

DECAY AND SCATTERING FOR THE CHERN-SIMONS-SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider the Chern-Simons-Schrödinger model in $1+2$ dimensions, and prove scattering for small solutions of the Cauchy problem in the Coulomb gauge. This model is a gauge covariant Schrödinger equation, with a potential decaying like r^{-1} at infinity. To overcome the difficulties due to this long range decay we start by performing L^2 -based estimates covariantly. This gives favorable commutation identities so that only curvature terms, which decay faster than r^{-1} , appear in our weighted energy estimates. We then select the Coulomb gauge to reveal a genuinely cubic null structure, which allows us to show sharp decay by Fourier methods.

1. INTRODUCTION

In this paper, we investigate the long term behavior and asymptotics of small solutions to the *Chern-Simons-Schrödinger* (CSS) system, which is a non-relativistic gauge field theory on \mathbb{R}^{1+2} taking the form

$$\begin{cases} \mathbf{D}_t \phi = i \mathbf{D}_\ell \mathbf{D}_\ell \phi + ig |\phi|^2 \phi, \\ F_{12} = -\frac{1}{2} |\phi|^2, \\ F_{01} = -\frac{i}{2} (\phi \overline{\mathbf{D}_2 \phi} - (\mathbf{D}_2 \phi) \bar{\phi}), \\ F_{02} = \frac{i}{2} (\phi \overline{\mathbf{D}_1 \phi} - (\mathbf{D}_1 \phi) \bar{\phi}). \end{cases} \quad (\text{CSS})$$

Here ϕ is a \mathbb{C} -valued function (Schrödinger field), $A = A_0 dx^0 + A_1 dx^1 + A_2 dx^2$ is a real-valued 1-form (gauge potential), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (curvature 2-form), $\mathbf{D}_\mu = \partial_\mu + iA_\mu$ (covariant derivative) and $g \in \mathbb{C}$. Our main theorem demonstrates that sufficiently small initial data lead to a global unique solution on \mathbb{R}^{1+2} which exhibits *linear scattering* to a free Schrödinger field in the Coulomb gauge ($\partial_1 A_1 + \partial_2 A_2 = 0$); we refer to Theorem 1.2 for a precise statement.

The (CSS) system was introduced by Jackiw-Pi [JP90-1, JP90-2], with an emphasis on its self-dual structure (for $g = 1$) and existence of (multi-)vortex solitons. It serves as a basic model for Chern-Simons dynamics on the plane, which is used to analyze various planar phenomena, e.g. anyonic statistics, fractional quantum Hall effect and high T_c superconductors. For a more thorough discussion on the physical relevance of (CSS) and self-duality, we refer the reader to [JP92, Du95, HZ09, Ya01] and the references therein.

Recently, there has been some work regarding (CSS) from the mathematical side. Various authors have contributed to the problem of local well-posedness [BDS95, Hu13, LST-p]; currently, the best result is due to Liu-Smith-Tataru [LST-p], who proved local well-posedness for $\phi(0) \in H_x^\varepsilon$ for any $\varepsilon > 0$. In the case of focusing self-interaction potential $g > 0$, Bergé-de Bouard-Saut [BDS95] showed finite time blow-up via a virial argument, and an explicit example was given later by Huh [Hu09]. Global existence of a weak solution (without uniqueness) with sub-threshold charge was also proved in [BDS95]. We also mention the interesting papers [DS07, DS09] on a closely related

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system proposed by Manton [Ma97]. On the other hand, to our best knowledge, there has been no prior work concerning the asymptotic behavior of global solutions to (CSS).

The major difficulty in studying the asymptotic behavior of solutions to (CSS) is the apparent long range behavior of the potentials A_j . To elaborate, we begin from the fact that (CSS) is *gauge invariant*, i.e., it is invariant under the gauge transform

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi, \quad \phi \rightarrow e^{-i\chi} \phi$$

where χ is a real-valued function on \mathbb{R}^{1+2} . Due to this fact, the initial value problem for (CSS) is well-posed only after fixing a specific representative of (A_μ, ϕ) , a procedure usually referred to as *gauge fixing*. For concreteness of our discussion, we shall choose the *Coulomb gauge*, which is a gauge defined by the condition

$$\partial_1 A_1 + \partial_2 A_2 = 0.$$

Expanding out $\mathbf{D}_\mu = \partial_\mu + iA_\mu$, (CSS) in the Coulomb gauge leads to the equation

$$(\partial_t - i\Delta)\phi = -2A_1\partial_1\phi - 2A_2\partial_2\phi - iA_0\phi + (\text{Higher order terms}). \quad (1.1)$$

Let us focus on the nonlinear terms with the most derivatives on ϕ , i.e., $-2A_j\partial_j\phi$. Under the Coulomb condition, A_j satisfies the equation $\Delta A_j = \frac{1}{2}\epsilon_{jk}\partial_k|\phi|^2$, where ϵ_{jk} is the unique anti-symmetric 2-form such that $\epsilon_{12} = 1$. Therefore, A_j is given by the Biot-Savart law

$$A_j(t, x) = -\frac{1}{2}\epsilon_{jk}\partial_k(-\Delta)^{-1}|\phi(t, x)|^2 = \frac{1}{4\pi}\epsilon_{jk} \int_{\mathbb{R}^2} \frac{(x-y)^k}{|x-y|^2} |\phi(t, y)|^2 dy. \quad (1.2)$$

Then, it is easy to see that the right-hand side of (1.2) has an r^{-1} tail as $r \rightarrow \infty$ for any non-trivial solution ϕ to (CSS), hence A_j is a *long range potential*. In principle, such long range potentials can lead to complicated asymptotic behaviors for even arbitrarily small solutions, such as modified scattering [HN98, KP11] or finite time blow-up [Joh79, Joh81].

The preceding discussion applies to other gauges as well, since the divergence-free part of A_j (according to the Hodge decomposition of 1-forms on \mathbb{R}^2) is always given by the formula (1.2). Indeed, the r^{-1} behavior of (a part of) A_j is present in the work [LST-p], in which a non-Coulomb gauge (more precisely, the heat gauge $A_t = \partial_\ell A_\ell$) was used.

Nevertheless, in this paper we are able to establish global existence of solutions of (CSS) for sufficiently small initial data, and prove *linear scattering*¹ of such solutions in the Coulomb gauge. Our proof relies on the following two main observations:

- 1) The (CSS) system does not exhibit any long range behavior when viewed *covariantly*, i.e., when phrased in terms of $\mathbf{D}_t, \mathbf{D}_j$ and $F_{\mu\nu}$ as in (CSS). Using the covariant charge identity (4.2) and covariant commutators such as $\mathbf{D}_j, \mathbf{J}_j := x_j + 2it\mathbf{D}_j$, we can establish global a priori L^2 bounds under the assumption that ϕ exhibits the linear decay rate, i.e., $|t|^{-1}$ as $t \rightarrow \pm\infty$. Unfortunately, such a method seems to fall short of retrieving the $|t|^{-1}$ decay rate, and for this purpose we turn to our second observation:
- 2) The (CSS) system exhibits a *strong, genuinely cubic null structure* in the Coulomb gauge. Remarkably, by (and only by) considering all cubic² terms $(-2A_j\partial_j\phi - iA_0\phi)$ together, we reveal a very strong null structure which effectively cancels out the long range effect of A_μ for the purpose of establishing the desired decay rate of ϕ . Combined with the a priori L^2 bounds obtained using covariant methods, we are able to conclude the $|t|^{-1}$ decay via an analysis of (1.1) in Fourier space.

¹By linear scattering, we mean convergence of the solution ϕ to some free Schrödinger field $e^{it\Delta}f_\infty$ as $t \rightarrow \infty$ in an appropriate topology; see (1.7) in Theorem 1.2 for a more precise statement.

²The terms $A_j\partial_j\phi$ and $A_0\phi$ are cubic in ϕ , as A_μ satisfies a Poisson equation with quadratic (and higher) terms in ϕ as a source. See (CSS-Coulomb) in §1.1

Broadly speaking, our method may be understood as a mix of two different existing approaches to small data scattering: The use of the covariant charge identity and commutators is akin to the tensor-geometric approach of Christodoulou-Klainerman [CK90, CK93], whereas the cubic null structure is identified and utilized within the Fourier-analytic framework of the works on space-time resonances [GMS09] and [PS13]. A more detailed description of our main ideas will be given in Section 2, after we state the main theorem in §1.1.

1.1. Statement of the main theorem. In the Coulomb gauge $\partial_1 A_1 + \partial_2 A_2 = 0$, which is the setting of our main theorem, (CSS) can be reformulated as

$$\left\{ \begin{array}{l} \partial_t \phi - i\Delta \phi = -iA_0 \phi - 2A_\ell \partial_\ell \phi - iA_\ell A_\ell \phi + ig|\phi|^2 \phi, \\ \Delta A_1 = \frac{1}{2} \partial_2 |\phi|^2, \\ \Delta A_2 = -\frac{1}{2} \partial_1 |\phi|^2, \\ \Delta A_0 = \frac{i}{2} \partial_1 (\phi \overline{\mathbf{D}_2 \phi} - (\mathbf{D}_2 \phi) \overline{\phi}) - \frac{i}{2} \partial_2 (\phi \overline{\mathbf{D}_1 \phi} - (\mathbf{D}_1 \phi) \overline{\phi}), \end{array} \right. \quad (\text{CSS-Coulomb})$$

where repeated indices are assumed to be summed, e.g., $-2A_\ell \partial_\ell \phi = -2 \sum_{\ell=1}^2 A_\ell \partial_\ell \phi$ etc.

We define the initial data set for (CSS-Coulomb) as follows.

Definition 1.1 (Coulomb initial data set). We say that a pair (a_j, ϕ_0) of a 1-form a_j and a \mathbb{C} -valued function ϕ on \mathbb{R}^2 is a *Coulomb initial data set* for (CSS) if it satisfies

$$\partial_1 a_1 + \partial_2 a_2 = 0, \quad (1.3)$$

$$\partial_1 a_2 - \partial_2 a_1 = -\frac{1}{2} |\phi_0|^2. \quad (1.4)$$

The condition (1.3) is the Coulomb gauge condition for the initial data, whereas (1.4) is the *constraint equation* imposed by the system (CSS). We remark that the div-curl system (1.3)–(1.4) is enough to uniquely specify a_j (with a mild condition at infinity) in terms of the (gauge invariant) amplitude of ϕ_0 , at least under our regularity assumptions below.

We are now ready to state the main theorem of this paper.

