The Likelihood Encoder for Lossy Source Compression

Eva C. Song Paul Cuff H. Vincent Poor Dept. of Electrical Eng., Princeton University, NJ 08544 {csong, cuff, poor}@princeton.edu

Abstract—In this work, a likelihood encoder is studied in the context of lossy source compression. The analysis of the likelihood encoder is based on a soft-covering lemma. It is demonstrated that the use of a likelihood encoder together with the soft-covering lemma gives alternative achievability proofs for classical source coding problems. The case of the rate-distortion function with side information at the decoder (i.e. the Wyner-Ziv problem) is carefully examined and an application of the likelihood encoder to the multi-terminal source coding inner bound (i.e. the Berger-Tung region) is outlined.

I. INTRODUCTION

Rate-distortion theory, founded by Shannon in [1] and [2], provides the fundamental limits of lossy source compression. The minimum rate required to represent an independent and identically distributed (i.i.d.) source sequence under a given tolerance of distortion is given by the rate-distortion function. Related problems such as source coding with side information available only at the decoder [3] and distributed source coding [4], [5], [6] have also been heavily studied in the past decades. Standard proofs [7], [8] of achievability for these ratedistortion problems often use joint-typicality encoding, i.e. the encoder looks for a codeword that is jointly typical with the source sequence. The distortion analysis involves bounding several "error" events which may come from either encoding or decoding. These bounds use the joint asymptotic equipartition principle (J-AEP) and its immediate consequences as the main tool. In the cases where there are multiple information sources, such as side information at the decoder, intricacies arise, such as the need for a Markov lemma [7] and [8]. These subtleties also lead to error-prone proofs involving the analysis of error caused by random binning, which have been pointed out in several existing works [9] [10].

In this paper, we propose using a likelihood encoder to achieve classical source coding results such as the Wyner-Ziv rate-distortion function and Berger-Tung inner bound. This encoder has been used in [11] to achieve the rate-distortion function for point-to-point communication and in [12] and [13] to achieve strong coordination. The advantage of the likelihood encoder over a joint-typicality encoder becomes crucial in secrecy systems [14].

Just as the joint-typicality encoder relies on the J-AEP, the likelihood encoder relies on the soft-covering lemma. The idea

This research was supported in part by the Air Force Office of Scientific Research under Grant FA9550-12-1-0196 and MURI Grant FA9550-09-05086 and in part by National Science Foundation under Grants CCF-1116013 and CNS-09-05086.

of soft-covering was first introduced in [15] and was later used in [16] for channel resolvability.

The application of the likelihood encoder together with the soft-covering lemma is not limited to only discrete alphabet. The proof for sources from continuous alphabets is readily included, since the soft-covering lemma imposes no restriction on alphabet size. Therefore, no extra work, i.e. quantization of the source, is needed to extend the standard proof for discrete sources to continuous sources as in [8]. This advantage becomes more desirable for the multi-terminal case, since generalization of the type-covering lemma and the Markov lemma to continuous alphabets is non-trivial. Strong versions of the Markov lemma on finite alphabets that can prove the Berger-Tung inner bound can be found in [8] and [17]. However, generalization to the continuous alphabets is still an ongoing research topic. Some work, such as [18], has been dedicated to making this transition, yet is not strong enough to be applied to the Berger-Tung case.

II. PRELIMINARIES

A. Notation

A sequence $X_1, ..., X_n$ is denoted by X^n . Limits taken with respect to " $n \to \infty$ " are abbreviated as " \to_n ". Inequalities with $\limsup_{n\to\infty} h_n \leq h$ and $\liminf_{n\to\infty} h_n \geq h$ are abbreviated as $h_n \leq_n h$ and $h_n \geq_n h$, respectively. When X denotes a random variable, x is used to denote a realization, \mathcal{X} is used to denote the support of that random variable, and Δ_{χ} is used to denote the probability simplex of distributions with alphabet \mathcal{X} . The symbol $|\cdot|$ is used to denote the cardinality. A Markov relation is denoted by the symbol –. We use \mathbb{E}_P , \mathbb{P}_P , and $I_P(X;Y)$ to indicate expectation, probability, and mutual information taken with respect to a distribution P; however, when the distribution is clear from the context, the subscript will be omitted. To keep the notation uncluttered, the arguments of a distribution are sometimes omitted when the arguments' symbols match the subscripts of the distribution, e.g. $P_{X|Y}(x|y) = P_{X|Y}$. We use a bold capital letter **P** to denote that a distribution P is random. We use $\mathbb R$ to denote the set of real numbers and \mathbb{R}^+ to denote the nonnegative

For a distortion measure $d: \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}^+$, we use $\mathbb{E}\left[d(X,Y)\right]$ to measure the distortion of X incurred by representing it as Y. The maximum distortion is defined as

$$d_{max} = \max_{(x,y)\in\mathcal{X}\times\mathcal{Y}} d(x,y).$$

The distortion between two sequences is defined to be the per-letter average distortion

$$d(x^n, y^n) = \frac{1}{n} \sum_{t=1}^n d(x_t, y_t).$$

B. Total Variation Distance

The total variation distance between two distributions P and Q on the same alphabet \mathcal{X} is defined as

$$||P - Q||_{TV} \triangleq \sup_{A} |P(A) - Q(A)|,$$

where A ranges over all subsets of the sample space.

