

Spectrum of Majorana Quantum Mechanics with $O(4)^3$ Symmetry

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Abstract

We study the quantum mechanics of 3-index Majorana fermions ψ^{abc} governed by a quartic Hamiltonian with $O(N)^3$ symmetry. Similarly to the Sachdev-Ye-Kitaev model, this tensor model has a solvable large N limit dominated by the melonic diagrams. For $N = 4$ the total number of states is 2^{32} , but they naturally break up into distinct sectors according to the charges under the $U(1) \times U(1)$ Cartan subgroup of one of the $O(4)$ groups. The biggest sector has vanishing charges and contains over 165 million states. Using a Lanczos algorithm, we determine the spectrum of the low-lying states in this and other sectors. We find that the absolute ground state is non-degenerate. If the $SO(4)^3$ symmetry is gauged, it is known from earlier work that the model has 36 states and a residual discrete symmetry. We study the discrete symmetry group in detail; it gives rise to degeneracies of some of the gauge singlet energies. We find all the gauge singlet energies numerically and use the results to propose exact analytic expressions for them.

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1 Introduction

In recent literature there has been considerable interest in the quantum mechanical models where the degrees of freedom are fermionic tensors of rank 3 or higher [1, 2]. Similarly to the Sachdev-Ye-Kitaev model [3–5], these models have solvable large N limits dominated by the so-called melonic diagrams [6–8]. In this limit they become solvable with the use of Schwinger-Dyson equations as were derived earlier for the SYK-like models [2, 4, 5, 9–12]. While this spectrum of eigenstates is discrete and bounded for finite N , the low-lying states become dense for large N leading to the (nearly) conformal behavior where it makes sense to calculate the operator scaling dimensions. In the SYK model, the number of states is $2^{N_{\text{SYK}}/2}$, and numerical calculations of spectra have been carried out for rather large values of N_{SYK} [13–15]. They reveal a smooth distribution of energy eigenvalues, which typically has no degeneracies and is almost symmetric under $E \rightarrow -E$.

The corresponding studies of spectra in the tensor models of [1] and [2] have been carried out in [16–23], but in these cases the numerical limitations have been more severe – the number of states grows as $2^{N^3/2}$ in the $O(N)^3$ symmetric model of [2] and as 2^{2N^3} in the $O(N)^6$ symmetric Gurau-Witten (GW) model [1]. The results have shown an interesting structure. For example, for the $N = 2$ GW model the exact values of the 140 $SO(2)^6$ invariant energies were found [22]. Due to the discrete symmetries, there are only 5 distinct $E < 0$ eigenvalues and each one squares to an integer (the singlet spectrum also contains 50 zero-energy states).

The $O(N)^3$ model [2], has the Hamiltonian ¹

$$H = \psi^{abc} \psi^{ab'c'} \psi^{a'bc'} \psi^{a'b'c} - \frac{1}{4} N^4 , \quad (1.1)$$

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'} \delta^{bb'} \delta^{cc'} , \quad a, b, c = 0, 1, \dots, N-1 . \quad (1.2)$$

For $N = 2$ there are only two gauge singlet states with $E = \pm 8$. For $N = 3$, as for any odd N , there are none [23]. While the complete spectra of (1.2) can be calculated for $N = 2$ and 3 using a laptop, this is no longer true for $N = 4$, where the total number of states is 2^{32} . However, they split into smaller sectors according to the charges (Q_0, Q_1) of the $U(1) \times U(1)$ Cartan subgroup of one of the $SO(4)$ groups. The most complicated and interesting is the $(0, 0)$ sector; it is the part of the 32 qubit spectrum at the "half-half-filling," i.e. where the first 16 qubits contain 8 zeros and 8 ones, and the same applies to the remaining 16 qubits. In particular, all the $SO(4)^3$ invariant states are in this subsector; their number, 36, was found using the gauged version of the free fermion theory [23]. Since there are over 165 million states at half-half-filling, the spectrum cannot be determined completely. However, using a Lanczos algorithm, we will be able to determine a number of low-lying eigenstates. We will also be able to find the complete spectrum of the 36 gauge singlet states, including their transformation properties under the residual discrete symmetries of the model where the $SO(4)^3$ symmetry is gauged. Thus, our work reveals the spectrum of a finite- N system without disorder, which is nearly conformal and solvable in the large- N limit, and identifies the discrete symmetries crucial for efficient numerical studies of such finite systems.

