A relative of Hadwiger's conjecture

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Abstract

Hadwiger's conjecture asserts that if a simple graph G has no K_{t+1} minor, then its vertex set $V(G)$ can be partitioned into t stable sets. This is still open, but we prove under the same hypotheses that $V(G)$ can be partitioned into t sets $X_1, \ldots, X_t$, such that for $1 \le i \le t$, the subgraph induced on X_i has maximum degree at most a function of t. This is sharp, in that the conclusion becomes false if we ask for a partition into $t-1$ sets with the same property.

1 Introduction

All graphs in this paper are finite and have no loops or multiple edges. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by edge-contraction. In 1943, Hadwiger [4] proposed the following, perhaps the most famous open problem in graph theory:

1.1 (Hadwiger's Conjecture.) *For all integers $t \geq 0$, and every graph G, if K_{t+1} is not a minor of G, then the chromatic number of G is at most t; that is, $V(G)$ can be partitioned*[1] *into t stable sets.*

This remains open, although it has been proved for all $t \leq 5$ (see [8]). It is best possible in that the result becomes false if we ask for a partition into $t-1$ stable sets.

There are several results proving weakenings of Hadwiger's conjecture (see section 3), but as far as we know, the result of this paper is the first which (under the same hypotheses as 1.1) asserts the existence of a partition of $V(G)$ into t sets with any non-trivial property. We prove the following. (If G is a graph, $\Delta(G)$ denotes the maximum degree of G, and if $X \subseteq V(G)$, we denote by $G|X$ the subgraph of G induced on X. If $X = \emptyset$, then $\Delta(G|X) = 0$.)

1.2 *For all integers $t \geq 0$ there is an integer s, such that for every graph G, if K_{t+1} is not a minor of G, then $V(G)$ can be partitioned into t sets $X_1, \ldots, X_t$, such that $\Delta(G|X_i) \leq s$ for $1 \leq i \leq t$.*

Such partitions (into subgraphs with bounded maximum degree) are called "defective colourings" in the literature – see for instance [3].

A reason for interest in 1.2 is that, despite being much weaker than the original conjecture of Hadwiger, it is still best possible in the same sense; if we ask for a partition into $t-1$ subgraphs each with bounded maximum degree, the result becomes false. Let us first see the latter assertion:

1.3 *For all integers $s \geq 0$ and $t \geq 1$, there is a graph $G = G(s,t)$, such that K_{t+1} is not a minor of G, and there is no partition $X_1, \ldots, X_{t-1}$ of $V(G)$ into $t-1$ sets such that $\Delta(G|X_i) \leq s$ for $1 \leq i \leq t-1$.*

Proof. If $t = 1$ we may take $G(s,t)$ to be a one-vertex graph. For $t \geq 2$, we proceed by induction on t. Take the disjoint union of $s+1$ copies $H_1, \ldots, H_{s+1}$ of $G(s, t-1)$, and add one new vertex v adjacent to every other vertex, forming $G = G(s,t)$. It follows that G has no K_{t+1} minor, since each H_i has no K_t minor. Assume that $X_1, \ldots, X_{t-1}$ is a partition of $V(G)$ into $t-1$ sets such that $\Delta(G|X_i) \leq s$ for $1 \leq i \leq t-1$. We may assume that $v \in X_{t-1}$. If $X_{t-1} \cap V(H_i) \neq \emptyset$ for all $i \in \{1, \ldots, s+1\}$, then the degree of v is greater than s in $G|X_{t-1}$, a contradiction; so we may assume that $X_{t-1} \cap V(H_1) = \emptyset$ say. Let $Y_i = X_i \cap V(H_1)$ for $1 \leq i \leq t-2$. Then $Y_1, \ldots, Y_{t-2}$ provides a partition of $V(H_1)$ into $t-2$ sets; and since H_1 is isomorphic to $G(s, t-1)$, it follows that $\Delta(H_1|Y_i) > s$ for some $i \in \{1, \ldots, t-2\}$, a contradiction. Thus there is no such partition $X_1, \ldots, X_{t-1}$. This proves 1.3. ∎

[1] A *partition* of a set V is a list of pairwise disjoint (possibly empty) subsets of V whose union equals V.

A warning: it is tempting to view 1.2 as supporting evidence for Hadwiger's conjecture. However, it is to the same degree "supporting evidence" for the false conjecture of Hajós [2], that every graph that contains no subdivision of K_{t+1} is t-colourable; because we could replace the hypothesis of 1.2 that G has no K_{t+1} minor by the weaker hypothesis that no subgraph of G is a subdivision of K_{t+1}, and the same proof (using an appropriate modification of 2.1) still works.

2 The proof

To prove 1.2 we use the following lemma, due to Kostochka [6, 7] and Thomason [9, 10].