Property 1 (Property 2 [14]). The total variation distance satisfies the following properties:

(a) Let $\varepsilon > 0$ and let f(x) be a function in a bounded range with width $b \in \mathbb{R}$. Then

$$||P - Q||_{TV} < \varepsilon \Longrightarrow |\mathbb{E}_P[f(X)] - \mathbb{E}_Q[f(X)]| < \varepsilon b.$$
 (1)

(b) Total variation satisfies the triangle inequality. For any $R \in \Delta_{\mathcal{X}}$,

$$||P - Q||_{TV} \le ||P - R||_{TV} + ||R - Q||_{TV}.$$
 (2)

(c) Let $P_X P_{Y|X}$ and $Q_X P_{Y|X}$ be two joint distributions on $\Delta_{\mathcal{X} \times \mathcal{Y}}$. Then

$$||P_X P_{Y|X} - Q_X P_{Y|X}||_{TV} = ||P_X - Q_X||_{TV}.$$
 (3)

(d) For any $P, Q \in \Delta_{\mathcal{X} \times \mathcal{Y}}$,

$$||P_X - Q_X||_{TV} \le ||P_{XY} - Q_{XY}||_{TV}.$$
 (4)

C. The Likelihood Encoder

We define the likelihood encoder, operating at rate R, which receives a sequence $x_1,...,x_n$ and maps it to a message $M \in [1:2^{nR}]$. In normal usage, a decoder then uses M to form an approximate reconstruction of the $x_1,...,x_n$ sequence.

The encoder is specified by a codebook of $y^n(m)$ sequences and a joint distribution P_{XY} . Consider the likelihood function for each codeword, with respect to a memoryless channel from Y to X, defined as follows:

$$\mathcal{L}(m|x^n) \triangleq P_{X^n|Y^n}(x^n|y^n(m)).$$

A likelihood encoder is a stochastic encoder that determines the message index with probability proportional to $\mathcal{L}(m|x^n)$, i.e.

$$P_{M|X^n}(m|x^n) = \frac{\mathcal{L}(m|x^n)}{\sum_{m' \in [1:2^nR]} \mathcal{L}(m'|x^n)} \propto \mathcal{L}(m|x^n).$$

D. Soft-Covering Lemma

Now we introduce the core lemma that serves as the foundation for this analysis. One can consider the role of the soft-covering lemma in analyzing the likelihood encoder as analogous to that of the J-AEP which is used for the analysis of joint-typicality encoders. The general idea of the soft-covering lemma is that the distribution induced by selecting uniformly from a random codebook and passing the codeword through a memoryless channel is close to an i.i.d. distribution as long as the codebook size is large enough.

Lemma 1 (Lemma 1.1 [11] and Lemma IV.1 [12]). Given a joint distribution P_{XY} , let $C^{(n)}$ be a random collection of sequences $Y^n(m)$, with $m=1,...,2^{nR}$, each drawn independently and i.i.d. according to P_Y . Denote by P_{X^n} the output distribution induced by selecting an index m uniformly at random and applying $Y^n(m)$ to the memoryless channel specified by $P_{X|Y}$. Then if R > I(X;Y),

$$\mathbb{E}_{\mathcal{C}^n} \| P_{X^n} - \prod_{t=1}^n P_X \|_{TV} \le \epsilon_n \to_n 0.$$

E. Approximation Lemma

Lemma 2. For a distribution P_{UVX} and $0 < \varepsilon < 1$, if $\mathbb{P}[U \neq V] \leq \varepsilon$, then

$$||P_{UX} - P_{VX}||_{TV} \le \varepsilon.$$

The proof is omitted due to a lack of space.

III. PROBLEM SETUP AND RESULT REVIEW

A. Wyner-Ziv Model Review

The source and side information (X^n, B^n) is distributed i.i.d. according to $(X_t, B_t) \sim \overline{P}_{XB}$. The system has the following constraints:

- Encoder $f_n: \mathcal{X}^n \mapsto \mathcal{M}$ (possibly stochastic).
- Decoder $g_n: \mathcal{M} \times \mathcal{B}^n \mapsto \mathcal{Y}^n$ (possibly stochastic).
- Compression rate: R, i.e. $|\mathcal{M}| = 2^{nR}$.

The system performance is measured according to the following distortion metric:

• Average distortion: $d(X^n, Y^n) = \frac{1}{n} \sum_{t=1}^n d(X_t, Y_t)$.

Definition 1. A rate distortion pair (R, D) is achievable if there exists a sequence of rate R encoders and decoders (f_n, g_n) , such that $\mathbb{E}[d(X^n, Y^n)] \leq_n D$.

Definition 2. The rate distortion function is $R(D) \triangleq \inf_{\{(R,D) \text{ is achievable}\}} R$.