Theorem 1.2 (Main Theorem). *Let $(A_j(0), \phi(0))$ be a Coulomb initial data set for (CSS) satisfying the smallness assumption*

$$\sum_{m=0}^2 \|\mathbf{D}^{(m)} \phi(0)\|_{L_x^2} + \| |x| \phi(0) \|_{L_x^2} + \| |x| \mathbf{D} \phi(0) \|_{L_x^2} + \| |x|^2 \phi(0) \|_{L_x^2} \leq \varepsilon_1. \quad (1.5)$$

Then, for sufficiently small ε_1 there exists a unique global solution $\phi \in C_t(\mathbb{R}; H_x^2)$ of (CSS-Coulomb) such that

$$\|\phi(t)\|_{L_x^\infty} \lesssim \varepsilon_1 (1 + |t|)^{-1}. \quad (1.6)$$

Moreover, for each sign \pm and $0 \leq s < 2$, there exists $f_{\pm\infty} \in H_x^s$ such that

$$e^{-it\Delta} \phi(t) \xrightarrow{t \rightarrow \pm\infty} f_{\pm\infty} \quad \text{in } H_x^s. \quad (1.7)$$

1.2. Notations and conventions.

- Unless otherwise specified, we adopt the convention of summing up all repeated indices, e.g., $\mathbf{D}_j \mathbf{D}_j = \mathbf{D}_1 \mathbf{D}_1 + \mathbf{D}_2 \mathbf{D}_2$.
- ϵ_{jk} denotes the anti-symmetric 2-form with $\epsilon_{12} = 1$
- We denote the ordinary and covariant spatial gradients by $D = (\partial_1, \partial_2)$ and $\mathbf{D} = (\mathbf{D}_1, \mathbf{D}_2)$, respectively. The m -fold ordinary and covariant spatial gradients will be denoted by $D^{(m)}$ and $\mathbf{D}^{(m)}$, respectively.

- We define the operators $\mathbf{J} = (\mathbf{J}_1, \mathbf{J}_2)$ and $J = (J_1, J_2)$, where

$$\mathbf{J}_k := x_k + 2it\mathbf{D}_k, \quad J_k := x_k + 2it\partial_k$$

Like \mathbf{D}, D , we denote the m -fold application of \mathbf{J}, J by $\mathbf{J}^{(m)}, J^{(m)}$, respectively.

- We adopt the convention $\mathcal{F}(\psi)(\xi) = \widehat{\psi}(\xi) := (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} \psi(x) dx$, for the Fourier transform.
- We denote the space of Schwartz functions on \mathbb{R}^2 by \mathcal{S}_x .

2. MAIN IDEAS

The purpose of this section is to provide a more detailed description of the main ideas of our proof of Theorem 1.2. In order to establish global existence and scattering for small solutions, the basic strategy is to prove estimates for Sobolev and weighted L^2 norms of ϕ , and a decay estimate with the same rate of a linear solution, i.e.

$$\|\phi(t)\|_{L_x^\infty} \lesssim \varepsilon_1 (1 + |t|)^{-1}. \quad (2.1)$$

Covariant charge estimates. To avoid the long range effect of A_μ in a fixed gauge, we derive apriori L^2 bounds using covariant methods. Our basic tool is the simple covariant charge identity (4.1)–(4.2). To bound higher derivatives we commute the covariant Schrödinger equation with \mathbf{D}_j , and for weighted L^2 bounds we commute with

$$\mathbf{J}_k := x_k + 2it\mathbf{D}_k,$$

which is the covariant version of the well-known operator $J_k = x_k + 2it\partial_k$ for the Schrödinger equation. Such commutations are rather straightforward to perform in the covariant setting, leading to nice (schematic) equations such as

$$\begin{aligned} (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell) \mathbf{D}_j \phi &= \phi \cdot \phi \cdot \mathbf{D} \phi, \\ (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell) \mathbf{J}_j \phi &= \phi \cdot \phi \cdot \mathbf{J} \phi + t(\phi \cdot \phi \cdot \mathbf{D} \phi). \end{aligned}$$

See (4.11)–(4.16) for the full list. Then, assuming the decay (2.1), we can bootstrap covariant L^2 bounds of the form

$$\|\mathbf{D}^{(m)} \phi(t)\|_{H_x^2} \lesssim \varepsilon_1, \quad \|\mathbf{J}^{(m)} \phi(t)\|_{L_x^2} \lesssim \varepsilon_1 (\log(2 + |t|))^m \quad (m = 0, 1, 2),$$

for ε_1 sufficiently small.

Transition of covariant bounds to gauge-dependent bounds. The next step consists in deriving L^2 bounds in the Coulomb gauge, such as

$$\|D^{(m)} \phi(t)\|_{L_x^2} \lesssim \varepsilon_1, \quad \|J^{(m)} \phi(t)\|_{L_x^2} \lesssim \varepsilon_1 (\log(2 + |t|))^m \quad (m = 0, 1, 2),$$

from the covariant L^2 bounds. The execution of this step depends on estimates obtained from the elliptic equation for A_μ in the Coulomb gauge. To eventually obtain the desired final result, it only remains to retrieve (2.1).

Asymptotic analysis in the Coulomb gauge. Combined with the apriori growth bound of the weighted L^2 norms of $J^{(m)} \phi(t)$, a standard lemma (Lemma 6.1) reduces the proof of decay (2.1) to establishing uniform boundedness of $\widehat{\phi}(t)$, i.e.,

$$\|\widehat{\phi}(t)\|_{L_x^\infty} \lesssim \varepsilon_1. \quad (2.2)$$

To achieve this, we shall use the Fourier-analytic framework of Germain-Masmoudi-Shatah [GMS09]: Defining $f := e^{-it\Delta} \phi$, the Schrödinger equation in the Coulomb gauge becomes

$$\partial_t \widehat{f} = e^{it|\xi|^2} \mathcal{F}[-2A_j \partial_j \phi - iA_0 \phi + (\text{Higher order terms})]$$

As $\|\widehat{f}\|_{L_\xi^\infty} = \|\widehat{\phi}\|_{L_\xi^\infty}$, we simply need to estimate the L_ξ^∞ norm of the above right-hand side. The contribution of the higher order terms are easily manageable, and we are only left to consider the cubic terms $-2A_j\partial_j\phi - iA_0\phi$.

The cubic null structure of (CSS) in the Coulomb gauge. We now describe the strong cubic null structure of these cubic terms in the Coulomb gauge, which is crucial to close the whole argument. According to the framework of space-time resonances [GMS09, PS13], a cubic expression in $f = e^{-it\Delta}\phi$ of the form

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{it\varphi(\eta, \sigma, \xi)} a(\eta, \sigma, \xi) \cdot D_{\eta, \sigma} \varphi(\eta, \sigma, \xi) \widehat{f}(t, \eta - \sigma) \widehat{f}(t, \sigma) \widehat{f}(t, \xi - \eta) d\eta d\sigma \quad (2.3)$$

for some coefficient $a(\eta, \sigma, \xi)$, is a null structure (the symbol of the interaction vanishes on the space resonant set). Indeed, there is a gain of a factor of t^{-1} upon integrating by parts in η and/or σ . This may be viewed as an alternative interpretation of the classical null condition due to Klainerman [Kla85a, Kla86].

It is not difficult to see that the contribution of each $-2A_j\partial_j\phi$ and $-iA_0\phi$ is a null structure of the form (2.3). However, the long range effect of A_μ is still manifest, in the sense that the coefficient $a(\eta, \sigma, \xi)$ has a singularity of the form $|\eta|^{-2}$ (from the inversion of Δ in the equation for A_μ). Nevertheless, looking at $-2A_j\partial_j\phi - iA_0\phi$ as a whole gives (after a change of variables, and up to an irrelevant constant)

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{it\varphi(\xi, \eta, \sigma)} \left(d_\eta \log |\eta| \wedge d_\eta \varphi(\xi, \eta, \sigma) \right) \widehat{f}(t, \xi - \sigma) \widehat{f}(t, \sigma + \eta - \xi) \widehat{f}(t, \xi - \eta) d\sigma d\eta$$

where $d_\eta g \wedge d_\eta h$ is a differential-forms notation for $\partial_{\eta_1} g \partial_{\eta_2} h - \partial_{\eta_2} g \partial_{\eta_1} h$; see §6.2 for details. This is the claimed *strong cubic null structure*: Not only is the singularity at $\eta = 0$ milder ($d_\eta \log |\eta| \sim |\eta|^{-1}$), but it vanishes completely when an extra d_η falls on $d_\eta \log |\eta|$ after integration by parts (this cancellation reduces exactly to the standard fact $d^2 = 0$ regarding exterior differential). Exploiting this null structure, as well as using the a priori L^2 bounds established earlier, we finally obtain (2.2) (for ε_1 small enough) and close the whole argument.

Structure of the paper. In Section 3, we reduce the main theorem (Theorem 1.2) to three propositions in accordance to the main ideas sketched above: Propositions 3.1, 3.2 and 3.3 concerning *covariant charge estimates*, *transition from covariant to gauge-dependent bounds*, and *asymptotic analysis in the Coulomb gauge*, respectively. Then in Sections 4, 5 and 6, we give proofs of these propositions in order.

3. REDUCTION OF THE MAIN THEOREM

In this section, we reduce the proof of Theorem 1.2 to establishing three statements, namely Propositions 3.1, 3.2 and 3.3. Before we state the propositions, we shall fix a terminology: By an H^2 solution to (CSS) in the Coulomb gauge on $(-T, T)$, we mean a solution (A_μ, ϕ) to (CSS-Coulomb) such that $\phi \in C_t((-T, T); H_x^2)$.

In the first proposition, we prove gauge covariant a priori L^2 bounds, under bootstrap assumptions which include the critical L_x^∞ decay assumption (3.4). We remind the reader that $\mathbf{J}_k := x_k + 2it\mathbf{D}_k$.

Proposition 3.1 (Covariant charge estimates). *Let (A_μ, ϕ) be an H^2 solution to (CSS) in the Coulomb gauge on $(-T, T)$, which satisfies the initial data estimate (1.5) and obeys the bootstrap*

assumptions

$$\|\phi(t)\|_{L_x^2} + \|\mathbf{D}\phi(t)\|_{L_x^2} + \|\mathbf{D}^{(2)}\phi(t)\|_{L_x^2} \leq B\varepsilon_1 \quad (3.1)$$

$$\|\mathbf{J}\phi(t)\|_{L_x^2} + \|\mathbf{J}\mathbf{D}\phi(t)\|_{L_x^2} \leq B\varepsilon_1 \log(2 + |t|) \quad (3.2)$$

$$\|\mathbf{J}^{(2)}\phi(t)\|_{L_x^2} \leq B\varepsilon_1 (\log(2 + |t|))^2 \quad (3.3)$$

$$\|\phi(t)\|_{L_x^\infty} \leq B\varepsilon_1 (1 + |t|)^{-1} \quad (3.4)$$

for $t \in (-T, T)$ and an absolute constant $B > 0$. Then, for B sufficiently large and $\varepsilon_1 > 0$ small enough, we improve (3.1), (3.2) and (3.3) on $(-T, T)$, respectively, to

$$\|\phi(t)\|_{L_x^2} + \|\mathbf{D}\phi(t)\|_{L_x^2} + \|\mathbf{D}^{(2)}\phi(t)\|_{L_x^2} \leq \frac{B}{200}\varepsilon_1 \quad (3.5)$$

$$\|\mathbf{J}\phi(t)\|_{L_x^2} + \|\mathbf{J}\mathbf{D}\phi(t)\|_{L_x^2} \leq \frac{B}{200}\varepsilon_1 \log(2 + |t|) \quad (3.6)$$

$$\|\mathbf{J}^{(2)}\phi(t)\|_{L_x^2} \leq \frac{B}{200}\varepsilon_1 (\log(2 + |t|))^2. \quad (3.7)$$

The second proposition is used to translate the gauge covariant bounds (3.5)–(3.7) to the corresponding bounds in the Coulomb gauge. Recall that $J_k := x_k + 2it\partial_k$.

