

Fermion-scalar conformal blocks

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ABSTRACT: We compute the conformal blocks associated with scalar-scalar-fermion-fermion 4-point functions in 3D CFTs. Together with the known scalar conformal blocks, our result completes the task of determining the so-called ‘seed blocks’ in three dimensions. Conformal blocks associated with 4-point functions of operators with arbitrary spins can now be determined from these seed blocks by using known differential operators.

KEYWORDS: Conformal and W Symmetry, Nonperturbative Effects

ARXIV EPRINT: [1511.01497](https://arxiv.org/abs/1511.01497)

Contents

1	Introduction	1
2	Mixed fermion-scalar conformal blocks	2
2.1	Embedding of spinor fields	2
2.2	The three-point function	4
2.3	The four-point function	4
2.4	Recursion relations	5
2.5	Selection rules	6
3	Results	7
3.1	Blocks in the limit $\Delta \rightarrow \infty$	8
3.2	Poles and residues	8
4	Summary and discussion	10
A	The Casimir equation	11
B	Determining OPE coefficients	14

1 Introduction

The conformal bootstrap [1–3] has recently resurfaced as a powerful numerical tool for constraining the algebra of local operators of conformal field theories (CFTs) in $D > 2$ spacetime dimensions [4–42]. After achieving a great degree of success in constraining four-point functions of scalar operators in various dimensions, in [40] the numerical bootstrap was first applied to four-point functions of spin-1/2 fermions in 3D CFTs. The preliminary numerical study of [40] suggests that the fermionic bootstrap may give access to CFTs that are otherwise hard to study by analyzing only correlators of scalars, such as the Gross-Neveu-Yukawa models or the $\mathcal{N} = 1$ supersymmetric Ising model.

One natural way to push this numerical program further is to study the bootstrap constraints associated with a mixed system of four-point functions of scalars and spin-1/2 fermions. Indeed, so far the numerical bootstrap has been used to place constraints on operators that appear in the operator product expansions (OPEs) of either two scalars or of two fermions. All the operators in these OPEs are bosonic. In contrast, the mixed scalar-fermion bootstrap would allow us to also place constraints on the fermionic operators that appear in the OPE of a scalar with a fermion, such as the supercurrent in a supersymmetric theory. The goal of this paper is to compute the conformal blocks that are necessary in order to apply the bootstrap to such a mixed system of correlators.

In general, the main theoretical obstruction to applying the bootstrap to four-point functions of operators with spin is that the associated conformal blocks are not always known. However, there has been some progress in our understanding of spinning conformal blocks since the original work [43–45] of Dolan and Osborn on conformal blocks of scalar four-point functions. In particular, at least in some cases it is now known that spinning conformal blocks can be obtained by acting with certain differential operators on the conformal blocks of a distinguished subset of four-point functions, which are known as “seed correlators.” It is straightforward to determine these differential operators [40, 46, 47], but the seed blocks are not known in general.¹

In $D \geq 4$ the full set of seed correlators is infinite, but in 3D, because of the simplicity of the Lorentz group, this set is finite. In fact, in 3D there are only two seed four-point functions: one with four scalars, and the other with two scalars and two spin-1/2 fermions. The blocks of the first seed correlator are well-known [13, 16, 27, 51], and can be used to determine the blocks corresponding to any four-point function of four bosonic or four fermionic operators using the differential operators constructed in [40, 46]. The blocks of the second seed correlator are of two types. The first type corresponds to an exchange of a bosonic operator that appears both in the OPE of the two scalars and in the OPE of the two fermions; these blocks, too, can be obtained from the scalar conformal blocks using the differential operators in [40, 46]. The second type of blocks corresponds to an exchange of a fermionic operator that appears in the OPE of a scalar with a fermion. These blocks are not currently known and will be determined in this paper. Using the same differential operators as in [40, 46], they can be used to further determine the blocks corresponding to a fermionic exchange in a four-point function of two bosonic and two fermionic operators of arbitrary spin.

As done in [16, 27] for scalar blocks, we will determine a recursion formula expression for our fermion-scalar blocks. We start by using the embedding space formalism, following the conventions presented in [40], to constrain the generic form of the three-point function $\langle \Psi(x_1)\Phi(x_2)\mathcal{O}_\ell(x) \rangle$ (section 2.2) and of the four-point function $\langle \Psi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Psi_4(x_4) \rangle$ (section 2.3). In sections 2.4–2.5, we obtain an ansatz for a recursion formula of the conformal blocks. In section 3, we iteratively solve each order of the recursion using the Casimir equations presented in appendix A. We end with a brief discussion in section 4 that includes the decomposition of a free theory four-point function in terms of our blocks. Finally, in appendix B, we relate the coefficients that appear in the conformal block expansion to products of OPE coefficients as defined through our normalization conventions.

2 Mixed fermion-scalar conformal blocks

2.1 Embedding of spinor fields

In this section we set up our conventions for embedding 3D spinor fields into a 5D space by following [40]. We work in flat 3D spacetime with metric $\eta^{\mu\nu} = \text{diag}(-1, 1, 1)$ and coordinates x^μ ($\mu, \nu = 0, 1, 2$). The 3D conformal group $\text{SO}(3, 2)$ acts linearly in 5D embedding

¹For examples of non-trivial seed blocks that are known, see [48–50].

space with metric $\eta^{AB} = \eta_{AB} = \text{diag}(-1, 1, 1, 1, -1)$ and coordinates X^A ($A, B = 0, \dots, 4$). We will use light-cone coordinates $X^\pm = X^4 \pm X^3$, and write the components of X from now on as $X = (X^\mu, X^+, X^-)$.

The 3D spacetime is embedded in 5D as the projective null cone, defined to be the set of points satisfying $X \cdot X = \eta_{AB} X^A X^B = 0$ and identified by $X^A \sim \lambda X^A$. The exact relation between x^μ and X^A can be written as

$$x^\mu = \frac{X^\mu}{X^+}, \quad X = X^+(x^\mu, 1, x^2), \tag{2.1}$$

where $x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu$.

Let $\mathcal{O}^{\alpha_1 \dots \alpha_{2\ell}}(x)$ be a dimension Δ primary operator in the spin- ℓ irrep of the $\text{Spin}(2, 1) \simeq \text{Sp}(2, \mathbf{R})$ double-cover of the 3D Lorentz group.² To suppress spinor indices on operators we introduce commuting polarizations s_α and define $\mathcal{O}_\ell(x, s) \equiv s_{\alpha_1} \dots s_{\alpha_{2\ell}} \mathcal{O}^{\alpha_1 \dots \alpha_{2\ell}}(x)$. The embedding space parent of $\mathcal{O}_\ell(x, s)$ is given by

$$\mathcal{O}_\ell(X, S) \equiv S_{I_1} \dots S_{I_{2\ell}} \mathcal{O}^{I_1 \dots I_{2\ell}}(X), \tag{2.2}$$

where S_I are embedding space spinor polarizations transforming in the fundamental of $\text{Sp}(4, \mathbf{R}) \simeq \text{Spin}(3, 2)$. The operator $\mathcal{O}_\ell(X, S)$ has the following homogeneity properties:

$$\mathcal{O}_\ell(aX, bS) = a^{-(\Delta+\ell)} b^{2\ell} \mathcal{O}_\ell(X, S). \tag{2.3}$$

The projection $\mathcal{O}_\ell(X, S) \rightarrow \mathcal{O}_\ell(x, s)$ is given by the prescription

$$\mathcal{O}_\ell(X, S) = \frac{1}{(X^+)^{\Delta}} \mathcal{O}_\ell(x, s), \tag{2.4}$$

where X is related to x according to (2.1), and S is related to s through

$$S_I = \sqrt{X^+} \begin{pmatrix} s_\alpha \\ -x^\beta \gamma s^\gamma \end{pmatrix}. \tag{2.5}$$

Note that S_I satisfies the transversality condition

$$S_I X^I{}_J = 0. \tag{2.6}$$

The form of correlation functions in embedding space is constrained by the $\text{SO}(3, 2)$ symmetry (which acts linearly on the embedding coordinates), homogeneity (2.3), and transversality (2.6).

² $\text{Sp}(2, \mathbf{R})$ indices $\alpha, \beta = 1, 2$ are raised and lowered with the symplectic form $\Omega_{\alpha\beta} = \Omega^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and are suppressed in contractions using the convention $\psi\chi \equiv \psi_\alpha \chi^\alpha$. We turn vectors into bi-spinors using $x^\alpha{}_\beta \equiv x^\mu (\gamma_\mu)^\alpha{}_\beta$, where $\gamma^\mu \equiv (i\sigma^2, \sigma^1, \sigma^3)$. Similarly, embedding $\text{Sp}(4, \mathbf{R})$ spinor indices, $I, J = 1, \dots, 4$, are raised and lowered with $\Omega_{IJ} = \Omega^{IJ} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$, and contracted as $\Psi\Lambda \equiv \Psi_I \Lambda^I$. Embedding vectors X^A are written as embedding bi-spinors defined by $X^I{}_J \equiv X^A (\Gamma_A)^I{}_J$, where $\Gamma^A \equiv (\gamma^2 \otimes \gamma^0, \mathbb{1} \otimes \gamma^1, \mathbb{1} \otimes \gamma^2, \gamma^0 \otimes \gamma^0, \gamma^1 \otimes \gamma^0)$.