The above mathematical formulation is illustrated in Fig. 1.

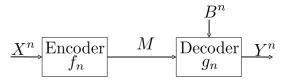


Fig. 1: The Wyner-Ziv problem: rate-distortion for source coding with side information at the decoder

B. Rate-Distortion Function of Wyner-Ziv

The solution to this source coding problem is given in [3]. The rate-distortion function with side information at the decoder is

$$R(D) = \min_{\overline{P}_{V|XB} \in \mathcal{M}(D)} I_{\overline{P}}(X; V|B), \tag{5}$$

where

$$\mathcal{M}(D) = \left\{ \overline{P}_{V|XB} : V - X - B, |\mathcal{V}| \le |\mathcal{X}| + 1, \right.$$
 and there exists

a function
$$\phi$$
 s.t. $\mathbb{E}\left[d(X,Y)\right] \leq D, Y \triangleq \phi(V,B)$. (6)

IV. ACHIEVABILITY PROOF USING THE LIKELIHOOD ENCODER

Our proof technique involves using the likelihood encoder and a channel decoder and showing that the behavior of the system is approximated by a well-behaved distribution. Exact bounds are obtained by using the soft-covering lemma to analyze how well the approximating distribution matches the system. For the readers' reference, a very short and simple achievability proof for point-to-point lossy compression was provided in [11], which will serve to familiarize the reader with the proof techniques in this paper using the likelihood encoder.

We will introduce a virtual message which is produced by the encoder but not physically transmitted to the receiver so that this virtual message together with the actual message gives a high enough rate for applying the soft-covering lemma. Then we show that this virtual message can be reconstructed with vanishing error probability at the decoder by using the side information. This is analogous to the technique of random binning.

Let R > R(D), where R(D) is from (5). We prove that R is achievable for distortion D. Let M' be a virtual message with rate R' which is not physically transmitted. By the rate-distortion formula (5), we can fix $\overline{P}_{V|XB} \in \mathcal{M}(D)$, $(\overline{P}_{V|XB} = \overline{P}_{V|X})$ such that $R + R' > I_{\overline{P}}(X;V)$ and $R' < I_{\overline{P}}(V;B)$. We will use the likelihood encoder derived from \overline{P}_{XV} and a random codebook $\{v^n(m,m')\}$ generated according to \overline{P}_V to prove the result. The decoder will first use the transmitted message M and the side information B^n to decode M' as \hat{M}' and reproduce $v^n(M,\hat{M}')$. Then the reconstruction Y^n is produced as a function of B^n and V^n .

The distribution induced by the encoder and decoder is

$$\frac{\mathbf{P}_{X^nB^nMM'\hat{M}'Y^n}}{\overline{P}_{X^nB^n}\mathbf{P}_{MM'|X^n}\mathbf{P}_{\hat{M}'|MB^n}\mathbf{P}_{Y^n|M\hat{M}'B^n}} \tag{7}$$

$$\triangleq \overline{P}_{X^n B^n} \mathbf{P}_{LE}(m, m'|x^n) \mathbf{P}_D(\hat{m}'|m, b^n) \mathbf{P}_{\Phi}(y^n|m, \hat{m}', b^n)$$
(8)

where \mathbf{P}_{LE} is the likelihood encoder; $\mathbf{P}_D(\hat{m}'|m,b^n)$ is the first part of the decoder that estimates m' as \hat{m}' ; and $\mathbf{P}_{\Phi}(y^n|m,\hat{m}',b^n)$ is the second part of the decoder that reconstructs the source sequence. Note that the distributions are random due to the random codebook.

We now concisely restate the behavior of the encoder and decoder, as components of the induced distribution.

Codebook generation: We independently generate $2^{n(R+R')}$ sequences in \mathcal{V}^n according to $\prod_{i=1}^n \overline{P}_V(v_i)$ and index by $(m,m')\in[1:2^{nR}]\times[1:2^{nR'}]$. We use $\mathcal{C}^{(n)}$ to denote the random codebook.

Encoder: The encoder $\mathbf{P}_{LE}(m, m'|x^n)$ is the likelihood encoder that chooses M and M' stochastically with probability proportional to the likelihood function given by

$$\mathcal{L}(m, m'|x^n) = \overline{P}_{X^n|V^n}(x^n|V^n(m, m')).$$

Decoder: The decoder has two steps. Let $\mathbf{P}_D(\hat{m}'|m,b^n)$ be a good channel decoder (e.g. the maximum likelihood decoder) with respect to the sub-codebook $\mathcal{C}^{(n)}(m) = \{v^n(m,a)\}_a$ and the memoryless channel $\overline{P}_{B|V}$. For the second part of the

decoder, let $\phi(\cdot,\cdot)$ be the function corresponding to the choice of $\overline{P}_{V|XB}$ in (6), that is $Y=\phi(V,B)$ and $\mathbb{E}_{\overline{P}}\left[d(X,Y)\right]\leq D$. Define $\phi^n(v^n,b^n)$ as the concatenation $\{\phi(v_t,b_t)\}_{t=1}^n$ and set the decoder \mathbf{P}_{Φ} to be the deterministic function

$$\mathbf{P}_{\Phi}(y^n|m,\hat{m}',b^n) \triangleq \mathbb{1}\{y^n = \phi^n(V^n(m,\hat{m}'),b^n)\}.$$