Proposition 3.2 (Transition from covariant to gauge-dependent bounds). *Let (A_μ, ϕ) be an H^2 solution to (CSS) in the Coulomb gauge on $(-T, T)$, which obeys the bootstrap assumption (3.4) and the improved covariant bounds (3.5)–(3.7). Then for $\varepsilon_1 > 0$ sufficiently small compared to B , the following gauge-dependent bounds hold on $(-T, T)$:*

$$\|\phi(t)\|_{L_x^2} + \|D\phi(t)\|_{L_x^2} + \|D^{(2)}\phi(t)\|_{L_x^2} \leq \frac{B}{100}\varepsilon_1 \quad (3.5')$$

$$\|J\phi(t)\|_{L_x^2} + \|JD\phi(t)\|_{L_x^2} \leq \frac{B}{100}\varepsilon_1 \log(2 + |t|) \quad (3.6')$$

$$\|J^{(2)}\phi(t)\|_{L_x^2} \leq \frac{B}{100}\varepsilon_1 (\log(2 + |t|))^2. \quad (3.7')$$

Finally, in the third proposition, we improve the L_x^∞ decay assumption. The argument takes place entirely in the Coulomb gauge, and relies crucially on the *cubic null structure* of (CSS) in this gauge.

Proposition 3.3 (Asymptotic analysis in the Coulomb gauge). *Let (A_μ, ϕ) be an H^2 solution to (CSS) in the Coulomb gauge on $(-T, T)$, which satisfies the initial data estimate (1.5). Assume furthermore that (A_μ, ϕ) obeys the bootstrap assumption (3.4) and the gauge-dependent bounds (3.5')–(3.7') for $t \in (-T, T)$.*

Then for B sufficiently large and $\varepsilon_1 > 0$ small enough, we improve (3.4) on $(-T, T)$ to

$$\|\phi(t)\|_{L_x^\infty} \leq \frac{B}{10}\varepsilon_1 (1 + |t|)^{-1}. \quad (3.8)$$

Moreover, for each sign \pm , there exists $\widehat{f}_{\pm\infty} \in L_\xi^\infty$ such that

$$e^{it|\xi|^2} \widehat{\phi}(t) \xrightarrow{t \rightarrow \pm\infty} \widehat{f}_{\pm\infty} \quad \text{in } L_\xi^\infty. \quad (3.9)$$

For a more precise statement regarding the scattering property (3.9), we refer the reader to Proposition 6.2, and in particular (6.2).

Assuming Propositions 3.1, 3.2 and 3.3 for the moment, we now prove Theorem 1.2.

Proof of Theorem 1.2. We begin with a few quick reductions. Let $(A_j(0), \phi(0))$ be a Coulomb initial data set satisfying (1.5). By the H^2 local well-posedness theorem due to Bergé-de Bouard-Saut [BDS95], there exists a unique solution ϕ to the initial value problem for (CSS) in the Coulomb

gauge on some $(-T, T)$ such that $\phi \in C_t((-T, T); H_x^2)$. Note that, by Proposition 3.2, the covariant bounds (3.1)–(3.3) and (3.4) imply the gauge-dependent estimates (3.5')–(3.7'); in particular, this implies that $\sup_{t \in (-T, T)} \|\phi(t)\|_{H_x^2} \lesssim \varepsilon_1$ for every $(-T, T)$, for ε_1 sufficiently small. This, by the same LWP theorem, establishes the global existence of ϕ , which is unique in $C_t(\mathbb{R}; H_x^2)$.

Therefore, to prove global existence and the decay rate (1.6), it suffices to prove (3.1)–(3.3) and (3.4) by a bootstrap argument; as this is rather standard, we will only give a brief sketch. It is obvious to see that the bootstrap assumptions (3.1)–(3.3) and (3.4) are satisfied for small $|t|$, simply by continuity. Next, for any $T > 0$, (3.1)–(3.3) on $(-T, T)$ are improved to (3.5)–(3.7) provided that $B > 0$ is chosen sufficiently large and $\varepsilon_1 > 0$ is small enough, thanks to Proposition 3.1. Applying Propositions 3.2 and 3.3 in order, we improve (3.4) to (3.8) as well, choosing $B > 0$ larger and ε_1 smaller if necessary. Thus, by a standard continuity argument, we conclude that (3.1)–(3.3) and (3.4) hold for any $T > 0$, thereby finishing the proof of Theorem 1.2.

We are now only left to establish the scattering property (1.7). By symmetry, it suffices to consider the case $t \rightarrow +\infty$. By the preceding bootstrap argument, especially the bound (3.5'), the H_x^2 norm of $e^{-it\Delta}\phi(t)$ is uniformly bounded, i.e.,

$$\sup_{t \in \mathbb{R}} \|e^{-it\Delta}\phi(t)\|_{H_x^2} \lesssim B\varepsilon_1. \quad (3.10)$$

By compactness in the weak-star topology, there exists $f'_\infty \in H_x^2$ and a sequence $t_k \in [0, \infty)$ with $t_k \rightarrow \infty$ such that, in particular,

$$e^{-it_k\Delta}\phi(t_k) \xrightarrow{k \rightarrow \infty} f'_\infty \quad \text{in the distributional sense.}$$

Then by (3.9) and uniqueness of distributional limit, we conclude that $f_\infty = f'_\infty \in H_x^2$.

Let $M > 2$ be a parameter to be chosen below. Given any $0 \leq s < 2$ and ψ such that $\psi \in H_x^2$ and $\widehat{\psi} \in L_\xi^\infty$, note that

$$\begin{aligned} \|\psi\|_{H_x^s} &\leq \|(1 + |\xi|)^s \psi\|_{L_\xi^2(|\xi| \geq M)} + \|(1 + |\xi|)^s \psi\|_{L_\xi^2(|\xi| \leq M)} \\ &\lesssim M^{s-2} \|\psi\|_{H_x^2} + M^{1+s} \|\widehat{\psi}\|_{L_\xi^\infty}. \end{aligned}$$

This implies $\psi \in H_x^s$ and, optimizing the choice of M , we obtain

$$\|\psi\|_{H_x^s} \lesssim \|\psi\|_{H_x^2}^{\frac{s+1}{3}} \|\widehat{\psi}\|_{L_\xi^\infty}^{\frac{2-s}{3}}.$$

Applying this with $\psi = f_\infty$, we first conclude that $f_\infty \in H_x^s$ for any $0 \leq s < 2$. Moreover, another application of the preceding inequality with $\psi = e^{-it\Delta}\phi(t) - f_\infty$, combined with (3.10), $\|f_\infty\|_{H_x^2} < \infty$ and (3.9), allow us to conclude (1.7) as desired. \square

4. COVARIANT CHARGE ESTIMATES

In this section, we prove Proposition 3.1 via covariant techniques to avoid the long-range potentials A_0, A_j . In §4.1, we formulate and prove the covariant charge estimate, which will be our basic tool. Then in §4.2, we derive various commutation formulae, which will be used later to derive the covariant Schrödinger equations satisfied by the various fields we are interested in, e.g., $\phi, \mathbf{D}\phi, \mathbf{D}^{(2)}\phi$. In §4.3, we prove covariant versions of the Gagliardo-Nirenberg inequality. Then finally, in §4.4, we put everything together and give a proof of Proposition 3.1.

4.1. Covariant charge identity. Consider an inhomogeneous covariant Schrödinger equation

$$(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell)\psi = N. \quad (4.1)$$

Multiplying the equation by $\overline{\psi}$, taking the real part and integrating by parts, we see that

$$\frac{1}{2} \int_{\{t=T_2\}} |\psi|^2 dx - \frac{1}{2} \int_{\{t=T_1\}} |\psi|^2 dx = \iint_{(T_1, T_2) \times \mathbb{R}^2} \operatorname{Re}(N\overline{\psi}) dt dx. \quad (4.2)$$

The identity (4.2), which we call the *covariant charge identity*, will be the basis for our proof of Proposition 3.1. From (4.2), the following lemma follows immediately.

Lemma 4.1. *Let ψ be a solution to (4.1) such that $\psi \in C_t L_x^2$. Then for all $t \geq 0$, we have*

$$\|\psi(t)\|_{L_x^2} \lesssim \|\psi(0)\|_{L_x^2} + \int_0^t \|N(t')\|_{L_x^2} dt'. \quad (4.3)$$

An analogous statement holds for $t \leq 0$.

4.2. Commutation formulae. The following lemma is the key computation of this section, and gives formulae for commuting \mathbf{D}_j , \mathbf{J}_j and the covariant Schrödinger operator of (CSS).

Lemma 4.2. *Let (A_μ, ϕ) be a H^2 solution to (CSS). Then the following commutation formulae hold:*

$$(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) \mathbf{D}_k \psi = \mathbf{D}_k (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) \psi + \epsilon_{k\ell} \left(|\phi|^2 \mathbf{D}_\ell \psi + \phi \overline{\mathbf{D}_\ell \phi} \psi \right), \quad (4.4)$$

$$(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) \mathbf{J}_k \psi = \mathbf{J}_k (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) \psi + 2it\epsilon_{k\ell} \left(|\phi|^2 \mathbf{D}_\ell \psi + \phi \overline{\mathbf{D}_\ell \phi} \psi \right). \quad (4.5)$$

Proof. We begin with (4.4). By (CSS), we have

$$\mathbf{D}_k \mathbf{D}_t \psi - \mathbf{D}_t \mathbf{D}_k \psi = -\frac{1}{2} \epsilon_{k\ell} (\phi \overline{\mathbf{D}_\ell \phi} - (\mathbf{D}_\ell \phi) \overline{\phi}) \psi$$

and

$$\begin{aligned} \mathbf{D}_k \mathbf{D}_\ell \mathbf{D}_\ell \psi &= \mathbf{D}_\ell \mathbf{D}_k \mathbf{D}_\ell \psi + iF_{k\ell} \mathbf{D}_\ell \psi \\ &= \mathbf{D}_\ell \mathbf{D}_\ell \mathbf{D}_k \psi + iF_{k\ell} \mathbf{D}_\ell \psi + i\mathbf{D}_\ell (F_{k\ell} \psi) \\ &= \mathbf{D}_\ell \mathbf{D}_\ell \mathbf{D}_k \psi - i\epsilon_{k\ell} \left(|\phi|^2 \mathbf{D}_\ell \psi + \frac{1}{2} (\phi \overline{\mathbf{D}_\ell \phi} + \mathbf{D}_\ell \phi \overline{\phi}) \psi \right). \end{aligned}$$

Thus

$$\mathbf{D}_k (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) \psi = (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) (\mathbf{D}_k \psi) - \epsilon_{k\ell} \left(|\phi|^2 \mathbf{D}_\ell \psi + \phi \overline{\mathbf{D}_\ell \phi} \psi \right),$$

which, upon rearranging the terms, gives (4.4).