2.2 The three-point function

Consider the 3-point function of a spin- $\frac{1}{2}$ fermion primary $\Psi(X_1, S_1)$ of dimension Δ_1 , a scalar $\Phi(X_2)$ of dimension Δ_2 , and a third fermion $\mathcal{O}_\ell(X_3, S_3)$ of half-odd integer spin ℓ and dimension Δ_3 . Conformal invariance restricts the form of this 3-point function to be

$$\langle \Psi(X_1, S_1) \Phi(X_2) \mathcal{O}_{\Delta_3, \ell}(X_3, S_3) \rangle = \frac{\lambda_{\Psi\Phi\mathcal{O}}^+ r_+ + \lambda_{\Psi\Phi\mathcal{O}}^- r_-}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3-\ell+1/2}{2}} X_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+\ell-1/2}{2}} X_{31}^{\frac{\Delta_3+\Delta_1-\Delta_2+\ell+1/2}{2}}}, \quad (2.7)$$

where $X_{ij} \equiv -2X_i \cdot X_j$, $\lambda_{\Psi\Phi\mathcal{O}}^\pm$ are independent OPE coefficients, and where the structures r_\pm can be chosen such that

$$r_+ = \frac{\langle S_1 S_3 \rangle \langle S_3 X_1 X_2 S_3 \rangle^{\ell-\frac{1}{2}}}{X_{12}^{\ell-\frac{1}{2}}}, \quad (2.8)$$

$$r_- = \frac{X_{13}^{1/2} \langle S_1 X_2 S_3 \rangle \langle S_3 X_1 X_2 S_3 \rangle^{\ell-\frac{1}{2}}}{X_{23}^{1/2} X_{12}^\ell}. \quad (2.9)$$

As in [40], the angle-brackets are defined by

$$\langle S_1 X_2 X_3 \dots S_n \rangle \equiv S_{1I} X_2^I X_3^J X_K^J \dots \Omega^{LM} S_{nM}. \quad (2.10)$$

The function multiplying $\lambda_{\Psi\Phi\mathcal{O}}^+$ is even under parity, $X_k \rightarrow -X_k$, while the one multiplying $\lambda_{\Psi\Phi\mathcal{O}}^-$ is odd. Throughout the rest of this paper, we will use \pm indices to track the parity of the 3-point structures, of the corresponding OPE coefficients, or of the conformal blocks. Furthermore, from unitarity, if the operators $\Psi(X_1, S_1)$ and $\mathcal{O}_{\Delta_3, \ell}(X_3, S_3)$ are real, then Fermi statistics requires that the OPE coefficients $\lambda_{\Psi\Phi\mathcal{O}}^\pm$ are pure imaginary.

Projecting down to $\mathbf{R}^{2,1}$ from embedding space, we can write

$$\langle \Psi_1(x_1, s_1) \Phi_2(x_2) \mathcal{O}_{\Delta, \ell}(x_3, s_3) \rangle = \lambda_{\Psi\Phi\mathcal{O}}^+ \mathcal{R}_{\Delta, \ell}^+ + \lambda_{\Psi\Phi\mathcal{O}}^- \mathcal{R}_{\Delta, \ell}^-, \quad (2.11)$$

where the structures $\mathcal{R}_{\Delta, \ell}^\pm$ are obtained by applying the procedure described in section 2.1 to the structures in (2.7). We explicitly write $\mathcal{R}_{\Delta, \ell}^\pm$ in appendix B. The $\mathcal{R}_{\Delta, \ell}^\pm$ depend on all the quantities $\Delta_1, \Delta_2, \Delta, \ell$, but we have chosen to only highlight their dependence on Δ, ℓ in order to avoid cluttering.

2.3 The four-point function

Let $\Psi_{1,4}$ be spin- $\frac{1}{2}$ fermions of dimension $\Delta_{1,4}$, and $\Phi_{2,3}$ be dimension $\Delta_{2,3}$ scalars. The 4-point function of these operators can be written as

$$\langle \Psi_1(X_1, S_1) \Phi_2(X_2) \Phi_3(X_3) \Psi_4(X_4, S_4) \rangle = \left(\frac{X_{24}}{X_{14}} \right)^{\frac{\Delta_{12}}{2}} \left(\frac{X_{13}}{X_{14}} \right)^{\frac{\Delta_{43}}{2}} \frac{\sum_I t_I g^I(u, v)}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}}, \quad (2.12)$$

where u and v are conformally-invariant cross ratios defined as

$$u \equiv \frac{X_{12} X_{34}}{X_{13} X_{24}}, \quad v \equiv \frac{X_{32} X_{14}}{X_{13} X_{24}}, \quad (2.13)$$

and where the t_I ($I = 1, \dots, 4$) form a basis for the four independent tensor structures allowed by conformal invariance. We choose our basis to consist of the two parity even 4-point structures

$$t_1 = i \frac{\langle S_1 S_4 \rangle}{\sqrt{X_{14}}}, \quad t_2 = -i \frac{\langle S_1 X_2 X_3 S_4 \rangle}{\sqrt{X_{12} X_{34} X_{23}}}, \quad (2.14)$$

and the two parity odd 4-point structures

$$t_3 = -i \frac{\langle S_1 X_2 S_4 \rangle}{\sqrt{X_{12} X_{24}}}, \quad t_4 = -i \frac{\langle S_1 X_3 S_4 \rangle}{\sqrt{X_{13} X_{34}}}. \quad (2.15)$$

It will sometimes be convenient to use the coordinates (r, η) introduced in [51] instead of (u, v) in (2.13). The relation between them is

$$u = \frac{16r^2}{(1+r^2+2r\eta)^2}, \quad v = \frac{(1+r^2-2r\eta)^2}{(1+r^2+2r\eta)^2}. \quad (2.16)$$

2.4 Recursion relations

The functions $g^I(u, v)$ can be expanded in conformal blocks as follows

$$g^I(u, v) = \sum_{\mathcal{O}} \sum_{a,b=\pm} \lambda_{\Psi_1 \Phi_2 \mathcal{O}}^a \lambda_{\Psi_4 \Phi_3 \mathcal{O}}^b g_{\Delta, \ell}^{I, ab}(u, v). \quad (2.17)$$

Here, Δ, ℓ label the dimension and spin of the exchanged operator \mathcal{O} . Note that ℓ is a half-odd integer in the present case. The indices a, b label three-point tensor structures (2.11). Furthermore, as described in appendix B, the OPE coefficients $\lambda_{\Psi_i \Phi_j \mathcal{O}}^a$ are the same as those appearing in the three-point function (2.7) for an appropriate normalization of the two point function of \mathcal{O} .

The conformal blocks are defined in terms of a sum over states in an individual conformal multiplet as

$$\begin{aligned} & \frac{1}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{\Delta_{12}}{2}} \left(\frac{X_{13}}{X_{14}} \right)^{\frac{\Delta_{43}}{2}} \sum_{I,a,b} \lambda_{\Psi_1 \Phi_2 \mathcal{O}}^a \lambda_{\Psi_4 \Phi_3 \mathcal{O}}^b t_I g_{\Delta, \ell}^{I, ab}(u, v) \\ &= \sum_{\alpha=\mathcal{O}, P\mathcal{O}, \dots} \frac{\langle \Psi_1(X_1, S_1) \Phi_2(X_2) | \alpha \rangle \langle \alpha | \Phi_3(X_3) \Psi_4(x_4, S_4) \rangle}{\langle \alpha | \alpha \rangle}, \end{aligned} \quad (2.18)$$

where α runs over primaries $|\alpha\rangle = |\mathcal{O}\rangle$ and descendants $|\alpha\rangle = P|\mathcal{O}\rangle, \dots$

This expression can be understood in terms of its analytic properties in Δ . Here, we follow the discussion and notation of [52]. Poles in Δ occur when one of the descendant states $|\alpha\rangle$ becomes a primary. Indeed, this is only possible if $|\alpha\rangle$ is null (otherwise acting on $|\alpha\rangle$ with a special conformal generator would produce a state with nonvanishing norm), so that the corresponding denominator on the right-hand side of (2.18) vanishes. Descendants of $|\alpha\rangle$ are then also null states and each gives a contribution to the residue of the pole.

Specifically, there exist a differential operator \mathcal{D}_A such that the descendant state

$$|\mathcal{O}_A, s\rangle \equiv \mathcal{D}_A \mathcal{O}(x, s) |0\rangle \quad (2.19)$$

becomes a null primary when $\Delta \rightarrow \Delta_A^*$,

$$\langle \mathcal{O}_A, s | \mathcal{O}_A, s' \rangle = (\Delta - \Delta_A^*) Q_A^{-1} \mathcal{I}_{\ell_A}(s, s') + O((\Delta - \Delta_A^*)^2). \quad (2.20)$$

The operator \mathcal{O}_A has spin ℓ_A and dimension $\Delta + n_A$, where $n_A \geq 1$ is the degree of \mathcal{D}_A in ∂_x . Here, \mathcal{I}_{ℓ_A} is a canonical tensor structure for a two-point function of operators with spin ℓ_A . The constant Q_A depends on the differential operator \mathcal{D}_A .