Analysis: We will need three distributions for the analysis, the induced distribution \mathbf{P} and two approximating distributions $\mathbf{Q}^{(1)}$ and $\mathbf{Q}^{(2)}$. The idea is to show that 1) the system has nice behavior for distortion under $\mathbf{Q}^{(2)}$; and 2) \mathbf{P} and $\mathbf{Q}^{(2)}$ are close in total variation (averaged over the random codebook) through $\mathbf{Q}^{(1)}$.

$$M$$
 C^n
 $V^n(M,M')$
 $\bar{P}_{XB|V}$
 B^n

Fig. 2: Auxiliary distribution with test channel $\overline{P}_{XB|V}$

Now we will design an auxiliary distribution Q through a test channel as shown in Fig. 2. The joint distribution under Q in Fig. 2 can be written as

$$\mathbf{Q}_{X^{n}B^{n}V^{n}MM'} = Q_{MM'}\mathbf{Q}_{V^{n}|MM'}\mathbf{Q}_{X^{n}B^{n}|MM'}
= \frac{1}{2^{n(R+R')}}\mathbb{I}\{v^{n} = V^{n}(m,m')\}\prod_{t=1}^{n}\overline{P}_{XB|V}(x_{t},b_{t}|V_{t}(m,m'))
= \frac{1}{2^{n(R+R')}}\mathbb{I}\{v^{n} = V^{n}(m,m')\}\prod_{t=1}^{n}\overline{P}_{X|V}(x_{t}|v_{t})\overline{P}_{B|X}(b_{t}|x_{t})$$
(9)

where (9) follows from the Markov chain under \overline{P} , V-X-B. In fact, the reason for choosing the likelihood encoder lies in

$$\mathbf{Q}_{MM'|X^n} = \mathbf{P}_{LE}.\tag{10}$$

Furthermore, it can be verified that

$$\mathbb{E}_{\mathcal{C}^{(n)}}\left[\mathbf{Q}_{X^nB^nV^n}(x^n,b^n,v^n)\right] = \overline{P}_{X^nB^nV^n}(x^n,b^n,v^n), (11)$$

where $\overline{P}_{X^nB^nV^n}$ denotes the i.i.d. distribution $\prod_{t=1}^n \overline{P}_{XBV}$. Define two distributions $\mathbf{Q}^{(1)}$ and $\mathbf{Q}^{(2)}$ based on \mathbf{Q} as follows:

$$\mathbf{Q}_{X^{n}B^{n}V^{n}MM'\hat{M}'Y^{n}}^{(1)} \triangleq \mathbf{Q}_{X^{n}B^{n}V^{n}MM'}\mathbf{P}_{D}\mathbf{P}_{\Phi}(y^{n}|m,\hat{m}',b^{n}) \quad (12)$$

$$\mathbf{Q}_{X^{n}B^{n}V^{n}MM'\hat{M}'Y^{n}}^{(2)} \triangleq \mathbf{Q}_{X^{n}B^{n}V^{n}MM'}\mathbf{P}_{D}\mathbf{P}_{\Phi}(y^{n}|m,m',b^{n}). \quad (13)$$

Notice that $\mathbf{Q}^{(2)}$ differs from $\mathbf{Q}^{(1)}$ by allowing the decoder to use m' rather than \hat{m}' when forming its reconstruction through ϕ^n .

Therefore, on account of (11),

$$\mathbb{E}_{\mathcal{C}^{(n)}}\left[\mathbf{Q}_{X^{n}B^{n}V^{n}Y^{n}}^{(2)}(x^{n},b^{n},v^{n},y^{n})\right] = \overline{P}_{X^{n}B^{n}V^{n}Y^{n}}(x^{n},b^{n},v^{n},y^{n}).$$

Consequently,

$$\mathbb{E}_{\mathcal{C}^{(n)}}\left[\mathbb{E}_{\mathbf{Q}^{(2)}}[d(X^n, Y^n)]\right] = \mathbb{E}_{\overline{P}}[d(X, Y)]. \tag{14}$$

Now applying the soft-covering lemma, since $R+R'>I_{\overline{P}}(B,X;V)=I_{\overline{P}}(X;V),$ we have

$$\mathbb{E}_{\mathcal{C}^{(n)}}\left[\left\|\overline{P}_{X^nB^n} - \mathbf{Q}_{X^nB^n}\right\|_{TV}\right] \le \epsilon_n \to_n 0.$$

And with (8), (10), (12), and Property 1(c), we obtain

$$\mathbb{E}_{\mathcal{C}^{(n)}}\left[\|\mathbf{P}_{X^nB^nMM'\hat{M}'Y^n} - \mathbf{Q}_{X^nB^nMM'\hat{M}'Y^n}^{(1)}\|_{TV}\right] \leq \epsilon_n \text{ (15)} \text{ and decoders } (f_{1n}, f_{2n}, g_n) \text{ such that }$$