Next, we compute the commutator arising from \mathbf{J}_k to prove (4.5). Using (4.4) and $\mathbf{D}_t(t\psi) = t\mathbf{D}_t \psi + \psi$, we see that

$$2it\mathbf{D}_k (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) \psi = (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) (2it\mathbf{D}_k \psi) - 2it\epsilon_{k\ell} \left(|\phi|^2 \mathbf{D}_\ell \psi + \phi \overline{\mathbf{D}_\ell \phi} \psi \right) - 2i\mathbf{D}_k \psi.$$

On the other hand, we easily compute

$$\begin{aligned} x_k (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) \psi &= \mathbf{D}_t (x_k \psi) - i\mathbf{D}_\ell (x_k \mathbf{D}_\ell \psi) + i\mathbf{D}_k \psi \\ &= (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}_\ell) (x_k \psi) + 2i\mathbf{D}_k \psi. \end{aligned}$$

Adding these up and rearranging the terms, we obtain (4.5). \square

Our next lemma gives a simple formula for the commutator between \mathbf{J}_j and \mathbf{D}_k .

Lemma 4.3. *Let (A_μ, ϕ) be an H^2 solution to (CSS). Then the following commutation formula holds.*

$$\mathbf{J}_j \mathbf{D}_k \psi - \mathbf{D}_k \mathbf{J}_j \psi = \delta_{jk} \psi + t\epsilon_{jk} |\phi|^2 \psi. \quad (4.6)$$

Proof. We compute

$$\mathbf{J}_j \mathbf{D}_k \psi - \mathbf{D}_k \mathbf{J}_j \psi = x_j \mathbf{D}_k \psi - \mathbf{D}_k (x_j \psi) + 2it(\mathbf{D}_j \mathbf{D}_k \psi - \mathbf{D}_k \mathbf{D}_j \psi) = \delta_{jk} \psi + t\epsilon_{jk} |\phi|^2 \psi. \quad \square$$

We end this subsection with a Leibniz rule for the cubic nonlinearity of the form $\psi \overline{\psi} \psi$.

Lemma 4.4. *Let (A_μ, ϕ) be an H^2 solution to (CSS). Then the following formulae hold:*

$$\mathbf{D}_k(\psi_1 \overline{\psi_2} \psi_3) = (\mathbf{D}_k \psi_1) \overline{\psi_2} \psi_3 + \psi_1 \overline{\mathbf{D}_k \psi_2} \psi_3 + \psi_1 \overline{\psi_2} \mathbf{D}_k \psi_3 \quad (4.7)$$

$$\mathbf{J}_k(\psi_1 \overline{\psi_2} \psi_3) = (\mathbf{J}_k \psi_1) \overline{\psi_2} \psi_3 - \psi_1 \overline{\mathbf{J}_k \psi_2} \psi_3 + \psi_1 \overline{\psi_2} \mathbf{J}_k \psi_3 \quad (4.8)$$

Proof. We shall only give a proof of (4.8); the other formula (4.7) can be proved similarly. Decompose $\mathbf{J}_k = (x_k - 2tA_k) + 2it\partial_k$. As the first term is real, we have

$$(x_k - 2tA_k)\psi_1 \overline{\psi_2} \psi_3 = (x_k - 2tA_k)\psi_1 \overline{\psi_2} \psi_3 - \psi_1 \overline{(x_k - 2tA_k)} \psi_2 \psi_3 + \psi_1 \overline{\psi_2} (x_k - 2tA_k) \psi_3.$$

On the other hand, for the second term, by Leibniz's rule, we have

$$2it\partial_k(\psi_1 \overline{\psi_2} \psi_3) = (2it\partial_k \psi_1) \overline{\psi_2} \psi_3 - \psi_1 \overline{(2it\partial_k \psi_2)} \psi_3 + \psi_1 \overline{\psi_2} (2it\partial_k \psi_3).$$

Adding these up, we obtain the lemma. \square

4.3. Covariant Gagliardo-Nirenberg inequalities. To deal with some of the error terms arising from commutation, we will need the following covariant version of the standard Gagliardo-Nirenberg inequality $\|D\psi\|_{L_x^4} \lesssim \|\psi\|_{L_x^\infty}^{1/2} \|\Delta\psi\|_{L_x^2}^{1/2}$.

Lemma 4.5. *For $\psi \in \mathcal{S}_x$ and $A_j \in \mathcal{S}_x$, we have*

$$\|\mathbf{D}\psi\|_{L_x^4} \lesssim \|\psi\|_{L_x^\infty}^{1/2} \|\mathbf{D}^{(2)}\psi\|_{L_x^2}^{1/2} \quad (4.9)$$

Proof. Using the identity $\partial_j(\psi^1 \overline{\psi^2}) = \mathbf{D}_j \psi^1 \overline{\psi^2} + \psi^1 \overline{\mathbf{D}_j \psi^2}$ and integrating by parts, we obtain

$$\begin{aligned} \sum_{j=1,2} \|\mathbf{D}_j \psi\|_{L_x^4}^4 &= \sum_{j=1,2} \int \mathbf{D}_j \psi \overline{\mathbf{D}_j \psi} \mathbf{D}_j \psi \overline{\mathbf{D}_j \psi} \, dx \\ &= - \sum_{j=1,2} \int \psi \overline{\mathbf{D}_j \mathbf{D}_j \psi} \mathbf{D}_j \psi \overline{\mathbf{D}_j \psi} \, dx - 2 \sum_{j=1,2} \int \psi \overline{\mathbf{D}_j \psi} \operatorname{Re}(\mathbf{D}_j \psi \overline{\mathbf{D}_j \mathbf{D}_j \psi}) \, dx. \end{aligned}$$

Then using Hölder, we estimate the last line by

$$\lesssim \|\psi\|_{L_x^\infty} \|\mathbf{D}^{(2)}\psi\|_{L_x^2} \left(\sum_{j=1,2} \|\mathbf{D}_j \psi\|_{L_x^4}^4 \right)^{1/2},$$

from which (4.9) follows. \square

We also need a Gagliardo-Nirenberg-type inequality for \mathbf{J} .

Lemma 4.6. *For $\psi \in \mathcal{S}_x$ and $A_j \in \mathcal{S}_x$, we have*

$$\|\mathbf{J}\psi\|_{L_x^4} \lesssim \|\psi\|_{L_x^\infty}^{1/2} \|\mathbf{J}^{(2)}\psi\|_{L_x^2}^{1/2} \quad (4.10)$$

Proof. Note the identities

$$\mathbf{J}_j = 2ite^{i|x|^2/4t} \mathbf{D}_j e^{-i|x|^2/4t}, \quad \mathbf{J}_j \mathbf{J}_k = -4t^2 e^{i|x|^2/4t} \mathbf{D}_j \mathbf{D}_k e^{-i|x|^2/4t}.$$

Thus, using Lemma 4.5, we estimate

$$\begin{aligned} \|\mathbf{J}_j \psi\|_{L_x^4}^2 &= \|2ite^{i|x|^2/4t} \mathbf{D}_j e^{-i|x|^2/4t} \psi\|_{L_x^4}^2 \\ &\lesssim \|\psi\|_{L_x^\infty} \|4t^2 \mathbf{D}^{(2)} e^{-i|x|^2/4t} \psi\|_{L_x^2} = \|\psi\|_{L_x^\infty} \|\mathbf{J}^{(2)}\psi\|_{L_x^2}. \end{aligned} \quad \square$$

4.4. Proof of Proposition 3.1. Using the lemmas proved so far, it is not difficult to prove Proposition 3.1.

Proof of Proposition 3.1. We restrict our attention to $t \geq 0$; the other case is symmetric. Furthermore, for notational simplicity, we introduce the following convention: We write $\psi_1 \cdot \psi_2 \cdot \psi_3$ for a linear combination of products of either ψ_j or $\overline{\psi_j}$ for $j = 1, 2, 3$. If ψ_j is vector-valued (e.g. $\mathbf{D}\phi$), then it may take any of its components or the corresponding complex conjugate. The constants may depend on $g \in \mathbb{C}$.

From Lemmas 4.2, 4.4 and (CSS), it is not difficult to derive the following schematic equations³:

$$(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\phi = \phi \cdot \phi \cdot \phi, \quad (4.11)$$

$$(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{D}_j\phi = \phi \cdot \phi \cdot \mathbf{D}\phi, \quad (4.12)$$

$$(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{D}_j\mathbf{D}_k\phi = \phi \cdot \phi \cdot \mathbf{D}^{(2)}\phi + \phi \cdot \mathbf{D}\phi \cdot \mathbf{D}\phi, \quad (4.13)$$

$$(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{J}_j\phi = \phi \cdot \phi \cdot \mathbf{J}\phi + t(\phi \cdot \phi \cdot \mathbf{D}\phi), \quad (4.14)$$

$$\begin{aligned} (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{J}_j\mathbf{D}_k\phi &= \phi \cdot \phi \cdot \mathbf{J}\mathbf{D}\phi + \phi \cdot \mathbf{D}\phi \cdot \mathbf{J}\phi \\ &\quad + t(\phi \cdot \phi \cdot \mathbf{D}^{(2)}\phi + \phi \cdot \mathbf{D}\phi \cdot \mathbf{D}\phi), \end{aligned} \quad (4.15)$$

$$\begin{aligned} (\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{J}_j\mathbf{J}_k\phi &= \phi \cdot \phi \cdot \mathbf{J}^{(2)}\phi + \phi \cdot \mathbf{J}\phi \cdot \mathbf{J}\phi \\ &\quad + t(\phi \cdot \phi \cdot \mathbf{J}\mathbf{D}\phi + \phi \cdot \phi \cdot \mathbf{D}\mathbf{J}\phi + \phi \cdot \mathbf{D}\phi \cdot \mathbf{J}\phi). \end{aligned} \quad (4.16)$$