When $\Delta = \Delta_A^*$, the operator \mathcal{O}_A transforms like a primary with dimension $\Delta_A \equiv \Delta_A^* + n_A$ and spin ℓ_A . Thus, its three-point function with $\Psi\Phi$ can be expressed in terms of our basis of three-point structures (2.11),

$$\mathcal{D}_A \mathcal{R}_{\Delta, \ell}^a \Big|_{\Delta = \Delta_A^*} = (M_A)_b^a \mathcal{R}_{\Delta_A, \ell_A}^b. \quad (2.21)$$

where $\mathcal{R}_{\Delta_A, \ell_A}^b$ depends on the scaling dimensions of both the fermionic (Ψ) and scalar (Φ) operators, while M_A depends on the difference in scaling dimensions, $\Delta_\Phi - \Delta_\Psi$.

The residue of a conformal block at $\Delta = \Delta_A^*$ comes from the contribution of \mathcal{O}_A and all of its descendants (which also become null as $\Delta \rightarrow \Delta_A^*$). Thus, it is proportional to a conformal block for \mathcal{O}_A . The constant of proportionality follows from (2.20) and (2.21),

$$g_{\Delta, \ell}^{I, ab} = \frac{(M_A^{(L)})_c^a Q_A (M_A^{(R)})_d^b}{\Delta - \Delta_A^*} g_{\Delta_A, \ell_A}^{I, cd} + O((\Delta - \Delta_A^*)^0), \quad (2.22)$$

where the superscripts L or R indicate that we compute M_A using the left three-point function $\langle \Psi_1 \Phi_2 \mathcal{O} \rangle$ or the right three-point function $\langle \Psi_4 \Phi_3 \mathcal{O} \rangle$. In our case, the left and right three-point functions involve the same kinds of operators, so the M_A are simply related by $\Psi_1 \Phi_2 \leftrightarrow \Psi_4 \Phi_3$. Since M_A only depends on the difference between the dimension of the scalar and fermion operator, we can write

$$M_A^{(L)} = M_A^{\Delta_{12}}, \quad M_A^{(R)} = M_A^{\Delta_{43}}. \quad (2.23)$$

In the case studied here, the only poles in the conformal blocks are simple poles of the form (2.22).³ Furthermore, after stripping off a factor r^Δ , the blocks have a finite limit as $\Delta \rightarrow \infty$. It follows that the blocks satisfy a recursion relation of the form

$$\begin{aligned} g_{\Delta, \ell}^{I, ab}(r, \eta) &= r^\Delta h_{\Delta, \ell}^{I, ab}(r, \eta), \\ h_{\Delta, \ell}^{I, ab}(r, \eta) &= h_{\infty, \ell}^{I, ab}(r, \eta) + \sum_A r^{n_A} \frac{(M_A^{(L)})_c^a Q_A (M_A^{(R)})_d^b}{\Delta - \Delta_A^*} h_{\Delta_A, \ell_A}^{I, cd}(r, \eta). \end{aligned} \quad (2.24)$$

(See (2.16) for the relation between (r, η) and the cross-ratios (u, v) defined in (2.13).)

2.5 Selection rules

Because we have chosen our tensor structures to have definite parity, the indices I, a, b satisfy selection rules, regardless of whether the CFT we are studying is invariant under parity. A parity even four-point structure $I_+ = 1, 2$ can only arise from a combination

³See [52] for a detailed discussion of what kind of poles can occur in more general situations.

of two parity even or two parity odd three-point structures ($ab = ++, --$). Similarly, a parity odd four-point structure $I_- = 3, 4$ can only arise from a combination of three-point structures of opposite parity ($ab = +-, -+$). Thus, the only nonzero blocks are

$$g_{\Delta,\ell}^{I_+,++}, \quad g_{\Delta,\ell}^{I_+,-}, \quad g_{\Delta,\ell}^{I_-,-}, \quad g_{\Delta,\ell}^{I_-,+}, \quad (I_+ = 1, 2, I_- = 3, 4). \quad (2.25)$$

The coefficients $(M_A)_b^a$ obey additional selection rules depending on the parity properties of the differential operator \mathcal{D}_A :

- If \mathcal{D}_A preserves parity, then only $(M_A)_+^+, (M_A)_-^-$ are nonzero.
- If \mathcal{D}_A changes parity, then only $(M_A)_-^+, (M_A)_+^-$ are nonzero.

For convenience, let us group the nonzero blocks into 2-vectors.

$$\mathbf{g}_{\Delta,\ell}^{I_+} \equiv \begin{pmatrix} g_{\Delta,\ell}^{I_+,++} \\ g_{\Delta,\ell}^{I_+,-} \end{pmatrix}, \quad \mathbf{g}_{\Delta,\ell}^{I_-} \equiv \begin{pmatrix} g_{\Delta,\ell}^{I_-,-} \\ g_{\Delta,\ell}^{I_-,+} \end{pmatrix}. \quad (2.26)$$

Our recursion relation then has the form

$$\mathbf{h}_{\Delta,\ell}^{I_\pm}(r, \eta) = \mathbf{h}_{\infty,\ell}^{I_\pm}(r, \eta) + \sum_A \frac{r^{n_A}}{\Delta - \Delta_A^*} \mathbf{c}_A^\pm \mathbf{h}_{\Delta_A,\ell_A}^{I_\pm}(r, \eta), \quad (2.27)$$

where $\mathbf{h}_{\Delta,\ell}^I$ is defined analogously to (2.26), and the 2×2 matrices \mathbf{c}_A^\pm are

$$\begin{aligned} \mathbf{c}_A^+ &\equiv Q_A \begin{pmatrix} (M_A^{\Delta_{12}})^+ (M_A^{\Delta_{43}})^+ & (M_A^{\Delta_{12}})^- (M_A^{\Delta_{43}})^+ \\ (M_A^{\Delta_{12}})^- (M_A^{\Delta_{43}})^- & (M_A^{\Delta_{12}})^- (M_A^{\Delta_{43}})^- \end{pmatrix}, \\ \mathbf{c}_A^- &\equiv Q_A \begin{pmatrix} (M_A^{\Delta_{12}})^+ (M_A^{\Delta_{43}})^- & (M_A^{\Delta_{12}})^+ (M_A^{\Delta_{43}})^+ \\ (M_A^{\Delta_{12}})^- (M_A^{\Delta_{43}})^+ & (M_A^{\Delta_{12}})^- (M_A^{\Delta_{43}})^+ \end{pmatrix}. \end{aligned} \quad (2.28)$$

If \mathcal{D}_A preserves parity then \mathbf{c}_A^\pm will be diagonal, while if \mathcal{D}_A changes parity then \mathbf{c}_A^\pm will be anti-diagonal.

3 Results

Since each operator in the four-point function (2.17) is an eigenvector of the Casimir operator of the conformal group, the functions $g^I(r, \eta)$ will have to satisfy a set of Casimir equations which we derive in appendix A. Due to our choice of structures, the Casimir equations decouple into two sets of equations (A.10), one corresponding to parity even blocks and one to parity odd blocks.

Our strategy for computing the blocks $g^{I,ab}(r, \eta)$ is as follows: using the recursion ansatz obtained from the state-operator correspondence (2.27), which we plug into (A.10), we firstly solve for $\mathbf{h}_{\infty,\ell}^I(r, \eta)$ and then compute the residue matrices \mathbf{c}_A^\pm to all orders in r .

3.1 Blocks in the limit $\Delta \rightarrow \infty$

As explained in appendix A, the functions $\mathbf{h}_{\infty,\ell}^I$ can be determined by solving the Casimir equation in the limit $\Delta \rightarrow \infty$. The equation is automatically satisfied at order Δ^2 . The r -dependence of the solution is fixed by solving the equation at order Δ^1 . The η -dependence can be determined by examining the equation at order Δ^0 and small r . This leaves two linearly independent solutions for $\mathbf{h}_{\infty,\ell}^{1,2}$ and two linearly independent solutions for $\mathbf{h}_{\infty,\ell}^{3,4}$. We fix the correct linear combinations by matching to the OPE limit $r \rightarrow 0, v \rightarrow 1$ in appendix B. The result is

