x_{*}\right)=0$, that $u$ is positive on the interval $\left[x_{*}, x_{*}+\varepsilon\right]$, and the values of $u$ on that interval are of the same order of magnitude. We claim that under these assumptions, $u$ has at least $N-2$ critical points on $\left(x_{*}, x_{*}+\varepsilon\right)$.

Indeed, for any $1 \leqslant i<N$, we have

$$
u^{\prime}\left(x_{i}\right)-u^{\prime}\left(x_{i+1}\right)=\int_{x_{i+1}}^{x_{i}} u^{\prime \prime}(x) d x=-\int_{x_{i+1}}^{x_{i}} K(x) u(x) d x
$$

and moreover we have $u^{\prime}\left(x_{N}\right)=u^{\prime}\left(x_{*}\right)=0$. Hence, by our assumptions, for each $1 \leqslant i<$ $N$, the sign of $u^{\prime}\left(x_{i}\right)=\left(u^{\prime}\left(x_{i}\right)-u^{\prime}\left(x_{i+1}\right)\right)+\ldots+\left(u^{\prime}\left(x_{N-1}\right)-u^{\prime}\left(x_{N}\right)\right)$ is positive if $i$ is
even, and is negative when $i$ is odd. Therefore for each $1 \leqslant i<N, u$ has a critical point $\xi_{i} \in\left(x_{i+1}, x_{i}\right)$, which is moreover isolated since $u^{\prime \prime}\left(\xi_{i}\right)=-K\left(\xi_{i}\right) u\left(\xi_{i}\right) \neq 0$. We conclude that $u$ has at least $N-2$ isolated critical points on the interval $\left(x_{*}, x_{*}+\varepsilon\right)$.

We can also consider a more general setting where the function $K$ has infinitely many rapidly decaying oscillations near some point $x_{*}$, and in this case we may conclude that $u$ has infinitely many isolated critical points. More precisely, it is enough to assume the following:
(1) We have some $x_{*} \in \mathbb{R}$ and a strictly decreasing sequence $\left(x_{i}\right)_{i=1}^{\infty}$, such that $\lim _{i \rightarrow \infty} x_{i}=x_{*}$.
(2) We have a smooth function $K: \mathbb{R} \rightarrow \mathbb{R}$ such that $\left.(-1)^{i} K\right|_{\left(x_{i+1}, x_{i}\right)}<0$ for every $i$, and

$$
\left|\int_{x_{i+1}}^{x_{i}} K(x) d x\right| \geq C \sum_{j=i+1}^{\infty}\left|\int_{x_{j+1}}^{x_{j}} K(x) d x\right|
$$

with some numerical constant $C>1$.
(3) $u: \mathbb{R} \rightarrow \mathbb{R}$ is a solution of $u^{\prime \prime}(x)+K(x) u(x)=0$ with $u\left(x_{*}\right)>0$ and $u^{\prime}\left(x_{*}\right)=0$.

Then $u$ has infinitely many isolated critical points on a small interval ( $x_{*}, x_{*}+\delta$ ).
We will apply this idea to the sequence of periodic functions $K_{i}(x)=\lambda_{i} Q(x)-m_{i}^{2}$ and the corresponding periodic solutions $u_{i}=F_{m_{i}}$ of the ODE (2).
2.3. The construction. Here we will construct a family of smooth periodic functions $q_{S, \tau}(x)$ having infinitely many "cascades" each consisting of an infinite number of rapidly decaying oscillations.
a) Let $\psi: \mathbb{R} \rightarrow[0,1]$ be a smooth even function such that $\operatorname{supp}(\psi) \subset(-1,-1 / 4) \cup(1 / 4,1)$ and such that $\psi=1$ on $[-3 / 4,-1 / 2] \cup[1 / 2,3 / 4]$. For every $t \in[0, \infty)$, denote $\psi_{t}(x)=$ $\psi\left(4^{t} x\right)$ and then define the 2-periodic function $\varphi_{t}(x)=1-\sum_{n \in \mathbb{Z}} \psi_{t}(x+2 n)$.
b) Choose a smooth positive function $q: \mathbb{R} \rightarrow(0, \infty)$ satisfying:

- $q(x)=q(-x), x \in \mathbb{R}$,
- $q(x+2)=q(x), x \in \mathbb{R}$,
- $q$ is increasing on $[-1,0]$ and decreasing on $[0,1]$.