We now claim that

$$\|(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{J}\phi(t)\|_{L_x^2} \lesssim (1+t)^{-1} B^3 \varepsilon_1^3 \quad (4.17)$$

$$\|(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{J}\mathbf{D}\phi(t)\|_{L_x^2} \lesssim (1+t)^{-1} B^3 \varepsilon_1^3 \quad (4.18)$$

$$\|(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{J}^{(2)}\phi(t)\|_{L_x^2} \lesssim (1+t)^{-1} \log(2+t) B^3 \varepsilon_1^3. \quad (4.19)$$

Indeed, using Hölder's inequality and Lemmas 4.5, 4.6, we can estimate:

$$\begin{aligned} \|(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{J}\phi(t)\|_{L_x^2} &\lesssim \|\phi\|_{L_x^\infty}^2 \|\mathbf{J}\phi\|_{L_x^2} + t\|\phi\|_{L_x^\infty}^2 \|\mathbf{D}\phi\|_{L_x^2} \\ &\lesssim B^3 \varepsilon_1^3 (1+t)^{-2} \log(2+t) + B^3 \varepsilon_1^3 t(1+t)^{-2} \\ &\lesssim B^3 \varepsilon_1^3 (1+t)^{-1}, \\ \|(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{J}\mathbf{D}\phi(t)\|_{L_x^2} &\lesssim \|\phi\|_{L_x^\infty}^2 \|\mathbf{J}\mathbf{D}\phi\|_{L_x^2} + \|\phi\|_{L_x^\infty} \|\mathbf{D}\phi\|_{L_x^4} \|\mathbf{J}\phi\|_{L_x^4} \\ &\quad + t\|\phi\|_{L_x^\infty}^2 \|\mathbf{D}^{(2)}\phi\|_{L_x^2} + t\|\phi\|_{L_x^\infty} \|\mathbf{D}\phi\|_{L_x^4}^2 \\ &\lesssim B^3 \varepsilon_1^3 (1+t)^{-2} \log(2+t) + B^3 \varepsilon_1^3 t(1+t)^{-2} \\ &\lesssim B^3 \varepsilon_1^3 (1+t)^{-1}, \\ \|(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell)\mathbf{J}^{(2)}\phi(t)\|_{L_x^2} &\lesssim \|\phi\|_{L_x^\infty}^2 \|\mathbf{J}^{(2)}\phi\|_{L_x^2} + \|\phi\|_{L_x^\infty} \|\mathbf{J}\phi\|_{L_x^4}^2 \\ &\quad + t\|\phi\|_{L_x^\infty}^2 \|\mathbf{J}\mathbf{D}\phi\|_{L_x^2} + t\|\phi\|_{L_x^\infty}^2 \|\mathbf{D}\mathbf{J}\phi\|_{L_x^2} \\ &\quad + t\|\phi\|_{L_x^\infty} \|\mathbf{D}\phi\|_{L_x^4} \|\mathbf{J}\phi\|_{L_x^4} \\ &\lesssim B^3 \varepsilon_1^3 (1+t)^{-2} (\log(2+t))^2 + B^3 \varepsilon_1^3 t(1+t)^{-2} \log(2+t) \\ &\lesssim B^3 \varepsilon_1^3 (1+t)^{-1} \log(2+t). \end{aligned}$$

Note that we have used $\|\mathbf{D}\mathbf{J}\phi\|_{L_x^2} \lesssim B\varepsilon_1 \log(2+t)$, which follows immediately from Lemma 4.3.

³We remark that the particular structure $\psi_1 \overline{\psi_2} \psi_3$ of the cubic nonlinearities arising from Lemma 4.2 is important for applying Lemma 4.4 to derive these equations. However, as it is not needed for applying the charge estimate after the equations are derived, we throw it away for notational simplicity.

Proceeding similarly, it is easy to also establish

$$\|(\mathbf{D}_t - i\mathbf{D}_\ell \mathbf{D}^\ell) \mathbf{D}^{(m)} \phi(t)\|_{L_x^2} \lesssim (1+t)^{-2} B^3 \varepsilon_1^3 \quad (4.20)$$

for $m = 0, 1, 2$. Then from (4.17)–(4.20), Proposition 3.1 follows by an application of the charge estimate (4.3). \square

5. FROM COVARIANT TO GAUGE-DEPENDENT BOUNDS

In this brief section, we prove Proposition 3.2 concerning the transition from the covariant estimates (3.5)–(3.7) to the gauge-dependent estimates (3.5')–(3.7') in the Coulomb gauge. Our basic tool is the following set of estimates for the Schrödinger field ϕ and gauge potential A_j in the Coulomb gauge.

Lemma 5.1 (Estimates in Coulomb gauge). *Let (A_μ, ϕ) be an H^2 solution to (CSS) in the Coulomb gauge on $(-T, T)$, which obeys (3.4) and (3.5). Then the following bounds hold for $t \in (-T, T)$:*

$$\|\phi(t)\|_{L_x^p} \leq B\varepsilon_1(1+|t|)^{-1+2/p} \quad \text{for } 2 \leq p \leq \infty, \quad (5.1)$$

$$\|A_j(t)\|_{L_x^p} \lesssim_p B^2 \varepsilon_1^2 (1+|t|)^{-1+2/p} \quad \text{for } 2 < p \leq \infty, \quad (5.2)$$

$$\|DA_j(t)\|_{L_x^p} \lesssim_p B^2 \varepsilon_1^2 (1+|t|)^{-2+2/p} \quad \text{for } 2 \leq p < \infty. \quad (5.3)$$

Proof. The first estimate (5.1) is an immediate consequence of interpolation between the L_x^∞ and L_x^2 bound on ϕ in (3.4) and (3.5), respectively. Next, in order to prove estimates for A_j , recall from (CSS-Coulomb) that A_j satisfies the following elliptic equation in the Coulomb gauge:

$$-\Delta A_j = \frac{1}{2} \varepsilon_{jk} \partial_k |\phi|^2.$$

Thus, $\partial_\ell A_j = (\varepsilon_{jk}/2) R_\ell R_k |\phi|^2$, where $R_j = \partial_j / \sqrt{-\Delta}$ is the Riesz transform. By the L^p boundedness of the Riesz transform, we have for $1 < p < \infty$

$$\|\partial_\ell A_j\|_{L_x^p} \lesssim_p \|\phi\|_{L_x^{2p}}^2.$$

On the other hand, by Hardy-Littlewood-Sobolev fractional integration, we have for $2 < p < \infty$

$$\|A_j\|_{L_x^p} \lesssim_p \|\phi\|_{L_x^{4p/(2+p)}}^2.$$

Thus, the desired estimates (5.2) and (5.3) for $p < \infty$ are an easy consequence of (5.1). On the other hand, the case $p = \infty$ of (5.2) follows from the Gagliardo-Nirenberg inequality $\|A_j\|_{L_x^\infty} \lesssim \|A_j\|_{L_x^4}^{1/2} \|DA_j\|_{L_x^4}^{1/2}$ and the case $p = 4$ of (5.2), (5.3). \square

Proof of Proposition 3.2. For simplicity, we restrict to $t \geq 0$. Without loss of generality, we may assume that $B\varepsilon_1 \leq 1$. Expanding out the covariant derivatives, we have

$$\begin{aligned} \mathbf{D}_j \phi &= \partial_j \phi + iA_j \phi, \\ \mathbf{D}_j \mathbf{D}_k \phi &= \partial_j \partial_k \phi + i(\partial_j A_k) \phi + iA_k \partial_j \phi + iA_j \partial_k \phi - A_j A_k \phi. \end{aligned}$$

Using Hölder, (3.4), (3.5) and (5.1)–(5.3) we obtain:

$$\begin{aligned} \|\mathbf{D}\phi(t) - D\phi(t)\|_{L_x^2} &\lesssim \|A(t)\|_{L_x^\infty} \|\phi(t)\|_{L_x^2} \lesssim B^3 \varepsilon_1^3 (1+t)^{-1}, \\ \|\mathbf{D}^{(2)}\phi(t) - D^{(2)}\phi(t)\|_{L_x^2} &\lesssim \|DA(t)\|_{L_x^\infty} \|\phi(t)\|_{L_x^\infty} + \|A(t)\|_{L_x^\infty} \|D\phi(t)\|_{L_x^2} + \|A(t)\|_{L_x^\infty}^2 \|\phi(t)\|_{L_x^2} \\ &\lesssim B^3 \varepsilon_1^3 (1+t)^{-1}, \end{aligned}$$

where on the last line, we additionally used $B\varepsilon_1 \leq 1$ and the estimate for $\|D\phi(t) - \mathbf{D}\phi(t)\|_{L_x^2}$ from the first line to estimate $\|D\phi(t)\|_{L_x^2}$.

Similarly, expanding out \mathbf{J} and \mathbf{D} , we have

$$\begin{aligned}\mathbf{J}_j\phi &= J_j\phi - 2tA_j\phi, \\ \mathbf{J}_j\mathbf{D}_k\phi &= J_j\partial_k\phi + iA_kJ_j\phi - 2t(\partial_jA_k)\phi - 2tA_j\partial_k\phi - 2itA_jA_k\phi, \\ \mathbf{J}_j\mathbf{J}_k\phi &= J_jJ_k\phi - 2tA_kJ_j\phi - 4it^2(\partial_jA_k)\phi - 2tA_jJ_k\phi + 4t^2A_jA_k\phi.\end{aligned}$$

Then, as before, we estimate via Hölder, (3.4)–(3.7) and (5.1)–(5.3):

$$\begin{aligned}\|\mathbf{J}\phi(t) - J\phi(t)\|_{L_x^2} &\lesssim t\|A(t)\|_{L_x^\infty}\|\phi(t)\|_{L_x^2} \lesssim B^3\varepsilon_1^3, \\ \|\mathbf{J}\mathbf{D}\phi(t) - JD\phi(t)\|_{L_x^2} &\lesssim \|A(t)\|_{L_x^\infty}\|J\phi(t)\|_{L_x^2} + t\|DA(t)\|_{L_x^2}\|\phi(t)\|_{L_x^\infty} \\ &\quad + t\|A(t)\|_{L_x^\infty}\|D\phi(t)\|_{L_x^2} + t\|A(t)\|_{L_x^\infty}^2\|\phi(t)\|_{L_x^2} \\ &\lesssim B^3\varepsilon_1^3, \\ \|\mathbf{J}^{(2)}\phi(t) - J^{(2)}\phi(t)\|_{L_x^2} &\lesssim t\|A(t)\|_{L_x^\infty}\|J\phi(t)\|_{L_x^2} + t^2\|DA(t)\|_{L_x^2}\|\phi(t)\|_{L_x^\infty} \\ &\quad + t^2\|A(t)\|_{L_x^\infty}^2\|\phi(t)\|_{L_x^2} \\ &\lesssim B^3\varepsilon_1^3\log(2+t).\end{aligned}$$

Taking $\varepsilon_1 > 0$ sufficiently small Proposition 3.2 follows. \square

6. DECAY FOR THE SCHRÖDINGER FIELD

In this section, we prove Proposition 3.3, thereby completing the proof of Theorem 1.2. In §6.1, we reduce the proof of Proposition 3.3 to establishing a uniform bound on $\|\widehat{\phi}(t)\|_{L_x^\infty}$; see Proposition 6.2. Then in §6.2, we rewrite the Schrödinger equation in the Coulomb gauge and reveal the *cubic null structure* of (CSS) in this gauge. Finally, in §6.3, we give a proof of Proposition 6.2.