$$\begin{aligned}
 \mathbf{h}_{\infty,\ell}^1(r, \eta) &= \frac{v^{\frac{1}{4}(\Delta_{12} + \Delta_{43})}}{\sqrt{1-r^2}\sqrt{1+r^2+2r\eta}} \begin{pmatrix} \frac{-P_{\ell-1/2}^{(0,1)}(\eta)}{1-r} - \frac{P_{\ell-1/2}^{(1,0)}(\eta)}{1+r} \\ \frac{-P_{\ell-1/2}^{(0,1)}(\eta)}{1-r} + \frac{P_{\ell-1/2}^{(1,0)}(\eta)}{1+r} \end{pmatrix}, \\
 \mathbf{h}_{\infty,\ell}^2(r, \eta) &= \frac{v^{\frac{1}{4}(\Delta_{12} + \Delta_{43})}}{\sqrt{1-r^2}\sqrt{1+r^2+2r\eta}} \begin{pmatrix} \frac{P_{\ell-1/2}^{(0,1)}(\eta)}{1-r} - \frac{P_{\ell-1/2}^{(1,0)}(\eta)}{1+r} \\ \frac{P_{\ell-1/2}^{(0,1)}(\eta)}{1-r} + \frac{P_{\ell-1/2}^{(1,0)}(\eta)}{1+r} \end{pmatrix}, \\
 \mathbf{h}_{\infty,\ell}^3(r, \eta) &= \frac{v^{\frac{1}{4}(\Delta_{12} + \Delta_{43})}}{\sqrt{1-r^2}\sqrt{1+r^2-2r\eta}} \begin{pmatrix} \frac{P_{\ell-1/2}^{(0,1)}(\eta)}{1+r} - \frac{P_{\ell-1/2}^{(1,0)}(\eta)}{1-r} \\ \frac{P_{\ell-1/2}^{(0,1)}(\eta)}{1+r} + \frac{P_{\ell-1/2}^{(1,0)}(\eta)}{1-r} \end{pmatrix}, \\
 \mathbf{h}_{\infty,\ell}^4(r, \eta) &= \frac{v^{\frac{1}{4}(\Delta_{12} + \Delta_{43})}}{\sqrt{1-r^2}\sqrt{1+r^2-2r\eta}} \begin{pmatrix} \frac{P_{\ell-1/2}^{(0,1)}(\eta)}{1+r} + \frac{P_{\ell-1/2}^{(1,0)}(\eta)}{1-r} \\ \frac{P_{\ell-1/2}^{(0,1)}(\eta)}{1+r} - \frac{P_{\ell-1/2}^{(1,0)}(\eta)}{1-r} \end{pmatrix}, \tag{3.1}
 \end{aligned}$$

where $P_n^{(a,b)}(\eta)$ are Jacobi polynomials.

3.2 Poles and residues

The Casimir equations (A.10) can be solved order by order in a series expansion in r and η . The full solution presented below was guessed based on the first few terms in this expansion. It was then verified that this guess solves the Casimir equations to a very high order in the expansion in r and η .

In practice, the procedure for solving (A.10) in a series expansion and extracting the poles and residues appearing in (2.27) is as follows. A solution of (A.10) for the functions $\mathbf{h}_{\Delta,\ell}^{I\pm} = r^{-\Delta} \mathbf{g}_{\Delta,\ell}^{I\pm}$ has an expansion

$$\mathbf{h}_{\Delta,\ell}^{I\pm}(r, \eta) = \sum_{k=0}^{\infty} \sum_{j=0}^{k+\ell-\frac{1}{2}} \mathbf{a}_{k,j}^{I\pm}(\Delta, \ell) r^k \eta^j, \tag{3.2}$$

where the dependence on the external operator dimensions Δ_{12} and Δ_{34} is suppressed. The zeroth order coefficients $\mathbf{a}_{0,0}^{I\pm}(\Delta)$ are fixed by the explicit solution for $\mathbf{h}_{\infty,\ell}^{I\pm}(r, \eta)$ given in (3.1). For a given ℓ we solve for the coefficients $\mathbf{a}_{k,j}^{I\pm}$ up to a finite order $k < \Lambda$, and rewrite the result in the form of the recursion formula (2.27). In particular, the locations

A	n_A	Δ_A^*	ℓ_A	parity(\mathcal{D}_A)	values of k
$(1, k)$	k	$1 - \ell - k$	$\ell + k$	+	$k = 1, 2, \dots,$
$(2, k)$	k	$(3 - k)/2$	ℓ	-	$k = 1, 3, 5, \dots,$
$(3, k)$	k	$2 + \ell - k$	$\ell - k$	+	$k = 1, 2, \dots, \ell - \frac{1}{2}.$

Table 1. Table of poles in Δ for the fermion-scalar conformal blocks. There are three series of poles, which we call $A = (1, k), (2, k), (3, k)$. The integer k ranges over the values shown. n_A is the level of the descendant corresponding to the pole, Δ_A^* gives the dimension at which the pole appears, and ℓ_A is the spin of the descendant associated with the pole. parity(\mathcal{D}_A) indicates whether the differential operator \mathcal{D}_A preserves or changes parity.

of the poles Δ_A^* are simply read off from the coefficients $\mathbf{a}_{k,j}^{\pm}(\Delta, \ell)$, while the spins ℓ_A are inferred from the degree of the polynomials in η that multiply each pole.

Following the above procedure, we found three series of poles (two infinite and one finite), in analogy with the case of scalar blocks [16, 27, 52]. They are listed in table 1. We now describe the differential operators and residues associated with each pole.⁴

- *First series of poles:* the differential operators with label $(1, k)$ have the form

$$\mathcal{D}_{1,k} \propto (s\partial_x s)^k, \quad (3.3)$$

where $s\partial_x s = s_\alpha(\partial_x)^\alpha \beta s^\beta$.⁵ By solving the Casimir equation, we find

$$\begin{aligned} Q_{1,k} &= -\frac{(-4)^k k \left(\frac{3}{2} + \ell\right)_k}{(k!)^2 (1 + \ell)_k}, \\ (M_{1,k}^\Delta)_+^\pm &= \left(\frac{-2k - 2\Delta + 1}{4}\right)_k, & (M_{1,k}^\Delta)_-^\pm &= \left(\frac{-2k - 2\Delta + 3}{4}\right)_k, \\ (M_{1,k}^\Delta)_-^\pm &= (M_{1,k}^\Delta)_+^\pm = 0, \end{aligned} \quad (3.4)$$

where the Pochhammer symbol is defined by $(a)_n = \Gamma(a + n)/\Gamma(a)$. The coefficients $(M_{1,k}^\Delta)_-^\pm, (M_{1,k}^\Delta)_+^\pm$ vanish because $\mathcal{D}_{1,k}$ is parity even. The residue matrices \mathbf{c}_1^\pm are obtained by plugging the above into (2.28). They are diagonal.

- *Second series of poles:* the differential operators with label $(2, k)$ have the form

$$\mathcal{D}_{2,k} \propto (s\partial_x \partial_s) \sum_m a_m \partial_x^{2m} ((s\partial_x s)(\partial_s \partial_x \partial_s))^{n-m}, \quad (3.5)$$

where the coefficients a_m could be determined by demanding that $\mathcal{D}_{2,k} \mathcal{O}_{\Delta, \ell}$ is primary when $\Delta = \Delta_{2,k}^*$. We find

$$Q_{2,k} = \frac{4^k k \Gamma\left(\frac{k}{2}\right)^2}{2\pi \Gamma\left(\frac{1+k}{2}\right)^2 \left(\frac{1}{2} + \ell - \frac{k}{2}\right)_k \left(\frac{3}{2} + \ell - \frac{k}{2}\right)_k},$$

⁴Because we obtained the residues from the Casimir equation, and not by explicitly constructing null descendants, we cannot immediately fix the normalization of the differential operators \mathcal{D}_A , and the corresponding values of Q_A, M_A . Different normalization conventions will rescale Q_A, M_A in such a way that the residue is unaffected. Our convention is such that Q_A, M_A are as simple as possible.

⁵The labels $(1, k), (2, k), (3, k)$ correspond to descendants of type I, III, II in [52].

$$\begin{aligned}
(M_{2,k}^\Delta)_-^\pm &= \left(\frac{-k + 2\ell - 2\Delta + 2}{4} \right)_{\frac{k+1}{2}} \left(\frac{-k + 2\ell + 2\Delta + 4}{4} \right)_{\frac{k-1}{2}}, \\
(M_{2,k}^\Delta)_+^\pm &= \left(\frac{-k - 2\ell - 2\Delta}{4} \right)_{\frac{k+1}{2}} \left(\frac{-k - 2\ell + 2\Delta + 2}{4} \right)_{\frac{k-1}{2}}, \\
(M_{2,k}^\Delta)_+^\pm &= (M_{2,k}^\Delta)_-^\pm = 0,
\end{aligned} \tag{3.6}$$

where the matrix \mathbf{c}_2^\pm is anti-diagonal for all values of k and ℓ .

- *Third series of poles:* the differential operators with label $(3, k)$ have the form

$$\mathcal{D}_{3,k} \propto (\partial_s \partial_x \partial_s)^k. \tag{3.7}$$

We find

$$\begin{aligned}
Q_{3,k} &= -\frac{(-4)^k k \left(\frac{1}{2} - k + \ell\right)_k}{(k!)^2 (1 - k + \ell)_k}, \\
(M_{3,k}^\Delta)_+^\pm &= \left(\frac{-2k - 2\Delta + 3}{4} \right)_k, & (M_{3,k}^\Delta)_-^\pm &= \left(\frac{-2k - 2\Delta + 1}{4} \right)_k, \\
(M_{3,k}^\Delta)_-^\pm &= (M_{3,k}^\Delta)_+^\pm = 0,
\end{aligned} \tag{3.8}$$

where the matrix \mathbf{c}_3^\pm is thus diagonal for all values of k and ℓ .

