ON GLOBAL SOLUTIONS OF A ZAKHAROV TYPE SYSTEM

THOMAS BECK, FABIO PUSATERI, PHIL SOSOE, AND PERCY WONG

Abstract. We consider a class of wave-Schrödinger systems in three dimensions with a Zakharov-type coupling. This class of systems is indexed by a parameter γ which measures the strength of the null form in the nonlinearity of the wave equation. The case $\gamma = 1$ corresponds to the well-known Zakharov system, while the case $\gamma = -1$ corresponds to the Yukawa system. Here we show that sufficiently smooth and localized Cauchy data lead to pointwise decaying global solutions which scatter, for any $\gamma \in (0, 1]$.

1. Introduction

1.1. **Statement of the problem and main result.** We will consider the following parametrized family of systems in three spatial dimensions:

$$\begin{cases} i\partial_t u + \Delta u = un \\ \Box n = \Lambda^{1+\gamma}|u|^2 \end{cases} \tag{1.1}$$

where $\Box := -\partial_t^2 + \Delta$, $\Lambda := |\nabla| = \sqrt{-\Delta}$, and $-1 \leq \gamma \leq 1$. Here we have

$$u : (x,t) \in \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}, \quad n : (x,t) \in \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}\,.$$

The case $\gamma = 1$ corresponds to the well-known Zakharov system, modeling propagation of Langmuir waves in an ionized plasma [44]. The case $\gamma = -1$ corresponds to the (massless) Yukawa system, which is a model for the interaction between a meson and a nucleon. (1.1) is a special case of the models introduced in [35] (see Section 3 there)

$$\begin{cases} i\partial_t u + L_1 u = un \\ L_2 n = L_3|u|^2\,, \end{cases} \tag{1.2}$$

where L_1, L_2 and L_3 are constant coefficient differential operators. This class of systems is referred to as Davey-Stewartson (DS) systems in the work of Zakahrov-Schulman [35, Section 3]. In the recent literature the name DS is associated to a specific 2 dimensional system of the form 1.2, modeling the evolution of weakly nonlinear water waves travelling predominantly in one direction, in which the wave amplitude is modulated slowly in two horizontal directions. See, for example, [4, 5].

Here we will consider (1.1) in the range $0 < \gamma \leq 1$ and prove the following

Theorem 1.1. *Let $\gamma \in (0, 1]$ be given. Then there exist $N = N(\gamma) \gg 1$, and a small constant $\varepsilon_0 = \varepsilon_0(\gamma) > 0$, such that for any initial data $(u_0, n_0, n_1) = (u, n, \partial_t n)(t = 0)$ satisfying*[1]

$$\|u_0\|_{H^{N+1}(\mathbb{R}^3)} + \|\langle x\rangle^2 u_0\|_{L^2(\mathbb{R}^3)} \leq \varepsilon_0\,, \tag{1.3}$$

$$\|(\Lambda n_0, n_1)\|_{H^{N-1}(\mathbb{R}^3)} + \|\langle\Lambda\rangle(\Lambda n_0, n_1)\|_{\dot{B}^0_{1,1}(\mathbb{R}^3)} + \left\|\langle x\rangle^2 (n_0, n_1)\right\|_{H^1(\mathbb{R}^3)} \leq \varepsilon_0\,, \tag{1.4}$$

The second author was partially supported by NSF grant DMS 1265875.

[1]Here $\langle x\rangle$ is used to denote $\sqrt{1 + |x|^2}$, and the Besov norm $\dot{B}^0_{1,1}$ is defined in (2.5).

there exists a unique global-in-time solution $(u,n)(t)$ to the Cauchy problem associated to (1.1). *Moreover, there exists $0 < \alpha < 1/6$*[2] *such that*

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \varepsilon_0 (1+t)^{-1-\alpha} \quad , \quad \|n(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \varepsilon_0 (1+t)^{-1} . \tag{1.5}$$

As a consequence, the solution $(u,n)(t)$ approaches a linear solution as $t \to \infty$.

The proof shows that the assumptions (1.4) on the initial data are somewhat stronger than necessary. We have chosen to display these conditions for simplicity. In Section 2, we will give the strategy of the proof, and describe some more of the properties satisfied by the solution $(u,n)(t)$.

1.2. **Motivation and previous results.**

1.2.1. *The Zakharov system.* The Zakharov system, ((1.1) with $\gamma = 1$),

$$\begin{cases} i\partial_t u + \Delta u = nu \\ \Box n = -\Delta |u|^2, \end{cases} \tag{Z}$$

was derived by V. Zakharov in [44] to model Langmuir waves in plasma, and has since then been under intensive investigation by physicists and mathematicians (see [32, 41] for some background). As we remarked above, it is a particular example of Zakharov-Schulman systems, introduced in [35] and studied in [32], [27], [28].

From the mathematical side, there has been considerable work on the local and global well-posedness of solutions with rough data through the works of Kenig, Ponce and Vega [27], Bourgain and Colliander [1], Ginibre, Tsutsumi and Velo [8], Bejenaru, Herr, Holmer and Tataru [2] and Bejenaru and Herr [3]. In particular, global well-posedness for small data in the energy space was obtained in [1] by combining local well-posedness and conservation laws. (See the references in the cited works for previous well-posedness results). Many works have also dealt with singular limits related to the Zakharov system and with the rigorous derivation of the system in various limiting regimes from other equations and vice versa; see for example [42], [31] and references therein.

