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Performance Bounds for Finite Moving Average Tests in Transient Change Detection

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***Abstract*—This paper deals with the problem of sequentially detecting statistical changes. In particular, the focus is on transient change detection, in which a probability minimizing optimal criterion is desirable. This is in contrast with the traditional minimization of the detection delay, proposed in quickest change detection problems. A finite moving average stopping time is proposed for the general transient change detection problem. The statistical performance of this stopping time is investigated and compared to other methods available in the literature. The proposed stopping time and theoretical findings are applied to quality monitoring, including reliability monitoring in industrial processes and signal quality monitoring in global navigation satellite systems. Numerical simulations are presented to assess the goodness of the presented theoretical results, and the performance of the considered stopping times. This will show the superiority of the proposed scheme.**

***Index Terms*—Transient Change Detection, Stopping Time, Finite Moving Average, GNSS, Signal Quality Monitoring.**

I. Introduction

The problem of detecting sudden statistical changes has many important applications, including fault detection, on-line monitoring of safety-critical infrastructures, detection of signals in radar and sonar signal processing, and segmentation of signals. An overview of these applications can be found in [1]. This kind of detection lies on the field of statistical change detection, including *quickest* change detection (QCD) and *transient* change detection (TCD). The traditional QCD problem deals with a change of infinite duration, whereas TCD deals with finite change duration.

The optimality criterion in QCD is to minimize the detection delay subject to a level of false alarms. Comprehensive overviews of this type of detection can be found in [1] and [2]. In the present work, we focus on non-Bayesian approaches in which the change time is modeled as being unknown but non-random. For these approaches, the CUSUM algorithm was first proposed as a continuous inspection scheme in the 1950s [3], but it was not until 1971 that its optimality for the QCD problem was established asymptotically (i.e. when false alarms go to zero) [4]. More than a decade later, Moustakides [5] proved that the CUSUM is also non-asymptotically optimal.

In contrast, in TCD problems a bounded detection delay is desired. Unfortunately, the traditional QCD criterion does not completely fit into this problem. In this case, we wish to minimize the probability of missed detection (unbounded delay) subject to a level of false alarms. This criterion was first adopted in [6] for the Bayesian approach, but without controlling the false alarm rate. In 2013, the authors of [7] considered a semi-Bayesian approach imposing a suitable constraint on the false alarm rate. But, it was not until 2014, that the first optimal results for the non-Bayesian case were provided by Moustakides [8]. Nevertheless, all of these works have considered the very particular case of a bounded delay of a single sample, which has limited practical application. Actually, the field of TCD is still under development.

There has been substantial recent interest in safety-critical applications such as navigation monitoring [9], water distribution monitoring [10] or cyber attacks on networked control systems [11]. In these applications it is desirable to detect abnormal situations with an established maximum tolerable delay m_{d}, so that detections declared after this maximum tolerable delay are actually considered as missed. In these applications, the above mentioned TCD criterion is appropriate. This is also the case in quality monitoring, including reliability monitoring in industrial processes [12] and signal quality monitoring (SQM) in global navigation satellite systems (GNSSs) [13]. These latter is of particular interest due to the recent interest in GNSS-based critical applications [14]. These applications are often associated with terrestrial environments, where local effects such as multipath, interference and spoofing abound. This effects can jeopardize the safety and trust of the end-user position and time, thus making it of paramount importance to promptly detect any possible anomaly or misleading behavior that could be endangering the received GNSS signal. It is here where SQM can be very helpful to improve the reliability of the system.

A. Related Literature

As mentioned previously, the related literature on QCD is extensive and applied to a wide range of fields [15]–[17]. In contrast, the related literature for the fundamentals of TCD is scarce (listed above). This dearth of theory is in contrast with the recent interest on TCD problems in areas as diverse as navigation and drinking water quality monitoring, or radar and sonar image processing, to mention a few. Several approaches for dealing with the TCD problem in these applications have been proposed. Standard solutions are based on the CUSUM algorithm [18]–[20]. Unfortunately, almost all available results are applicable to off-line detection on finite

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observation intervals, and they adopt the traditional criterion of QCD. Exceptions, adopting the probability minimization criterion above mentioned, are [9] and [10]. On the GNSS side, the application of QCD is limited. Based on this observation, we already addressed the problem of detecting local degrading effects in a QCD framework (see [21]–[26]). However, for SQM purposes, a bounded delay is desirable and then we should rely on the framework of TCD.

SQM is linked with the concept of GNSS integrity, which is intended to provide timely warnings to the user when GNSS should not be used due to miss-performance of the system. This concept has its origins on civil aviation, and it has been addressed by means of augmentation systems [27] and receiver autonomous integrity monitoring (RAIM) techniques [28]–[30]. These approaches are feasible in civil aviation, where local effects have a controlled influence on the GNSS signal, but it is not the case in terrestrial environments, where a plethora of disturbing effects may be present, so that the signal quality is strongly affected. For this reason, a large number of contributions addressing SQM have appeared in the last decade, adopting either a block-wise detection framework [31]–[33], or the use of external information like maps, sensors, or cameras [13], [34]. Unfortunately, these contributions do not consider the timely detection of local threats. To this end, the TCD framework may be useful, and it is why we consider its application to SQM as an example of an application of our theoretical findings.

B. Contributions

As noted above, the TCD field is still under development. Indeed, for the non-Bayesian case, the only optimal result was provided a few years ago in [8] for the case of a maximum tolerable delay of one sample (i.e. $m_{\mathrm{d}} = 1$), but the problem is still open for finite delays greater than one sample. In this line, the recent paper [10], proposing a windowed-limited CUSUM (WLC) solution (i.e. using a number of samples equal to the maximum tolerable delay m_{d}), is of interest. The reason is that it was shown that a finite moving average (FMA) [1], [3] stopping time is the solution of the Gaussian mean change when the WLC is optimized. The optimality criterion was to find the threshold of the WLC that provides, for a Gaussian mean change, the minimum bound on the probability of missed detection with a constraint on the false alarms. Since the FMA is the optimal solution for the Gaussian mean change, we propose its use for the general TCD problem, although its optimality with respect to the above criterion is no longer guaranteed.

The contribution of this work is twofold. Firstly, we theoretically investigate the statistical performance of the FMA stopping time, based on the TCD framework, in the general case. This leads to the provision of novel bounds valid for any kind of change and not restricted to the Gaussian mean change. These bounds are more tight than those available in the literature for other approaches, convenient to be used in practice. These bounds were briefly introduced in [35]. In this work, however, we provide a more extensive and complete proof. Secondly, and importantly in terms of comparison, we show with numerical results that the FMA stopping time outperforms other methods proposed in the literature.

For the sake of exemplification, most of the numerical results are carried out in the setting of SQM in GNSS. This was also briefly introduced in [35] for the carrier-to-noise ratio (C/N_0) metric. Here, we provide a more extensive analysis including a brief description of integrity algorithms and its connection to TCD. Moreover, the three multipath detection techniques presented in [26], including all possible changes in a Gaussian distribution (i.e. mean or/and variance changes), are analyzed. It is worth clarifying that the application of the presented theoretical results to GNSS is without loss of generalization, that is, they are valid to the general theory and they are not restricted to GNSS neither to Gaussian distributions. To support this claim, we also consider the example of detecting a change in the rate parameter of an exponential distribution.

The rest of the paper is organized as follows: Section II provides background on statistical change detection, including both QCD and TCD, and introducing the proposed FMA stopping time. Next, Section III investigates the statistical properties of the FMA stopping time. Then, Section IV investigates the application of the obtained theoretical results to different study cases. Further, Section V introduces concepts about SQM and integrity in GNSS and its connection to TCD. Finally, Section VI deals with our numerical results, while Section VII concludes the paper.

II. Background on Statistical Change Detection

A change detection algorithm, including QCD and TCD, is completely defined by its stopping time T at which the change is declared. In general, a change detection algorithm can be modeled as follows: Let $\{x_n\}_{n\geq 1}$ be a random sequence observed sequentially, and let v be the instant (in samples) when the change in distribution appears. We consider a family $\{\mathbb{P}_v | v \in [1, 2, \ldots, \infty]\}$ of probability measures, such that, under $\mathbb{P}_v$, $x_1, \ldots, x_{v-1}$ and $x_{v+m}, \ldots, x_\infty$, with m the change duration, are independent and identically distributed (iid) with a fixed marginal probability density function (pdf) f_0, corresponding to the normal conditions (i.e. $\mathcal{H}_0$). On the other hand, $x_v, \ldots, x_{v+m-1}$ are iid with another marginal pdf $f_1 \neq f_0$, corresponding to the abnormal conditions (i.e. the change is present, $\mathcal{H}_1$). Next, we briefly recall the problem of QCD. Secondly, we introduce the problem of TCD. Finally, the idea of windowed solutions and the FMA stopping time proposed in this paper are presented.