For $t \in[0, \infty)$, denote $q_{t}(x)=\varphi_{t}(x)\left(q(x)-q\left(4^{-t}\right)\right)+q\left(4^{-t}\right)$. We may assume that the function $q$ is "flat" enough at $x=0$ so that $q_{t} \rightarrow q$ in the $C^{\infty}$ topology, when $t \rightarrow \infty$. This is possible since $q_{t}-q=\left(\varphi_{t}-1\right)\left(q-q\left(4^{-t}\right)\right)$. The first factor vanishes outside of the interval $\left(4^{-t-1}, 4^{-t}\right)$. Given a function $\varphi_{t}$, we choose the function $q$ so flat at the origin


Figure 1.


Figure 2.
that the smallness of $q-q\left(4^{-t}\right)$ and its derivatives will compensate the size of $\varphi_{t}$ and its derivatives on the interval $\left[4^{-t-1}, 4^{-t}\right]$.
c) Fix a smooth even function $h: \mathbb{R} \rightarrow \mathbb{R}$ with supp $h \subset(-3 / 4,-1 / 2) \cup(1 / 2,3 / 4)$, such that for some $x_{\infty} \in(1 / 2,3 / 4)$ and for a strictly decreasing sequence $\left(x_{i}\right)_{i=1}^{\infty}$, $x_{i} \in\left(x_{\infty}, 3 / 4\right)$, with $\lim _{i \rightarrow \infty} x_{i}=x_{\infty}$, we have $\left.(-1)^{i} h\right|_{\left(x_{i+1}, x_{i}\right)}<0$ for every $i$, and that moreover, for every $i$ we have

$$
\left|\int_{x_{i+1}}^{x_{i}} h(x) d x\right| \geq C \sum_{j=i+1}^{\infty}\left|\int_{x_{j+1}}^{x_{j}} h(x) d x\right|
$$

with some numerical constant $C>1$. For each $t \in[0, \infty)$, denote $h_{t}(x)=h\left(4^{t} x\right)$, and then $H_{t}(x)=\sum_{n \in \mathbb{Z}} h_{t}(x+2 n)$.
d) For any finite 1 -separated set $S \subset[0, \infty)$ (1-separation means that $|s-t| \geqslant 1$ for every $s \neq t \in S$ ), and for any function $\tau: S \rightarrow \mathbb{R}$, define the function $q_{S, \tau}: \mathbb{R} \rightarrow \mathbb{R}$ as follows.


Figure 3.
We set $q_{\varnothing, \tau}=q$, and if $S \subset[0, \infty)$ is a non-empty finite set, then we choose some $t \in S$, denote $S^{\prime}=S \backslash\{t\}$ and $\tau^{\prime}=\tau_{\mid S^{\prime}}$, and define $q_{S, \tau}(x)=\varphi_{t}(x)\left(q_{S^{\prime}, \tau^{\prime}}(x)-q_{S^{\prime}, \tau^{\prime}}\left(4^{-t}\right)\right)+$ $q_{S^{\prime}, \tau^{\prime}}\left(4^{-t}\right)+\tau(t) H_{t}(x)$. For each $t \in S$, this creates a cascade of oscillations around the


Figure 4.
point $4^{-t} x_{\infty}$ of intensity $\tau(t)$. Note that $\left\{x: q_{S, \tau}(x) \neq q(x)\right\} \subset \bigcup_{t \in S}\left(4^{-t-1}, 4^{-t}\right)$ (the intervals on the RHS are disjoint), and that, for $t \in S, q_{S, \tau}\left(4^{-t} x_{\infty}\right)=q\left(4^{-t}\right)$.
e) Choose a smooth positive function $\widetilde{q}: \mathbb{R} \rightarrow \mathbb{R}$ such that:

- $\widetilde{q}(x+2)=\widetilde{q}(x), x \in \mathbb{R}$,
- $\widetilde{q}(-x)=\widetilde{q}(x), x \in \mathbb{R}$,
- $\widetilde{q}(x)=q(5 x), x \in[0,1 / 5]$,
- $\widetilde{q}(x)<q(4 x), x \in[1 / 5,2 / 9]$,
- $\widetilde{q}(x)<q(1), x \in[2 / 9,1]$.