Concerning the scattering question, most of the previous work has been carried out for the final value problem, i.e. data at $t = \infty$, as in the papers of Ozawa and Tsutsumi [33], Ginibre and Velo [9], and Shimomura [38]. Similar work has also been dedicated to other coupled systems of Schrödinger and wave equations, see for example the works of Ginibre and Velo [10, 11, 12], Shimomura [39, 40], and references therein. The first work that deals with scattering for the Cauchy problem of the Zakharov system (or any other Wave-Schrödinger systems in 3 dimensions) is by Guo and Nakanishi [14], where they considered small radial solutions in the energy space. In [18], the second author, Hani and Shatah proved pointwise decay and scattering for sufficiently smooth and localized solutions of (Z). Global dynamics below the ground state, under radial symmetry, have been analyzed in [15]. The results in [14] and [18] were then strengthened in [16], where the authors used a generalized Strichartz estimate to obtain scattering for data in the energy space with additional angular regularity.

1.2.2. *Parameter range; The Yukawa System.* The restriction $0 < \gamma \leq 1$ in Theorem 1.1 is due to our methods, but it is conceivable that similar techniques can apply for some $\gamma \leq 0$[3]. We note that in the case $\gamma = -1$ (the Yukawa Wave-Schrödinger system), the expectation is to have modified scattering, that is, the behavior of the solutions for large t does not coincide with that of linear solutions. This was proven to be the case for the final value problem in [10, 11, 12], [39]

[2] One can choose $\alpha \sim 1/6$ for $\gamma \geq 1/2$.

[3] One important issue that arises in the case $\gamma < 0$ is the presence of a "singularity" in the nonlinear term of the second equation of (1.1), which is apparent when (1.1) is written as the first order system (Z_0) below. While some of the arguments we present are still valid in this case, some others break down and would require substantial modifications to yield the same final result.

and [13]. Because of this expectation, we decide here to pursue a proof of global existence and scattering for (1.1) based on the use of weighted spaces, rather than the approach of [14] based on radial Strichartz estimates[4]. Indeed, our approach allows us to extract more precise information about the pointwise time decay of solutions, which is an essential part of the analysis when dealing with nonlinear asymptotic behavior[5]. In the future we hope to refine our techniques, and possibly combine them with some of the recent advances in the area, see [22, 23] for example, to further push the admissible range of γ towards -1.

1.2.3. *Techniques.* We briefly discuss the technical features of our argument. For a more detailed discussion, see Section 2.2. The strategy follows the general scheme of much recent work on small global solutions of dispersive systems, see for example [6, 7, 17, 22], and [18] which is more closely related to the problem we are considering. The vector fields method of Klainerman [29] cannot be applied to deal with (1.1), because of the lack of space-time transformations leaving the combined Schrödinger and wave equations invariant. On the other hand, we use the observation, which appeared in [18], that Δ in the right hand side of second equation of (Z), plays the role of a Klainerman null form [30], allowing us to integrate by parts to gain decay[6]; see Section 2 and the identity (2.18). We show how this type of argument can be still used for (1.1), where $\gamma > 0$ can be arbitrarily small, if one combines it with a careful exploitation of the improved low frequency behavior of solutions of the linear wave equation. An important role is also played by the use of the pseudo-scaling identity (2.19) which allows to integrate by parts and estimate weighted norms of the Schrödinger component. Another key point is the spatial profile decomposition, also used in [18], to obtain decay for the wave component.

2. Preliminary setup

Writing $w_\pm = \Lambda^{-1}(i\partial_t \pm \Lambda)n$, the system (Z) becomes

$$\begin{cases} i\partial_t u + \Delta u = \frac{1}{2}(w_+ u - w_- u) \\ i\partial_t w_\pm \mp \Lambda w_\pm = \Lambda^\gamma |u|^2 . \end{cases} \tag{Z$_0$}$$

Let $f = e^{-it\Delta}u$ and $g_\pm = e^{\pm it\Lambda}w_\pm$ denote the profiles, and let $\widehat{f} = \mathcal{F}f$ and $\widehat{g} = \mathcal{F}g$ denote their Fourier transforms. Duhamel's formula in Fourier space then reads

$$\widehat{f}(\xi,t) = \widehat{f}(\xi,0) + \sum_\pm \mp i \int_0^t \int_{\mathbb{R}^3} e^{is\phi_\pm(\xi,\eta)} \widehat{f}(\xi-\eta,s)\widehat{g}_\pm(\eta,s)\mathrm{d}\eta\mathrm{d}s \tag{2.1a}$$

$$\widehat{g}_\pm(\xi,t) = \widehat{g}_\pm(\xi,0) - i\int_0^t \int_{\mathbb{R}^3} |\xi|^\gamma e^{is\psi_\pm(\xi,\eta)} \widehat{f}(\xi-\eta,s)\overline{\widehat{f}}(\eta,s)\mathrm{d}\eta\mathrm{d}s\,, \tag{2.1b}$$

where the phases are

$$\phi_\pm(\xi,\eta) = |\xi|^2 - |\xi-\eta|^2 \pm |\eta| = 2\xi\cdot\eta - |\eta|^2 \pm |\eta| \tag{2.2a}$$

$$\psi_\pm(\xi,\eta) = \mp|\xi| - |\xi-\eta|^2 + |\eta|^2 = \mp|\xi| - |\xi|^2 + 2\xi\cdot\eta\,. \tag{2.2b}$$

[4]We believe it is likely that the techniques used in [14] apply to (1.1) in the parameter range that we consider here, to obtain scattering for small data in the energy space under the assumption of radial symmetry.