A. Quickest change detection (QCD): CUSUM stopping time

The statistical model for QCD is described as follows:

$$x_n \sim \begin{cases} \mathcal{H}_0: & f_0(x) \quad \text{if } n < v \\ \mathcal{H}_1: & f_1(x) \quad \text{if } n \geq v \end{cases}, \tag{1}$$

where in this case $m = \infty$. Denoting $\mathbb{E}_v$ as the expectation under the probability measure $\mathbb{P}_v$, the effectiveness of QCD has been traditionally quantified by the minimization of [4]

$$d(T) = \sup_{v\geq 1} \operatorname{essup} \mathbb{E}_v \left[(T - v + 1)^+ | x_1, \ldots, x_{v-1} \right] \tag{2}$$

among all stopping times T satisfying $\mathbb{E}_\infty(T) \geq \gamma$, where $(x)^+ = \max(0, x)$, essup denotes the essential supremum, and $\gamma > 0$ a finite constant. That is, we seek a stopping time T that minimizes the delay $d(T)$ within a lower-bound constraint on the mean time between false alarms $\mathbb{E}_\infty(T)$. The following solution was proposed for the QCD in [3]:

$$T_\mathrm{C}(h) \doteq \inf\left\{n \geq 1 : \max_{1\leq k\leq n} S_k^n \geq h\right\}; S_k^n \doteq \sum_{i=k}^{n} \mathrm{LLR}(i), \tag{3}$$

where $\mathrm{LLR}(i) \doteq \ln(f_1(x_i)/f_0(x_i))$ is the log-likelihood ratio (LLR) of the observation x_i and h is the detection threshold.

The above solution is known as the CUSUM stopping time, and its first optimal results, in the sense of the criterion in (2), were shown in [4] and [5], in an asymptotic (i.e. $\gamma \to \infty$) and non-asymptotic (i.e. for all finite γ) way, respectively. However, as shown in [36], the requirement of having large values of $\mathbb{E}_\infty(T)$ does not guarantee small values of the probability of false alarm $\mathbb{P}_\infty(l \leq T < l + m_\alpha)$ within a fixed interval of length m_α, for all $l \geq 1$. As a result, [36] proposed to replace the traditional constraint $\mathbb{E}_\infty(T) \geq \gamma$ by the following constraint on the worst-case probability of false alarm within any interval of length m_α, which is more convenient for safety-critical applications:

$$\mathbb{P}_\mathrm{fa}(T, m_\alpha) \doteq \sup_{l\geq 1} \mathbb{P}_\infty(l \leq T < l + m_\alpha) \leq \widetilde{\alpha}, \tag{4}$$

with $\widetilde{\alpha} \in (0, 1)$ a given constant value. It was shown in [36] that the CUSUM stopping time T_C asymptotically minimizes (as $\mathbb{P}_\mathrm{fa} \to 0$) the detection delay, over all stopping times T satisfying (4), if h fulfills the following equation:

$$\mathbb{P}_\mathrm{fa}(T_\mathrm{C}, m_\alpha) \leq \widetilde{\alpha} = m_\alpha e^{-h}. \tag{5}$$

B. Transient change detection (TCD): Shewhart stopping time

Unlike QCD, in which the change duration m is assumed to be infinite, the change duration in TCD problems is assumed to be finite. This is modeled as

$$x_n \sim \begin{cases} \mathcal{H}_0 : & f_0(x) \quad \text{if } n < v \text{ or } n \geq v + m \\ \mathcal{H}_1 : & f_1(x) \quad \text{if } v \leq n < v + m \end{cases}. \tag{6}$$

As discussed in [10], there are two types of TCD problems. The first type involves the detection of suddenly arriving signals of random unknown duration. In this case m denotes the unknown duration of the change. The second type involves safety-critical applications where a maximum tolerable detection delay is a priori fixed to a pre-established value m_d, and then m is considered to be known. This article belongs to this second class of TCD problems; in this case a detection with a delay greater than m_d is considered as missed, even if $m > m_\mathrm{d}$. On the other hand, if the duration of the change m is smaller than m_d, then such a change is considered less dangerous because its impact on the system is limited or negligible. It is for this reason that the duration m is considered to be known henceforth and equal to m_d (i.e. $m = m_\mathrm{d}$).

We observe from (2), though, that no hard limit is imposed on the detection delay; consequently, this quantity can become arbitrarily large. In this sense, the optimality criteria for the TCD problem should be modified, with respect to the one used in QCD, in order to seek a small probability of missed detection given an acceptable false alarm rate. In other words, we wish to have $v \leq T < v + m$. Stopping within the prescribed interval constitutes a desirable event while stopping at $T \geq v + m$ is considered a missed detection. This criterion involves the following minimization:

$$\inf_{T\in C_\alpha} \left\{\mathbb{P}_\mathrm{md}(T, m) \doteq \sup_{v\geq 1} \mathbb{P}_v(T \geq v + m | T \geq v)\right\} \tag{7}$$

among all stopping times $T \in C_\alpha$ satisfying

$$C_\alpha = \{T : \mathbb{P}_\mathrm{fa}(T, m_\alpha) \leq \widetilde{\alpha}\}, \tag{8}$$

where $\mathbb{P}_\mathrm{md}$ and $\mathbb{P}_\mathrm{fa}$ stand for the worst-case probabilities of missed detection and false alarm within any interval of length m_α, respectively, with $\mathbb{P}_\mathrm{fa}$ defined as in (4). Very recently, the optimal solution of this class of TCD problem, for $m = 1$, was shown to be the Shewhart test [8]

$$T_\mathrm{S}(h) \doteq \inf\{n \geq 1 : \mathrm{LLR}(n) \geq h\}. \tag{9}$$

Indeed, this is the only available optimal result for the criterion in (7)–(8). Nevertheless, for the case of finite $m > 1$ no optimal solution is available, so that the problem is still open.

C. Windowed solutions: WLC and FMA stopping times

Since there is no optimal solution available in the literature of TCD for a finite $m > 1$, we propose a windowed solution based on the following idea: We know that the optimal solution for QCD, that is for $m = \infty$, is the CUSUM test [5], which uses information about all the past samples. On the other hand, the Shewhart test, which uses information of one sample, is established to be optimal for the non-Bayesian TCD problem with $m = 1$ [8]. Hence, it is intuitive to think that the optimal solution for $1 < m < \infty$ would be some test statistic between these two techniques, and particularly, a test statistic using information about m samples (i.e. windowed). In this context, a WLC solution is proposed in [10] by using at each moment the m last observations only, given by

$$T_\mathrm{WLC}(h) \doteq \inf\left\{n \geq m : \max_{n-m+1\leq k\leq n} S_k^n \geq h\right\}. \tag{10}$$

It is assumed that the WLC is not operational during the first $m - 1$ observations.

It is also shown that after certain optimization process the WLC, for a Gaussian mean change, leads to

$$T^*_\mathrm{WLC}\left(\widetilde{h}\right) = \inf\left\{n \geq m : \sum_{i=n-m+1}^{n} x_i \geq \widetilde{h}\right\}, \tag{11}$$

where $\widetilde{h}$ denotes the chosen threshold after the optimization.

It is important to note that the previous stopping time is equivalent to an FMA test (i.e. comparison of a moving average of m LLRs to a threshold) for the case of a Gaussian mean change. Inspired by this result and the idea of windowed solution, in this paper we propose the use of an FMA stopping time for any general TCD problem, which becomes

$$T_\mathrm{F}(h) \doteq \inf\{n \geq m : S_n \geq h\}; S_n \doteq S_{n-m+1}^n. \tag{12}$$

Furthermore, the use of the FMA stopping time is motivated by the fact, as we will see, that we can obtain tight bounds for both the probabilities of missed detection and false alarm. On the contrary, the bounds available in the literature for these probabilities for the CUSUM and WLC are not that tight.

III. Statistical Performance of the FMA Stopping Time

The goal of this section is to theoretically investigate the statistical performance of the FMA stopping time $T_{\mathrm{F}}(h)$; that is, to determine the worst-case probability of missed detection $\mathbb{P}_{\mathrm{md}}\left(T_{\mathrm{F}}(h), m\right)$ and the worst-case probability of false alarm for a given duration m_α, $\mathbb{P}_{\mathrm{fa}}\left(T_{\mathrm{F}}(h), m_\alpha\right)$. The exact calculation of these probabilities is very complicated, and this is why the existence of tight-enough bounds is of practical interest. These bounds are stated in the next theorem.

Theorem 1. Let us consider the FMA stopping time $T_{\mathrm{F}}(h)$ in (12) with $S_m \doteq \sum_{i=1}^{m} \mathrm{LLR}(i)$; then the worst-case probability of false alarm for a given duration m_α is bounded as

$$\mathbb{P}_{\mathrm{fa}}\left(T_{\mathrm{F}}(h), m_\alpha\right) \leq \alpha\left(h, m_\alpha\right), \tag{13}$$

where

$$\alpha\left(h, m_\alpha\right) = 1 - \left[\mathbb{P}_\infty\left(S_m < h\right)\right]^{m_\alpha}. \tag{14}$$

On the other hand, the worst-case probability of missed detection is bounded as

$$\mathbb{P}_{\mathrm{md}}\left(T_{\mathrm{F}}(h), m\right) \leq \beta(h, m), \tag{15}$$

where

$$\beta(h, m) = \mathbb{P}_1\left(S_m < h\right). \tag{16}$$

Proof: The proof is given in Appendix A. ∎

In practice, values $\widetilde{\alpha}$ for $\mathbb{P}_{\mathrm{fa}}$ are imposed, so that we have to guarantee that $\mathbb{P}_{\mathrm{fa}} \leq \widetilde{\alpha}$. Thus, the threshold h has to be selected in order to satisfy this constraint, and then $\mathbb{P}_{\mathrm{md}}$ turns out to be a function of the fixed $\widetilde{\alpha}$ (i.e. $\mathbb{P}_{\mathrm{md}}\left(T_{\mathrm{F}}(h(\widetilde{\alpha})), m\right)$). In some sense, $\mathbb{P}_{\mathrm{md}}\left(T_{\mathrm{F}}(h(\widetilde{\alpha})), m\right)$ plays the same role in the TCD theory as the Cramér-Rao lower bound in estimation theory, or as the receiver operating characteristic (ROC) in classical detection. Moreover, this kind of ROC allow us to compare the performance of different algorithms in terms of the optimality criterion in (7). This relation between $\mathbb{P}_{\mathrm{md}}$ and $\mathbb{P}_{\mathrm{fa}}$ is given in the following corollary.