Since by definition
$$\mathbf{Q}_{X^nB^nMM'\hat{M}'}^{(1)}=\mathbf{Q}_{X^nB^nMM'\hat{M}'}^{(2)},$$

$$\Upsilon \triangleq \mathbb{P}_{\mathbf{Q}^{(1)}}[\hat{M}' \neq M'] = \mathbb{P}_{\mathbf{Q}^{(2)}}[\hat{M}' \neq M'].$$

Also, since R' < I(V; B), the codebook is randomly generated, and M' is uniformly distributed under Q, it is well known that the maximum likelihood decoder P_D (as well as a variety of other decoders) will drive the error probability to zero as n goes to infinity. Specifically,

$$\mathbb{E}_{\mathcal{C}^{(n)}}\left[\mathbb{P}_{\mathbf{Q}^{(1)}}[M' \neq \hat{M}']\right] \leq \delta_n \to_n 0.$$

Applying Lemma 2, we obtain

$$\mathbb{E}_{\mathcal{C}^{(n)}} \| \mathbf{Q}_{X^n B^n M \hat{M}'}^{(1)} - \mathbf{Q}_{X^n B^n M M'}^{(2)} \|_{TV} \le \mathbb{E}_{\mathcal{C}^{(n)}} [\Upsilon] \le \delta_n. (16)$$

Thus by Property 1(c) and definitions (12) and (13),

$$\mathbb{E}_{\mathcal{C}^{(n)}} \left[\| \mathbf{Q}_{X^n B^n M \hat{M}' Y^n}^{(1)} - \mathbf{Q}_{X^n B^n M M' Y^n}^{(2)} \|_{TV} \right] \le \delta_n. \quad (17)$$

Combining (15) and (17) and using Property 1(b) (d), we have

$$\mathbb{E}_{\mathcal{C}^{(n)}}\left[\|\mathbf{P}_{X^nY^n} - \mathbf{Q}_{X^nY^n}^{(2)}\|_{TV}\right] \le \epsilon_n + \delta_n,\tag{18}$$

where ϵ_n and δ_n are the error terms introduced from the softcovering lemma and channel coding, respectively.

Using Property 1(a) and (14) and (18), we have

$$\mathbb{E}_{\mathcal{C}^{(n)}}\left[\mathbb{E}_{\mathbf{P}}[d(X^n, Y^n)]\right] \le \mathbb{E}_{\overline{P}}[d(X, Y)] + d_{max}(\epsilon_n + \delta_n).$$
(19)

Therefore, there exists a codebook under which

$$\mathbb{E}_P[d(X^n,Y^n)] \leq_n D.$$

V. EXTENSION TO DISTRIBUTED LOSSY SOURCE COMPRESSION

The application of the likelihood encoder can go beyond single-user communications. In this section, we will outline an alternative proof for achieving the Berger-Tung inner bound.

A. Berger-Tung Model Review

We now assume a pair of correlated sources (X_1^n, X_2^n) , distributed i.i.d. according to $(X_{1t}, X_{2t}) \sim \overline{P}_{X_1X_2}$, independent encoders, and a joint decoder, satisfying the following constraints:

- Encoder 1 $f_{1_n}: \mathcal{X}_1^n \mapsto \mathcal{M}_1$ (possibly stochastic).
- Encoder $2 f_{2n} : \mathcal{X}_2^n \mapsto \mathcal{M}_2$ (possibly stochastic). Decoder $g_n : \mathcal{M}_1 \times \mathcal{M}_2 \mapsto \mathcal{Y}_1^n \times \mathcal{Y}_2^n$ (possibly
- Compression rates: R_1, R_2 , i.e. $|\mathcal{M}_1| = 2^{nR_1}, |\mathcal{M}_2| =$

The system performance is measured according to the following distortion metric:

• $\mathbb{E}[d_k(X_k{}^n,Y_k{}^n)]=\frac{1}{n}\sum_{t=1}^n d_k(X_{kt},Y_{kt}),\ k=1,2,$ where $d_k(\cdot,\cdot)$ can be different distortion measures for different k.

Definition 3. (R_1, R_2) is achievable under distortion level (D_1, D_2) if there exists a sequence of rate (R_1, R_2) encoders

$$\mathbb{E}[d_1(X_1^n, Y_1^n)] \leq_n D_1,$$

$$\mathbb{E}[d_2(X_2^n, Y_2^n)] \le_n D_2.$$

The achievable rate region is not yet known in general. But an inner bound, reproduced below, was given in [4] and [5] and is known as the Berger-Tung inner bound. The rates (R_1, R_2) are achievable if

$$R_1 > I_{\overline{P}}(X_1; U_1|U_2), \tag{20}$$

$$R_2 > I_{\overline{P}}(X_2; U_2|U_1), \tag{21}$$

$$R_1 + R_2 > I_{\overline{P}}(X_1, X_2; U_1, U_2)$$
 (22)

for some $\overline{P}_{U_1X_1X_2U_2}=\overline{P}_{X_1X_2}\overline{P}_{U_1|X_1}\overline{P}_{U_2|X_2},$ and functions $\phi_k(\cdot,\cdot)$ such that $\mathbb{E}[d_k(X_k,Y_k)] \leq D_k$, where $Y_k \triangleq$ $\phi_k(U_1, U_2), k = 1, 2.$ ¹