[5]For some examples of modified scattering results in weighted spaces, see the papers [19, 20, 21, 26, 24] which deal with semilinear equations, [43, 40] for results on the final value problem for field equations with long-range potentials, e.g. Maxwell-Schrödinger, and [25] for a recent example involving a quasilinear system, the water waves equations.

[6]The integration by parts argument in Fourier space is related to the space-time resonance method [6, 7], and was used to deal with wave equations satisfying a nonresonance condition, akin to Klainerman's null condiditon [30], in [34].

We define the functions $F_\pm$ and $G_\pm$ by,

$$F_\pm(\xi,t) \overset{def}{=} \mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}^3} e^{is\phi_\pm(\xi,\eta)}\widehat{f}(\xi-\eta,s)\widehat{g}_\pm(\eta,s)\mathrm{d}\eta\mathrm{d}s\,, \tag{2.3}$$

$$G_\pm(\xi,t) \overset{def}{=} \mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}^3} |\xi|^\gamma e^{is\psi_\pm(\xi,\eta)}\widehat{f}(\xi-\eta,s)\overline{\widehat{f}}(\eta,s)\mathrm{d}\eta\mathrm{d}s\,. \tag{2.4}$$

2.1. **Norms and a priori bounds.** We denote by $\dot{B}^s_{p,q}$ the Besov space defined by the norm

$$\|u\|_{\dot{B}^s_{p,q}} := \left\|2^{sk}\|P_k u\|_{L^p_x(\mathbb{R}^3)}\right\|_{l^q_k(\mathbb{Z})} \tag{2.5}$$

where P_k denotes the Littlewood-Paley projection onto frequencies $|\xi|\sim 2^k$.

Given $\gamma>0$, we choose $\delta,\alpha>0$ and $N\gg 1$ such that[7]

$$\begin{aligned} &5N^{-1}\le\delta \quad,\quad \delta\ll 1\,,\\ &3\delta<\alpha\le\gamma/2-10\delta \quad,\quad \alpha\le 1/6-10\delta\,. \end{aligned} \tag{2.6}$$

The proof of Theorem 1.1 follows a bootstrap argument in the Banach space X defined by the norm[8] [9]:

$$\begin{aligned} \|(u,w_\pm)\|_X \overset{def}{=} \sup_t \Big(& t^{-\delta}\|f(t)\|_{H^{N+1}} + t^{-\delta}\|xf(t)\|_{L^2} + t^{-1+2\alpha+\delta}\||x|^2 f(t)\|_{L^2} \\ & + \|g_\pm(t)\|_{H^N} + t\,\|e^{\mp it\Lambda}g_\pm(t)\|_{\dot{B}^0_{\infty,1}}\Big)\,. \end{aligned} \tag{2.7}$$

We choose $\varepsilon_1=\varepsilon_0^{2/3}$ and assume a priori bounds on the quantities appearing in the $\|\cdot\|_X$ norm:

$$\|f(t)\|_{H^{N+1}}\le\varepsilon_1 t^\delta\,,\quad \|xf(t)\|_{L^2}\le\varepsilon_1 t^\delta\,,\quad \||x|^2f(t)\|_{L^2}\le\varepsilon_1 t^{1-2\alpha-\delta}\,, \tag{2.8}$$

and

$$\|g_\pm(t)\|_{H^N}\le\varepsilon_1\,,\quad \|e^{\mp it\Lambda}g_\pm(t)\|_{\dot{B}^0_{\infty,1}}\le\varepsilon_1 t^{-1}\,. \tag{2.9}$$

As an intermediate step, we include the additional a priori bounds for $G_\pm$:

$$\|e^{\mp it\Lambda}x\Lambda G_\pm(t)\|_{L^{4/(1+\gamma)}}\le\varepsilon_1 t^{-1/4+3\gamma/4-2\alpha-3\delta}\,, \tag{2.10}$$

$$\|e^{\mp it\Lambda}\Lambda^{-1}G_\pm(t)\|_{L^3}\le\varepsilon_1 t^{-2\alpha-3\delta}\,,\quad \|\Lambda^{1/2}xG_\pm(t)\|_{L^2}\le\varepsilon_1\,. \tag{2.11}$$

Remark 2.1. *In contrast to* [18]*, we do not place an a priori bound on $x^2G_\pm(t)$. Instead, we make greater use of the linear dispersive estimates for the wave group, and place an a priori bound on $e^{\mp it\Lambda}x\Lambda G_\pm(t)$ and $e^{\mp it\Lambda}\Lambda^{-1}G_\pm(t)$ in suitable L^p spaces. This greatly simplifies many of the estimates in* [18]*, while yielding the same exact conclusions for $\gamma\ge 1/3$.*

To obtain our result we will then show

$$\|(F_\pm,G_\pm)\|_X\lesssim\varepsilon_1^2\,,$$

which, together with the initial assumptions (1.3)-(1.4) (see also (2.16) below), will give

$$\|(u,w_\pm)\|_X\lesssim\varepsilon_0+\|(F_\pm,G_\pm)\|_X\lesssim\varepsilon_0+\varepsilon_1^2\lesssim\varepsilon_0+\varepsilon_0^{4/3}\,,$$

and guarantee a global-in-time solution belonging to X, provided ε_0 is chosen small enough.