Using our numerical results we are able to infer the exact expressions for all the singlet eigenvalues. In particular, the ground state energy,² which is numerically $E_0 \approx -160.140170$, agrees well with $E_0 = -\sqrt{32(447 + \sqrt{125601})}$. Other gauge singlet energies either have similar expressions or are simply square roots of integers. This suggests that the Hamiltonian can be diagonalized exactly analytically.

¹ Compared to [2, 23] we have set the overall dimensionful normalization constant g to 4 in order to simplify the equations.

² For some results on the ground states in the SYK and related models see [13, 14, 24].

2 Discrete symmetries acting on the gauge singlets

For any even N , if we gauge the $SO(N)^3$ symmetry, there remain some gauge singlet states [23], which are annihilated by the symmetry charges

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}] , \quad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}] , \quad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}] . \quad (2.1)$$

These states may still have degeneracies due to the residual discrete symmetries. Indeed, each $O(N)$ group contains a Z_2 parity symmetry which is an axis reflection. For example, inside $O(N)_1$ there is parity symmetry P_1 which send $\psi^{0bc} \rightarrow -\psi^{0bc}$ for all b, c and leaves all other components invariant. The corresponding generator is

$$P_1 = P_1^\dagger = 2^{N^2/2} \prod_{bc} \psi^{0bc} . \quad (2.2)$$

One can indeed check that

$$P_1 \psi^{abc} P_1^\dagger = (-1)^{\delta_{a,0} + N^2} \psi^{abc} . \quad (2.3)$$

Similarly, there are Z_2 generators P_2 and P_3 inside $O(N)_2$ and $O(N)_3$.

It is also useful to introduce unitary operators P_{ij} associated with permutations of the $O(N)_i$ and $O(N)_j$ groups:

$$\begin{aligned} P_{23} &= P_{23}^\dagger = i^{n(n-1)/2} \prod_a \prod_{b>c} (\psi^{abc} - \psi^{acb}) , \\ P_{12} &= P_{12}^\dagger = i^{n(n-1)/2} \prod_c \prod_{a>b} (\psi^{abc} - \psi^{bac}) , \end{aligned} \quad (2.4)$$

where $n = N^2(N-1)/2$ is the number of fields in the product. They satisfy

$$P_{23} \psi^{abc} P_{23}^\dagger = (-1)^{N^2(N-1)/2} \psi^{acb} , \quad P_{12} \psi^{abc} P_{12}^\dagger = (-1)^{N^2(N-1)/2} \psi^{bac} . \quad (2.5)$$

These permutations flip the sign of H [23, 25]:

$$P_{23} H P_{23}^\dagger = -H , \quad P_{12} H P_{12}^\dagger = -H . \quad (2.6)$$

This explains why the spectrum is symmetric under $E \rightarrow -E$.

We now define the cyclic permutation operator $P = P_{12}P_{23}$ such that

$$P\psi^{abc}P^\dagger = \psi^{cab} , \quad PHP^\dagger = H , \quad P^3 = I . \quad (2.7)$$

Thus, P is the generator of the Z_3 symmetry of the Hamiltonian. Applying the Z_3 symmetry to the parity reflections P_i we see that

$$PP_1P^\dagger = P_2 , \quad PP_2P^\dagger = P_3 , \quad PP_3P^\dagger = P_1 . \quad (2.8)$$