Corollary 1. Let F_i, with $i = \{0, 1\}$, be the cumulative distribution function (cdf) of S_m under $\mathcal{H}_i$ and let h be selected so that

$$\mathbb{P}_{\mathrm{fa}}(T_{\mathrm{F}}(h), m_\alpha) \leq \widetilde{\alpha}, \tag{17}$$

with $\widetilde{\alpha}$ a desirable constant value for the probability of false alarm. A possible threshold h satisfying (17) is given by

$$h\left(\widetilde{\alpha}\right) = F_0^{-1}\left[\left(1 - \widetilde{\alpha}\right)^{1/m_\alpha}\right], \tag{18}$$

where F_0^{-1} is the inverse of F_0, and thus

$$\beta\left(h\left(\widetilde{\alpha}\right), m\right) = F_1\left[F_0^{-1}\left[\left(1 - \widetilde{\alpha}\right)^{1/m_\alpha}\right]\right]. \tag{19}$$

Moreover,

$$\mathbb{P}_{\mathrm{md}}\left(T_{\mathrm{F}}\left(\widetilde{h}\right), m\right) \leq \beta\left(h\left(\widetilde{\alpha}\right), m\right), \tag{20}$$

with $\widetilde{h}$ the threshold for which the exact $\mathbb{P}_{\mathrm{fa}}$ fulfills $\mathbb{P}_{\mathrm{fa}}\left(T_{\mathrm{F}}\left(\widetilde{h}\right), m_\alpha\right) = \widetilde{\alpha}$.

Proof: It is worth noting that $\mathbb{P}_j(S_m < h) = F_i(h)$, with $j = \{1, \infty\}$ and $i = \{1, 0\}$, is the cdf of S_m under $\mathcal{H}_i$, respectively, evaluated at h. This is because $\mathbb{P}_j(S_m < h)$ denotes the probability that the random variable S_m is below the threshold h when the change in distribution appears at time $j = 1$ or $j = \infty$, that is, at the first sample or never, respectively. This means that we are evaluating the probability that the sum of m LLR samples under $\mathcal{H}_1$ and $\mathcal{H}_0$, respectively, is below h, which actually is the definition of $F_i(h)$, with $i = \{1, 0\}$, respectively. Hence, solving the equation $\alpha(h, m_\alpha) = \widetilde{\alpha}$ for h from (14), a possible threshold can be selected as

$$h(\widetilde{\alpha}) = F_0^{-1}\left[(1 - \widetilde{\alpha})^{1/m_\alpha}\right], \tag{21}$$

which leads to $\mathbb{P}_{\mathrm{fa}}\left(T_{\mathrm{F}}(h), m_\alpha\right) \leq \widetilde{\alpha}$ thanks to (13), and (18) thus follows. The proof of (19) follows immediately by the definition of the cdf F_1 and by substituting (21) into (16). In order to prove (20) it is important to see that the threshold $\widetilde{h}$, such that $\mathbb{P}_{\mathrm{fa}}\left(T_{\mathrm{F}}\left(\widetilde{h}\right), m_\alpha\right) = \widetilde{\alpha}$, is lower than $h(\widetilde{\alpha})$ (i.e. $\widetilde{h} < h(\widetilde{\alpha})$) and that

$$\mathbb{P}_{\mathrm{md}}\left(T_{\mathrm{F}}\left(\widetilde{h}\right), m\right) \leq \beta\left(\widetilde{h}, m\right) \leq \beta\left(h, m\right), \tag{22}$$

where the last inequality follows because $\beta(h, m)$ in (16) is a monotonically increasing function on h, so that (20) follows, completing the proof of the corollary. ∎

The previous results are valid for the general FMA stopping time. That is, they are not restricted to the Gaussian mean change, as in [10], but they are valid for any kind of change. Moreover, as we will see later, these bounds are tighter than bounds for other available methods in the literature. Unfortunately, we cannot establish the optimality of the proposed FMA stopping time in the class C_α. To do so we should analyze the speed of convergence of the term $F_0^{-1}[(1 - \widetilde{\alpha})^{1/m_\alpha}]$, which is beyond the scope of this paper. Nonetheless, in [10] the optimality of the FMA stopping time is shown for the particular case of a Gaussian mean change. In addition, we will show later how the proposed FMA stopping time outperforms other available methods in the literature. This makes it evident that the FMA stopping time is a good candidate for TCD problems.

IV. Study Cases

The goal of this section is to theoretically evaluate the performance of the FMA stopping time for several problems of practical interest. In particular, we show the statistical characterization of the LLR for these problems, so that the corresponding results given by Corollary 1 can be obtained. To do so, we will denote the parameters that govern the distribution under $\mathcal{H}_1$ by $\boldsymbol{\lambda}$. Since these parameters may be unknown in practice, we will first formulate the LLR for a particular change parameter denoted by $\widetilde{\boldsymbol{\lambda}}$, which will be given by the minimum change parameter we want to detect. Then, the statistical characterization of the obtained LLR will be evaluated taking into account the true distribution with parameters given by $\boldsymbol{\lambda}$. This is done with the aim of obtaining the actual performance of the detectors.

In the following, we will start with the problem of detecting the change in the rate parameter of an exponential distribution. Next, we will continue with the case of a change in a Gaussian distribution. We will consider first the simpler cases of having a change in either the mean or variance of a Gaussian distribution. Finally, we present the most general case of having changes in both the mean and variance of a Gaussian distribution. Indeed, these cases represent a wide-range of practical TCD problems in which the Gaussian distribution appears. Nevertheless, it is worth clarifying that the theoretical results in Section III are not restricted to Gaussian distributions, but they are valid for the general TCD theory. This will be confirmed with the evaluation of the exponential distribution problem.

A. *Exponential rate parameter change*

In this section, we evaluate the statistical characterization of the LLR for the problem of detecting a change in the rate parameter of an exponential distribution. This problem is of particular interest in reliability theory, in which it is often desired to react to increasing (or decreasing) failure rates. This is the case when we wish to detect the onset of deterioration of reliability in the course of production in industrial processes. Times between failures are usually assumed to have an exponential distribution or, more generally, a Weibull distribution [4]. Let us consider the following exponential-based statistical model:

$$x_n \sim \begin{cases} \mathcal{H}_0 : \lambda_0 e^{-\lambda_0 x_n} & \text{if } n < v \text{ or } n \geq v+m \\ \mathcal{H}_1 : \lambda_1 e^{-\lambda_1 x_n} & \text{if } v \leq n < v+m \end{cases}, \quad (23)$$

with x_n the observation samples (e.g. magnitude of the process to be monitored), λ_0 the known rate parameter under $\mathcal{H}_0$, and λ_1 the unknown rate parameter under $\mathcal{H}_1$ (i.e. after deterioration). With this model, we have the following result.

Corollary 2. Let $\widetilde{\lambda}_1 = \lambda_0 + \theta$ be the minimum change parameter we want to detect, with $\theta > 0$. Therefore, the LLR for the model in (23) is

$$\mathrm{LLR_e}(n) = \ln\left(\frac{\widetilde{\lambda}_1}{\lambda_0}\right) - \theta \cdot x_n. \quad (24)$$

Now, let $T_{\mathrm{F,e}}(h)$ be the FMA stopping time in (12) with LLR in (24) and threshold h so that $\mathbb{P}_{\mathrm{fa}}(T_{\mathrm{F,e}}(h), m_\alpha) \leq \widetilde{\alpha}$. Furthermore, let $\Gamma(x; a, b)$ denote the cdf of the gamma distribution with shape and scale parameters given by a and b, respectively; and let λ_1 be the actual change parameter. Hence

$$h(\widetilde{\alpha}) = B - \theta \cdot \Gamma^{-1}\left(A; m, \lambda_0^{-1}\right), \quad (25)$$

$$\mathbb{P}_{\mathrm{md}}(T_{\mathrm{F,e}}(h), m) \leq 1 - \Gamma\left(\frac{B-h}{\theta}; m, \lambda_1^{-1}\right), \quad (26)$$

$$\mathbb{P}_{\mathrm{fa}}(T_{\mathrm{F,e}}(h), m_\alpha) \leq 1 - \left[1 - \Gamma\left(\frac{B-h}{\theta}; m, \lambda_0^{-1}\right)\right]^{m_\alpha}, \quad (27)$$

with $A = 1 - (1-\widetilde{\alpha})^{1/m_\alpha}$ and $B = m \cdot \ln(\widetilde{\lambda}_1/\lambda_0)$.