4 Summary and discussion

Our main result in this work is an explicit recursion relation for the conformal blocks corresponding to the exchange of a half-odd integer spin conformal primary that can be used to compute the blocks numerically. This recursion formula takes the form (2.27), with the various quantities appearing in this formula determined in section 3.

As a check on our results, let us decompose a simple free theory four-point function into conformal blocks. Consider the theory of a free Majorana fermion ψ and free scalar ϕ in three space-time dimensions. We have

$$\langle \Psi(X_1, S_1) \Phi(X_2) \Phi(X_3) \Psi(X_4, S_4) \rangle = \frac{i \langle S_1 S_4 \rangle}{X_{14}^{3/2} X_{23}^{1/2}} = \frac{t_1}{X_{14} X_{23}^{1/2}}, \tag{4.1}$$

so that

$$g_{\text{free}}^1(u, v) = \frac{u^{3/4}}{v^{1/2}}, \quad g_{\text{free}}^2(u, v) = 0. \tag{4.2}$$

By matching several orders in an expansion in r , we find

$$g_{\text{free}}^{I_+}(u, v) = \sum_{s=0}^{\infty} p_s g_{\frac{3}{2}+s, \frac{1}{2}+s}^{I_+, ++}(u, v), \quad (I_+ = 1, 2), \tag{4.3}$$

$$p_s = -4(s+1). \tag{4.4}$$

Thus, the free four-point function can be expanded in conformal blocks for higher spin currents

$$J_s^{\mu_1 \dots \mu_s} = \psi \partial^{\mu_1} \dots \partial^{\mu_s} \phi, \quad (s \geq 0). \tag{4.5}$$

Furthermore, only the parity even OPE coefficients $\lambda_{\Psi\Phi J_s}^+$ are nonzero. Each coefficient is pure imaginary since $p_s = (\lambda_{\Psi\Phi J_s}^+)^2$ is negative, as expected in a unitary theory.

With the conformal blocks for mixed correlators between scalars and fermions in hand, one can now investigate the constraints that crossing symmetry and unitarity impose on the space of CFTs with both spin-0 and spin-1/2 conformal primary operators. We hope to report on such a study in a future publication.

Acknowledgments

We thank João Penedones, Emilio Trevisani, and Masahito Yamazaki for discussions. This work was supported by the US NSF under grant No. PHY-1418069 (LI, SSP, and RY), DOE grant number DE-SC0009988 (DSD), and NSF grant PHY-1350180 (FK and DP). DSD is supported in part by a William D. Loughlin Membership at the Institute for Advanced Study. DP is supported in part by a Martin A. and Helen Chooljian Founders' Circle Membership at the Institute for Advanced Study.

A The Casimir equation

In this appendix we derive the Casimir equations satisfied by the conformal blocks and solve them at large Δ .

For a scalar field $\Phi(x)$ whose embedding space parent (defined on the 5D light cone) is $\Phi(X) = \frac{1}{(X^+)^{\Delta_\Phi}} \Phi(x)$, we can write the action of the 5D Lorentz generators as

$$i[J^{AB}, \Phi(X)] = \left(X^B \frac{\partial}{\partial X_A} - X^A \frac{\partial}{\partial X_B} \right) \Phi(X). \quad (\text{A.1})$$

Upon reduction to 3D, (A.1) reduces to the action of the conformal generators on the conformal primary field $\Phi(x)$ of dimension Δ_Φ .

For a 3D spinor field $\Psi^\alpha(x)$ whose embedding space parent is $\Psi^I(X)$, as defined in the main text through $S_I \Psi^I(X) = \frac{1}{(X^+)^{\Delta_\Psi}} s_\alpha \Psi^\alpha(x)$, the action of the conformal generators is more subtle. The subtlety comes from the fact that, because the 5D spinor polarizations S_I are transverse, $S_I X^I{}_J = 0$, the parent spinor field $\Psi^I(X)$ is defined only modulo shifts of the form

$$\Psi^I(X) \rightarrow \Psi^I(X) + X^I{}_J \Theta^J(X), \quad (\text{A.2})$$

where $\Theta^J(X)$ is an arbitrary spinor on the 5D light cone. To remove this ambiguity, one can define

$$\tilde{\Psi}^I(X) = X^I{}_J \Psi^J(X). \quad (\text{A.3})$$

The 5D $\tilde{\Psi}^I(X)$ does not suffer from the ambiguity in (A.2) because $X^I{}_J X^J{}_K = 0$ on the 5D light cone. On $\tilde{\Psi}$, the 5D Lorentz generators have the usual action in the spinor representation

$$i[J^{AB}, \tilde{\Psi}(X)] = \left(X^B \frac{\partial}{\partial X_A} - X^A \frac{\partial}{\partial X_B} + \frac{1}{4} [\Gamma^A, \Gamma^B] \right) \tilde{\Psi}(X), \quad (\text{A.4})$$

where we have suppressed all spinor indices. One can check that, once traced through all the definitions, eq. (A.4) implies that the 3D spinor field $\Psi^\alpha(x)$ is a conformal primary of dimension Δ_Ψ .

Let us now analyze the conformal block decomposition of the mixed 4-point function $\langle \Psi_1(X_1)\Phi_2(X_2)\Phi_3(X_3)\Psi_4(X_4) \rangle$ in (2.12). Using (A.3) and (2.12)–(2.15), we can write

$$\langle \tilde{\Psi}_1^J(X_1)\Phi_2(X_2)\Phi_3(X_3)\tilde{\Psi}_4^K(X_4) \rangle = \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{X_{13}}{X_{14}}\right)^{\frac{\Delta_{43}}{2}} \frac{\sum_I (\tilde{t}_I)^{JK} g^I(u, v)}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}}, \quad (\text{A.5})$$

where

$$\begin{aligned} (\tilde{t}_1)^{JK} &= -i \frac{(X_1 X_4)^{JK}}{\sqrt{X_{14}}}, & (\tilde{t}_2)^{JK} &= i \frac{(X_1 X_2 X_3 X_4)^{JK}}{\sqrt{X_{12} X_{34} X_{23}}}, \\ (\tilde{t}_3)^{JK} &= i \frac{(X_1 X_2 X_4)^{JK}}{\sqrt{X_{12} X_{24}}}, & (\tilde{t}_4)^{JK} &= i \frac{(X_1 X_3 X_4)^{JK}}{\sqrt{X_{13} X_{34}}}. \end{aligned} \quad (\text{A.6})$$

Let $(J_1)^{KL}$ and $(J_2)^{KL}$ be the 5D Lorentz generators acting on $\tilde{\Psi}_1$ and Φ_2 according to (A.4) and (A.1), respectively, and let us denote their sum squared as

$$L^2 = \frac{1}{2} (J_1 + J_2)^{KL} (J_1 + J_2)_{KL}. \quad (\text{A.7})$$

It can be shown that the contribution to (A.5) coming from the conformal multiplet of a primary $\mathcal{O}_{\Delta, \ell}$ must obey the eigenvalue equation

$$(L^2 - C_{\Delta, \ell}) \left(\left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{X_{13}}{X_{14}}\right)^{\frac{\Delta_{43}}{2}} \sum_I \frac{\tilde{t}_I g_{\Delta, \ell}^I(u, v)}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}} \right) = 0, \quad (\text{A.8})$$

with eigenvalue

$$C_{\Delta, \ell} = \Delta(\Delta - 3) + \ell(\ell + 1). \quad (\text{A.9})$$

Using the explicit expressions in (A.4) and (A.1), one can write down the equations for $g_{\Delta, \ell}^I$ implied by (A.8) as

$$\begin{aligned} & \left[\begin{pmatrix} \mathcal{L}_D^+ & \mathcal{L}_A^+ \\ \mathcal{L}_A^+ & \mathcal{L}_D^+ \end{pmatrix} + \begin{pmatrix} 0 & \frac{4r(\Delta_{12}+\Delta_{43})}{1+r^2-2r\eta} \\ 0 & \frac{4r(\eta-2r+r^2\eta)(\Delta_{12}+\Delta_{43})}{(1+r^2-2r\eta)^2} \end{pmatrix} \right] \begin{pmatrix} g_{\Delta, \ell}^1 \\ g_{\Delta, \ell}^2 \end{pmatrix} = C_{\Delta, \ell} \begin{pmatrix} g_{\Delta, \ell}^1 \\ g_{\Delta, \ell}^2 \end{pmatrix}, \\ & \left[\begin{pmatrix} \mathcal{L}_D^- & \mathcal{L}_A^- \\ \mathcal{L}_A^- & \mathcal{L}_D^- \end{pmatrix} + \begin{pmatrix} \frac{4r(\eta+2r+r^2\eta)\Delta_{43}}{(1+r^2+2r\eta)^2} & -\frac{4r\Delta_{12}}{1+r^2+2r\eta} \\ -\frac{4r\Delta_{43}}{1+r^2+2r\eta} & \frac{4r(\eta+2r+r^2\eta)\Delta_{12}}{(1+r^2+2r\eta)^2} \end{pmatrix} \right] \begin{pmatrix} g_{\Delta, \ell}^3 \\ g_{\Delta, \ell}^4 \end{pmatrix} = C_{\Delta, \ell} \begin{pmatrix} g_{\Delta, \ell}^3 \\ g_{\Delta, \ell}^4 \end{pmatrix}, \end{aligned} \quad (\text{A.10})$$