Forming all the possible products of I, P, P_1, P_2, P_3 , we find that the full discrete symmetry group contains 24 elements. Using the explicit representation (2.2) for P_1 , and the analogous ones for P_2 and P_3 , we note that the three parity operators commute with each other. Furthermore,

$$[\Pi, P] = 0 , \quad \Pi = P_1P_2P_3 , \quad \Pi^2 = I . \quad (2.9)$$

Therefore, Π commutes with all the group elements, so that the group has a Z_2 factor with elements I and Π . The symmetry group turns out to be $A_4 \times Z_2$, and the 12 elements of the alternating group A_4 are

$$I , P_1 , P_2 , P_1P_2 , P , P^2 , P_1P , P_2P , P_1P_2P , P_1P^2 , P_2P^2 , P_1P_2P^2 . \quad (2.10)$$

Each of them can be associated with a sign preserving permutation of 4 ordered elements, and the action is

$$\begin{aligned} P_1(a_0, a_1, a_2, a_3) &= (a_1, a_0, a_3, a_2) , \\ P_2(a_0, a_1, a_2, a_3) &= (a_2, a_3, a_0, a_1) , \\ P_3(a_0, a_1, a_2, a_3) &= (a_3, a_2, a_1, a_0) , \\ P(a_0, a_1, a_2, a_3) &= (a_0, a_3, a_1, a_2) . \end{aligned} \quad (2.11)$$

The degenerate $SO(N)^3$ invariant states of a given non-zero energy form irreducible representations of $A_4 \times Z_2$. For even N we can choose a basis where all the wavefunctions and matrix elements of the Hamiltonian are real. In this case we should study the representation of the symmetry group over the field \mathbb{R} . The degrees of the irreducible representations of A_4 over that field are 1, 2, 3. The Z_2 factor does not change the degrees since both irreducible representations of Z_2 , the trivial one and the sign one, have degree 1.

Let us discuss the representations of A_4 in more detail. Using a reference eigenstate $|\psi_0\rangle$ not invariant under the Z_3 subgroup I, P, P^2 , we can form a triplet of states

$$|\psi_0\rangle, \quad P|\psi_0\rangle, \quad P^2|\psi_0\rangle. \quad (2.12)$$

If the parities (P_1, P_2, P_3) of the state $|\psi_0\rangle$ are the same, then we can form a linear combination which transforms trivially under the Z_3 ,

$$|\psi\rangle = \frac{1}{\sqrt{3}}(1 + P + P^2)|\psi_0\rangle, \quad P|\psi\rangle = |\psi\rangle, \quad (2.13)$$

while the remaining 2 linear combination form the degree 2 representation of Z_3 ,

$$P|\psi_1\rangle = |\psi_2\rangle, \quad P|\psi_2\rangle = -|\psi_1\rangle - |\psi_2\rangle, \quad (2.14)$$

where $|\psi_1\rangle = \frac{1}{\sqrt{3}}|\psi_0\rangle - |\psi\rangle$. Because of this, some eigenstates have degeneracy 2.

If the parities (P_1, P_2, P_3) of the state $|\psi_0\rangle$ are not equal, then the triplet representation (2.12) of the full discrete group is irreducible. For example for $(P_1, P_2, P_3) = (+, +, -)$, i.e.

$$P_1|\psi_0\rangle = |\psi_0\rangle, \quad P_2|\psi_0\rangle = |\psi_0\rangle, \quad P_3|\psi_0\rangle = -|\psi_0\rangle, \quad (2.15)$$

we find that the parities of the states $P|\psi_0\rangle$ and $P^2|\psi_0\rangle$ are given by the cyclic permutations of $(+, +, -)$. Indeed, using (2.8), we find that the parities of the state $P|\psi_0\rangle$ are

$$P_1P|\psi_0\rangle = -P|\psi_0\rangle, \quad P_2P|\psi_0\rangle = P|\psi_0\rangle, \quad P_3P|\psi_0\rangle = P|\psi_0\rangle. \quad (2.16)$$

Thus, each of the states in the triplet (2.12) has a distinct set of parities. Then it is impossible to form linear combinations which are eigenstates of the parities, and we have an irreducible representation of A_4 of degree 3. In this situation we find that an energy eigenvalue has degeneracy 3.

We also note the relations

$$P_{23}P_1P_{23}^\dagger = (-1)^{N(N^2-1)/2}P_1, \quad P_{12}P_1P_{12}^\dagger = (-1)^{N(N^2-1)/2}P_2, \quad P_{13}P_1P_{12}^\dagger = (-1)^{N(N^2-1)/2}P_3, \quad (2.17)$$

and their cyclic permutations. Since an operator P_{ij} maps an eigenstate of energy E into an eigenstate of energy $-E$, we see that such mirror states have the same parities when $N/2$ is even, but opposite parities when $N/2$ is odd.