Figure 5.

We claim that for any finite 1-separated set $S \subset[1, \infty)$ and for any sufficiently small function $\tau: S \rightarrow \mathbb{R}$, we have $\widetilde{q} \leqslant q_{S, \tau} \leqslant q$. The upper bound is true for $\tau$ small enough by the definition of the function $q_{S, \tau}$. To check the lower bound, it is enough to show that, for $q_{S}:=q_{S, 0}$ (where $0: S \rightarrow \mathbb{R}$ is the zero function), we have $q_{S}(x)>\widetilde{q}(x)$ for any $x \in(0,1]$. Let us check this.

Note that, by the construction of $q_{S}$, we have $q_{S}(x) \geqslant q(4 x)$ for $x \in[0,1 / 4]$, and $q_{S}(x) \geqslant q(1)$ for $x \in[1 / 4,1]$. Take any $x \in(0,1]$. Then

Case 1: $x \in(0,1 / 5]$. In that case we have $q_{S}(x)-\widetilde{q}(x) \geqslant q(4 x)-\widetilde{q}(x)=q(4 x)-q(5 x)>0$.
Case 2: $x \in[1 / 5,2 / 9]$. Then $q_{S}(x)-\widetilde{q}(x) \geqslant q(4 x)-\widetilde{q}(x)>0$.

Case 3: $x \in[2 / 9,1]$. Then $q_{S}(x)-\widetilde{q}(x) \geqslant q(1)-\widetilde{q}(x)>0$.
2.4. Two lemmas. Here we will bring two lemmas on the solutions of the periodic ODE (2). The first lemma will guarantee that under a certain choice of $\lambda=\Lambda_{m, Q}$ the linear space of solutions to ODE (2) is two-dimensional. This will ensure existence of a solution $F_{m}$ with vanishing derivative at a given point $x_{*}$, which is needed for generation of oscillations of $F_{m}$ nearby $x_{*}$.

Let $Q: \mathbb{R} \rightarrow(0, \infty)$ be a 2 -periodic positive smooth even function. For every $m \in \mathbb{N}$, consider the Dirichlet problem

$$
\left\{\begin{array}{l}
u:[0,4] \rightarrow \mathbb{R}  \tag{3}\\
u^{\prime \prime}(x)+\left(\lambda Q(x)-m^{2}\right) u(x)=0 \\
u(0)=u(4)=0
\end{array}\right.
$$

We denote by $\Lambda_{Q, m}$ the first eigenvalue and by $U_{Q, m}$ the first eigenfunction of this Dirichlet problem, extend $U_{Q, m}$ to $[0,8]$ by $U_{Q, m}(x)=-U_{Q, m}(8-x)$ for $x \in[4,8]$, and then extend $U_{Q, m}$ to the whole $\mathbb{R}$ by making it 8-periodic. Then $U_{Q, m}: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth solution of

$$
\left\{\begin{array}{l}
u: \mathbb{R} \rightarrow \mathbb{R} \text { is } 8 \text {-periodic }  \tag{4}\\
u^{\prime \prime}(x)+\left(\lambda Q(x)-m^{2}\right) u(x)=0
\end{array}\right.
$$

Lemma 2. The space of solutions to (4) with $\lambda=\Lambda_{Q, m}$ is two-dimensional.
Proof of Lemma R Recall that $Q$ is even and 2-periodic. We have $U_{Q, m}^{\prime}(2)=0$, since otherwise $v(x):=U_{Q, m}(4-x)$ is another solution of (3) with $\lambda=\Lambda_{Q, m}$, which is linearly independent with $\left.U_{Q, m}\right|_{[0,4]}$, while the first eigenvalue of the Dirichlet problem (3) must be simple. Therefore we get: $U_{Q, m}(0)=0, U_{Q, m}^{\prime}(0) \neq 0, U_{Q, m}(2) \neq 0, U_{Q, m}^{\prime}(2)=0$. Hence if we define $V_{Q, m}=U_{Q, m}(x+2)$, then $V_{Q, m}(x)$ is a solution of (4) and is linearly independent with $U_{Q, m}$. We conclude that the space of solutions of (4) with $\lambda=\Lambda_{Q, m}$ is two-dimensional (obviously, the dimension cannot be greater than 2).