6.1. Reduction of Proposition 3.3. The first step in the proof of the sharp $|t|^{-1}$ decay of ϕ is given by the following standard lemma:

Lemma 6.1. *For $\psi \in C_t\mathcal{S}_x$ and $|t| \geq 1$ we have*

$$\|\psi(t)\|_{L_x^\infty} \lesssim \frac{1}{|t|}\|\widehat{\psi}(t)\|_{L_x^\infty} + \frac{1}{|t|^{5/4}}\sum_{m=0}^2\|J^{(m)}\psi(t)\|_{L_x^2}. \quad (6.1)$$

For a proof of the above, we refer to [HN98]. Thanks to this, one can easily see that establishing Proposition 3.3 can be reduced to the following proposition:

Proposition 6.2. *Let (A_μ, ϕ) be an H^2 solution to (CSS) in the Coulomb gauge which satisfies the initial data estimate*

$$\sum_{m=0}^2\|\mathbf{D}^{(m)}\phi(0)\|_{L_x^2} + \||x|\phi(0)\|_{L_x^2} + \||x|\mathbf{D}\phi(0)\|_{L_x^2} + \||x|^2\phi(0)\|_{L_x^2} \leq \varepsilon_1. \quad (1.5)$$

Assume furthermore that (A_μ, ϕ) obeys the bootstrap assumption

$$\|\phi(t)\|_{L_x^\infty} \leq B\varepsilon_1(1+|t|)^{-1} \quad (3.4)$$

and the improved bounds

$$\|\phi(t)\|_{L_x^2} + \|D\phi(t)\|_{L_x^2} + \|D^{(2)}\phi(t)\|_{L_x^2} \leq \frac{B}{100}\varepsilon_1 \quad (3.5')$$

$$\|J\phi(t)\|_{L_x^2} + \|JD\phi(t)\|_{L_x^2} \leq \frac{B}{100}\varepsilon_1\log(2+|t|) \quad (3.6')$$

$$\|J^{(2)}\phi(t)\|_{L_x^2} \leq \frac{B}{100}\varepsilon_1(\log(2+|t|))^2 \quad (3.7')$$

for $t \geq 0$. Then, for any $0 \leq t_1 \leq t_2$ we have

$$\|e^{it_2|\xi|^2}\widehat{\phi}(t_2, \xi) - e^{it_1|\xi|^2}\widehat{\phi}(t_1, \xi)\|_{L_\xi^\infty} \lesssim B^3 \varepsilon_1^3 (1 + t_1)^{-1/10}. \quad (6.2)$$

In particular, given $\delta > 0$, choosing B sufficiently large and ε_1 small enough, we have

$$\|\widehat{\phi}(t)\|_{L_\xi^\infty} \leq \delta B \varepsilon_1, \quad (6.3)$$

and, moreover, there exists $\widehat{f}_\infty \in L_\xi^\infty$ such that

$$\|e^{it|\xi|^2}\widehat{\phi}(t) - \widehat{f}_\infty\|_{L_\xi^\infty} \lesssim B^3 \varepsilon_1^3 (1 + t)^{-1/10}, \quad (6.4)$$

for all $t \geq 0$. An analogous statement holds for $t \leq 0$.

In the rest of this section, we will be concerned with the proof of Proposition 6.2

6.2. Cubic null structure in the Coulomb gauge. We first split A_0 into its quadratic and quartic parts, i.e., $A_0 = A_{0,1} + A_{0,2}$, where

$$\begin{aligned} A_{0,1} &= \frac{i}{2}(-\Delta)^{-1} \left(-\partial_1(\phi\overline{\partial_2\phi} - \partial_2\phi\overline{\phi}) + \partial_2(\phi\overline{\partial_1\phi} - \partial_1\phi\overline{\phi}) \right), \\ A_{0,2} &= -(-\Delta)^{-1} \left(\partial_1(A_2|\phi|^2) - \partial_2(A_1|\phi|^2) \right). \end{aligned}$$

Then we may write the Schrödinger equation in the Coulomb gauge as

$$\partial_t \phi - i\Delta \phi = \mathcal{N} + \mathcal{R} + \mathcal{T},$$

where

$$\begin{aligned} \mathcal{N} &:= -iA_{0,1}\phi - 2A_\ell\partial_\ell\phi \\ \mathcal{R} &:= -iA_{0,2}\phi - iA_\ell A_\ell\phi, \\ \mathcal{T} &:= ig|\phi|^2\phi. \end{aligned}$$

In words, \mathcal{N} and \mathcal{R} are the cubic and quintic terms arising from the covariant Schrödinger operator $(\mathbf{D}_t - i\mathbf{D}^\ell\mathbf{D}_\ell)$, respectively, and \mathcal{T} is the cubic self-interaction term $g|\phi|^2\phi$. We may write \mathcal{N} and \mathcal{R} more explicitly as follows:

$$\begin{aligned} \mathcal{N} &= (-\Delta)^{-1} \left(-\partial_1\phi\overline{\partial_2\phi} + \partial_2\phi\overline{\partial_1\phi} \right) \phi \\ &\quad + (-\Delta)^{-1} (\partial_2|\phi|^2)\partial_1\phi - (-\Delta)^{-1} (\partial_1|\phi|^2)\partial_2\phi, \\ \mathcal{R} &= i(-\Delta)^{-1} \left(\partial_1(A_2|\phi|^2) - \partial_2(A_1|\phi|^2) \right) \phi - iA_\ell A_\ell\phi. \end{aligned}$$

Define $f(t, x) := (e^{-it\Delta}\phi(t))(t, x)$. Then

$$\partial_t f(t) = e^{-it\Delta} (\mathcal{N}(t) + \mathcal{R}(t) + \mathcal{T}(t)),$$

and thus taking the Fourier transform,

$$\partial_t \widehat{f}(t) = e^{it|\xi|^2} (\widehat{\mathcal{N}}(t) + \widehat{\mathcal{R}}(t) + \widehat{\mathcal{T}}(t)). \quad (6.5)$$

Then, in order to estimate $|\widehat{\phi}(t)| = |\widehat{f}(t)|$, we estimate the right-hand side of (6.5), viz. $\widehat{\mathcal{N}}$, $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{T}}$ in L_ξ^∞ . With this goal in mind, we shall now demonstrate the *cubic null structure* of \mathcal{N} . We

start by writing \mathcal{N} in the Fourier space as follows:

$$\begin{aligned}\widehat{\mathcal{N}}(t, \xi) &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\eta|^{-2} [(\eta_1 - \sigma_1)\sigma_2 - (\eta_2 - \sigma_2)\sigma_1] \widehat{\phi}(t, \eta - \sigma) \widehat{\phi}(t, \sigma) \widehat{\phi}(t, \xi - \eta) \, d\sigma d\eta \\ &\quad + \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\eta|^{-2} [-\eta_2(\xi_1 - \eta_1) + \eta_1(\xi_2 - \eta_2)] \widehat{\phi}(t, \eta - \sigma) \widehat{\phi}(t, \sigma) \widehat{\phi}(t, \xi - \eta) \, d\sigma d\eta \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\eta|^{-2} [(\xi_2 + \sigma_2)\eta_1 - (\xi_1 + \sigma_1)\eta_2] \widehat{\phi}(t, \eta - \sigma) \widehat{\phi}(t, \sigma) \widehat{\phi}(t, \xi - \eta) \, d\sigma d\eta.\end{aligned}$$

Next, we change variables ($\sigma \rightarrow \sigma + \eta - \xi$), and write the above expression in terms of f :

$$\begin{aligned}4\pi^2 \widehat{\mathcal{N}}(t, \xi) &= e^{-it|\xi|^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{it\varphi(\eta, \sigma)} |\eta|^{-2} m(\eta, \sigma) \widehat{f}(t, \xi - \sigma) \widehat{f}(t, \sigma + \eta - \xi) \widehat{f}(t, \xi - \eta) \, d\sigma d\eta, \\ \text{where } \varphi(\eta, \sigma) &:= |\xi|^2 - |\xi - \sigma|^2 + |\sigma + \eta - \xi|^2 - |\xi - \eta|^2 = 2\eta \cdot \sigma, \\ m(\eta, \sigma) &:= \sigma_2\eta_1 - \sigma_1\eta_2 = \frac{1}{2}(\eta_1\partial_{\eta_2}\varphi - \eta_2\partial_{\eta_1}\varphi).\end{aligned}$$

The identity relating m and φ above is a null structure and we can use it to integrate by parts in frequency. Indeed, using the identities

$$\partial_{\eta_j}(-\log|\eta|) = -\eta_j|\eta|^{-2} \quad \text{and} \quad \partial_{\eta_j}e^{it\varphi(\eta, \sigma)} = it\partial_{\eta_j}\varphi(\eta, \sigma)e^{it\varphi(\eta, \sigma)},$$

we see that $e^{it|\xi|^2}\widehat{\mathcal{N}}(t, \xi)$ is given by

$$-\frac{1}{8\pi^2 it} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (d_\eta(-\log|\eta|) \wedge d_\eta e^{it\varphi(\eta, \sigma)}) \widehat{f}(t, \xi - \sigma) \widehat{f}(t, \sigma + \eta - \xi) \widehat{f}(t, \xi - \eta) \, d\sigma d\eta \quad (6.6)$$

where $d_\eta f \wedge d_\eta g$ is a shorthand for $\partial_{\eta_1}f\partial_{\eta_2}g - \partial_{\eta_2}f\partial_{\eta_1}g$. Notice the crucial gain of a power of t^{-1} . Moreover, when d_η is integrated by parts off from $e^{it\varphi(\eta, \sigma)}$, the contribution of $d_\eta(-\log|\eta|)$ is zero, thanks to the fact that $\partial_{\eta_1}\partial_{\eta_2}(-\log|\eta|) - \partial_{\eta_2}\partial_{\eta_1}(-\log|\eta|) = 0$. This special cancellation is the aforementioned *strong, genuinely cubic null structure* of \mathcal{N} .

6.3. Uniform boundedness of $\widehat{\phi}$ in the Coulomb gauge. In this subsection, we prove Proposition 6.2, which concludes the proof of Theorem 1.2.