For the states at zero energy, the discrete symmetry group is enhanced to 48 elements because the permutation generators P_{ij} map them into themselves. Using the relations (2.17) we find

$$P_{12}\Pi P_{12}^\dagger = (-1)^{N(N^2-1)/2}\Pi, \quad (2.18)$$

which implies that $\Pi = P_1 P_2 P_3$ commutes or anti-commutes with other elements depending on the value of N . Focusing on the case where $N(N^2 - 1)/2$ is even and the sign above is positive (this includes $N = 4$ which is our main interest in this paper), we find that Π commutes with all other generators, so that the group has a Z_2 factor with elements I and Π . The symmetry group for $E = 0$ turns out to be $S_4 \times Z_2$, which is the full cube group. Its subgroup S_4 is formed out of the products of $I, P_1, P_2, P_{12}, P_{23}, P_{13}$. The parity generators are realized in the same way as in (2.11), while the permutations act by the natural embedding $S_3 \subset S_4$:

$$\begin{aligned} P_{12}(a_0, a_1, a_2, a_3) &= (a_0, a_2, a_1, a_3), \\ P_{23}(a_0, a_1, a_2, a_3) &= (a_0, a_1, a_3, a_2), \\ P_{13}(a_0, a_1, a_2, a_3) &= (a_0, a_3, a_2, a_1). \end{aligned} \quad (2.19)$$

The degrees of the irreducible representations of S_4 are 1, 1, 2, 3, 3.

3 Diagonalization of the Hamiltonian

The Majorana fermions ψ^{abc} may be thought of as generators of the Clifford algebra in N^3 -dimensional Euclidean space. Restricting to the cases where N is even, the dimension of the Hilbert space is $2^{N^3/2}$, and the states may be represented by series of $N^3/2$ ‘‘qubits’’ $|s\rangle$, where $s = 0$ or 1. It is convenient to introduce operators [20, 23]

$$\begin{aligned} \bar{c}_{abk} &= \frac{1}{\sqrt{2}} (\psi^{ab(2k)} + i\psi^{ab(2k+1)}), & c_{abk} &= \frac{1}{\sqrt{2}} (\psi^{ab(2k)} - i\psi^{ab(2k+1)}), \\ \{c_{abk}, c_{a'b'k'}\} &= \{\bar{c}_{abk}, \bar{c}_{a'b'k'}\} = 0, & \{\bar{c}_{abk}, c_{a'b'k'}\} &= \delta_{aa'}\delta_{bb'}\delta_{kk'}, \end{aligned} \quad (3.1)$$

where $a, b = 0, 1, \dots, N-1$, and $k = 0, \dots, \frac{1}{2}N-1$. In this basis the $O(N)^2 \times U(N/2)$ symmetry is manifest, and the Hamiltonian is [20, 23]

$$H = 2(\bar{c}_{abk}\bar{c}_{ab'k'}c_{a'b'k}c_{a'b'k} - \bar{c}_{abk}\bar{c}_{a'b'k'}c_{ab'k'}c_{a'b'k}). \quad (3.2)$$

If we number the qubits from 0 to $\frac{1}{2}N^3 - 1$, then operators c_{abk}, \bar{c}_{abk} correspond to qubit number $N^2k + Nb + a$.

In the basis (3.1) the parity operators P_i corresponding to i -th group $O(N)$ are

$$P_1 = \prod_{b=0}^{N-1} \prod_{k=0}^{N/2-1} [\bar{c}_{0bk}, c_{0bk}], \quad P_2 = \prod_{a=0}^{N-1} \prod_{k=0}^{N/2-1} [\bar{c}_{a0k}, c_{a0k}], \quad P_3 = \prod_{a=0}^{N-1} \prod_{b=0}^{N-1} (\bar{c}_{ab0} + c_{ab0}). \quad (3.3)$$

The operator P_3 implements charge conjugation on the $k=0$ operators, i.e. it acts to interchange \bar{c}_{ab0} and c_{ab0} . This conjugation is a symmetry of H . In fact, for each k the Hamiltonian is symmetric under the interchange of \bar{c}_{abk} and c_{abk} .