In the second lemma we estimate the size of $\Lambda_{Q, m}$ chosen according to Lemma 2 .
Lemma 3. Let $Q: \mathbb{R} \rightarrow(0, \infty)$ be a 2-periodic positive smooth function with $Q(0)>Q(x)$, $x \notin 2 \mathbb{Z}$. Then we have

$$
\Lambda_{Q, m}>\frac{m^{2}}{Q(0)}
$$

for each $m$, and for every $\varepsilon>0$, we have

$$
\Lambda_{Q, m}<\frac{m^{2}}{Q(0)-\varepsilon}
$$

when $m$ is large enough.
Proof of Lemma 3. The variational principle applied to the Dirichlet problem (3) (see, for instance, [5, Ch. VI, §1]) says that

$$
\Lambda_{Q, m}=\min \frac{\int_{0}^{4}\left(u^{\prime}(x)\right)^{2}+m^{2}(u(x))^{2} \mathrm{~d} x}{\int_{0}^{4} Q(x)(u(x))^{2} \mathrm{~d} x}
$$

where the minimum is taken over all smooth functions $u:[0,4] \rightarrow \mathbb{R}$ with $u(0)=u(4)=0$. From here we clearly have

$$
\Lambda_{Q, m}>\frac{m^{2}}{\max Q}=\frac{m^{2}}{Q(0)}
$$

Now, choose $\delta>0$ small enough, and pick a smooth function $u:[0,4] \rightarrow \mathbb{R}$ with $\operatorname{supp}(u) \subset(2-\delta, 2+\delta)$ and $u \not \equiv 0$. Then for every $m$ we have

$$
\Lambda_{Q, m} \leqslant \frac{\int_{0}^{4}\left(u^{\prime}(x)\right)^{2}+m^{2}(u(x))^{2} \mathrm{~d} x}{\int_{0}^{4} Q(x)(u(x))^{2} \mathrm{~d} x} \leqslant \frac{\int_{0}^{4}\left(u^{\prime}(x)\right)^{2}+m^{2}(u(x))^{2} \mathrm{~d} x}{\left(\min _{[2-\delta, 2+\delta]} Q\right) \cdot \int_{0}^{4}(u(x))^{2} \mathrm{~d} x}=\frac{C+m^{2}}{\min _{[-\delta, \delta]} Q},
$$

where

$$
C=\frac{\int_{0}^{4}\left(u^{\prime}(x)\right)^{2} d x}{\int_{0}^{4}(u(x))^{2} d x}
$$

From here we see that by choosing $\delta>0$ so small that $\min _{[-\delta, \delta]} Q>Q(0)-\varepsilon$, we obtain

$$
\Lambda_{Q, m}<\frac{m^{2}}{Q(0)-\varepsilon}
$$

for large enough $m$.
2.5. The main argument. We need to find an increasing sequence ( $m_{i}$ ) of positive integers and a 1 -separated set $S=\left(t_{i}\right)$ so that for any fast decaying sequence $\tau\left(t_{i}\right)$, letting $Q=q_{S, \tau}$, we get $\Lambda_{Q, m_{i}} Q\left(4^{-t_{i}} x_{\infty}\right)=m_{i}^{2}, i=1,2, \ldots$, where $\Lambda_{Q, m_{i}}$ is the first eigenvalue of the Dirichlet problem (3). Note that $Q\left(4^{-t_{i}} x_{\infty}\right)=q_{S, \tau}\left(4^{-t_{i}} x_{\infty}\right)=q\left(4^{-t_{i}}\right)$.

Using Lemma 3 and a continuous dependence of the first eigenvalue $\Lambda_{Q, m}$ on the function $Q$, it is not difficult to find a positive integer $m$ such that, for any sufficiently small $\tau$, there exists $t$ (depending on $m$ and $\tau$ such that $\Lambda_{q_{\{t\}, \tau}} q\left(4^{-t}\right)=m^{2}$. This will yield the existence of a metric on $\mathbb{T}^{2}$ with one Laplace eigenfunction with infinitely many critical points. To construct a metric with a sequence of eigenfunctions having infinitely many critical points we will use the following proposition.