2.1 *There exists $C > 0$ such that for all integers $t \geq 0$ and all graphs G, if K_{t+1} is not a minor of G then G has at most $C(t+1)(\log(t+1))^{\frac{1}{2}}|V(G)|$ edges.*

We use that to prove two more lemmas:

2.2 *Let $t \geq 0$ be an integer, let C be as in 2.1, and let $r \geq C(t+1)(\log(t+1))^{\frac{1}{2}}$. Let G be a graph such that K_{t+1} is not a minor of G, and let $A \subseteq V(G)$ be a stable set of vertices each of degree at least t. Then*

$$|E(G \setminus A)| + |A| \leq r|V(G \setminus A)|.$$

Proof. We proceed by induction on $|A|$. By 2.1, we may assume that $A \neq \emptyset$. Let $v \in A$. Since v has degree at least t and G has no K_{t+1} subgraph, v has two neighbours x, y which are non-adjacent to each other. Let $G' = (G \setminus v) + xy$ and $A' = A \setminus \{v\}$. Since G' is a minor of G and so K_{t+1} is not a minor of G', it follows from the inductive hypothesis that $|E(G' \setminus A')| + |A'| \leq r|V(G' \setminus A')| = r|V(G \setminus A)|$. But $|E(G' \setminus A')| = |E(G \setminus A)| + 1$ and $|A'| = |A| - 1$. This proves 2.2. ∎

2.3 *Let $t \geq 0$ be an integer, let C be as in 2.1, and let $r \geq C(t+1)(\log(t+1))^{\frac{1}{2}}$ and $r > t/2$. Let s be the least integer greater than $r(2r-t+2)$. Let G be a non-null graph, such that K_{t+1} is not a minor of G. Then either*

- *some vertex has degree less than t, or*
- *there are two adjacent vertices, both with degree less than s.*

Proof. We may assume that $t \geq 2$, for if $t \leq 1$ the result is trivially true. Let A be the set of all vertices with degree less than s, and $B = V(G) \setminus A$. We may assume that every vertex in A has degree at least t, for otherwise the first outcome holds. We may also assume that no two vertices of A are adjacent because otherwise the second outcome holds. Consequently, by summing all the degrees, we deduce that $2|E(G)| \geq t|A| + s|B|$. On the other hand, by 2.1, $|E(G)| \leq r(|A| + |B|)$. It follows that $t|A| + s|B| \leq 2r(|A| + |B|)$, that is,

$$|A| \geq \frac{s-2r}{2r-t}|B|,$$

since $2r > t$. But by 2.2, $|A| \leq r|B|$. Since G is a non-null graph, $|B| \neq 0$ and so $r \geq (s-2r)/(2r-t)$, that is, $s \leq r(2r-t+2)$, a contradiction. This proves 2.3. ∎

Now we prove 1.2, in the following sharpened form.

2.4 *Let $t \geq 0$ be an integer, and let s be as in 2.3. For every graph G, if K_{t+1} is not a minor of G, then $V(G)$ can be partitioned into t sets $X_1, \ldots, X_t$, such that $\Delta(G|X_i) < s$ for $1 \leq i \leq t$.*

Proof. We proceed by induction on $|V(G)| + |E(G)|$. If some vertex v of G has degree less than t, the result follows from the inductive hypothesis by deleting v (find a partition by induction and add v to some set X_i that contains no neighbour of v). If some edge e has both ends of degree at most s, then the result follows from the inductive hypothesis by deleting e (find a partition by induction, and note that replacing e will not cause either of the ends of e to have degree too large). Thus the result follows from 2.3. This proves 2.4 and hence 1.2. ∎

3 Remark

Kawarabayashi and Mohar [5] proved the following.

3.1 *There is a function $f(t)$ such that, if G is a graph with no K_{t+1} minor, then $V(G)$ can be partitioned into $f(t)$ sets, inducing subgraphs in which every component is of bounded size.*

Kawarabayashi and Mohar proved that taking $f(t) = \lceil 15.5(t+1) \rceil$ works; and Wood [11] improved this, showing that taking $f(t) = \lceil 3.5t + 2 \rceil$ works, using an unpublished result of Norin and Thomas on large $(t+1)$-connected graphs with no K_t minor (announced about 2008). (This has recently been improved to $f(t) = 2(t+1)$ by Norin [unpublished].) That suggests a nice open question – can we prove the same with $f(t) = t$? This would then give a common extension of these results and 1.2.

Here is a way to improve the Kawarabayashi-Mohar result, showing that $f(t) = 4t$ works (not quite as good as Wood's result, but easier). Alon et al. [1, Theorem 6.6] proved that for all integers $t \geq 0$ and Δ, there exists s such that for every graph G, if K_{t+1} is not a minor of G and $\Delta(G) \leq \Delta$, then $V(G)$ can be partitioned into four sets $X_1, \ldots, X_4$ such that every component of $G|X_1$, ..., $G|X_4$ has at most s vertices. By combining this with 1.2, we obtain a partition of $V(G)$ into $4t$ sets each inducing a graph with no large component.

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