The $U(1)^{N/2}$ subgroup of the $U(N/2)$ symmetry is realized simply. The corresponding charges,

$$Q_k = \sum_{a,b} \frac{1}{2} [\bar{c}_{abk}, c_{abk}], \quad k = 0, \dots, \frac{1}{2}N-1, \quad (3.4)$$

are the Dynkin labels of a state of the third $SO(N)$ group, and the spectrum separates into sectors according to their values. The oscillator vacuum state satisfies

$$c_{abk} |\text{vac}\rangle = 0, \quad Q_k |\text{vac}\rangle = -\frac{N^2}{2} |\text{vac}\rangle, \quad (3.5)$$

and other states are obtained by acting on it with some number of \bar{c}_{abk} .

For $N=4$ the total number of states is $2^{32} = 4294967296$, but they break up into $17^2 = 289$ smaller sectors due to the conservation of the $U(1) \times U(1)$ charges Q_0 and Q_1 . The biggest sector is $(Q_0, Q_1) = (0, 0)$; it consists of $\frac{(16!)^2}{(8!)^4} = 165636900$ states. The next biggest are the 4 sectors $(\pm 1, 0)$ and $(0, \pm 1)$; each of them contains 147232800 states. The smallest 4 sectors are $(\pm 8, \pm 8)$, and each one consists of just 1 state; each of these states has $E=0$. In general, the spectrum in the (q, q') sector is the same as in (q', q) due to the symmetry of H under interchange of the c_{ab0} and c_{ab1} oscillators.

Let us first study the $(0, 0)$ sector. These states are obtained by acting on $|\text{vac}\rangle$ with 8 raising operators \bar{c}_{ab0} and 8 raising operators \bar{c}_{ab1} . In the qubit notation, both the first

16 qubits, and the second 16 qubits, have equal number, 8, of zeros and ones. Clearly, all the $SO(4)^3$ invariant states are in this sector.³ While the numbers of such “half-half-filled” states is still very large, they turn out to be tractable numerically because the matrix we need to diagonalize is rather sparse. This has allowed us to study the low-lying eigenvalues of H , which occur in various representations of $SO(4)^3$. To find the gauge singlet energies, we study the operator proportional to $H + 100 \sum_{i=1}^3 C_2^i$, where the quadratic Casimir of the $SO(N)_1$ symmetry is $C_2^1 = \frac{1}{2} Q_1^{aa'} Q_1^{aa'}$, and analogously for $SO(N)_2$ and $SO(N)_3$. The Lanczos algorithm allows us to identify the lowest eigenvalues of this operator, which all correspond to $SO(4)^3$ invariant states; the non-singlets receive large additive contributions due to the second term.

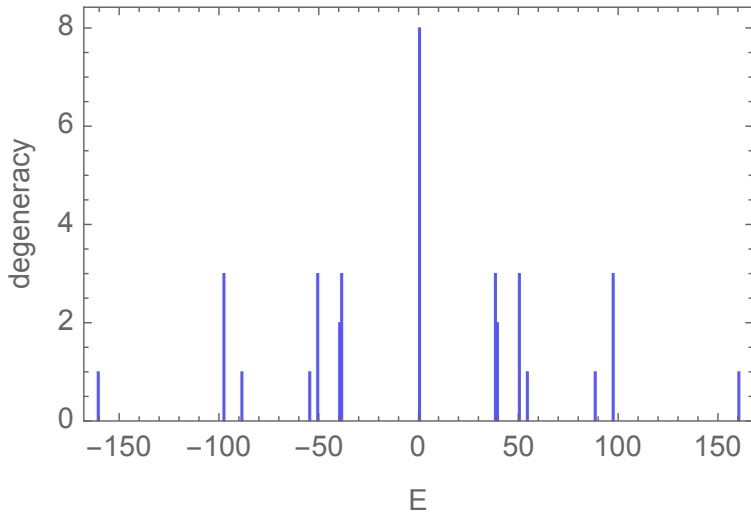


Figure 1: Spectrum of gauge singlets in the $O(4)^3$ model

In table 1 we list the energies and parities of all 36 $SO(4)^3$ invariant states. In order to identify the values of P_i , we calculated the low-lying spectrum of operator

$$H + 100 \sum_{i=1}^3 C_2^i + \sum_{i=1}^3 a_i P_i , \quad (3.6)$$

where a_i are unequal small coefficients.⁴ The biggest degeneracy is found for the $E = 0$ states; it corresponds to the 2^3 independent choices of the three parities. Since the discrete group acting on the $E = 0$ states is $S_4 \times Z_2$, which is the full cube group, we find two different

³ There are additional constraints on the gauge singlet wave functions, but we will not discuss them explicitly here.