Proposition 4. There exist sequences: $m_{1}<m_{2}<\ldots$ of positive integers, $\varepsilon_{1}, \varepsilon_{2}, \ldots \in$ $(0, \infty)$, and $M_{0}=0, M_{1}, M_{2}, \ldots \in[0, \infty)$, with $M_{i+1}>M_{i}+1$ such that, for any $k \in \mathbb{N}$ and any $\tau_{1}, \tau_{2}, \ldots, \tau_{k} \in \mathbb{R}$ with $\left|\tau_{i}\right| \leqslant \varepsilon_{i}$, one can find $t_{1}, \ldots, t_{k}$ with $t_{i} \in\left[M_{i-1}+1, M_{i}\right]$, $i=1,2, \ldots k$, with the following property:

For $S=\left\{t_{1}, t_{2}, \ldots, t_{k}\right\}$ and for the function $\tau: S \rightarrow \mathbb{R}, \tau\left(t_{i}\right)=\tau_{i}, i=1,2, \ldots, k$, the function $Q=q_{S, \tau}$ satisfies $\Lambda_{Q, m_{i}} Q\left(4^{-t_{i}} x_{\infty}\right)-m_{i}^{2}=0$ for $i=1,2, \ldots, k$, where $\Lambda_{Q, m_{i}}$ is the first eigenvalue of the problem (3) with $m=m_{i}$.

Remark 5. As we have already mentioned, by construction of the function $q_{S, \tau}$ in 2.3 d , in the setting of the proposition we always have $Q\left(4^{-t_{i}} x_{\infty}\right)=q_{S, \tau}\left(4^{-t_{i}} x_{\infty}\right)=q\left(4^{-t_{i}}\right)$ for $i=1,2, \ldots, k$.

Proof of Proposition 4 First, we choose sequences $\left(m_{k}\right),\left(\varepsilon_{k}\right),\left(M_{k}\right)$ inductively as follows (the values $t_{1}, \ldots, t_{k}$ will be chosen in the very end of the proof).
$\underline{k=1}$ : By Lemma 3, there exists $m_{1} \in \mathbb{N}$ such that

$$
\frac{m_{1}^{2}}{q(0)}<\Lambda_{\tilde{q}, m_{1}}<\frac{m_{1}^{2}}{q(1 / 4)}
$$

(Recall that $\max \widetilde{q}=\widetilde{q}(0)=q(0))$. Then again by the lemma and by the monotonicity property for eigenvalues $(q \geqslant \widetilde{q})$, we have

$$
\frac{m_{1}^{2}}{q(0)}<\Lambda_{q, m_{1}} \leqslant \Lambda_{\tilde{q}, m_{1}}<\frac{m_{1}^{2}}{q(1 / 4)} .
$$

Let $M_{1}>1$ be such that

$$
q\left(4^{-M_{1}}\right)=\frac{m_{1}^{2}}{\Lambda_{q, m_{1}}} .
$$

Now choose $\varepsilon_{1}$ such that for any $t_{1} \in\left[1, M_{1}\right]$ and $\tau_{1} \in\left[-\varepsilon_{1}, \varepsilon_{1}\right]$ we have $\widetilde{q} \leqslant q_{S, \tau} \leqslant q$ on $\mathbb{R}$, where $S=\left\{t_{1}\right\}, \tau: S \rightarrow \mathbb{R}, \tau\left(t_{1}\right)=\tau_{1}$.

The inductive step: Assume that we have chosen

$$
m_{1}, \ldots, m_{k-1}, \quad \varepsilon_{1}, \ldots, \varepsilon_{k-1}, \quad M_{1}, \ldots, M_{k-1}
$$

and now choose $m_{k}, \varepsilon_{k}, M_{k}$. By Lemma 3, there exists $m_{k} \in \mathbb{N}$ such that

$$
\frac{m_{k}^{2}}{q(0)}<\Lambda_{\tilde{q}, m_{k}}<\frac{m_{k}^{2}}{q\left(4^{-1-M_{k-1}}\right)}
$$