Proof: The expression for the LLR in (24) follows after simple calculus from (23) and the definition of the LLR. With this LLR we have

$$S_m \doteq \sum_{n=1}^{m} \mathrm{LLR_e}(n) = B - \theta \cdot \sum_{n=1}^{m} x_n = B - \theta \cdot Y, \quad (28)$$

with Y a gamma random variable with shape parameter m and scale parameter equal to λ_0^{-1} and λ_1^{-1} under $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Hence, taking into account that for $\theta > 0$

$$\mathbb{P}_i\left(S_m < h\right) = \mathbb{P}_i\left(Y > \frac{B-h}{\theta}\right), \quad (29)$$

with $i = \{\infty, 1\}$, (25)–(27) thus follows by virtue of Theorem 1 and Corollary 1. Similar results follow for $\theta < 0$. ■

B. *Gaussian mean change*

In this section, we analyze the statistical characterization of the LLR for the problem of a change in the mean of a Gaussian distribution. This problem appears in applications such as radar or communication systems [37], in which under $\mathcal{H}_0$ the received signal follows a Gaussian distribution with certain mean that changes under $\mathcal{H}_1$. First, let us introduce the statistical model for this problem given by

$$x_n \sim \begin{cases} \mathcal{H}_0 : & f(x_n; \mu_{,0}, \sigma^2) \\ \mathcal{H}_1 : & f(x_n; \mu_1, \sigma^2) \end{cases}, \quad (30)$$

where $f(x; \mu, \sigma^2) \doteq (1/\sqrt{2\pi\sigma^2}) \exp(-(x-\mu)^2/2\sigma^2)$ denotes the Gaussian pdf with mean μ and variance σ^2, x_n stands for the sequential observations used for detection (e.g. received signal samples), μ_0 and σ^2 the known mean and variance of x_n under $\mathcal{H}_0$, and μ_1 the unknown change parameter, denoting the mean of x_n under $\mathcal{H}_1$. For the sake of notation simplicity, we omit here and henceforth the time conditions in each hypothesis (i.e. if $n < v$ or $n > v+m$, $\ldots$) corresponding to the TCD model. With the above statistical model, the following result for the Gaussian mean change problem is obtained.

Corollary 3. Let $\widetilde{\mu}_1$ and μ_1 be the minimum change parameter we want to detect and the actual change parameter, respectively, and let μ_0 and σ be known. Therefore, the LLR for the Gaussian mean change problem in (30) is given by

$$\mathrm{LLR_m}(n) = y_n = \frac{\widetilde{\mu}_1 - \mu_0}{\sigma^2}\left(x_n - \frac{\widetilde{\mu}_1 + \mu_0}{2}\right), \quad (31)$$

with mean and variance equal to

$$\begin{aligned} \mu_{y,0} &= -\frac{(\widetilde{\mu}_1 - \mu_0)^2}{2\sigma^2}, \quad \sigma_y^2 = -2\mu_{y,0}, \\ \mu_{y,1} &= \frac{\widetilde{\mu}_1 - \mu_0}{\sigma^2}\left(\mu_1 - \frac{\widetilde{\mu}_1 + \mu_0}{2}\right). \end{aligned} \quad (32)$$

Now, let $T_{\mathrm{F,m}}(h)$ be the FMA stopping time in (12) with LLR in (31) and threshold h so that $\mathbb{P}_{\mathrm{fa}}(T_{\mathrm{F,m}}(h), m_\alpha) \leq \widetilde{\alpha}$, and let $\Phi(x)$ denote the cdf of the standard Gaussian distribution.

Hence

$$h(\widetilde{\alpha}) = \sqrt{m \cdot \sigma_y^2} \cdot \Phi^{-1}\left[(1-\widetilde{\alpha})^{1/m_\alpha}\right] + m \cdot \mu_{y,0}, \quad (33)$$

$$\mathbb{P}_{\mathrm{md}}(T_{\mathrm{F,m}}(h,m) \leq \Phi\left(\frac{h - m \cdot \mu_{y,1}}{\sqrt{m \cdot \sigma_y^2}}\right), \quad (34)$$

$$\mathbb{P}_{\mathrm{fa}}(T_{\mathrm{F,m}}(h), m_\alpha) \leq 1 - \left[\Phi\left(\frac{h - m \cdot \mu_{y,0}}{\sqrt{m \cdot \sigma_y^2}}\right)\right]^{m_\alpha}. \quad (35)$$

Proof: From the definition of the Gaussian pdf and the statistical model for the Gaussian mean change in (30), (31) follows after simple calculus. Thereby, it is trivial to see, from (31) and (30), that the LLR for the Gaussian mean change problem is Gaussian distributed in both hypotheses, with mean and variance as in (32). Hence, $S_m^{(\mathrm{m})} = \sum_{i=1}^{m} \mathrm{LLR}_{\mathrm{m}}(i)$ is Gaussian as well, but with mean and variance scaled by a factor m, and (33)–(35) thus follows by direct application of Theorem 1 and Corollary 1. ■

C. Gaussian variance change

Now, we analyze the characterization of the LLR for the problem of a change in the variance of a Gaussian distribution. This problem is relevant for applications like spectrum sensing in cognitive radio [17]. In these cases the sequential observations follow a Gaussian distribution with a given variance under $\mathcal{H}_0$, whereas under $\mathcal{H}_1$ the variance changes. Moreover, in the above applications, the Gaussian distribution under both $\mathcal{H}_0$ and $\mathcal{H}_1$ has zero mean, so that the statistical model for the variance Gaussian change is given by

$$x_n \sim \begin{cases} \mathcal{H}_0: & f(x_n; 0, \sigma_0^2) \\ \mathcal{H}_1: & f(x_n; 0, \sigma_1^2) \end{cases}, \quad (36)$$

with x_n the sequential observations used for detection (e.g. received signal power), σ_0^2 the known variance under $\mathcal{H}_0$, and σ_1^2 the unknown variance under $\mathcal{H}_1$. Thereby, we obtain the following result for the above Gaussian variance change.

Corollary 4. Let $\widetilde{\sigma}_1^2$ be the minimum change parameter we want to detect, and let σ_0 be known. Therefore, the LLR for the Gaussian variance change in (36) is given by

$$\mathrm{LLR}_{\mathrm{v}}(n) = a \cdot x_n^2 + c, \quad (37)$$

with

$$a = \frac{\widetilde{\sigma}_1^2 - \sigma_0^2}{2\sigma_0^2 \cdot \widetilde{\sigma}_1^2} \text{ and } c = \ln\left(\frac{\sigma_0}{\widetilde{\sigma}_1}\right). \quad (38)$$

Now, let $T_{\mathrm{F,v}}(h)$ be the FMA stopping time for the Gaussian variance change with LLR in (37) and threshold h so that $\mathbb{P}_{\mathrm{fa}}(T_{\mathrm{F,v}}(h), m_\alpha) \leq \widetilde{\alpha}$; and let σ_1^2 be the actual change parameter. Furthermore, let $\Upsilon_m(x)$ denote the cdf of the Chi-squared distribution with m degrees of freedom, and $k_i = \sigma_i^2 \cdot a$, with $i = \{0, 1\}$. Hence,

$$h(\widetilde{\alpha}) = k_0 \cdot \Upsilon_m^{-1}\left[(1-\widetilde{\alpha})^{1/m_\alpha}\right] + m \cdot c, \quad (39)$$

$$\mathbb{P}_{\mathrm{md}}(T_{\mathrm{F,v}}(h), m) \leq \Upsilon_m\left(\frac{h - m \cdot c}{k_1}\right), \quad (40)$$

$$\mathbb{P}_{\mathrm{fa}}(T_{\mathrm{F,v}}(h), m_\alpha) \leq 1 - \left[\Upsilon_m\left(\frac{h - m \cdot c}{k_0}\right)\right]^{m_\alpha}. \quad (41)$$

Proof: From the definition of the Gaussian pdf and the statistical model for the Gaussian variance change in (36), (37) follows. Thereby, under $\mathcal{H}_1$, and denoting $S_m^{(\mathrm{v})} \doteq \sum_{i=1}^{m} \mathrm{LLR}_{\mathrm{v}}(i)$, we have

$$\begin{aligned} S_m^{(\mathrm{v})} | \mathcal{H}_1 &= a \sum_{n=1}^{m} \sigma_1^2 \left(\frac{x_n}{\sigma_1}\right)^2 + m \cdot c \\ &= k_1 \sum_{n=1}^{m} X_n^2 + m \cdot c, \end{aligned} \quad (42)$$

with $X_n \sim f(X_n; 0, 1)$ a standard Gaussian random variable. A similar result is obtained under $\mathcal{H}_0$, and then we can write $S_m^{(\mathrm{v})} | \mathcal{H}_i = k_i \widetilde{X} + m \cdot c$, with $i = \{0, 1\}$ and $\widetilde{X}$ a Chi-squared random variable with m degrees of freedom. Hence,

$$\mathcal{H}_i : \frac{S_m^{(\mathrm{v})} - m \cdot c}{k_i} \sim \chi_m^2, \quad (43)$$

where χ_m^2 stands for the Chi-squared pdf with m degrees of freedom, and (39)–(41) thus follow by direct application of Theorem 1 and Corollary 1. ■