B. Proof Sketch Using the Likelihood Encoder

simplicity, we will focus on $(I_{\overline{P}}(X_1; U_1), I_{\overline{P}}(X_2; U_2|U_1))$ $C_2 \triangleq (I_{\overline{P}}(X_1; U_1|U_2), I_{\overline{P}}(X_2; U_2)), \text{ of the region given}$ in (20) through (22) and use convexity to claim the complete region. Below we demonstrate how to achieve C_1 . The point

 C_2 follows by symmetry. Fix a $\overline{P}_{U_1U_2|X_1X_2} = \overline{P}_{U_1|X_1}\overline{P}_{U_2|X_2}$ and functions $\phi_k(\cdot,\cdot)$ such that $Y_k = \phi_k(U_1,U_2)$ and $\mathbb{E}_{\overline{P}}\left[d_k(X_k,Y_k)\right] < D_k$. Note that $U_1-X_1-X_2-U_2$ forms a Markov chain under \overline{P} . We must show that any rates (R_1,R_2) satisfying $R_1 > I_{\overline{P}}(X_1; U_1)$ and $R_2 > I_{\overline{P}}(X_2; U_2|U_1)$ are achievable.

First we will use the likelihood encoder derived from $\overline{P}_{X_1U_1}$ and a random codebook $\{u_1^n(m_1)\}$ generated according to \overline{P}_{U_1} for Encoder 1. Then we will use the likelihood encoder derived from $\overline{P}_{X_2U_2}$ and another random codebook $\{u_2^n(m_2, m_2')\}$ generated according to \overline{P}_{U_2} for Encoder 2. The decoder will use the transmitted message M_1 to decode U_1^n , as in the point-to-point case, and use the transmitted message M_2 along with the decoded U_1^n to decode M_2' as M_2' , as in the Wyner-Ziv case, and reproduce $u_2^n(M_2, \hat{M}_2')$. Finally, the decoder outputs the reconstructions \tilde{Y}_k^n as functions of U_1^n and U_2^n .

The distribution induced by the encoders and decoder is

$$\mathbf{P}_{X_1{}^nX_2{}^nU_1{}^nM_1M_2M_2'\hat{M}_2'Y_1{}^nY_2{}^n} = \overline{P}_{X_1{}^nX_2{}^n}\mathbf{P}_1\mathbf{P}_2$$

$$\mathbf{P}_{1} \triangleq \mathbf{P}_{M_{1}|X_{1}^{n}}\mathbf{P}_{U_{1}^{n}|M_{1}} \tag{23}$$

$$\mathbf{P}_{2} \triangleq \mathbf{P}_{M_{2}M_{2}'|X_{2}^{n}} \mathbf{P}_{\hat{M}_{2}'|M_{2}U_{1}^{n}} \prod_{k=1,2} \mathbf{P}_{Y_{k}^{n}|U_{1}^{n}M_{2}\hat{M}_{2}'}(24)$$

$$\triangleq \mathbf{P}_{M_2 M_2' | X_2^n} \mathbf{P}_D \prod_{k=1,2} \mathbf{P}_{\Phi,k}, \tag{25}$$

where again M'_2 plays the role of the virtual message that is not physically transmitted as in the Wyner-Ziv case.

¹This region, after optimizing over auxiliary variables, is in fact not convex, so it can be improved to the convex hull through time-sharing.

Codebook generation: We independently generate 2^{nR_1} sequences in \mathcal{U}_1^n according to $\prod_{t=1}^n \overline{P}_{U_1}(u_{1t})$ and index them by $m_1 \in [1:2^{nR_1}]$, and independently generate $2^{n(R_2+R_2')}$ sequences in \mathcal{U}_2^n according to $\prod_{t=1}^n \overline{P}_{U_2}(u_{2t})$ and index them by $(m_2,m_2') \in [1:2^{nR_2}] \times [1:2^{nR_2'}]$. We use $\mathcal{C}_1^{(n)}$ and $\mathcal{C}_2^{(n)}$ to denote the two random codebooks, respectively.