Then, again by the lemma and by the monotonicity property for eigenvalues, we have

$$
\frac{m_{k}^{2}}{q(0)}<\Lambda_{q, m_{k}} \leqslant \Lambda_{\tilde{q}, m_{k}}<\frac{m_{k}^{2}}{q\left(4^{-1-M_{k-1}}\right)} .
$$

Now let $M_{k}>M_{k-1}+1$ be such that

$$
q\left(4^{-M_{k}}\right)=\frac{m_{k}^{2}}{\Lambda_{q, m_{k}}},
$$

and let $\varepsilon_{k}>0$ be such that for any $t_{k} \in\left[M_{k-1}+1, M_{k}\right]$ and any $\tau_{k} \in\left[-\varepsilon_{k}, \varepsilon_{k}\right]$ we have $\widetilde{q} \leqslant q_{S, \tau} \leqslant q$ on $\mathbb{R}$, where $S=\left\{t_{k}\right\}$ and $\tau: S \rightarrow \mathbb{R}, \tau\left(t_{k}\right)=\tau_{k}$. This completes inductive construction of the sequences $\left(m_{k}\right),\left(\varepsilon_{k}\right),\left(M_{k}\right)$.

Recall that $M_{0}=0$. Let $k \in \mathbb{N}$, and let

$$
P:=\left[M_{0}+1, M_{1}\right] \times\left[M_{1}+1, M_{2}\right] \times \cdots \times\left[M_{k-1}+1, M_{k}\right] \subset \mathbb{R}^{k} .
$$

Let $\tau_{1}, \ldots, \tau_{k} \in \mathbb{R}$ be such that $\left|\tau_{i}\right| \leqslant \varepsilon_{i}, i=1,2, \ldots, k$. Now define the map $F: P \rightarrow \mathbb{R}^{k}$ as follows. For any $\left(t_{1}, \ldots, t_{k}\right) \in P$, set $S:=\left\{t_{1}, \ldots, t_{k}\right\}$, let $\tau: S \rightarrow \mathbb{R}$ be the function such that $\tau\left(t_{i}\right)=\tau_{i}, i=1,2, \ldots k$, and denote $Q=q_{S, \tau}$. We have $\widetilde{q} \leqslant Q \leqslant q$, hence for each $1 \leqslant i \leqslant k$,

$$
\frac{m_{i}^{2}}{q\left(4^{-M_{i}}\right)}=\Lambda_{q, m_{i}} \leqslant \Lambda_{Q, m_{i}} \leqslant \Lambda_{\tilde{q}, m_{i}}<\frac{m_{i}^{2}}{q\left(4^{-1-M_{i-1}}\right)},
$$

and therefore there exists a unique $s_{i} \in\left(M_{i-1}+1, M_{i}\right]$ such that

$$
q\left(4^{-s_{i}}\right)=\frac{m_{i}^{2}}{\Lambda_{Q, m_{i}}} .
$$

Then we put $F\left(t_{1}, \ldots, t_{k}\right)=\left(s_{1}, \ldots, s_{k}\right)$.
Note first, that $F$ is continuous (this follows from the continuous dependence of the first Dirichlet eigenvalue $\Lambda_{Q, m}$ on the function $Q$ ). Secondly, for any $\left(t_{1}, \ldots, t_{k}\right) \in P$ we have $\left(s_{1}, \ldots, s_{k}\right)=F\left(t_{1}, \ldots, t_{k}\right) \in P$. That is, $F(P) \subset P$. Therefore, by the Brower fixed point theorem, there exists a point $\left(t_{1}, \ldots, t_{k}\right) \in P$ for which $F\left(t_{1}, \ldots, t_{k}\right)=\left(t_{1}, \ldots, t_{k}\right)$, and hence by Remark 5 we have

$$
Q\left(4^{-t_{i}} x_{\infty}\right)=q\left(4^{-t_{i}}\right)=\frac{m_{i}^{2}}{\Lambda_{Q, m_{i}}}
$$