[7]Note that the first inequality in (2.6) places a greater restriction on the size of α precisely when $0<\gamma<1/3$, whereas for $\gamma\ge 1/3$ we can choose $\alpha>0$ arbitrarily close to $1/6$.

[8]Local existence in time of solutions belonging to weighted Sobolev spaces can be established by standard techniques.

[9]Without loss of generality we can restrict our attention to times $t\ge 1$.

Remark 2.2 (Linear dispersive estimates)**.** *From the linear estimates for the Schrödinger group*

$$\|e^{it\Delta}f\|_{L^6} \lesssim \frac{1}{t}\|xf\|_{L^2} \quad , \quad \|e^{it\Delta}f\|_{L^\infty} \lesssim \frac{1}{t^{\frac{3}{2}}}\|xf\|_{L^2}^{\frac{1}{2}}\|x^2 f\|_{L^2}^{\frac{1}{2}}\,, \tag{2.12}$$

we deduce that the X norm bounds

$$\|e^{it\Delta}f\|_{L^6} \lesssim \frac{1}{t^{1-\delta}}\|u\|_X\,, \tag{2.13}$$

$$\|e^{it\Delta}f\|_{L^\infty} \lesssim \frac{1}{t^{1+\alpha}}\|u\|_X\,. \tag{2.14}$$

Moreover, by the linear dispersive estimate for the wave equation

$$\|e^{it\Lambda}h\|_{\dot{B}^0_{p,r}} \lesssim \frac{1}{t^{1-2/p}}\|h\|_{\dot{B}^{2(1-2/p)}_{p',r}} \quad , \quad p \geq 2\,, \tag{2.15}$$

(cf. for example [37]*), and the fact that $g_\pm(0) = \Lambda^{-1}in_1 \pm n_0$, we see that* (1.4) *implies*

$$\|e^{\mp it\Lambda}g_\pm(0)\|_{\dot{B}^0_{\infty,1}} \lesssim \frac{\varepsilon_0}{t}\,. \tag{2.16}$$

Finally, we note that (2.15) *with $r = 2$, and embeddings between Besov and Sobolev spaces, gives*

$$\|e^{it\Lambda}h\|_{L^p} \lesssim \frac{1}{t^{1-2/p}}\left\|\Lambda^{2(1-2/p)}h\right\|_{L^{p'}}\,. \tag{2.17}$$

2.2. **Strategy of the proof.** From the definition of the X-norm, we see that in order to close our argument we need to obtain estimates on high Sobolev norms of $(u, w_\pm)$ and weighted norms of f, and pointwise bounds for $w_\pm$. Bounds on high Sobolev norms follow via standard energy estimates. To show decay for $w_\pm$ we use the weighted L^2 bounds on $f_\pm$. To eventually bound weighted norms of $f_\pm$ we use the intermediate estimates (2.10)-(2.11) on $G_\pm$. A key aspect is that the system (2.1) has null resonances, which we can use in combination with the space time resonance method to obtain weighted and pointwise bounds.

2.2.1. *Estimates for $G_\pm$.* We notice that the phase $\psi_\pm(\xi,\eta)$ satisfies,

$$|\xi| = \frac{1}{2}\frac{\xi}{|\xi|}\cdot\nabla_\eta\psi_\pm(\xi,\eta)\,. \tag{2.18}$$

This means that the factor of $|\xi|^\gamma$ in (2.4) gives the equation a resonant structure, although we see that this becomes weaker as γ decreases towards 0. We can thus use (2.18) to integrate by parts in η and gain decay in s. In particular, using this together with Sobolev embedding and the a priori bounds on f, we can obtain a uniform in time estimate for $x\Lambda^{1/2}G_\pm$ in L^2. Carefully exploiting the linear dispersive estimates for wave group gives the estimates for $e^{\mp it\Lambda}x\Lambda G_\pm$ and $e^{\mp it\Lambda}\Lambda^{-1}G_\pm$. These estimates are presented in Section 3.

2.2.2. *L^∞ bounds.* These are obtained similarly to [18]. The pointwise decay of $e^{it\Delta}f$, see (2.14), is a direct consequence of the weighted estimates for xf and $|x|^2 f$. To obtain a t^{-1} pointwise decay for $e^{\mp it\Lambda}G_\pm$, we use the improved small frequency behavior of solutions of the linear wave equation, see (2.15), the identity (2.18), and a similar argument to [18]. Some of the details for the estimate of $e^{\mp it\Lambda}G_\pm$ are presented in Section 3.2.

2.2.3. *Weighted L^2 estimates for $F_\pm$.* To obtain the desired weighted estimates for $F_\pm$, we use the (pseudo-scaling) identity

$$\nabla_\xi\phi_\pm = -2\eta = -2\frac{\eta}{|\eta|}\left(\frac{\eta}{|\eta|}\cdot\nabla_\eta\phi_\pm\right) - 2\frac{\phi_\pm}{|\eta|}\frac{\eta}{|\eta|}\,. \tag{2.19}$$

More specifically, calculating $xF_\pm$ in Fourier space involves applying a derivative in ξ to $\widehat{F}_\pm$. When this derivative is applied to the factor of $e^{is\phi_\pm}(\xi,\eta)$ we obtain a factor of $\nabla_\xi\phi_\pm(\xi,\eta)$. We can then use (2.19) to express $\nabla_\xi\phi_\pm(\xi,\eta)$ in terms of $\nabla_\eta\phi_\pm(\xi,\eta)$ and $\phi_\pm(\xi,\eta)$. This allows us to integrate by parts one time in both space and time, and the estimate for $xF_\pm$ in L^2 then follows from the a priori estimates for f and $G_\pm$.