⁴The states at ± 39.191836 are doubly degenerate and have identical parities; these states form the degree 2 representation of the Z_3 subgroup of A_4 . To split such double degeneracies we added a small amount of noise to the Hamiltonian.

Table 1: The list of all the $SO(4)^3$ invariant states including their parities P_i .

E	P_1	P_2	P_3	E	P_1	P_2	P_3
-160.140170	1	1	1	160.140170	1	1	1
-97.019491	1	1	-1	97.019491	1	1	-1
-97.019491	-1	1	1	97.019491	-1	1	1
-97.019491	1	-1	1	97.019491	1	-1	1
-88.724292	-1	-1	-1	88.724292	-1	-1	-1
-54.434603	1	1	1	54.434603	1	1	1
-50.549167	1	1	-1	50.549167	1	1	-1
-50.549167	-1	1	1	50.549167	-1	1	1
-50.549167	1	-1	1	50.549167	1	-1	1
-39.191836	1	1	1	39.191836	1	1	1
-39.191836	1	1	1	39.191836	1	1	1
-38.366652	1	-1	-1	38.366652	1	-1	-1
-38.366652	-1	1	-1	38.366652	-1	1	-1
-38.366652	-1	-1	1	38.366652	-1	-1	1
0.000000	1	1	1	0.000000	-1	-1	-1
0.000000	-1	1	1	0.000000	1	-1	-1
0.000000	1	-1	1	0.000000	-1	1	-1
0.000000	1	1	-1	0.000000	-1	-1	1

irreducible representations of S_4 : the trivial one of degree 1 and the standard one of degree 3. We may associate the eight $E = 0$ states with the vertices of a cube. The energies of the gauge singlet states and their degeneracies are plotted in Figure 1.

Some of the energies agree within the available precision with square roots of integers: $8\sqrt{23} \approx 38.366652$, $8\sqrt{24} \approx 39.191836$, and $8\sqrt{123} \approx 88.724292$. Furthermore, the 4 eigenvalues with parities $(1, 1, 1)$, ± 160.140170 and ± 54.434603 , are approximations to the analytic expressions $\pm\sqrt{32(447 \pm \sqrt{125601})}$, while the triplet eigenvalues, ± 97.019491 and ± 50.549167 , are approximations to $\pm\sqrt{32(187 \pm \sqrt{11481})}$. To demystify these exact results, we note that there are only two $SO(4)^3$ invariant states with $P_1 = P_2 = P_3 = -1$ (see Table 1). Since the Hamiltonian has symmetry under $E \rightarrow -E$, the eigenvalue equation in this subsector must have the form of the second order even polynomial: $E^2 - A_1 = 0$. This explains why some of the eigenvalues are simply square roots. On the other hand, there are four $SO(4)^3$ and Z_3 invariant states with $P_1 = P_2 = P_3 = 1$. Thus, the eigenvalue equation in this subsector must have the form

$$E^4 + 2A_2E^2 + A_3 = 0, \quad (3.7)$$

and this explains why some of the energies satisfy $E^2 = -A_2 \pm \sqrt{A_2^2 - A_3}$. Similar symmetry considerations explain the form of all the gauge singlet energies in terms of the square roots. We leave exact derivation of the parameters A_i for future work.

Table 2: The low-lying energies in the $(0, 0)$ sector, i.e. at half-half filling, including the values of the quadratic Casimirs of each $SO(N)$ group. When the C_2^i are not all equal, there are additional states of the same energy with their values obtained by a cyclic permutation.