D. General Gaussian change

Previously, we have analyzed the characterization of the LLR for the particular cases of having a change in either the mean or variance of a Gaussian distribution. In this section, we analyze the characterization of the LLR for the most general case of having a change in both mean and variance of a Gaussian distribution. This problem appears in diverse applications such as neuron receptive fields modeling [38] or financial problems dealing with portfolio losses of CDO pricing [39], just to mention a few. The statistical model for the Gaussian mean and variance change is given by

$$x_n \sim \begin{cases} \mathcal{H}_0: & f(x_n; \mu_0, \sigma_0^2) \\ \mathcal{H}_1: & f(x_n; \mu_1, \sigma_1^2) \end{cases}, \quad (44)$$

with x_n the sequential observations used for detection (e.g. neuron movement), $\{\mu_0, \sigma_0^2\}$ the known mean and variance of x_n under $\mathcal{H}_0$, respectively, and $\{\mu_1, \sigma_1^2\}$ the unknown change parameters, denoting the mean and variance of x_n under $\mathcal{H}_1$, respectively. Thereby, from the definition of the Gaussian pdf and LLR, after some manipulations, we can write

$$\mathrm{LLR}_{\mathrm{g}}(n) = a \cdot x_n^2 + b \cdot x_n + c, \quad (45)$$

with

$$\begin{aligned} a &= \frac{\widetilde{\sigma}_1^2 - \sigma_0^2}{2\sigma_0^2 \cdot \widetilde{\sigma}_1^2}; \quad b = \frac{\sigma_0^2 \cdot \widetilde{\mu}_1 - \widetilde{\sigma}_1^2 \cdot \mu_0}{\sigma_0^2 \cdot \widetilde{\sigma}_1^2}; \\ c &= \ln\left(\frac{\sigma_0}{\widetilde{\sigma}_1}\right) + \frac{\widetilde{\sigma}_1^2 \cdot \mu_0^2 - \sigma_0^2 \cdot \widetilde{\mu}_1^2}{2\sigma_0^2 \cdot \widetilde{\sigma}_1^2}, \end{aligned} \quad (46)$$

where $\widetilde{\mu}_1$ and $\widetilde{\sigma}_1$ are the minimum change parameters we want to detect.

In this case we cannot find the distribution of $\mathrm{LLR}_{\mathrm{g}}(n)$ in an straightforward way as for the previous cases. Here, in order to find the pdf of $S_m^{(\mathrm{g})} \doteq \sum_{i=1}^{m} \mathrm{LLR}_{\mathrm{g}}(i)$, we make use of the so-called Edgeworth series expansion [40] and extreme value theory (EVT) [41], which provide a very tight closed-form expression for the bounds of the FMA stopping time for the general Gaussian change [42]. For the sake of notation clarity, let us write $S_m^{(\mathrm{g})}$ as the random variable Z (i.e. $Z = S_m^{(\mathrm{g})}$). Thereby, we can state the following result.

Corollary 5. Let $T_{\mathrm{F,g}}(h)$ be the FMA stopping time for the Gaussian mean and variance change with LLR in (45) and threshold h so that $\mathbb{P}_{\mathrm{fa}}(T_{\mathrm{F,g}}(h), m_\alpha) \leq \widetilde{\alpha}$, and let $\phi(x)$ be the standard Gaussian pdf. Hence, we have

$$h\left(\widetilde{\alpha}\right) = \delta - \frac{\ln\left(-\ln\left(1-\widetilde{\alpha}\right)\right)}{\gamma}, \tag{47}$$

$$\mathbb{P}_{\mathrm{md}}\left(T_{\mathrm{F,g}}\left(h\right), m\right) \leq F_1\left(h\right), \tag{48}$$

$$\mathbb{P}_{\mathrm{fa}}\left(T_{\mathrm{F,g}}\left(h\right), m_\alpha\right) \leq 1 - \exp\left(-e^{-\gamma(h-\delta)}\right), \tag{49}$$

with

$$\begin{aligned} \delta &= F_0^{-1}\left(1 - \frac{1}{m_\alpha}\right), \\ \gamma &= m_\alpha \cdot f_0\left(\delta\right), \end{aligned} \tag{50}$$

and

$$\begin{aligned} F_i(z) &= \Phi(\widetilde{z}_i) - \sigma_{z_\mathrm{s},i} \cdot \phi(\widetilde{z}_i) \sum_{k\in\mathcal{A}} C_{k,\mathcal{H}_i} \cdot H_{k-1}(\widetilde{z}_i), \\ f_{\mathrm{s},0}(z) &= \phi(\widetilde{z}_0)\left[1 + \sum_{k\in\mathcal{A}} C_{k,\mathcal{H}_0} \cdot H_k(\widetilde{z}_0)\right], \end{aligned} \tag{51}$$

where $\mathcal{A} = \{3,4,6\}$, $C_{k,\mathcal{H}_i}$, with $i = \{0,1\}$, are the coefficients C_k (expressions can be found in Appendix B) under $\mathcal{H}_i$, $H_k(z)$ is the Hermite polynomial of degree k evaluated at z and $\widetilde{z}_i = (z - \mu_{z_\mathrm{s},i})/\sigma_{z_\mathrm{s},i}$, with

$$\begin{aligned} \mu_{z_\mathrm{s},i} &= m\left[a\left(\sigma_i^2 + \mu_i^2\right) + b\cdot\mu_i + c\right], \\ \sigma^2_{z_\mathrm{s},i} &= m\left[\sigma_i^2\left[2a\left(a\cdot\sigma_i^2 + 2a\cdot\mu_i^2 + b\cdot\mu_i + b^2\right)\right]\right], \end{aligned} \tag{52}$$

where $\{\mu_i, \sigma_i^2\}$, with $i = \{0,1\}$, are the actual mean and variance under hypothesis $\mathcal{H}_i$, and $\{a,b,c\}$ defined as in (46).

V. Application to Signal Quality Monitoring

This section shows the application of TCD to SQM in GNSS and its link to GNSS integrity. This is done for the sake of exemplification; the previously presented theoretical results for TCD are not restricted to GNSS, but they could be used in any TCD problem. It is important to clarify that the application of SQM is beneficial for GNSS integrity, but its use in current integrity algorithms is not direct by the presented TCD framework. The application of TCD here is devoted to providing a novel mathematical framework for SQM, so that it can be used for integrity monitoring.

A. SQM: TCD framework

The quality of the received GNSS signal can be jeopardized, particularly in terrestrial environments, by local effects such as multipath, interference and spoofing, so that traditional integrity algorithms cannot be applied. It is for this reason that the use of SQM may play a prominent role for the design of future integrity algorithms in terrestrial environments. The key point is that SQM can compute metrics to monitor the quality of the signal from features of the received signal, measurable within the GNSS receiver, without need of external information. In the following, we show the application of TCD to SQM, so that a bounded detection is possible. We will focus on detecting three different effects threatening the quality of the signal: (i) a change in the mean of the $\mathrm{C/N_0}$ metric [26], [43], (ii) in the variance of the so-called code discriminator output (DLLout) metric [24], [26], and (iii) in the mean and/or variance of the slope asymmetry metric (SAM) [23], [26].

The first step to apply TCD is to statistically characterize the above mentioned metrics, so that the LLR can be computed. The distribution of the considered metrics can be fairly modelled as Gaussian according to the Central Limit Theorem. The metrics are computed by averaging many correlation values obtained in a GNSS receiver, and experimental results using real GNSS signals corroborated the validity of the approximation [26]. Thereby, we model the SQM problem as a change on the parameters of a Gaussian distribution, that is

$$x_n \sim \left\{ \begin{array}{ll} \mathcal{H}_0: & f(x_n;\mu_0,\sigma_0^2) \\ \mathcal{H}_1: & f(x_n;\mu_1,\sigma_1^2) \end{array} \right., \tag{53}$$

where x_n contains the series of values of metrics obtained from the underlying signal (e.g. $\mathrm{C/N_0}$, DLLout and SAM). It is worth clarifying that this model can be regarded as a simplification of real multipath effects, and it is used here for the sake of exemplification. Nevertheless, for operational SQM algorithms more sophisticated multipath models are needed. With the statistical model at hand, the next step is to calculate the LLR of the proposed metrics. The resulting LLR is one of those obtained in the previous section for the Gaussian changes, namely (31), (37) and (45) for the $\mathrm{C/N_0}$, DLLout and SAM metrics, respectively. Thus, Corollaries 3, 4 and 5 can be used for the statistical evaluation of the detectors.

B. Link to GNSS integrity

An integrity failure is defined as a positioning error that exceeds a certain threshold, called the alert limit (AL), during a maximum tolerable time collectively known as time to alert (TTA). The cornerstone of integrity algorithms is the so-called *integrity risk*, defined as the probability that a failure is present without warning the user within the TTA [30]. Now, let t_{tta} be the TTA in samples and let $\{\mathcal{P}_{\mathrm{fa}}(z_n, t_{m_\alpha}), \mathcal{P}_{\mathrm{risk}}(z_n, t_{\mathrm{tta}})\}$ be the probability of false alarm within an interval of t_{m_α} samples and integrity risk resulting from the test $z_n \geq h$, respectively. Integrity requirements of certain applications are given in the form of TTA, and fixed values of integrity risk and probability of false alarm given by $\{\widetilde{\alpha}, \widetilde{\beta}\}$ [44]. To fulfill these requirements, integrity algorithms compare the test statistic z_n with a threshold h in order to decide whether a failure is present or not. The threshold h is selected so that $\mathcal{P}_{\mathrm{fa}}(z_n, t_{m_\alpha}) \leq \widetilde{\alpha}$. When the rest of the requirements are fulfilled we say that the integrity algorithm is available.