Proof of Proposition 6.2. For simplicity, we again restrict to $t \geq 0$. Under the apriori assumptions (3.4), (3.5')–(3.7'), we claim that it suffices to show

$$\|\widehat{\mathcal{N}}(t)\|_{L_\xi^\infty} \lesssim B^3 \varepsilon_1^3 (1+t)^{-9/8} \quad (6.7)$$

$$\|\widehat{\mathcal{R}}(t)\|_{L_\xi^\infty} \lesssim B^5 \varepsilon_1^5 (1+t)^{-11/10} \quad (6.8)$$

$$\|\widehat{\mathcal{T}}(t)\|_{L_\xi^\infty} \lesssim B^5 \varepsilon_1^5 (1+t)^{-11/10}. \quad (6.9)$$

Indeed, integrating in t the identity (6.5), the bounds (6.7)–(6.9) immediately imply (6.2). Moreover since the initial data bound (1.5) implies

$$\|\widehat{\phi}(0)\|_{L_\xi^\infty} \lesssim \|\phi(0)\|_{L_x^1} \lesssim \|(1+|x|^2)\phi(0)\|_{L_x^2} \lesssim \varepsilon_1, \quad (6.10)$$

we easily see how (6.3) follows.

Before we proceed to establish the claim, we point out a few consequences of our apriori assumptions which will be useful later. Thanks to the logarithmic growth in (3.6')–(3.7'), for any $p_0 > 0$ we have

$$\|J\phi(t)\|_{L_x^2} + \|JD\phi(t)\|_{L_x^2} + \|J^{(2)}\phi(t)\|_{L_x^2} \lesssim_{p_0} B\varepsilon_1(1+t)^{p_0}.$$

Since x_j conjugates to J_j via $e^{it\Delta}$ (i.e., $J_j\phi = e^{it\Delta}(x_j f)$), we have

$$\|f(t)\|_{L_x^2} + \||x|^2 f(t)\|_{L_x^2} \lesssim_{p_0} B\varepsilon_1(1+t)^{p_0}.$$

Moreover, proceeding as in (6.10), we see that

$$\|\widehat{\phi}(t)\|_{L_\xi^\infty} = \|\widehat{f}(t)\|_{L_\xi^\infty} \lesssim \|(1 + |x|^2)f(t)\|_{L_x^2} \lesssim_{p_0} B\varepsilon_1(1+t)^{p_0}.$$

In what follows, we fix $0 < p_0 < 1/20$.

6.4. Estimate of the cubic null form \mathcal{N} . Here, we shall prove (6.7). We begin by integrating (6.6) by parts in η_1 and η_2 . Then, since $\partial_{\eta_1}(\eta_2|\eta|^{-2}) - \partial_{\eta_2}(\eta_1|\eta|^{-2}) = 0$, we can write

$$\begin{aligned} e^{it|\xi|^2}\widehat{\mathcal{N}}(t, \xi) &= \frac{1}{8\pi^2 it} (N_1(t, \xi) + N_2(t, \xi) + N_3(t, \xi) + N_4(t, \xi)), \\ N_1(t, \xi) &:= - \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{it\varphi(\eta, \sigma)} |\eta|^{-2} \eta_1 \widehat{f}(t, \xi - \sigma) \partial_{\eta_2} \widehat{f}(t, \sigma + \eta - \xi) \widehat{f}(t, \xi - \eta) \, d\sigma d\eta, \\ N_2(t, \xi) &:= \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{it\varphi(\eta, \sigma)} |\eta|^{-2} \eta_2 \widehat{f}(t, \xi - \sigma) \partial_{\eta_1} \widehat{f}(t, \sigma + \eta - \xi) \widehat{f}(t, \xi - \eta) \, d\sigma d\eta, \\ N_3(t, \xi) &:= - \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{it\varphi(\eta, \sigma)} |\eta|^{-2} \eta_1 \widehat{f}(t, \xi - \sigma) \widehat{f}(t, \sigma + \eta - \xi) \partial_{\eta_2} \widehat{f}(t, \xi - \eta) \, d\sigma d\eta, \\ N_4(t, \xi) &:= \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{it\varphi(\eta, \sigma)} |\eta|^{-2} \eta_2 \widehat{f}(t, \xi - \sigma) \widehat{f}(t, \sigma + \eta - \xi) \partial_{\eta_1} \widehat{f}(t, \xi - \eta) \, d\sigma d\eta. \end{aligned} \quad (6.11)$$

Let us fix $\chi : [0, \infty) \rightarrow [0, 1]$ a smooth function supported in $[0, 2]$ and equal to 1 in $[0, 1]$. For $M > 0$ we let $P_{\leq M}$ denote the projection operator defined by the Fourier multiplier $\xi \rightarrow \chi(|\xi|M^{-1})$, i.e. $(\mathcal{F}P_{\leq M}f)(\xi) = \chi(|\xi|M^{-1})\widehat{f}(\xi)$. We split N_1 as $N_1 = M_1 + M_2$, where

$$\begin{aligned} M_1(t, \xi) &:= - \int e^{it\varphi(\eta, \sigma)} |\eta|^{-2} \chi(\eta(1+t)^{1/4}) \eta_1 \widehat{f}(t, \xi - \sigma) \partial_{\eta_2} \widehat{f}(t, \sigma + \eta - \xi) \widehat{f}(t, \xi - \eta) \, d\sigma d\eta, \\ M_2(t, \xi) &:= \int e^{it\varphi(\eta, \sigma)} |\eta|^{-2} [\chi(\eta(1+t)^{1/4}) - 1] \eta_1 \widehat{f}(t, \xi - \sigma) \partial_{\eta_2} \widehat{f}(t, \sigma + \eta - \xi) \widehat{f}(t, \xi - \eta) \, d\sigma d\eta. \end{aligned}$$

We then estimate

$$\begin{aligned} |M_1(t, \xi)| &\lesssim \|\partial_2 \widehat{f}(t)\|_{L^2} \|\widehat{f}(t)\|_{L^2} \int |\eta|^{-1} \chi(\eta(1+t)^{1/4}) |\widehat{f}(t, \xi - \eta)| \, d\eta \\ &\lesssim \|\partial_2 \widehat{f}(t)\|_{L^2} \|\widehat{f}(t)\|_{L^2} \|\widehat{f}(t)\|_{L^\infty} (1+t)^{-1/4} \lesssim B^3 \varepsilon_1^3 (1+t)^{-1/4+2p_0}, \end{aligned}$$

and

$$\begin{aligned} |M_2(t, \xi)| &\lesssim \left\| P_{\geq (1+t)^{-1/4}} \partial_1 \Delta^{-1} (e^{-it\Delta} x_2 \overline{f}(t) \phi(t)) \right\|_{L^2} \|f(t)\|_{L^2} \\ &\lesssim (1+t)^{1/4} \|x_2 f(t)\|_{L^2} \|\phi(t)\|_{L^\infty} \|f(t)\|_{L^2} \lesssim B^3 \varepsilon_1^3 (1+t)^{-3/4+p_0}. \end{aligned}$$

The term N_2 can be estimated in the same way as N_1 .

To estimate N_3 we perform the same splitting as above, but we will need slightly different estimates. We write $N_3 = M_3 + M_4$, where

$$\begin{aligned} M_3(t, \xi) &:= - \int e^{it\varphi(\eta, \sigma)} |\eta|^{-2} \eta_1 \chi(\eta(1+t)^{1/4}) \widehat{f}(t, \xi - \sigma) \widehat{f}(t, \sigma + \eta - \xi) \partial_{\eta_2} \widehat{f}(t, \xi - \eta) \, d\sigma d\eta, \\ M_4(t, \xi) &:= \int e^{it\varphi(\eta, \sigma)} |\eta|^{-2} \eta_1 [\chi(\eta(1+t)^{1/4}) - 1] \widehat{f}(t, \xi - \sigma) \widehat{f}(t, \sigma + \eta - \xi) \partial_{\eta_2} \widehat{f}(t, \xi - \eta) \, d\sigma d\eta. \end{aligned}$$

We then estimate

$$\begin{aligned} |M_3(t, \xi)| &\lesssim \|\widehat{f}(t)\|_{L^2} \|\widehat{f}(t)\|_{L^2} \int |\eta|^{-1} \chi(\eta(1+t)^{1/4}) |\partial_2 \widehat{f}(t, \xi - \eta)| \, d\eta \\ &\lesssim \|f(t)\|_{L^2}^2 \|\partial_2 \widehat{f}(t)\|_{L^6} (1+t)^{-1/6} \lesssim B^2 \varepsilon_1^2 \|(1 + |x|^2)f(t)\|_{L^2} (1+t)^{-1/6} \\ &\lesssim B^3 \varepsilon_1^3 (1+t)^{-1/6+p_0}. \end{aligned}$$

The second term is bounded as follows:

$$\begin{aligned} |M_4(t, \xi)| &\lesssim \left\| P_{\geq(1+t)^{-1/4}} \partial_1 \Delta^{-1}(\phi(t) \bar{\phi}(t)) \right\|_{L^2} \|\partial_2 \widehat{f}(t)\|_{L^2} \\ &\lesssim (1+t)^{1/4} \|\phi(t)\|_{L^2} \|\phi(t)\|_{L^\infty} \|x_2 f(t)\|_{L^2} \lesssim B^3 \varepsilon_1^3 (1+t)^{-3/4+p_0}. \end{aligned}$$

The term N_4 can be estimated identically. We can then conclude that $|N_j| \lesssim (1+t)^{-1/8}$, for all $j = 1, \dots, 4$. In view of (6.11) we obtain the desired bound (6.7) for $\widehat{\mathcal{N}}(t)$.