Encoders: Encoder 1 $\mathbf{P}_{M_1|X_1^n}$ is the likelihood encoder according to $\overline{P}_{X_1^n U_1^n}$ and $\mathcal{C}_1^{(n)}$. Encoder 2 $\mathbf{P}_{M_2 M_2' | X_2^n}$ is

the likelihood encoder according to $\overline{P}_{X_2^nU_2^n}$ and $C_2^{(n)}$. **Decoder:** First, let $\mathbf{P}_{U_1|M_1}$ be a $C_1^{(n)}$ codeword lookup decoder. Then, let $\mathbf{P}_D(\hat{m}_2'|m_2,u_1^n)$ be a good channel decoder with respect to the sub-codebook $\underline{C}_2^{(n)}(m_2) = \{u_2^n(m_2,a)\}_a$ and the memoryless channel $\overline{P}_{U_1|U_2}$. Last, define $\phi_k^n(u_1^n, u_2^n)$ as the concatenation $\{\phi_k(u_{1t}, u_{2t})\}_{t=1}^n$ and set the decoders $\mathbf{P}_{\Phi,k}$ to be the deterministic functions

$$\mathbf{P}_{\Phi,k}(y_k^n|u_1^n, m_2, \hat{m}_2') \triangleq \mathbb{1}\{y_k^n = \phi_k^n(u_1^n, U_2^n(m_2, \hat{m}_2'))\}.$$

Analysis: We will need the following distributions: the induced distribution P and auxiliary distributions Q_1 and \mathbf{Q}_{1}^{*} . The general idea of the proof is as follows: Encoder 1 makes P and Q_1 close in total variation. Distribution Q_1^* (random only with respect to the second codebook $\mathcal{C}_2^{(n)}$) is the expectation of \mathbf{Q}_1 over the random codebook $\mathcal{C}_1^{(n)}$. This is really the key step in the proof. By considering the expectation of the distribution with respect to $C_1^{(n)}$, we effectively remove Encoder 1 from the problem and turn the message from Encoder 1 into memoryless side information at the decoder. Hence, the two distortions (averaged over $C_1^{(n)}$) under **P** are roughly the same as the distortions under \mathbf{Q}_1^* , which is a much simpler distribution. We then recognize \mathbf{Q}_1^* as precisely \mathbf{P} in (8) from the Wyner-Ziv proof of the previous section, with a source pair (X_1, X_2) , a pair of reconstructions (Y_1, Y_2) and U_1 as the side information.

1) The auxiliary distribution Q_1 takes the following form:

$$\mathbf{Q}_{1X_1{}^nX_2{}^nU_1{}^nM_1M_2M_2'\hat{M}_2'Y_1{}^nY_2{}^n} = \mathbf{Q}_{1M_1U_1{}^nX_1{}^nX_2{}^n}\mathbf{P}_2$$

$$\mathbf{Q}_{1M_{1}U_{1}^{n}X_{1}^{n}X_{2}^{n}}(m_{1}, u_{1}^{n}, x_{1}^{n}, x_{2}^{n})
= \frac{1}{2^{nR_{1}}} \mathbb{1}\{u_{1}^{n} = U_{1}^{n}(m_{1})\} \overline{P}_{X_{1}^{n}|U_{1}^{n}}(x_{1}^{n}|u_{1}^{n})
\overline{P}_{X_{2}^{n}|X_{1}^{n}}(x_{2}^{n}|x_{1}^{n})$$
(26)

where P_2 was defined earlier in (25). Applying the softcovering lemma, since $R_1 > I_{\overline{D}}(X_1; U_1)$,

$$\mathbb{E}_{\mathcal{C}_1^{(n)}} \left[\| \mathbf{Q}_{1X_1^n} - \overline{P}_{X_1^n} \|_{TV} \right] \le \epsilon_{1n} \to_n 0.$$

Consequently,

$$\mathbb{E}_{\mathcal{C}_1^{(n)}}\left[\|\mathbf{Q}_1 - \mathbf{P}\|_{TV}\right] \le \epsilon_{1n},\tag{27}$$

where \mathbf{Q}_1 and \mathbf{P} are distributions over random variables $X_1{}^n, X_2{}^n, U_1{}^n, M_1, M_2, M_2', \hat{M}_2', Y_1{}^n$, and $Y_2{}^n$.

2) Taking the expectation over codebook $\mathcal{C}_1^{(n)}$, we define

$$\mathbf{Q}_{1X_{1}^{n}X_{2}^{n}U_{1}^{n}M_{2}M_{2}'\hat{M}_{2}'Y_{1}^{n}Y_{2}^{n}}^{*} \triangleq \mathbb{E}_{\mathcal{C}_{1}^{(n)}} \left[\mathbf{Q}_{1X_{1}^{n}X_{2}^{n}U_{1}^{n}M_{2}M_{2}'\hat{M}_{2}'Y_{1}^{n}Y_{2}^{n}} \right]. \tag{28}$$

Note that under this definition of \mathbf{Q}_1^* , we have

$$\begin{split} \mathbf{Q}_{1X_{1}^{n}X_{2}^{n}U_{1}^{n}M_{2}M_{2}'\hat{M}_{2}'Y_{1}^{n}Y_{2}^{n}}({x_{1}^{n},x_{2}^{n},u_{1}^{n},m_{2},m_{2}',\hat{m}_{2}',y_{1}^{n},y_{2}^{n}}) \\ &= \overline{P}_{X_{1}^{n}X_{2}^{n}U_{1}^{n}}({x_{1}^{n},x_{2}^{n},u_{1}^{n}})\mathbf{P}_{2}(m_{2},m_{2}',\hat{m}_{2}',y_{1}^{n},y_{2}^{n}|{x_{2}^{n},u_{1}^{n}}). \end{split}$$