for $i=1,2, \ldots, k$. This finishes the proof of the proposition.
2.6. Completing the proof of Theorem 1. We fix the sequences $\left(m_{i}\right),\left(M_{i}\right)$, and $\left(\varepsilon_{i}\right)$ as in Proposition 4 For any $k \in \mathbb{N}$ and any $\tau_{1}, \ldots, \tau_{k}$ with $\left|\tau_{i}\right| \leq \varepsilon_{i}$, Proposition 4 provides us with the values $t_{i}(k) \in\left[M_{i-1}+1, M_{i}\right], 1 \leq i \leq k$ such that, putting $S(k)=$ $\left\{t_{1}(k), \ldots, t_{k}(k)\right\}, Q_{k}=q_{S(k), \tau}$, we get

$$
\Lambda_{Q_{k}, m_{i}} Q_{k}\left(4^{-t_{i}} x_{\infty}\right)=m_{i}^{2}, \quad i=1, \ldots, k
$$

Now, we let $k \rightarrow \infty$. Using a diagonal argument, we assume that, for each $i \in \mathbb{N}$, $t_{i}(k) \rightarrow t_{i} \in\left[M_{i-1}+1, M_{i}\right]$. Then we set $S=\left\{t_{1}, t_{2}, \ldots\right\}$. Making the values $\tau_{i}$ sufficiently small, we note that $Q_{k}=q_{S(k), \tau} \rightarrow q_{S, \tau}$ in the $C^{\infty}$-topology, and let $Q=q_{S, \tau}$. At last, recalling that the first eigenvalue $\Lambda_{Q, m}$ of the Dirichlet problem (3) continuously depends on $Q$, we arrive at the following corollary:

Corollary 6. There exists a sequence $m_{1}, m_{2}, \ldots \in \mathbb{N}$, an infinite set $S=\left\{t_{1}, t_{2}, \ldots\right\} \subset$ $[1, \infty)$, with $t_{i+1} \geqslant t_{i}+1$, and a function $\tau: S \rightarrow \mathbb{R} \backslash\{0\}$, such that the function $Q:=$ $q_{S, \tau}: \mathbb{R} \rightarrow \mathbb{R}$ is $C^{\infty}$-smooth and satisfies

$$
\Lambda_{Q, m_{i}} Q\left(4^{-t_{i}} x_{\infty}\right)-m_{i}^{2}=0, \quad i \in \mathbb{N}
$$

Now we will readily finish off the proof of Theorem 1. Let $Q$ be as in the corollary. Recall that the function $h$ makes infinitely many rapidly decreasing oscillations near $x_{\infty}$ (the function $h$ and the point $x_{\infty}$ were chosen in Section 2.3, c)). Hence for each $i$, the function $Q(x)$ makes infinitely many rapidly decreasing oscillations near $4^{-t_{i}} x_{\infty}$.

Fix $i$. Since the space of solutions of (4) is 2 -dimensional (by Lemma 2), we can always find a solution of

$$
\left\{\begin{array}{l}
u: \mathbb{R} \rightarrow \mathbb{R} \text { is 8-periodic }  \tag{5}\\
u^{\prime \prime}(x)+\left(\Lambda_{Q, m_{i}} Q(x)-m_{i}^{2}\right) u(x)=0,
\end{array}\right.
$$

denoted by $u_{i}(x)$, such that $u_{i}\left(4^{-t_{i}} x_{\infty}\right)>0$ and $u_{i}^{\prime}\left(4^{-t_{i}} x_{\infty}\right)=0$. Applying our main observation (Section 2.2) with $K_{i}(x)=\Lambda_{Q, m_{i}} Q(x)-m_{i}^{2}$ and $x_{*}=4^{-t_{i}} x_{\infty}$, we conclude that $u_{i}$ has infinitely many isolated critical points near $4^{-t_{i}} x_{\infty}$. This finishes the proof of Theorem 1 .