The equality in (2.19) is also used in the estimate of $x^2F_\pm$. Applying two derivatives in ξ to $\widehat{F}_\pm(\xi,s)$, we find that one term contains a factor of $(\nabla_\xi\phi_\pm(\xi,\eta))^2$. If, as in [18], we use (2.19) to integrate by parts twice, we end up with a term containing a factor of $x^2G_\pm$. However, we no longer have an a priori estimate on $x^2G_\pm$, so we do not proceed in this way. Instead, for this term in $x^2F_\pm$, we only integrate by parts in η once, and make use of the L^p estimates (2.10)-(2.11) on $e^{\mp it\Lambda}x\Lambda G_\pm(t)$ and $e^{\mp it\Lambda}\Lambda^{-1}G_\pm(t)$. We carry out these estimates in Section 4.

2.3. **Energy Estimates and high frequency cutoff.** We have the following:

Proposition 2.3. *Let $F_\pm$ and $G_\pm$ be given by* (2.3) *and* (2.4) *respectively. Then, for $\|(u,w_\pm)\|_X\leq\varepsilon_1$, we have*

$$\|G_\pm(t)\|_{H^N}+t^{-\delta}\|F_\pm(t)\|_{H^{N+1}}\lesssim\varepsilon_1^2\,.$$

We do not provide details on how to obtain the above bounds, since they are fairly easy to show and can be proved as in [18].

We can use the a priori bounds on high Sobolev norms to reduce all of our estimates to frequencies smaller than s^{δ_N}, where $\delta_N\ll 1$ is chosen small depending on N. To see this, let us assume in what follows that at least one of the frequencies η or $\xi-\eta$ in the expressions for $F_\pm$ and $G_\pm$, see (2.3) and (2.4), has size larger than $s^{2/(N-2)}$.

Observe that for all $k\geq 0$ one has $\|P_{\geq k}v(s)\|_{L^2}\lesssim 2^{-kl}\|v(s)\|_{H^l}$, and therefore, for frequencies $2^k\gtrsim s^{2/(N-2)}\gtrsim 1$, we have from the apriori assumptions (2.8)-(2.9),

$$\begin{aligned}\|P_{\geq k}u(s)\|_{H^3}&\lesssim 2^{-k(N-2)}\|u(s)\|_{H^{N+1}}\lesssim\varepsilon_1 s^{-2+\delta}\\ \|P_{\geq k}w_\pm(s)\|_{H^2}&\lesssim 2^{-k(N-2)}\|w_\pm(s)\|_{H^N}\lesssim\varepsilon_1 s^{-2}\,.\end{aligned}\tag{2.20}$$

These estimates are already sufficient to bound all norms that do not involve weights.

To establish bounds on L^p norms involving weights, we are going to apply ∇_ξ to the bilinear terms $\widehat{F}_\pm$ and $\widehat{G}_\pm$. We start by noticing that the action of weights on Littlewood-Paley projections (on high-frequencies) is harmless, and only gives terms that are easier to treat than any other term that we will have to deal with. We now briefly discuss how to estimate all the other contributions that result from applying derivatives to $\widehat{F}_\pm$ and $\widehat{G}_\pm$ in the case when $\max\{|\eta|,|\xi-\eta|\}\gtrsim s^{2/(N-2)}$.

High frequency contributions in $F_\pm$. Looking at (2.3), we see that applying ∇_ξ twice to $\widehat{F}_\pm$ gives three types of contributions. The first are those where ∇_ξ^2 hits the phase $e^{is\phi_\pm}$: these terms will contain powers of s but will not involve weights on the inputs f and g. Since at least one frequency has size larger than $s^{2/(N-2)}$, then all these terms can be estimated directly using (2.20). The second type of terms that arise are those with ∇_ξ^2 hitting the input $\widehat{f}(\xi-\eta)$. If this happens, the same estimates that we are going to perform below in section 4 will work, regardless of the size of frequencies. The third type of contribution is where one ∇_ξ hits the phase $e^{is\phi_\pm}$, and the other hits the input $\widehat{f}(\xi-\eta)$. In this case, if η is the largest frequency, the term can be estimated directly using the second bound in (2.20). If instead $\xi-\eta$ is the largest frequency, one can write $\nabla_\xi\widehat{f}(\xi-\eta)=\nabla_\eta\widehat{f}(\xi-\eta)$ and integrate by parts in η. This will generate one term with losses in powers of s similar to the first type of term discussed above, plus a term where the inputs are $\widehat{f}(\xi-\eta)$ and $\nabla_\eta\widehat{g}_\pm(\eta)$. Since we are assuming that $|\xi-\eta|\gtrsim s^{2/(N-2)}$, we can use Sobolev's embedding, the second apriori assumption in (2.11), and the first inequality in (2.20), to estimate directly this term and obtain the desired bound without resorting to further manipulations.