This availability is evaluated by means of the so-called protection level (PL) [27], [30], which is the value that upper bounds the true error with probability $\widetilde{\beta}$. Therefore, at each epoch, if PL≤AL the integrity algorithm is declared available, otherwise the algorithm is declared unavailable because it cannot guarantee that $\mathcal{P}_{\mathrm{risk}}(z_n, t_{\mathrm{tta}}) \leq \widetilde{\beta}$. The PL is computed based on the definition of integrity risk, given by

$$\mathcal{P}_{\mathrm{risk}}(z_n, t_{\mathrm{tta}}) \doteq \Pr\left\{z_n < h, n > v + t_{\mathrm{tta}} | \mathcal{H}_1\right\}, \tag{54}$$

where v denotes the unknown time in which a failure appears. Current integrity algorithms rely on RAIM and augmentation systems and a substantial literature is available for further details [27]–[30]. Along this line, it is important to note that when using the test $S_n \geq h$ instead of $z_n \geq h$, $t_{m_\alpha} = m_\alpha$, and $t_{\mathrm{tta}} = m$, the probability measure in (54) could be written as the $\mathbb{P}_{\mathrm{md}}$ in TCD problems (see (7)). Hence, the integrity problem is closely connected to TCD. Notwithstanding, further work is needed to fully connect the proposed SQM framework and current integrity algorithms. The presented application of TCD is mainly intended to give a mathematical framework aimed at minimizing the integrity risk for SQM.

VI. Numerical Results

The aim of this section is to first compare the proposed FMA stopping time with other approaches in the literature of TCD. This is done, without loss of generality of the presented theoretical results, in the setting of SQM by considering the C/N$_0$, DLLout and SAM metrics. In order to justify the generality of the results, we will also consider the problem of a change in the rate parameter of an exponential distribution. Secondly, the availability of the SQM algorithm is analyzed.

A. Evaluation of the probability minimization criterion

Here, we compare the FMA stopping time with those stopping times currently available in the literature of TCD. This comparison is done with both simulated and available theoretical results. The simulated results include Monte-Carlo simulations (10^6 runs) of the exact worst-case probability of missed detection $\mathbb{P}_{\mathrm{md}}(T_{\mathrm{F}}(h), m)$ as a function of the exact worst-case probability of false alarm $\mathbb{P}_{\mathrm{fa}}(T_{\mathrm{F}}(h), m_\alpha)$, henceforth referred to as the ROC. Regarding the theoretical results, on the one hand they include those obtained in Section IV for the FMA stopping time of the different considered study cases in this work, stated in Corollaries 2–5. On the other hand, they also include the bounds available in the literature for the CUSUM and WLC. For the probability of missed detection of these two methods we have an upper bound in the form of (16) (see [9] and [10] for the WLC and CUSUM, respectively). For the false alarm probability, we use the upper bound given by (4)–(5), which holds for both the CUSUM and WLC.

For the representation of the theoretical results shown in Figures 1–4, the procedure is as follows: (i) For the FMA bound (dashed line), the detection threshold is fixed as given in the corollaries for the corresponding study case (i.e. (25), (33), (39) and (47)) using the $\mathbb{P}_{\mathrm{fa}}$ value indicated on the x-axis; and (ii) then the obtained threshold is substituted into the formula for the bound $\beta(h, m)$ in the corollary pertaining to the corresponding study case (i.e. (26), (34), (40), (48)), giving the value depicted in the y-axis. Similarly, for the CUSUM and WLC bounds (solid line), the same formulas for $\beta(h, m)$ are used, but the detection threshold is computed with (5). For instance, for the case of detecting a change in the rate parameter of an exponential distribution, results in Corollary 2 apply and hence (25) is used to obtain the threshold $h(\mathbb{P}_{\mathrm{fa}})$, which is further applied to (26) leading to the value depicted on the y-axis. This procedure is indicated in the figure's legend

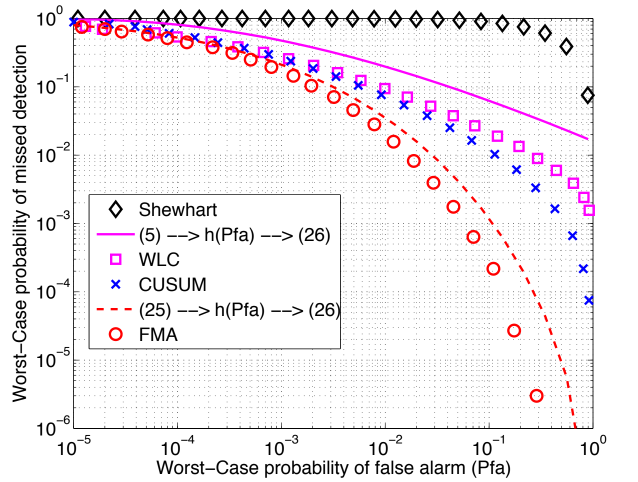

Fig. 1. Simulated ROC of the FMA stopping time and its competitors (markers) for the case of a change in the rate parameter of an exponential distribution. Comparison with the theoretical results (lines) given in (25)–(26) and those obtained with (5) for the bound of the probability of false alarms.

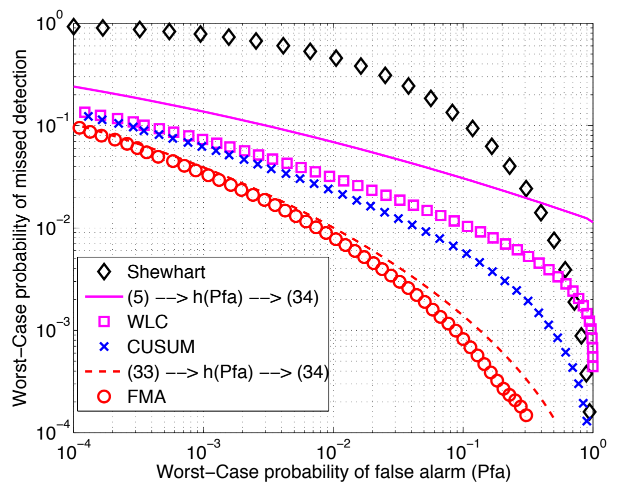

Fig. 2. Simulated ROC of the FMA stopping time and its competitors (markers) for the case of SQM with the C/N$_0$ metric. Comparison with the theoretical results (lines) given in (33)–(34) and those obtained with (5) for the bound of the probability of false alarms.

TABLE I
Simulation parameters for the considered problems.

Exponential		C/N$_0$		DLLout		SAM	
$\boldsymbol{\lambda_0}$	1	$\boldsymbol{\mu_0}$	$10^{4.4}$	$\boldsymbol{\sigma_0^2}$	$1.11 \cdot 10^{-5}$	$\boldsymbol{\mu_0}$	0.1
		$\boldsymbol{\mu_1}$	$10^{3.7}$			$\boldsymbol{\sigma_0^2}$	$1.14 \cdot 10^{-3}$
$\boldsymbol{\lambda_1}$	7	$\boldsymbol{\sigma^2}$	$2.5 \cdot 10^5$	$\boldsymbol{\sigma_1^2}$	$5.44 \cdot 10^{-4}$	$\boldsymbol{\mu_1}$	0.2
						$\boldsymbol{\sigma_1^2}$	$2.03 \cdot 10^{-3}$
$\boldsymbol{m}$	10	6					
$\boldsymbol{m_\alpha}$	60					300	

(see Fig. 1) as (25)$\rightarrow h(\mathbb{P}_{\mathrm{fa}}) \rightarrow$(26) for the FMA bound, and as (5)$\rightarrow h(\mathbb{P}_{\mathrm{fa}}) \rightarrow$(26) for the WLC and CUSUM bound. For the sake of comparison, we assume that the actual change parameters have been used to formulate the LLR (i.e. $\widetilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda}$). This is done by considering the parameters shown in Table I.

Similar results are given in Fig. 2, 3 and 4 for the case of SQM in GNSS when using the C/N$_0$, DLLout and SAM

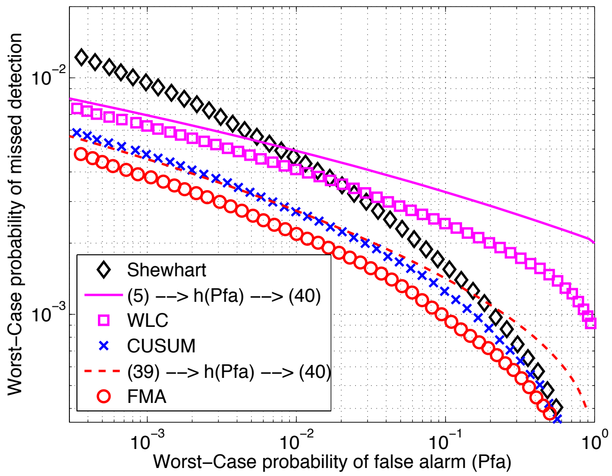

Fig. 3. Simulated ROC of the FMA stopping time and its competitors (markers) for the case of SQM with the DLLout metric. Comparison with the theoretical results (lines) given in (39)–(40) and those obtained with (5) for the bound of the probability of false alarms.