where

$$\begin{aligned} \mathcal{L}_D^\pm &= r^2 \partial_r^2 + (\eta^2 - 1) \partial_\eta^2 \\ &+ \left(\frac{4r^2 \eta (1 - r^2) (\Delta_{12} + \Delta_{43})}{(1 + r^2 - 2r\eta)(1 + r^2 + 2r\eta)} - \frac{r(1 + 3r^2)}{1 - r^2} - \frac{r(1 - r^2)(1 + r^2 \mp 2r\eta)}{(1 + r^2 + 2r\eta)(1 + r^2 - 2r\eta)} \right) \partial_r \\ &+ \left(\frac{4(\eta^2 - 1)(r^3 + r)(\Delta_{12} + \Delta_{43})}{(1 + r^2 + 2\eta r)(1 + r^2 - 2\eta r)} + \frac{[3\eta(1 + r^2) \pm 2r(4\eta^2 - 1)](1 + r^2 \mp 2r\eta)}{(1 + r^2 + 2r\eta)(1 + r^2 - 2r\eta)} \right) \partial_\eta \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{3}{4} - \frac{4r\Delta_{12}\Delta_{43}(\eta + 2r + r^2\eta)}{(1+r^2+2r\eta)^2} \right), \\
 \mathcal{L}_A^\pm & = \frac{2r^2}{1-r^2} \partial_r \pm \partial_\eta,
 \end{aligned} \tag{A.11}$$

and the coordinates r and η are defined in (2.16). Note that the parity-even blocks, $g_{\Delta,\ell}^1$ and $g_{\Delta,\ell}^2$, decouple from the parity-odd blocks, $g_{\Delta,\ell}^3$ and $g_{\Delta,\ell}^4$.

At large Δ , these equations can be solved as follows. From (2.24), we expect

$$g_{\Delta,\ell}^I(r, \eta) = r^\Delta [h_{\infty,\ell}^I(r, \eta) + O(1/\Delta, r)], \tag{A.12}$$

for some functions $h_{\infty,\ell}^I$ independent of Δ that we now find. Note that the term denoted by $O(1/\Delta, r)$ in (A.12) decays not only at large Δ but also at small r . (Recall that $n_A \geq 1$ in (2.24).) Plugging (A.12) into (A.10) and expanding up to order Δ^1 at large Δ , we find

$$\begin{aligned}
 \begin{pmatrix} \mathcal{D}^+ & \frac{2r^2}{1-r^2} \\ \frac{2r^2}{1-r^2} & \mathcal{D}^+ \end{pmatrix} \begin{pmatrix} h_{\infty,\ell}^1 \\ h_{\infty,\ell}^2 \end{pmatrix} & = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
 \begin{pmatrix} \mathcal{D}^- & \frac{2r^2}{1-r^2} \\ \frac{2r^2}{1-r^2} & \mathcal{D}^- \end{pmatrix} \begin{pmatrix} h_{\infty,\ell}^3 \\ h_{\infty,\ell}^4 \end{pmatrix} & = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
 \end{aligned} \tag{A.13}$$

with

$$\begin{aligned}
 \mathcal{D}^\pm & = 2r\partial_r + 2 + \frac{4r^2\eta(1-r^2)(\Delta_{12} + \Delta_{43})}{(1+r^2-2r\eta)(1+r^2+2r\eta)} - \frac{r(1+3r^2)}{1-r^2} \\
 & \quad - \frac{r(1-r^2)(1+r^2 \mp 2r\eta)}{(1+r^2+2r\eta)(1+r^2-2r\eta)}.
 \end{aligned} \tag{A.14}$$

These are first order ordinary differential equations in r . Their solution is

$$\begin{aligned}
 h_{\infty,\ell}^1(r, \eta) & = \frac{v^{\frac{1}{4}(\Delta_{12}+\Delta_{43})}}{\sqrt{1-r^2}\sqrt{1+r^2+2r\eta}} \left(\frac{-c_1(\eta)}{1-r} - \frac{c_2(\eta)}{1+r} \right), \\
 h_{\infty,\ell}^2(r, \eta) & = \frac{v^{\frac{1}{4}(\Delta_{12}+\Delta_{43})}}{\sqrt{1-r^2}\sqrt{1+r^2+2r\eta}} \left(\frac{c_1(\eta)}{1-r} - \frac{c_2(\eta)}{1+r} \right), \\
 h_{\infty,\ell}^3(r, \eta) & = \frac{v^{\frac{1}{4}(\Delta_{12}+\Delta_{43})}}{\sqrt{1-r^2}\sqrt{1+r^2-2r\eta}} \left(\frac{c_3(\eta)}{1+r} - \frac{c_4(\eta)}{1-r} \right), \\
 h_{\infty,\ell}^4(r, \eta) & = \frac{v^{\frac{1}{4}(\Delta_{12}+\Delta_{43})}}{\sqrt{1-r^2}\sqrt{1+r^2-2r\eta}} \left(\frac{c_3(\eta)}{1+r} + \frac{c_4(\eta)}{1-r} \right),
 \end{aligned} \tag{A.15}$$

where $c_i(\eta)$ are integration constants that are arbitrary functions of η .

To determine $c_i(\eta)$, one has to expand the Casimir equations (A.10) to next-to-leading order in $1/\Delta$, and the result to order r^0 . Such an expansion makes sense because the terms denoted by $O(\Delta^0, r)$ in (A.12) do not contribute in this limit. We find that the c_i 's must obey the second order differential equations

$$\begin{aligned}
 \left[(1-\eta^2)\partial_\eta^2 + (1-3\eta)\partial_\eta + \left(\ell + \frac{3}{2} \right) \left(\ell - \frac{1}{2} \right) \right] \begin{pmatrix} c_1(\eta) \\ c_3(\eta) \end{pmatrix} & = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
 \left[(1-\eta^2)\partial_\eta^2 + (1+3\eta)\partial_\eta + \left(\ell + \frac{3}{2} \right) \left(\ell - \frac{1}{2} \right) \right] \begin{pmatrix} c_2(\eta) \\ c_4(\eta) \end{pmatrix} & = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
 \end{aligned} \tag{A.16}$$

Each differential equation in (A.16) has only one normalizable solution:

$$\begin{aligned} c_1(\eta) &= C_1 P_{\ell-1/2}^{(0,1)}(\eta), & c_2(\eta) &= C_2 P_{\ell-1/2}^{(1,0)}(\eta), \\ c_3(\eta) &= C_3 P_{\ell-1/2}^{(0,1)}(\eta), & c_4(\eta) &= C_4 P_{\ell-1/2}^{(1,0)}(\eta), \end{aligned} \quad (\text{A.17})$$

with C_i being arbitrary constants and $P_n^{(a,b)}(\eta)$ being Jacobi polynomials. In (3.1), we choose $C_1 = C_3 = 1$ and $C_2 = C_4 = 1$ (top entry of each two-component vector) or $C_2 = C_4 = -1$ (bottom entry of each two-component vector).

B Determining OPE coefficients

The OPE coefficients $\lambda_{\Psi\Phi\mathcal{O}}^\pm$ appearing in the three-point function (2.7) are the same as those appearing in the four-point function (2.17) only for a specific normalization of the operator \mathcal{O} . If we write the two point function of \mathcal{O} as

$$\langle \mathcal{O}^{\alpha_1 \dots \alpha_{2\ell}}(x_1) \mathcal{O}^{\beta_1 \dots \beta_{2\ell}}(x_2) \rangle = i^{2\ell} c_{\mathcal{O}} \frac{(x_{12} i \sigma_2)^{\alpha_1 (\beta_1} \dots (x_{12} i \sigma_2)^{|\alpha_{2\ell} \beta_{2\ell}|}}{|x_{12}|^{2\Delta+2\ell}}, \quad (\text{B.1})$$

where $x_{ij} = x_i - x_j$ and $\alpha_1, \dots, \alpha_{2\ell}$ and $\beta_1, \dots, \beta_{2\ell}$ are spinor indices, our task is therefore to determine $c_{\mathcal{O}}$.⁶