6.5. Estimates for the quintic terms \mathcal{R} . Under our apriori assumptions we now want to prove:

$$\left| \mathcal{F} \left(\Delta^{-1} (\partial_1 (A_2 |\phi|^2) - \partial_2 (A_1 |\phi|^2)) \phi \right) \right| \lesssim B^5 \varepsilon_1^5 (1+t)^{-11/10}, \quad (6.12)$$

$$\left| \mathcal{F} (A^\ell A_\ell \phi) \right| \lesssim B^5 \varepsilon_1^5 (1+t)^{-11/10}. \quad (6.13)$$

Let us start by estimating the first contribution in the right-hand side of (6.12):

$$\begin{aligned} \left| \mathcal{F} \left(\Delta^{-1} \partial_1 (A_2 |\phi|^2) \phi \right) \right| &\lesssim \left| \mathcal{F} \left(\Delta^{-1} \partial_1 (A_2 |\phi|^2) \right) (\xi) * \widehat{\phi}(\xi) \right| \\ &\lesssim \left\| \mathcal{F} \left(\Delta^{-1} \partial_1 (A_2 |\phi|^2) \right) \right\|_{L^{6/5}} \|\widehat{\phi}(\xi)\|_{L^6} \\ &\lesssim [I(t) + II(t)] B \varepsilon_1 (1+t)^{p_0}, \end{aligned}$$

having defined

$$I(t) = \left\| \mathcal{F} \left(\Delta^{-1} \partial_1 (A_2(t) |\phi(t)|^2) \right) \right\|_{L^{6/5}(|\xi| \geq 1)} \quad (6.14)$$

$$II(t) = \left\| \mathcal{F} \left(\Delta^{-1} \partial_1 (A_2(t) |\phi(t)|^2) \right) \right\|_{L^{6/5}(|\xi| \leq 1)}. \quad (6.15)$$

Using an $L^3 \times L^2$ Hölder's inequality we can bound

$$\begin{aligned} I(t) &= \left\| \xi_1 |\xi|^{-2} \mathcal{F} (A_2(t) |\phi(t)|^2) \right\|_{L^{6/5}(|\xi| \geq 1)} \lesssim \left\| |\xi|^{-1} \right\|_{L^3(|\xi| \geq 1)} \left\| \mathcal{F} (A_2(t) |\phi(t)|^2) \right\|_{L^2} \\ &\lesssim \|A_2(t)\|_{L^\infty} \|\phi(t)\|_{L^2} \|\phi(t)\|_{L^\infty} \lesssim B^4 \varepsilon_1^4 (1+t)^{-2}. \end{aligned}$$

To estimate the contribution coming from low frequencies we use again Hölder followed by the Hausdorff-Young inequality:

$$\begin{aligned} II(t) &= \left\| \xi_1 |\xi|^{-2} \mathcal{F} (A_2(t) |\phi(t)|^2) \right\|_{L^{6/5}(|\xi| \leq 1)} \lesssim \left\| |\xi|^{-1} \right\|_{L^{3/2}(|\xi| \leq 1)} \left\| \mathcal{F} (A_2(t) |\phi(t)|^2) \right\|_{L^6} \\ &\lesssim \left\| A_2(t) |\phi(t)|^2 \right\|_{L^{6/5}} \lesssim \|A_2(t)\|_{L^\infty} \|\phi(t)\|_{L^2} \|\phi(t)\|_{L^3} \lesssim B^4 \varepsilon_1^4 (1+t)^{-4/3}. \end{aligned}$$

This shows that

$$\left| \mathcal{F} \left(\Delta^{-1} \partial_1 (A_2 |\phi|^2) \phi \right) \right| \lesssim B^5 \varepsilon_1^5 (1+t)^{-11/10}.$$

Since an identical estimate can be obtained if we exchange the indices 1 and 2, we have shown that (6.12) holds.

To prove (6.13) we first bound

$$\left| \mathcal{F} (A^\ell A_\ell \phi) \right| = \left| \widehat{A}^\ell * \widehat{A}_\ell * \widehat{\phi} \right| \lesssim \|\widehat{A}^\ell\|_{L^1} \|\widehat{A}_\ell\|_{L^1} \|\widehat{\phi}\|_{L^\infty}.$$

To obtain the desired bound it then suffices to show

$$\|\widehat{A}_\ell(t)\|_{L^1} \lesssim B^2 \varepsilon_1^2 (1+t)^{-2/3}. \quad (6.16)$$

Since the two cases $\ell = 1, 2$ are identical we only look at $\ell = 2$. We can estimate

$$\|\widehat{A}_2(t)\|_{L^1} \lesssim \left\| \mathcal{F} \left(\Delta^{-1} \partial_1 |\phi|^2 \right) (t) \right\|_{L^1} \lesssim III(t) + IV(t),$$

where

$$III(t) = \left\| |\xi|^{-2} \mathcal{F}(\partial_1 |\phi|^2)(t) \right\|_{L^1(|\xi| \geq 1)}, \quad (6.17)$$

$$IV(t) = \left\| \xi_1 |\xi|^{-2} \mathcal{F} |\phi|^2(t) \right\|_{L^1(|\xi| \leq 1)}. \quad (6.18)$$

It is clear that

$$III(t) \lesssim \left\| \mathcal{F}(\partial_1 |\phi(t)|^2) \right\|_{L^2} \lesssim \|\phi(t)\|_{H^1} \|\phi(t)\|_{L^\infty} \lesssim B^2 \varepsilon_1^2 (1+t)^{-1},$$

which is more than sufficient. Furthermore, using Hölder's inequality we have

$$IV(t) \lesssim \left\| |\xi|^{-1} \mathcal{F} |\phi(t)|^2 \right\|_{L^1(|\xi| \leq 1)} \lesssim \left\| \mathcal{F} |\phi(t)|^2 \right\|_{L^3} \lesssim \|\phi(t)^2\|_{L^{3/2}} \lesssim B^2 \varepsilon_1^2 (1+t)^{-2/3}.$$

This gives us (6.16) and concludes the proof of (6.13).

6.6. Estimate for the cubic term \mathcal{T} . Finally, we shall establish (6.9). We begin by estimating

$$\begin{aligned} |\mathcal{F}(|\phi|^2 \phi)(t, \xi)| &= |\mathcal{F}(e^{-it\Delta} |\phi|^2 \phi)(t, \xi)| \lesssim \left\| \mathcal{F}(e^{-it\Delta} |\phi|^2 \phi)(t) \right\|_{H_\xi^2} \\ &\lesssim \left\| |\phi|^2 \phi(t) \right\|_{L^2} + \left\| x^2 e^{-it\Delta} (|\phi|^2 \phi)(t) \right\|_{L^2}. \end{aligned}$$

By an $L^2 \times L^\infty \times L^\infty$ estimate, the first summand above is easily seen to satisfy a bound of the form $B^3 \varepsilon_1^3 (1+t)^{-2}$. For the second one, we use the fact that $x e^{-it\Delta} = e^{-it\Delta} J$, and the Leibniz rule (4.8) for J , to see that

$$\begin{aligned} \left\| x^2 e^{-it\Delta} (|\phi|^2 \phi) \right\|_{L^2} &= \left\| J^{(2)}(|\phi|^2 \phi) \right\|_{L^2} \lesssim \left\| J^{(2)} \phi \right\|_{L^2} \|\phi\|_{L^\infty}^2 + \left\| J \phi \right\|_{L^4}^2 \|\phi\|_{L^\infty} \\ &\lesssim B^3 \varepsilon_1^3 (1+t)^{-2} \log^2(2+t), \end{aligned}$$

having used the Gagliardo-Nirenberg inequality (4.10) with $A = 0$ and the apriori bounds (3.4), (3.6') and (3.7') in the last inequality. This completes the proof of (6.9). \square

REFERENCES

- [BDS95] Bergé, L., De Bouard, A. and Saut, J.-C. Blowing up time-dependent solutions of the planar, Chern-Simons gauged nonlinear Schrödinger equation. *Nonlinearity* 8 (1995), no. 2, 235-253.
- [CK90] Christodoulou, D. and Klainerman, S. Asymptotic properties of linear field equations in Minkowski space. *Comm. Pure Appl. Math.* 43 (1990), no. 2, 137-199.
- [CK93] Christodoulou, D. and Klainerman, S. The global nonlinear stability of the Minkowski space. *Princeton Mathematical Series* 41. Princeton University Press, Princeton, NJ, 1993. x+514 pp.
- [DS07] Demoulini, S. Global existence for a nonlinear Schrödinger-Chern-Simons system on a surface. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (2007), no. 2, 207-225.
- [DS09] Demoulini, S. and Stuart, D. Adiabatic limit and the slow motion of vortices in a Chern-Simons-Schrödinger system. *Comm. Math. Phys.* 290 (2009), no. 2, 597-632.
- [Du95] Dunne, G. Self-Dual Chern-Simons Theories. Series: Lecture Notes in Physics Monographs, M 36, X, 222 p, 1995.
- [GMS09] Germain, P., Masmoudi, N. and Shatah, J. Global solutions for 3-d quadratic Schrödinger equations. *Int. Math. Res. Notices* (2009), no. 3, 414-432.
- [HN98] Hayashi, N. and Naumkin, P. Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations. *Amer. J. Math.* 120 (1998), 369-389.
- [HZ09] Horvathy, P. and Zhang, P. Vortices in (Abelian) Chern-Simons gauge theory. *Phys. Rep.* 481 (2009), no. 5-6, 83-142.
- [Hu09] Huh, H. Blow-up solutions of the Chern-Simons-Schrödinger equations. *Nonlinearity* 22 (2009), no. 5, 967-974.

- [Hu13] Huh, H. Energy solution to the Chern-Simons-Schrödinger equations. *Abstr. Appl. Anal.* (2013), Art. ID 590653, 7 pp.
- [JP90-1] Jackiw, R. and Pi, S.-Y. Soliton solutions to the gauged nonlinear Schrödinger equation on the plane. *Phys. Rev. Lett.* 64 (1990), no. 25, 2969-2972.
- [JP90-2] Jackiw, R. and Pi, S.-Y. Classical and quantal nonrelativistic Chern-Simons theory. *Phys. Rev. D* 42 (1990), no. 10, 3500-3513.
- [JP92] Jackiw, R. and Pi, S.-Y. Self-dual Chern-Simons solitons. Low-dimensional field theories and condensed matter physics (Kyoto, 1991). *Progr. Theoret. Phys. Suppl.* (1992), no. 107, 1-40.
- [Joh79] John, F. Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Math.* 28 (1979), no. 1-3, 235-268.
- [Joh81] John, F. Blow-up for quasilinear wave equations in three space dimensions. *Comm. Pure Appl. Math.* 34 (1981), no. 1, 29-51.
- [KP11] Kato, J. and Pusateri, F. A new proof of long range scattering for critical nonlinear Schrödinger equations. *Diff. Int. Equations* 24 (2011), no. 9-10, 923-940.
- [Kla85a] Klainerman, S. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.* 38 (1985), no. 3, 321-332.
- [Kla86] Klainerman, S. The null condition and global existence for systems of wave equations. Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 293-326. *Lectures in Appl. Math.* 23, Amer. Math. Soc., Providence, RI, 1986.
- [LST-p] Liu, B., Smith, P. and Tataru, D. Local wellposedness of Chern-Simons-Schrödinger. *arXiv:1212.1476*, (2013). To appear in *Int. Math. Res. Notices*, doi: 10.1093/imrn/rnt161.
- [Ma97] Manton, N. S. First order vortex dynamics. *Ann. Physics* 256 (1997), no. 1, 114-131.
- [PS13] Pusateri, F. and Shatah, J. Space-time resonances and the null condition for first order systems of wave equations. *Comm. Pure Appl. Math.* 66 (2013), no. 10, 1495-1540.
- [Ya01] Yang, Y. Solitons in field theory and nonlinear analysis. *Springer Monographs in Mathematics*. Springer-Verlag, New York, 2001. xxiv+553 pp. ISBN: 0-387-95242-X.

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