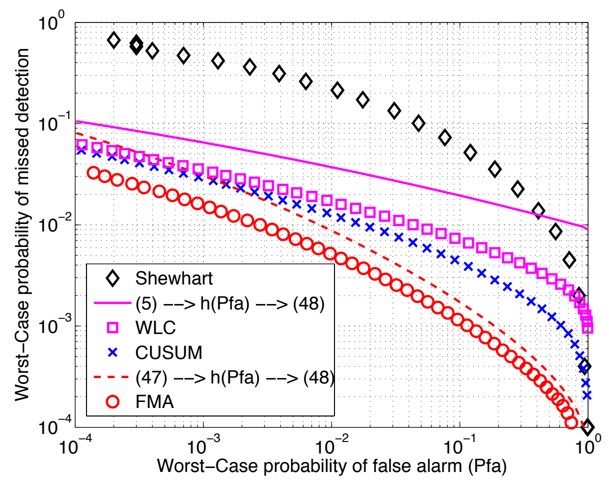

Fig. 4. Simulated ROC of the FMA stopping time and its competitors (markers) for the case of SQM with the SAM. Comparison with the theoretical results (line) given in (47)–(48) and those obtained with (5) for the bound of the probability of false alarms.

metrics. In this case, the parameters shown in Table I were considered in [23] and [26]. Results in Corollaries 3, 4 and 5 apply, respectively, so that {(33),(39),(47)} and {(34),(40),(48)} are used to obtain the threshold and probability of missed detection bound, respectively. It can be concluded from the obtained results that the FMA stopping time outperforms (for all the study cases), in the sense of the optimality criterion in (7)–(8), all the other stopping times considered. Moreover, we see in Figures 1–4 how the improvement of the FMA bounds with respect to those available in the literature for other stopping times is quite significant, providing between half- to two-orders-of-magnitude improvement. This is important not only for a theoretical study but also to set the threshold in practice and provide a level of performance that is close to the desired one. Finally, we also see that the Shewhart stopping time is not giving the best results for any of the considered problems, thus losing its optimality properties for $m > 1$.

At this point, it is worth noting that the curves corresponding to the WLC are those for the WLC without optimization (i.e. given by (10)).We do so for two reasons. First of all, if the WLC is optimized, then the expression of the FMA test is exactly obtained, and there is nothing to add or to compare since both of them coincide. This would be the case for the Gaussian mean change, for the other cases no expression is available. Second, the optimization done in [10] contains some ad hoc elements (this is why we have denoted it as optimization) because it includes the possibility that the threshold is a function of the window size, which is equivalent to biasing the LLRs with a window-size-dependent constant. We have considered the conventional WLC formulation with a window-size-independent to enrich the comparisons.

B. Numerical example: SQM availability

This section is intended to show the behavior of the considered stopping times in terms of availability of SQM. As before, the simulation parameters for each metric are presented in Table I, but here we will consider different values for the parameters under $\mathcal{H}_1$, stated below.

CASE 1: C/N$_0$ metric

Let us start with the C/N$_0$ metric assuming that we have a tolerable error equivalent to a mean change in the C/N$_0$ of 7dB; thus we fix the change parameter as $\widetilde{\mu}_1 = 10^{3.7}$, but the actual change parameter is $\mu_1 = 10^{3.4}$. Therefore, for the case of using the FMA stopping time, fixing the detection threshold from (33) to $h = 2.92$ so that $\mathbb{P}_{\text{fa}}(T_{\text{F,m}}(h), m_\alpha) \leq \widetilde{\alpha} = 10^{-1}$, and substituting the previous values in (32) and (34), the integrity risk is bounded as $\mathbb{P}_{\text{md}}(T_{\text{F,m}}(h), m) \leq \beta(h, m) = 6.97 \cdot 10^{-4}$. For the CUSUM or WLC stopping time we fix the threshold $\widetilde{h}$ from (5), which for $\widetilde{\alpha} = 10^{-1}$ gives $\widetilde{h} = 6.40$, and thus from (34) we get $\mathbb{P}_{\text{md}}(T_{\text{WLC}}(\widetilde{h}), m) \leq \beta(\widetilde{h}, m) = 4.56 \cdot 10^{-3}$. Now, suppose the maximum allowed integrity risk is $\widetilde{\beta} = 10^{-2}$; then since $\beta(h) < \beta(\widetilde{h}) < \widetilde{\beta}$ the SQM algorithm will be available in the case of using any of the analyzed stopping times. On the other hand, we suppose we need $\widetilde{\alpha} = 10^{-2}$ so from (33) and (5) we get $h = 3.59$ for the FMA and $\widetilde{h} = 8.70$ for the CUSUM and WLC, respectively. Thus, from (34) we have that $\beta(h) = 1.02 \cdot 10^{-3}$ and $\beta(\widetilde{h}) = 1.33 \cdot 10^{-2}$. Hence, in this case, SQM will be available only if the FMA stopping time is used. Otherwise, it will not be available since $\beta(\widetilde{h}) > \widetilde{\beta}$, showing the improvements of the FMA in terms of SQM availability.

CASE 2: DLLout metric

For the DLLout, we evaluate the effect of using the actual change parameter on the availability of SQM when a threat is present and we use the reference parameter $\widetilde{\sigma}_1^2$. To do so, imagine that a change is present with $\sigma_1^2 = 5.44 \cdot 10^{-4}$, but the maximum tolerable error in the measured range within the GNSS receiver for each satellite is equal to 14.65 m. For a GPS signal, 14.65 m of error is equivalent to a variation of ± 0.05 chips, which converted to DLLout variance as in [26] gives a minimum detectable change parameter of $\widetilde{\sigma}_1^2 = 2.78 \cdot 10^{-4}$. Assuming we want $\mathbb{P}_{\text{fa}}(T(h), m_\alpha) \leq \widetilde{\alpha} = 10^{-2}$, from (39), we have $h = 3.14$ for the FMA and, from (5), $\widetilde{h} = 8.70$ for the

CUSUM and WLC. Thereby, if we fix the actual parameter as $\sigma_1^2 = \widetilde{\sigma}_1^2$, we get from (40) $\beta(h,m) = 1.70 \cdot 10^{-2}$ and $\beta(\widetilde{h},m) = 4.25 \cdot 10^{-2}$, and then, since they are above $\widetilde{\beta}$, SQM is not available. On the other hand, if we would know the actual change parameter, from (40) we would have $\beta(h,m) = 2.74\cdot 10^{-3}$ and $\beta(\widetilde{h},m) = 7.41\cdot 10^{-3}$, which are bellow $\widetilde{\beta}$ and thus SQM would be available using either the FMA, CUSUM or WLC stopping time. With this result we corroborate the improvements on the availability by knowing the real change parameter in (40).

VII. Conclusions

This work has investigated the problem of TCD. Specifically, we have proposed the use of an FMA stopping time, inspired by the fact that the optimal WLC for the case of a Gaussian mean change is the FMA stopping time. This optimality refers to the choice of the threshold that minimizes the missed detection probability bound of the WLC, when treating a change in the mean of a Gaussian distribution. The statistical performance of the general FMA stopping time has been theoretically investigated and compared by numerical simulations to different methods available in the literature. These experiments have confirmed that the proposed solution outperforms other solutions available in the literature of TCD. It is for this reason that the proposed solution contributes to the general theory of TCD by providing the best solution currently available, in terms of both optimal properties and the goodness of the proposed bounds. Finally, it is worth mentioning that part of the numerical analysis has been done in the setting of GNSS SQM. Nevertheless, this is done without loss of generality, as shown by the considered problem of a change in an exponential distribution. This is because the presented theoretical results were obtained without any assumption on a particular application or on the statistical distribution, so that they are valid for the general theory of TCD.

Appendix A
Proof of Theorem 1

The proof of Theorem 1 is divided in two parts. Firstly, we prove the bound for the probability of false alarm given in (13)–(14). Secondly, we show the proof of the bound for the probability of missed detection given by (15)–(16).

A. Probability of false alarm $\mathbb{P}_{\text{fa}}(T_{\text{F}}(h), m_\alpha)$

We first introduce an important result, stated in the following lemma, that will be very useful to prove Theorem 1.

Lemma 1. Let $S_n \doteq \sum_{i=n-m+1}^{n} \text{LLR}(i)$, $k \geq m$ and $N > k$ be integers, then

$$\mathbb{P}_\infty\left(\bigcap_{i=k}^{k+N-1} \{S_i < h\}\right) \geq \left[\mathbb{P}_\infty\left(S_m < h\right)\right]^N. \tag{55}$$

Proof: Let $y_i = \text{LLR}(i)$, then from (12), for $n \geq m$ we can write

$$S_n = \sum_{i=n-m+1}^{n} y_i = \sum_{i=1}^{n} c_{n-i} y_i, \tag{56}$$

with

$$c_i = \begin{cases} 1 & \text{if } 0 \leq i \leq m-1 \\ 0 & \text{if } i \geq m \end{cases}, \tag{57}$$

so that S_n is written as a monotonically increasing function of $\{y_1, \ldots, y_n\}$ (since $c_i \geq 0$). Therefore, since $y_1, y_2, \ldots$ are iid under $\mathbb{P}_\infty$, from Theorem 5.1 of [45], we have that

$$\mathbb{P}_\infty\left(\bigcap_{i=k}^{k+N-1} \{S_i < h\}\right) \geq \prod_{i=k}^{k+N-1} \mathbb{P}_\infty\left(S_i < h\right), \tag{58}$$

and the inequality (55) thus follows from the fact that the distribution of S_i, under $\mathbb{P}_\infty$, is the same for any $i \geq m$. ■