High frequency contributions in $G_\pm$. We now briefly describe how to obtain the bound (2.10) and the second bound in (2.11) for $G_\pm$, in the case of high frequencies. Since the inputs in (2.4) are symmetric, up to complex conjugation (which leaves our norms invariant), we can assume that $|\eta| \gtrsim |\xi-\eta|$ and $|\eta| \gtrsim s^{2/(N-2)}$. When applying derivatives to $\widehat{G}_\pm$ we then obtain two types of contributions, similar to the ones discussed in the previous paragraph. The first contribution is the one where ∇_ξ hits the phase $e^{is\psi_\pm}$. This will cause a loss of a power of s which can be overcome directly using the decay given by (2.20). The second type of term will contain $\nabla_\xi \widehat{f}(\xi-\eta)$ as an input. Since we have already reduced ourselves to the case when η is the largest frequency, we can again use (2.20), and the apriori bound (2.8) on xf, to get the desired estimates in a straightforward fashion.

The above discussion shows that in estimating weighted norms of the bilinear terms $F_\pm$ and $G_\pm$ in (2.3) and (2.4), we can always reduce our analysis to frequencies $|\xi-\eta|, |\eta| \lesssim s^{2/(N-2)}$, for otherwise all the desired bounds can be shown to hold true without too much effort. We therefore agree on the following:

Convention 1. *In the rest of the paper, we assume that all frequencies, $\xi-\eta$ and η, appearing in the estimates of the bilinear terms* (2.3) *and* (2.4)*, have size bounded above by s^{δ_N}, where $\delta_N := \frac{2}{N-2}$ and the integer $N \gg 1$ is determined in the course of our proof by several upperbounds on δ_N. In particular, expressions such as $|\xi|$ or $\nabla_\xi \psi_\pm(\xi,\eta)$ will be constantly replaced by a factor of s^{δ_N}.*

We will also adopt the following additional notational convention:

Convention 2. *To make notations lighter, we will often drop the $\pm$ indices, and omit the dependence on the time t. Moreover, in the estimates of the bilinear terms F and G in* (2.3) *and* (2.4)*, we will often only consider the contribution of the integrals from 1 to ∞. All of the contributions coming from integrating between 0 and 1 are bounded in a straightforward fashion by Sobolev's embedding, and our control of high Sobolev norms of the solution (u,w).*

3. Estimates for G

We recall that we have the following a priori assumptions on $f = e^{it\Delta}u$:

$$\|xf\|_{L^2} \le \varepsilon_1 t^{\delta} \quad , \quad \|x^2 f\|_{L^2} \le \varepsilon_1 t^{1-2\alpha-\delta} \quad , \quad \|f\|_{H^N} t^{\delta} \le \varepsilon_1 \,. \tag{3.1}$$

As a consequence, the following dispersive bounds for u hold:

$$\|u\|_{L^\infty} \lesssim \varepsilon_1 t^{-1-\alpha} \quad , \quad \|u\|_{L^6} \lesssim \varepsilon_1 t^{-1+\delta} \,. \tag{3.2}$$

3.1. **Weighted estimates for G.** In this section we are going to prove the following:

Proposition 3.1. *Let G be the bilinear term defined in* (2.4)*:*

$$G = \mathcal{F}^{-1} \int_1^t \int_{\mathbb{R}^3} |\xi|^\gamma e^{is\psi(\xi,\eta)} \widehat{f}(\xi-\eta,s) \overline{\widehat{f}}(\eta,s) \,\mathrm{d}\eta \mathrm{d}s \,. \tag{3.3}$$

Under the a priori assumptions (3.1) *and* (3.2)*, we have*

$$\|\Lambda^{1/2} xG\|_{L^2} \lesssim \varepsilon_1^2 \,,$$
$$\|e^{it\Lambda} x\Lambda G\|_{L^{4/(1+\gamma)}} \lesssim t^{-1/4+3\gamma/4-2\alpha-3\delta} \varepsilon_1^2 \,,$$
$$\|e^{it\Lambda} \Lambda^{-1} G\|_{L^3} \lesssim t^{-2\alpha-3\delta} \varepsilon_1^2 \,.$$

The proof of the above proposition is split into Lemma 3.2, 3.3 and 3.4 below. We recall the following assumptions on the relative sizes of γ and α from (2.6),

$$3\delta < \alpha \le \gamma/2 - 10\delta \quad , \quad \alpha \le 1/6 - 10\delta. \tag{3.4}$$

Lemma 3.2. *Let G be the bilinear term defined in* (3.3) *and let α satisfy* (3.4)*. Then,*

$$\|\Lambda^{1/2} xG\|_{L^2} \lesssim \varepsilon_1^2 \,.$$

To prove this, it is crucial to notice that low frequencies play the role of a special null resonant structure in the nonlinear term G, see (2.18),

$$|\xi| = \frac{1}{2}\frac{\xi}{|\xi|} \cdot \nabla_\eta \psi \,. \tag{3.5}$$

Proof of Lemma 3.2. Applying $|\xi|^{1/2}\nabla_\xi$ to $\widehat{G}$ gives the terms:

$$\int_1^t \int_{\mathbb{R}^3} e^{is\psi(\xi,\eta)} |\xi|^{1/2+\gamma} \nabla_\xi \widehat{f}(\xi-\eta,s)\overline{\widehat{f}}(\eta,s) \mathrm{d}\eta \mathrm{d}s \tag{3.6}$$

$$\int_1^t \int_{\mathbb{R}^3} s\nabla_\xi \psi e^{is\psi(\xi,\eta)} |\xi|^{1/2+\gamma} \widehat{f}(\xi-\eta,s)\overline{\widehat{f}}(\eta,s) \mathrm{d}\eta \mathrm{d}s \,, \tag{3.7}$$

plus an easier term when ∇_ξ hits the symbol $|\xi|^\gamma$. (3.6) is easily estimated by Hölder's inequality together with the estimates on f and u in (3.1) and (3.2), and using the first condition in (3.4):

$$\|(3.6)\|_{L^2} \lesssim \int_1^t s^{2\delta_N} \|xf\|_{L^2} \|e^{is\Delta} f\|_{L^\infty} \,\mathrm{d}s \lesssim \int_1^t s^{2\delta_N} s^\delta \frac{1}{s^{1+\alpha}} \, ds \lesssim 1 \,.$$