C_2^1	C_2^2	C_2^3	E
0	0	0	-160.140170
0	4	8	-136.559039
0	0	12	-136.417554
0	0	24	-128.490197
4	4	4	-122.553686
0	0	12	-121.606040
4	8	8	-121.552284
4	8	8	-120.699077
4	8	8	-119.685636
0	8	12	-119.659802
0	12	8	-119.204505
0	8	4	-118.699780
0	4	16	-118.541049
4	4	4	-116.774758

The list of all the low-lying energy levels in the $(0, 0)$ sector, singlets and non-singlets, and the corresponding values of quadratic Casimirs C_2^i , is shown in table 2. In order to identify the values of C_2^i , we have calculated the low-lying spectrum of $H + \sum_{i=1}^3 a_i C_2^i$ where a_i are unequal small coefficients. When the C_2^i are not all equal, there are also states of the same energy with their values obtained by a cyclic permutation. For example, at $E = -136.559039$ we find states with $(C_2^1, C_2^2, C_2^3) = (0, 4, 8), (4, 8, 0), (8, 0, 4)$.

Absent from the list in table 2 is the lowest possible value of the quadratic Casimir, $C_2 = 3$, which corresponds to the $(1/2, 0) + (0, 1/2)$ irrep, i.e. fundamental representation 4 of $SO(4)$. Let us proceed to the sectors adjacent to one-particle and one-hole sectors, $(\pm 1, 0)$ and $(0, \pm 1)$, which contain some of the additional representations, including the $(4, 4, 4)$ of $SO(4)^3$. The refined bound [23] for this representation gives $|E_{(4,4,4)}| < 72\sqrt{5} \approx 160.997$, while the actual lowest state in this representation has $E \approx -140.743885$. The low-lying states in the sectors $(\pm 1, 0)$ and $(0, \pm 1)$ are given in table 3. We have also calculated the energies in other charge sectors. We find that the absolute ground state lies in the $(0, 0)$

Table 3: The low-lying states in the sectors $(\pm 1, 0)$ and $(0, \pm 1)$, i.e. with one extra hole (h) or particle (p) added to half-half-filling. The energies are the same within the accuracy shown, which is a good test of our diagonalization procedure. When the C_2^i are not all equal, there are additional states of the same energy with their values obtained by a cyclic permutation.

C_2^1	C_2^2	C_2^3	$E_h = E_p$
3	3	3	-140.743885
3	3	9	-128.059272
3	3	15	-124.547555
3	9	9	-118.371087
3	3	9	-117.798571
3	3	19	-115.861910
3	9	9	-114.885221
3	3	15	-114.660576
3	3	9	-114.539928

sector: as the magnitudes of charges increase, the energies tend to get closer to 0.

4 Supplementary Material for "Spectrum of Majorana Quantum Mechanics with $O(4)^3$ Symmetry"

For $N = 4$ we can use the explicit representation in terms of direct products of 32 2×2 matrices:

$$\begin{aligned}
\sqrt{2}\psi^{000} &= X \otimes \mathbb{1} \otimes \mathbb{1} \dots \otimes \mathbb{1}, & \sqrt{2}\psi^{001} &= Y \otimes \mathbb{1} \otimes \mathbb{1} \dots \otimes \mathbb{1}, \\
\sqrt{2}\psi^{100} &= Z \otimes X \otimes \mathbb{1} \dots \otimes \mathbb{1}, & \sqrt{2}\psi^{101} &= Z \otimes Y \otimes \mathbb{1} \dots \otimes \mathbb{1}, \\
&\dots & & \\
\sqrt{2}\psi^{232} &= Z \dots \otimes Z \otimes X \otimes \mathbb{1}, & \sqrt{2}\psi^{233} &= Z \dots \otimes Z \otimes Y \otimes \mathbb{1}, \\
\sqrt{2}\psi^{332} &= Z \dots \otimes Z \otimes Z \otimes X, & \sqrt{2}\psi^{333} &= Z \dots \otimes Z \otimes Z \otimes Y,
\end{aligned} \tag{4.1}$$

where X, Y, Z stand for the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$. Their action on a qubit is

$$\begin{aligned}
X|0\rangle &= |1\rangle, & Y|0\rangle &= -i|1\rangle, & Z|0\rangle &= -|0\rangle, \\
X|1\rangle &= |0\rangle, & Y|1\rangle &= i|0\rangle, & Z|1\rangle &= |1\rangle.
\end{aligned} \tag{4.2}$$