## 3. Eigenfunctions with a level set having infinitely many connected COMPONENTS

Here we outline changes in our construction needed to obtain a sequence of eigenfunctions having a level set with infinitely many connected components. For this, we replace condition (2) in Section 2.2 by a stronger one:
$\left(2^{\prime}\right) K: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that $\left.(-1)^{i} K\right|_{\left(x_{i+1}, x_{i}\right)}<0$ for every $i$, and

$$
\left|\int_{x_{i+1}}^{x_{i}}\left(x_{i}-x\right) K(x) d x\right| \geq C \sum_{j=i+1}^{\infty}\left|\int_{x_{j+1}}^{x_{j}} K(x) d x\right|
$$

with some numerical constant $C>1$.
As above, we denote by $u: \mathbb{R} \rightarrow \mathbb{R}$ a solution to $u^{\prime \prime}+K u=0$ with $u\left(x_{*}\right)>0, u^{\prime}\left(x_{*}\right)=0$. Choose $j_{0}$ so that $x_{j_{0}}-x_{*}<1$, and for every $x, y \in\left[x_{*}, x_{j_{0}}\right], 1 / C^{\prime} \leq u(x) / u(y) \leq C^{\prime}$ with $1<C^{\prime}<C$. Then it is not difficult to check that, for $j \geq j_{0}$, we have
(i) $\operatorname{sign}\left(u^{\prime}\left(x_{j}\right)\right)=(-1)^{j}$;
(ii) each interval $\left(x_{j+1}, x_{j}\right)$ contains exactly one critical point $\xi_{j}$ of $u$; it is a local maximum of $u$ if $j$ is odd, and a local minimum otherwise;
(iii) $\operatorname{sign}\left(u\left(x_{j}\right)-u\left(x_{j+1}\right)\right)=(-1)^{j}$;
(iv) $\operatorname{sign}\left(u\left(\xi_{j}\right)-u\left(x_{*}\right)\right)=(-1)^{j-1}$.

We conclude that if $\left(\eta_{j}\right)_{j \geq j_{0}}$ are solutions to the equation $u=u\left(x_{*}\right)$ on the interval $\left(x_{*}, x_{j_{0}}\right.$ ], then the sequences $\left(\eta_{j}\right)$ and $\left(\xi_{j}\right)$ interlace, i.e., $\eta_{j_{0}}>\xi_{j_{0}}>\eta_{j_{0}+1}>\xi_{j_{0}+1}>\eta_{j_{0}+2}>\ldots$

Now, let the sequences $\left(m_{i}\right)$ and $\left(t_{i}\right)$, the function $Q$, and the values $\Lambda_{Q, m_{i}}$ be the same as in Corollary 6. By $F_{m_{i}}$ we denote a solution of the 8 -periodic problem $F^{\prime \prime}+$ $\left(\Lambda_{Q, m_{i}} Q-m_{i}^{2}\right) F=0$ satisfying $F_{m_{i}}\left(4^{-t_{i}} x_{\infty}\right)>0$, and $F_{m_{i}}^{\prime}\left(4^{-t_{i}} x_{\infty}\right)=0$. Then the functions $\varphi_{i}(x, y)=F_{m_{i}}(x) \cos m_{i} y$ are Laplacian eigenfunctions on the torus $\mathbb{T}^{2}$ equipped with the Riemannian metric $\mathrm{d} s^{2}=Q(x)\left(\mathrm{d} x^{2}+\mathrm{d} y^{2}\right)$. Then, it is not difficult to see that, for each of these eigenfunctions, the level sets $\varphi_{i}=L_{i}$ with $L_{i}=F_{m_{i}}\left(4^{-t_{i}} x_{\infty}\right)$, have infinitely many connected components.

Indeed, fix $i$, take an odd $j \geq j_{0}(i)$, and consider the rectangle $\mathcal{R}_{j}=\left\{\eta_{j+1} \leq x \leq\right.$ $\left.\eta_{j},|y| \leq \pi /\left(2 m_{i}\right)\right\}$. Note that $\varphi_{i}>L_{i}$ on the interval $\left\{\eta_{j+1}<x<\eta_{j}, y=0\right\}$, and that $\varphi_{i}=L_{i}$ at the end points of this interval. Furthermore, for each $x \in\left[\eta_{j+1}, \eta_{j}\right]$, the function $y \mapsto \varphi_{i}(x, y)$ decays to 0 when $|y|$ increases from 0 to $\pi /\left(2 m_{i}\right)$. We conclude that, for each odd $j \geq j_{0}(i)$, the set $\left\{\varphi_{i}=L_{i}\right\} \cap \mathcal{R}_{j}$ is a topological circle. Clearly, these sets are disjoint for distinct odd $j \geq j_{0}(i)$. We are done.

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