Using the identity (3.5) and integrating by parts in η, (3.7) gives terms of the form

$$\int_1^t \int_{\mathbb{R}^3} e^{is\psi(\xi,\eta)} m_1(\xi,\eta) |\xi|^{-1/2+\gamma} \nabla_\eta \widehat{f}(\xi-\eta,s)\overline{\widehat{f}}(\eta,s) \mathrm{d}\eta ds,$$

together with symmetric or easier terms. Here $m_1(\xi,\eta)$ is a symbol satisfying homogeneous bounds of order 1 for large frequencies, and is otherwise harmless. By Plancherel, the L^2-norm of this term is bounded by

$$\int_1^t s^{\delta_N} \|\Lambda^{-1/2+\gamma} e^{is\Delta}(xf) e^{is\Delta} f\|_{L^2} \,\mathrm{d}s.$$

For $1/2 \leq \gamma < 1$, we can estimate this by,

$$\begin{aligned}\int_1^t s^{3\delta_N/2} \|e^{is\Delta}(xf)e^{is\Delta} f\|_{L^2} \, ds &\lesssim \int_1^t s^{3\delta_N/2} \|xf\|_{L^2} \|e^{is\Delta} f\|_{L^\infty} \,\mathrm{d}s \\ &\lesssim \int_1^t s^{3\delta_N/2+\delta} s^{-1-\alpha} \,\mathrm{d}s \lesssim 1,\end{aligned}$$

since $\alpha > 3\delta$. For $0 \leq \gamma < 1/2$, applying Sobolev embedding, we can estimate this by

$$\begin{aligned}\int_1^t s^{\delta_N} \|e^{is\Delta}(xf)e^{is\Delta} f\|_{L^{3/(2-\gamma)}} \,\mathrm{d}s &\lesssim \int_1^t s^{\delta_N} \|xf\|_{L^2} \|e^{is\Delta} f\|_{L^{6/(1-2\gamma)}} \,\mathrm{d}s \\ &\lesssim \int_1^t s^{\delta_N} s^\delta \|e^{is\Delta} f\|_{L^{6/(1-2\gamma)}} \,\mathrm{d}s.\end{aligned}$$

Interpolating between the L^6 and L^∞ estimates on $u = e^{it\Delta} f$ from (3.2), and choosing $\delta > 0$ sufficiently small, we can ensure that this integral has an $O(1)$ bound. □

Lemma 3.3. *Let G be the bilinear term defined in* (3.3) *and let α satisfy* (3.4)*. Then,*

$$\|e^{it\Lambda} x\Lambda G\|_{L^{4/(1+\gamma)}} \lesssim t^{-1/4+3\gamma/4-2\alpha-3\delta} \varepsilon_1^2 \,.$$

Proof. Applying $\nabla_\xi|\xi|$ to $\widehat{G}$ gives the terms:

$$\int_1^t \int_{\mathbb{R}^3} e^{is\psi(\xi,\eta)}|\xi|^{1+\gamma}\nabla_\xi \widehat{f}(\xi-\eta,s)\overline{\widehat{f}}(\eta,s)\mathrm{d}\eta\mathrm{d}s \tag{3.8}$$

$$\int_1^t \int_{\mathbb{R}^3} s\nabla_\xi\psi e^{is\psi(\xi,\eta)}|\xi|^{1+\gamma}\widehat{f}(\xi-\eta,s)\overline{\widehat{f}}(\eta,s)\mathrm{d}\eta\mathrm{d}s\,, \tag{3.9}$$

plus an easier term when ∇_ξ hits the symbol $|\xi|^{1+\gamma}$.

We now use (3.5) and integrate by parts in η to write (3.9) as terms of the form

$$\int_1^t \int_{\mathbb{R}^3} e^{is\psi(\xi,\eta)}m_1(\xi,\eta)|\xi|^{\gamma}\nabla_\eta \widehat{f}(\xi-\eta,s)\overline{\widehat{f}}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s, \tag{3.10}$$

together with symmetric or easier terms. Here, as before, $m_1(\xi,\eta)$ is a symbol satisfying homogeneous bounds of order 1 for large frequencies, and is otherwise harmless.