By Property 1(b),

$$\mathbb{E}_{\mathcal{C}_{1}^{(n)}}\left[\mathbb{E}_{\mathbf{P}}\left[d_{k}(X_{k}^{n}, Y_{k}^{n})\right]\right]$$

$$\leq \mathbb{E}_{\mathcal{C}_{1}^{(n)}}\left[\mathbb{E}_{\mathbf{Q}_{1}}\left[d_{k}(X_{k}^{n}, Y_{k}^{n})\right]\right] + d_{max}\epsilon_{1n}$$
 (29)

$$= \mathbb{E}_{\mathbf{Q}_{1}^{*}} \left[d_{k}(X_{k}^{n}, Y_{k}^{n}) \right] + d_{max} \epsilon_{1n}. \tag{30}$$

Note that \mathbf{Q}_1^* is exactly of the form of the induced distribution P in the Wyner-Ziv proof of the previous section, with the inconsequential modification that there are two reconstructions and two distortion functions. With the same techniques as (12)through (19), we obtain

$$\mathbb{E}_{\mathcal{C}_{2}^{(n)}} \left[\mathbb{E}_{\mathbf{Q}_{1}^{*}} \left[d_{k}(X_{k}^{n}, Y_{k}^{n}) \right] \right]$$

$$\leq \mathbb{E}_{\overline{P}} \left[d_{k}(X_{k}, Y_{k}) \right] + d_{max}(\epsilon_{2n} + \delta_{n}),$$
(31)

where ϵ_{2n} and δ_n are error terms introduced from the softcovering lemma and channel decoding, respectively.

Finally, taking the expectation over $C_1^{(n)}$ and using (30) and

$$\mathbb{E}_{\mathcal{C}_2^{(n)}}\left[\mathbb{E}_{\mathcal{C}_1^{(n)}}\left[\mathbb{E}_{\mathbf{P}}\left[d_k(X_k^n,Y_k^n)\right]\right]\right] \leq D_k + d_{max}(\epsilon_{1n} + \epsilon_{2n} + \delta_n).$$

- [1] C. E. Shannon, "A mathematical theory of communication," Bell Sys. Tech. Journal, vol. 27, pp. 379-423, 623-656, 1948.
- [2] C. E. Shannon, "Coding theorems for a discrete source with a fidelity criterion," IRE National Convention Record, Part 4, pp. 142-163, 1959.
- [3] A. Wyner and J. Ziv, "The rate-distortion function for source coding with side information at the decoder," IEEE Transactions on Information Theory, vol. 22, no. 1, pp. 1-10, 1976.
- S.-Y. Tung, Multiterminal Source Coding. PhD thesis, Cornell University, Ithaca, NY, May, 1978.
- [5] T. Berger, "Multiterminal source coding," The Information Theory Approach to Communications, vol. 229, pp. 171–231, 1977.
 [6] T. Berger and R. W. Yeung, "Multiterminal source encoding with one
- distortion criterion," IEEE Transactions on Information Theory, vol. 35, no. 2, pp. 228-236, 1989.
- T. M. Cover and J. A. Thomas, Elements of Information Theory. John Wiley & Sons. 2012.
- A. El Gamal and Y.-H. Kim, Network Information Theory. Cambridge University Press, 2011.
- P. Minero, S. H. Lim, and Y.-H. Kim, "Hybrid coding: An interface for joint source-channel coding and network communication," arXiv preprint arXiv:1306.0530, 2013.
- [10] A. Lapidoth and S. Tinguely, "Sending a bivariate gaussian over a gaussian mac," IEEE Transactions on Information Theory, vol. 56, pp. 2714–2752, June 2010.
- [11] P. Cuff and E. C. Song, "The likelihood encoder for source coding," in Proc. IEEE Information Theory Workshop (ITW), 2013.
- [12] P. Cuff, "Distributed channel synthesis," IEEE Transactions on Information Theory, vol. 59, no. 11, pp. 7071-7096, 2013.
- P. W. Cuff, H. H. Permuter, and T. M. Cover, "Coordination capacity," IEEE Transactions on Information Theory, vol. 56, no. 9, pp. 4181-4206, 2010.
- [14] C. Schieler and P. Cuff, "Rate-distortion theory for secrecy systems," CoRR, vol. abs/1305.3905, 2013.
- [15] A. D. Wyner, "The common information of two dependent random variables," IEEE Transactions on Information Theory, vol. 21, no. 2, pp. 163-179, 1975.
- [16] T. Han and S. Verdú, "Approximation theory of output statistics," IEEE Transactions on Information Theory, vol. 39, no. 3, pp. 752-772, 1993.
- [17] P. Cuff, H. Permuter, and T. Cover, "Coordination capacity," IEEE Transactions on Information Theory, vol. 56, pp. 4181–4206, Sept 2010.
- J. Jeon, "A generalized typicality for abstract alphabets," arXiv preprint arXiv:1401.6728, 2014.