Now, we aim to prove first another useful result to get (13)–(14), that is

$$\mathbb{P}_{\text{fa}}\left(T_{\text{F}}(h), m_\alpha\right) = \mathbb{P}_\infty\left(m \leq T_{\text{F}}(h) < m + m_\alpha\right). \tag{59}$$

To do so, in a similar way as in [10], from (8) we can write

$$\mathbb{P}_{\text{fa}}(T_{\text{F}}(h), m_\alpha) = \sup_{l \geq m} \sum_{k=l}^{l+m_\alpha-1} \mathbb{P}_\infty(T_{\text{F}}(h) = k). \tag{60}$$

Now denoting $V_l = \mathbb{P}_\infty(l \leq T_{\text{F}}(h) < l + m_\alpha)$ for $l \geq m$ and $U_k = \mathbb{P}_\infty(T_{\text{F}}(h) = k)$, we have that

$$\mathbb{P}_{\text{fa}}(T_{\text{F}}(h), m_\alpha) = \sup_{l \geq m} V_l = \sup_{l \geq m} \sum_{k=l}^{l+m_\alpha-1} U_k. \tag{61}$$

It is easy to verify, from the definition of $T_{\text{F}}(h)$ in (12), that

$$U_m = \mathbb{P}_\infty\left(S_m \geq h\right) \tag{62}$$

and

$$\begin{aligned} U_{m+1} &= \mathbb{P}_\infty\left(\{S_m < h\} \bigcap \{S_{m+1} \geq h\}\right) \\ &\leq \mathbb{P}_\infty\left(\{S_{m+1} \geq h\}\right) \overset{(a)}{=} \mathbb{P}_\infty\left(\{S_m \geq h\}\right) = U_m, \end{aligned} \tag{63}$$

where (a) follows because S_n has the same distribution, under $\mathbb{P}_\infty$, for $n \geq m$ (i.e. the distribution of the sum of m LLRs under $\mathcal{H}_0$). Similarly, for $k > m$, we have

$$\begin{aligned} U_{k+1} &= \mathbb{P}_\infty\left(\bigcap_{n=m}^{k} \{S_n < h\} \bigcap \{S_{k+1} \geq h\}\right) \\ &\leq \mathbb{P}_\infty\left(\bigcap_{n=m+1}^{k} \{S_n < h\} \bigcap \{S_{k+1} \geq h\}\right) \\ &= \mathbb{P}_\infty\left(\bigcap_{n=m}^{k-1} \{S_n < h\} \bigcap \{S_k \geq h\}\right) = U_k. \end{aligned} \tag{64}$$

Thus, $\{U_k\}_{k \geq m}$ is a non-increasing sequence, and then

$$V_l - V_{l+1} = \sum_{k=l}^{l+m_\alpha-1} U_k - \sum_{k=l+1}^{l+m_\alpha} U_k = U_l - U_{l+m_\alpha} \geq 0, \tag{65}$$

so that $\{V_l\}_{l \geq m}$ is a non-increasing sequence as well. Hence, from (61) and the definition of V_l,

$$\mathbb{P}_{\text{fa}}\left(T_{\text{F}}(h), m_\alpha\right) = \sup_{l \geq m} V_l = \mathbb{P}_\infty\left(m \leq T_{\text{F}}(h) < m + m_\alpha\right), \tag{66}$$

and thus (59) follows.

Now, we can proceed with the calculation of $\mathbb{P}_{\text{fa}}(T_{\text{F}}(h), m_\alpha)$. However, the exact calculation from (66) is complicated to obtain, and then an upper bound is proposed instead. From (66) and since $T_{\text{F}}(h) \geq m$ from the definition in (12), we can write

$$\mathbb{P}_{\text{fa}}\left(T_{\text{F}}(h), m_\alpha\right) = 1 - \mathbb{P}_\infty\left(T_{\text{F}}(h) \geq m + m_\alpha\right), \quad (67)$$

with

$$\mathbb{P}_\infty\left(T_{\text{F}}(h) \geq m + m_\alpha\right) = \mathbb{P}_\infty\left(\bigcap_{n=m}^{m+m_\alpha-1} \{S_n < h\}\right). \quad (68)$$

So, (13) and (14) follows by direct application of (55) to (68).

B. Probability of missed detection $\mathbb{P}_{\text{md}}(T_{\text{F}}(h), m)$

Applying the Bayes rule in (7) we have

$$\begin{aligned}\mathbb{P}_{\text{md}}(T_{\text{F}}(h), m) &= \sup_{v>m} \frac{\mathbb{P}_v\left(T_{\text{F}}(h) \geq v + m\right)}{\mathbb{P}_v\left(T_{\text{F}}(h) \geq v\right)} \\ &= \sup_{v>m} \frac{\mathbb{P}_v\left(\bigcap_{n=m}^{m+v-1} \{S_n < h\}\right)}{\mathbb{P}_v\left(\bigcap_{n=m}^{v-1} \{S_n < h\}\right)},\end{aligned} \quad (69)$$

where the last equality follows from the definition of $T_{\text{F}}(h)$ in (12). Due to the windowed behavior of $T_{\text{F}}(h)$ we have assumed that $v > m$. As for $\mathbb{P}_{\text{fa}}(T_{\text{F}}(h), m_\alpha)$, the exact calculation of $\mathbb{P}_{\text{md}}(T_{\text{F}}(h), m)$ from (69) is quite difficult, and then we propose the derivation of an upper bound.

Now, denoting the event $\mathcal{A}_n = \{S_n < h\}$, with $n \geq m$, it is clear that $\mathcal{A}_{v-1}$ and $\mathcal{A}_{m+v-1}$ are independent because they do not share any samples, thus

$$\mathbb{P}_v\left(\bigcap_{n=m}^{m+v-1} \mathcal{A}_n\right) \leq \mathbb{P}_v\left(\mathcal{A}_{m+v-1}\right) \mathbb{P}_v\left(\bigcap_{n=m}^{v-1} \mathcal{A}_n\right), \quad (70)$$

since in the left side we evaluate more events than in the right side. So, applying this result to (69) we have that

$$\begin{aligned}\mathbb{P}_{\text{md}}\left(T_{\text{F}}(h), m\right) &\leq \sup_{v>m} \mathbb{P}_v\left(S_{m+v-1} < h\right) \\ &= \mathbb{P}_1\left(S_m < h\right),\end{aligned} \quad (71)$$

where the equality follows because S_{m+v-1} is identically distributed under $\mathbb{P}_v$, and (15)–(16) thus follow, completing the proof of Theorem 1.

APPENDIX B
CALCULATION OF COEFFICIENTS C_k

A complete proof of the results in Corollary 5 can be found in [42]. Here we only give the expression for C_k so that the bounds in Corollary 5 can be calculated. The coefficients $C_{k,\mathcal{H}_i}$ are obtained as

$$\begin{aligned}C_3 &= \frac{\xi_{z_s,3} - 3\mu_{z_s,i} \cdot \xi_{z_s,2} + 2\mu_{z_s,i}^3}{\sigma_{z_s,i}^3}, \\ C_4 &= \frac{\xi_{z_s,4} - 4\mu_{z_s,i}\xi_{z_s,3} + 6\mu_{z_s,i}^2\xi_{z_s,2} - 3\mu_{z_s,i}^4}{\sigma_{z_s,i}^4} - 3, \\ C_6 &= 10C_3^2,\end{aligned} \quad (72)$$

with $\xi_{z_s,n}$ the moments of Z, under $\mathcal{H}_i$, given by

$$\begin{aligned}\xi_{z_s,2} &= \mu_{z_s,i}^2 + \sigma_{z_s,i}^2, \\ \xi_{z_s,3} &= m\left[\xi_{y,3} + (m-1)\left(\Delta + (m-2)\mu_{z_s,i}^3\right)\right], \\ \xi_{z_s,4} &= m\left[\xi_{y,4} + (m-1)\left(\Omega + (m-2)\left(\Gamma + \Lambda\right)\right)\right],\end{aligned} \quad (73)$$

where $\{\mu_{z_s,i}, \sigma_{z_s,i}^2\}$ are given by (52), $\Delta = 3\mu_{z_s,i} \cdot \xi_{y,2}$, $\Omega = 4\mu_{z_s,i} \cdot \xi_{y,3} + 3\xi_{y,2}^2$, $\Gamma = 6\mu_{z_s,i}^2 \cdot \xi_{y,2}$, $\Lambda = (m-3)\mu_{z_s,i}^4$, and

$$\xi_{y,n} = \sum_{i=0}^{n} A(i), \quad (74)$$

with

$$A(i) = \sum_{j=0}^{i} \binom{n}{i}\binom{i}{j} a_s^{n-i} \cdot b_s^{i-j} \cdot c_s^j \cdot \xi_{x_s,2n-i-j}, \quad (75)$$

where $\binom{n}{i} \doteq (n!)/(i!(n-i)!)$ is the binomial coefficient, $\xi_{x_s,k} \doteq \mathrm{E}_i(x_n^k)$ is the moment of k-th order of x_n under hypothesis $\mathcal{H}_i$ and $\{a_s, b_s, c_s\}$ are defined in (46).

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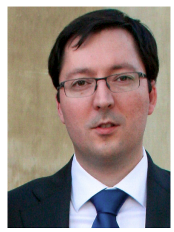

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