The Hamiltonian becomes

$$\begin{aligned}
H = & 2(\bar{c}_{ab0}\bar{c}_{ab'0}c_{a'b0}c_{a'b'0} - \bar{c}_{ab0}\bar{c}_{a'b0}c_{ab'0}c_{a'b'0}) + 2(\bar{c}_{ab1}\bar{c}_{ab'1}c_{a'b1}c_{a'b'1} - \bar{c}_{ab1}\bar{c}_{a'b1}c_{ab'1}c_{a'b'1}) \\
& + 4(\bar{c}_{ab0}\bar{c}_{ab'1}c_{a'b1}c_{a'b'0} - \bar{c}_{ab0}\bar{c}_{a'b1}c_{ab'1}c_{a'b'0}), \tag{4.3}
\end{aligned}$$

where in the first line we find two copies of the Hamiltonian of the $O(4)^2 \times O(2)$ model, which was solved in [23]. Each of these systems contains 16 qubits, and the second line creates a coupling between the two systems.

The expressions for the parity operators are, omitting the direct product signs,

$$\begin{aligned}
P_1 &= Z111Z111Z111Z111Z111Z111Z111Z111, \\
P_2 &= ZZZZ11111111111111ZZZZ111111111111, \\
P_3 &= YXYXYXYXYXYXYXYXYX1111111111111111. \tag{4.4}
\end{aligned}$$

The operator P_3 implements, up to a sign, the particle-hole conjugation on the first 16 qubits. These parity operators may be used only on the $SO(4)^3$ invariant states. For example, a rotated form of $P_3 = 2^8 \prod_{ab} \psi^{ab0}$ is

$$\tilde{P}_3 = 2^8 \prod_{ab} \psi^{ab1} = XYXYXYXYXYXYXYXY1111111111111111. \tag{4.5}$$

It has the same eigenvalues as P_3 on the singlets because

$$\tilde{P}_3 P_3 = ZZZZZZZZZZZZZZZZZZZ1111111111111111, \tag{4.6}$$

which is equal to 1 when acting on the singlet states, where the first 16 qubits are half-filled.

The ground state energy we find is close to the lower bound [23] $E_{bound} = -\frac{1}{4}N^3(N+2)\sqrt{N-1} \approx -166.277$. The ratio of E_0 and E_{bound} can be calculated in the large N limit using the exact propagator, and it is found to be ≈ 0.41 [23]. Since we find $E_0/E_{bound} \approx 0.96$, this suggests that large N approximations cannot be applied for $N = 4$.

In some charge sectors it is not hard to determine the energy spectrum. For example, in the $(-7, -7)$ sector, which contains 256 states $\bar{c}_{ab0}\bar{c}_{a'b'1}|\text{vac}\rangle$, only the second line in the Hamiltonian (4.3) acts non-trivially, and we find $E = \pm 16$ with multiplicity 15, and $E = 0$ with multiplicity 226. Due to the conjugation symmetry the same spectrum is found in the $(-7, 7)$ sector. In each of the $(\pm 6, \pm 6)$ sectors, some of the energies are square roots of integers, including the ground state $E = -24\sqrt{5}$. Finally, let us note that in each sector

of the form $(q, \pm 8)$ or $(\pm 8, q)$ the Hamiltonian is isomorphic to that of the $O(4)^2 \times O(2)$ model, which was solved in [23], and therefore has the same integer spectrum. The lowest and highest energies, occurring for $q = 0$, are ± 64 .

From the value of quadratic Casimir we may infer the (j_1, j_2) representation of $SO(4) \sim SU(2) \times SU(2)$ using the formula

$$C_2(j_1, j_2) = 2(j_1(j_1 + 1) + j_2(j_2 + 1)) . \quad (4.7)$$

For example, $C_2 = 4$ corresponds to the $(1, 0) + (0, 1)$ irrep of dimension 6; $C_2 = 8$ corresponds to the $(1, 1)$ irrep of dimension 9; $C_2 = 12$ corresponds to the $(2, 0) + (0, 2)$ irrep of dimension 10; etc.

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