Recall that the 3-point function $\Psi_1 \Phi_2 \mathcal{O}$ is given in (2.11). Explicitly, we have

$$\begin{aligned} &\langle \Psi_1^\beta(x_1) \Phi_2(x_2) \mathcal{O}^{\alpha_1 \alpha_2 \dots \alpha_{2\ell}}(x_3) \rangle \\ &= \lambda_{\Psi_1 \Phi_2 \mathcal{O}}^+ \frac{(x_{13} i \sigma_2)^{\beta(\alpha_1} (x_{31} x_{12} x_{23} i \sigma_2)^{\alpha_2 \alpha_3} \dots (x_{31} x_{12} x_{23} i \sigma_2)^{\alpha_{2\ell-1} \alpha_{2\ell}}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta + \ell - \frac{1}{2}} |x_{23}|^{\Delta_2 + \Delta - \Delta_1 + \ell - \frac{1}{2}} |x_{31}|^{\Delta + \Delta_1 - \Delta_2 + \ell + \frac{1}{2}}} \\ &\quad - \lambda_{\Psi_1 \Phi_2 \mathcal{O}}^- \frac{(x_{12} x_{23} i \sigma_2)^{\beta(\alpha_1} (x_{31} x_{12} x_{23} i \sigma_2)^{\alpha_2 \alpha_3} \dots (x_{31} x_{12} x_{23} i \sigma_2)^{\alpha_{2\ell-1} \alpha_{2\ell}}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta + \ell + \frac{1}{2}} |x_{23}|^{\Delta_2 + \Delta - \Delta_1 + \ell + \frac{1}{2}} |x_{31}|^{\Delta + \Delta_1 - \Delta_2 + \ell - \frac{1}{2}}}. \end{aligned} \quad (\text{B.2})$$

In the OPE limit $x_1 \rightarrow x_2$, the three-point function (B.2) becomes

$$\begin{aligned} &\langle \Psi_1^\beta(x_1) \Phi_2(x_2) \mathcal{O}^{\alpha_1 \alpha_2 \dots \alpha_{2\ell}}(x_3) \rangle \\ &\approx \lambda_{\Psi_1 \Phi_2 \mathcal{O}}^+ (i\sigma_2 x_{12})_{\beta_1 \beta_2} \dots (i\sigma_2 x_{12})_{\beta_{2\ell-2} \beta_{2\ell-1}} \frac{(x_{23} i \sigma_2)^{\beta(\alpha_1} (x_{23} i \sigma_2)^{|\beta_1 \alpha_2} \dots (x_{23} i \sigma_2)^{|\beta_{2\ell-1} \alpha_{2\ell}}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta + \ell - \frac{1}{2}} |x_{23}|^{2\Delta+2\ell}} \\ &\quad - \lambda_{\Psi_1 \Phi_2 \mathcal{O}}^- (i\sigma_2 x_{12})_{\beta_1 \beta_2} \dots (i\sigma_2 x_{12})_{\beta_{2\ell-2} \beta_{2\ell-1}} \frac{(x_{12} x_{23} i \sigma_2)^{\beta(\alpha_1} (x_{23} i \sigma_2)^{|\beta_1 \alpha_2} \dots (x_{23} i \sigma_2)^{|\beta_{2\ell-1} \alpha_{2\ell}}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta + \ell + \frac{1}{2}} |x_{23}|^{2\Delta+2\ell}}. \end{aligned} \quad (\text{B.3})$$

From (B.3), we can extract the contribution of \mathcal{O} to the $\Psi_1 \times \Phi_2$ OPE:

$$\begin{aligned} &\Psi_1^\beta(x_1) \Phi_2(x_2) \\ &= \dots + \left(- \frac{i^{2\ell} \lambda_{\Psi_1 \Phi_2 \mathcal{O}}^+}{c_{\mathcal{O}}} (i\sigma_2 x_{12})_{\alpha_1 \beta_1} \dots (i\sigma_2 x_{12})_{\alpha_{\ell-\frac{1}{2}} \beta_{\ell-\frac{1}{2}}} \frac{\mathcal{O}^{\beta \alpha_1 \beta_1 \dots \alpha_{\ell-\frac{1}{2}} \beta_{\ell-\frac{1}{2}}}(x_2)}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta + \ell - \frac{1}{2}}} \right. \\ &\quad \left. + \frac{i^{2\ell} \lambda_{\Psi_1 \Phi_2 \mathcal{O}}^-}{c_{\mathcal{O}}} (i\sigma_2 x_{12})_{\alpha_1 \beta_1} \dots (i\sigma_2 x_{12})_{\alpha_{\ell-\frac{1}{2}} \beta_{\ell-\frac{1}{2}}} (x_{12})^\beta_\gamma \frac{\mathcal{O}^{\gamma \alpha_1 \beta_1 \dots \alpha_{\ell-\frac{1}{2}} \beta_{\ell-\frac{1}{2}}}(x_2)}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta + \ell + \frac{1}{2}}} \right) + \dots \end{aligned} \quad (\text{B.4})$$

⁶In a reflection positive theory, the OPE coefficients $\lambda_{\Psi_i \Phi_j \mathcal{O}}$ will be pure imaginary and $c_{\mathcal{O}}$ will be positive. In a theory that violates reflection positivity, our conventions are such that $c_{\mathcal{O}}$ should still be fixed to the positive value determined in this appendix, after which the coefficients $\lambda_{\Psi_i \Phi_j \mathcal{O}}$ may no longer be pure imaginary.

Now we can examine the mixed 4-point function in the OPE limit. As we take $x_3 \rightarrow x_4$, using (B.4) and (B.1) and the fact that eq. (2.7) implies $\lambda_{\Psi_4\Phi_3\mathcal{O}}^b = (-1)^{\ell-\frac{1}{2}}\lambda_{\Phi_3\Psi_4\mathcal{O}}^b$, one can write the four-point function as

$$\langle \Psi_1^\alpha(x_1)\Phi_2(x_2)\Phi_3(x_3)\Psi_4^\beta(x_4) \rangle = \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}} \left(\frac{|x_{13}|}{|x_{14}|}\right)^{\Delta_{43}} \sum_{\mathcal{O}} \sum_{a,b=\pm} \frac{\lambda_{\Psi_1\Phi_2\mathcal{O}}^a \lambda_{\Psi_4\Phi_3\mathcal{O}}^b (g_{\mathcal{O}}^{ab}(x_i))^{\alpha\beta}}{|x_{12}|^{\Delta_1+\Delta_2} |x_{34}|^{\Delta_3+\Delta_4}}, \quad (\text{B.5})$$

with

$$(g_{\mathcal{O}}^{++})^{\alpha\beta}(x_i) = \left(\frac{|x_{14}|}{|x_{24}|}\right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{43}} \frac{(-1)^{\ell+1}}{c_{\mathcal{O}}} (i\sigma_2 x_{34})_{\alpha_1\alpha_2} \cdots (i\sigma_2 x_{34})_{\alpha_{2\ell-2}\alpha_{2\ell-1}} \quad (\text{B.6})$$

$$\times \frac{(x_{23}i\sigma_2)^{\alpha(\beta} (x_{31}x_{12}x_{23}i\sigma_2)^{\alpha_1\alpha_2} \cdots (x_{31}x_{12}x_{23}i\sigma_2)^{\alpha_{2\ell-2}\alpha_{2\ell-1}}}{|x_{12}|^{-\Delta+\ell-\frac{1}{2}} |x_{23}|^{\Delta-\Delta_{12}+\ell-\frac{1}{2}} |x_{31}|^{\Delta+\Delta_{12}+\ell+\frac{1}{2}} |x_{34}|^{-\Delta+\ell-\frac{1}{2}}},$$

$$(g_{\mathcal{O}}^{--})^{\alpha\beta}(x_i) = \left(\frac{|x_{14}|}{|x_{24}|}\right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{43}} \frac{(-1)^\ell}{c_{\mathcal{O}}} (i\sigma_2 x_{34})_{\alpha_1\alpha_2} \cdots (i\sigma_2 x_{34})_{\alpha_{2\ell-2}\alpha_{2\ell-1}} \quad (\text{B.7})$$

$$\times (x_{34})_\gamma^\beta \frac{(x_{12}x_{23}i\sigma_2)^{\alpha(\gamma} (x_{31}x_{12}x_{23}i\sigma_2)^{\alpha_1\alpha_2} \cdots (x_{31}x_{12}x_{23}i\sigma_2)^{\alpha_{2\ell-2}\alpha_{2\ell-1}}}{|x_{12}|^{-\Delta+\ell+\frac{1}{2}} |x_{23}|^{\Delta-\Delta_{12}+\ell+\frac{1}{2}} |x_{31}|^{\Delta+\Delta_{12}+\ell-\frac{1}{2}} |x_{34}|^{-\Delta+\ell+\frac{1}{2}}},$$

$$(g_{\mathcal{O}}^{+-})^{\alpha\beta}(x_i) = \left(\frac{|x_{14}|}{|x_{24}|}\right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{43}} \frac{(-1)^{\ell+1}}{c_{\mathcal{O}}} (i\sigma_2 x_{34})_{\alpha_1\alpha_2} \cdots (i\sigma_2 x_{34})_{\alpha_{2\ell-2}\alpha_{2\ell-1}} \quad (\text{B.8})$$

$$\times (x_{34})_\gamma^\beta \frac{(x_{23}i\sigma_2)^{\alpha(\gamma} (x_{31}x_{12}x_{23}i\sigma_2)^{\alpha_1\alpha_2} \cdots (x_{31}x_{12}x_{23}i\sigma_2)^{\alpha_{2\ell-2}\alpha_{2\ell-1}}}{|x_{12}|^{-\Delta+\ell-\frac{1}{2}} |x_{23}|^{\Delta-\Delta_{12}+\ell-\frac{1}{2}} |x_{31}|^{\Delta+\Delta_{12}+\ell+\frac{1}{2}} |x_{34}|^{-\Delta+\ell+\frac{1}{2}}},$$

$$(g_{\mathcal{O}}^{-+})^{\alpha\beta}(x_i) = \left(\frac{|x_{14}|}{|x_{24}|}\right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{43}} \frac{(-1)^\ell}{c_{\mathcal{O}}} (i\sigma_2 x_{34})_{\alpha_1\alpha_2} \cdots (i\sigma_2 x_{34})_{\alpha_{2\ell-2}\alpha_{2\ell-1}} \quad (\text{B.9})$$

$$\times \frac{(x_{12}x_{23}i\sigma_2)^{\alpha(\beta} (x_{31}x_{12}x_{23}i\sigma_2)^{\alpha_1\alpha_2} \cdots (x_{31}x_{12}x_{23}i\sigma_2)^{\alpha_{2\ell-2}\alpha_{2\ell-1}}}{|x_{12}|^{-\Delta+\ell+\frac{1}{2}} |x_{23}|^{\Delta-\Delta_{12}+\ell+\frac{1}{2}} |x_{31}|^{\Delta+\Delta_{12}+\ell-\frac{1}{2}} |x_{34}|^{-\Delta+\ell-\frac{1}{2}}}.$$

It is sufficient to write these expressions for a convenient choice of the 4 coordinates x_i :

$$\begin{aligned} x_1 &= (0, -1, 0), \\ x_2 &= (0, 1, 0), \\ x_3 &= (0, r \cos \theta, r \sin \theta), \\ x_4 &= (0, -r \cos \theta, -r \sin \theta). \end{aligned} \quad (\text{B.10})$$

With this choice, the OPE limit $x_3 \rightarrow x_4$ is realized as $r \rightarrow 0$. In this limit, the structures that appear in (B.6)–(B.9) can be approximated as

$$\begin{aligned} i\sigma_2 x_{34} &\approx -2r(-\sigma_3 \cos \theta + \sigma_1 \sin \theta), & x_{23}i\sigma_2 &\approx -\sigma_3, \\ x_{12}x_{23}i\sigma_2 &\approx 2i\sigma_2, & x_{31}x_{12}x_{23}i\sigma_2 &\approx 2\sigma_3, \end{aligned} \quad (\text{B.11})$$

while $|x_{12}| \approx 2$, $|x_{34}| \approx 2r$, $x_{14} \approx x_{23} \approx \sqrt{1+r^2-2r\cos\theta}$, and $x_{24} \approx x_{13} \approx \sqrt{1+r^2+2r\cos\theta}$. Thus (B.6)–(B.9) become

$$(g_{\mathcal{O}}^{++})^{\alpha\beta}(x_i) \approx (-1)^{\ell+1} \frac{(4r)^\Delta}{c_{\mathcal{O}}} (M_\theta)_{\alpha_1\alpha_2} \cdots (M_\theta)_{\alpha_{2\ell-2}\alpha_{2\ell-1}} (-\sigma_3)^{\alpha(\beta} M_0^{\alpha_1\alpha_2} \cdots M_0^{\alpha_{2\ell-2}\alpha_{2\ell-1}}),$$

$$(g_{\mathcal{O}}^{--})^{\alpha\beta}(x_i) \approx (-1)^\ell \frac{(4r)^\Delta}{c_{\mathcal{O}}} (M_\theta)_{\alpha_1\alpha_2} \cdots (M_\theta)_{\alpha_{2\ell-2}\alpha_{2\ell-1}} (\tilde{M}_\theta)_\gamma^\beta (-i\sigma_2)^{\alpha(\gamma} M_0^{\alpha_1\alpha_2} \cdots M_0^{\alpha_{2\ell-2}\alpha_{2\ell-1}}),$$

$$\begin{aligned}
 (g^{+-})_{\mathcal{O}}^{\alpha\beta}(x_i) &\approx (-1)^{\ell+1} \frac{(4r)^\Delta}{c_{\mathcal{O}}} (M_\theta)_{\alpha_1\alpha_2} \cdots (M_\theta)_{\alpha_{2l-2}\alpha_{2l-1}} (\tilde{M}_\theta)_{\gamma}^\beta (-\sigma_3)^{\alpha(\gamma} M_0^{\alpha_1\alpha_2} \cdots M_0^{\alpha_{2l-2}\alpha_{2l-1}}), \\
 (g^{-+})_{\mathcal{O}}^{\alpha\beta}(x_i) &\approx (-1)^\ell \frac{(4r)^\Delta}{c_{\mathcal{O}}} (M_\theta)_{\alpha_1\alpha_2} \cdots (M_\theta)_{\alpha_{2l-2}\alpha_{2l-1}} (-i\sigma_2)^{\alpha(\beta} M_0^{\alpha_1\alpha_2} \cdots M_0^{\alpha_{2l-2}\alpha_{2l-1}}), \quad (\text{B.12})
 \end{aligned}$$

where $M_\theta = \sigma_3 \cos \theta - \sigma_1 \sin \theta$ and $\tilde{M}_\theta = \sigma_3 \sin \theta + \sigma_1 \cos \theta$.

This expression should be compared with our 4-point function in a similar limit. Projected down to 3D, the 4-point function (2.12) takes the form:

$$\langle \Psi_1^\alpha(x_1) \Phi_2(x_2) \Phi_3(x_3) \Psi_4^\beta(x_4) \rangle = \left(\frac{|x_{24}|}{|x_{14}|} \right)^{\Delta_{12}} \left(\frac{|x_{13}|}{|x_{14}|} \right)^{\Delta_{43}} \frac{\sum_I t_I^{\alpha\beta} g^I(u, v)}{|x_{12}|^{\Delta_1+\Delta_2} |x_{34}|^{\Delta_3+\Delta_4}}, \quad (\text{B.13})$$

with the basis elements being

$$\begin{aligned}
 t_1^{\alpha\beta} &= i \frac{(x_{14} i \sigma_2)^{\alpha\beta}}{|x_{14}|}, & t_2^{\alpha\beta} &= -i \frac{(x_{12} x_{23} x_{34} i \sigma_2)^{\alpha\beta}}{|x_{12}| |x_{23}| |x_{34}|}, \\
 t_3^{\alpha\beta} &= i \frac{(x_{12} x_{24} i \sigma_2)^{\alpha\beta}}{|x_{12}| |x_{24}|}, & t_4^{\alpha\beta} &= i \frac{(x_{13} x_{34} i \sigma_2)^{\alpha\beta}}{|x_{13}| |x_{34}|}.
 \end{aligned} \quad (\text{B.14})$$

Considering the point choice (B.10) and taking $r \rightarrow 0$, we have

$$\begin{aligned}
 t_1 &\approx i\sigma_3, & t_2 &\approx i(-\sigma_3 \cos \theta + \sigma_1 \sin \theta), \\
 t_3 &\approx \sigma_2, & t_4 &\approx \sigma_2 \cos \theta - i\mathbf{1}_2 \sin \theta.
 \end{aligned} \quad (\text{B.15})$$

In this limit, the functions appearing in the four-point function (B.13) are simply given $g_{\Delta,\ell}^I(r, \eta) = r^\Delta h_{\Delta,\ell}^{(\infty)I}(r, \eta) + O(r^{\Delta+1})$. Using the first order expansion in r for $h_{\infty,\ell}^{I+}(r, \eta)$ and for $h_{\infty,\ell}^{I-}(r, \eta)$ (3.1) in the limit $r \rightarrow 0$, together with (B.15) and (B.12), we find that in the OPE limit

$$\begin{pmatrix} (g^{++})_{\mathcal{O}}^{\alpha\beta}(x_i) \\ (g^{--})_{\mathcal{O}}^{\alpha\beta}(x_i) \end{pmatrix} \approx \frac{4^{\Delta-1}(\ell + \frac{1}{2})!}{(\frac{1}{2})_{\ell+\frac{1}{2}}} \frac{1}{c_{\mathcal{O}}} \sum_{I_+} t_{I_+}^{\alpha\beta} \mathbf{g}_{\Delta,\ell}^{I_+}(u, v), \quad (\text{B.16})$$

$$\begin{pmatrix} (g^{+-})_{\mathcal{O}}^{\alpha\beta}(x_i) \\ (g^{-+})_{\mathcal{O}}^{\alpha\beta}(x_i) \end{pmatrix} \approx \frac{4^{\Delta-1}(\ell + \frac{1}{2})!}{(\frac{1}{2})_{\ell+\frac{1}{2}}} \frac{1}{c_{\mathcal{O}}} \sum_{I_-} t_{I_-}^{\alpha\beta} \mathbf{g}_{\Delta,\ell}^{I_-}(u, v), \quad (\text{B.17})$$

where $\mathbf{g}_{\Delta,\ell}^I \approx r^\Delta \mathbf{h}_{\infty,\ell}^I$ with $\mathbf{h}_{\infty,\ell}^I$ given by (3.1). In order for the OPE coefficients appearing in the three (2.7) and four-point (2.17) functions to be identical, we must choose

$$c_{\mathcal{O}} = \frac{4^{\Delta-1}(\ell + \frac{1}{2})!}{(\frac{1}{2})_{\ell+\frac{1}{2}}}. \quad (\text{B.18})$$

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