Using the linear dispersive estimate, the contribution from (3.8)-(3.9) can thus be bounded by

$$\begin{aligned}
&\int_0^t \frac{1}{(t-s)^{1/2-\gamma/2}} s^{\delta}\|\Lambda^{-\gamma}\Lambda^{\gamma}e^{is\Delta}(xf)(e^{is\Delta}f)\|_{L^{4/(3-\gamma)}}\,\mathrm{d}s\\
&\lesssim \int_0^t \frac{1}{(t-s)^{1/2-\gamma/2}} s^{\delta}\|xf\|_{L^2}\|e^{is\Delta}f\|_{L^{4/(1-\gamma)}}\,\mathrm{d}s\\
&\lesssim \int_0^t \frac{1}{(t-s)^{1/2-\gamma/2}} s^{2\delta}s^{-(3/4+3\gamma/4)(1-\delta)}\,\mathrm{d}s.
\end{aligned}$$

If $0<\gamma<1/3$, then we can bound this integral by $t^{-1/4-\gamma/4+4\delta}$. If $1/3\le\gamma<1$, then instead, we obtain a bound of $t^{-1/2+\gamma/2+4\delta}$. By the assumptions on α from (3.4), these estimates are sufficient. □

Lemma 3.4. *Let G be the bilinear term defined in* (3.3) *and let α satisfy* (3.4)*. Then,*

$$\|e^{it\Lambda}\Lambda^{-1}G\|_{L^3}\lesssim \varepsilon_1^2 t^{-2\alpha-3\delta}\,.$$

Proof. Using (3.3), we can write,

$$e^{it\Lambda}\Lambda^{-1}G(t)=\int_1^t e^{i(t-s)\Lambda}\Lambda^{-1+\gamma}|u(s)|^2\,\mathrm{d}s.$$

Thus, by the linear dispersive estimate,

$$\|e^{it\Lambda}\Lambda^{-1}G\|_{L^3}\lesssim \int_1^t \frac{1}{(t-s)^{1/3}}\|\Lambda^{-1/3+\gamma}G\|_{L^{3/2}}\,\mathrm{d}s. \tag{3.11}$$

Suppose first that $\gamma\ge 1/3$. Then, we can use the bounds from (3.1) to estimate (3.11) by

$$\int_1^t \frac{1}{(t-s)^{1/3}}s^{\delta/N}\|u\|_{L^3}^2\,\mathrm{d}s\lesssim \int_1^t \frac{1}{(t-s)^{1/3}}s^{\delta/N}\frac{1}{s^{1-\delta}}\,\mathrm{d}s\lesssim \frac{1}{t^{1/3}}t^{2\delta}.$$

By the third assumption on α from (3.4), this gives us the desired bound. For $0<\gamma<1/3$, we first apply Sobolev embedding to estimate (3.11) by

$$\int_1^t \frac{1}{(t-s)^{1/3}}\|u\|^2_{L^{18/(7-3\gamma)}}\,\mathrm{d}s.$$

Using the bounds from (3.2), we thus obtain,

$$(3.11)\lesssim \int_1^t \frac{1}{(t-s)^{1/3}}s^{-(2/3+\gamma)(1-\delta)}\,\mathrm{d}s\lesssim t^{-\gamma+2\delta}.$$

By the second assumption on α from (3.4), this gives us the desired bound. □

3.2. **Decay estimate for G.** In this section we want to show the following:

Proposition 3.5. *Let $G_\pm$ be the bilinear term defined in* (2.4). *Under the apriori assumptions* (3.1) *and* (3.2) *we have*

$$\|e^{it\Lambda}G_\pm\|_{\dot{B}^0_{\infty,1}} \lesssim \varepsilon_1^2(1+t)^{-1}\,.$$

The proof of the above proposition is analogous to the one in section 6 of [18]. We provide some of the details below.

Proof of Proposition 3.5. Let us split G into two parts, depending on the localization of the inputs. More precisely, we let $G = G_1 + G_2$ where

$$\begin{aligned} G_1 &:= G(f_{\leq s^{1/8}}, \overline{f}) + G(f, \overline{f}_{\leq s^{1/8}}) \\ G_2 &:= G(f_{\geq s^{1/8}}, \overline{f}_{\geq s^{1/8}})\,. \end{aligned}$$

The component G_1 can be shown to be bounded in a weighted Sobolev space stronger than $\dot{B}^2_{1,1}$; this directly gives the desired bound on $e^{it\Lambda}G_1$. The decay of $e^{it\Lambda}G_2$ will instead be proven using the null structure (3.5) in conjuction with the improved small frequency behavior of the dispersive estimate for linear wave equation. We will crucially use the fact that the L^2 norm of $f_{\geq s^{1/8}}$ decays in L^2. Since the two terms in the definition of G_1 are similar we can reduce to consider G_1 and G_2 given by

$$\widehat{G}_1 = \int_1^t \int_{\mathbb{R}^3} |\xi|^\gamma e^{is\psi(\xi,\eta)} \widehat{f_{\leq s^{1/8}}}(\xi-\eta,s)\overline{\widehat{f}}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s \tag{3.12}$$

$$\widehat{G}_2 = \int_1^t \int_{\mathbb{R}^3} |\xi|^\gamma e^{is\psi(\xi,\eta)} \widehat{f_{\geq s^{1/8}}}(\xi-\eta,s)\overline{\widehat{f_{\geq s^{1/8}}}}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s\,. \tag{3.13}$$

Decay estimate for $e^{it\Lambda}G_1$. To show that G_1 is bounded in $\dot{B}^2_{1,1}$ we will interpolate weighted L^2 norms inside the time integral. One can then exploit the "small" support of $f_{\leq s^{1/8}}$ to get improvements on these weighted norms, and on the decay of $e^{is\Delta}f_{\leq s^{1/8}}$. Recalling that we are only considering frequencies k such that $2^k \leq s^{\delta_N}$, we aim to prove

$$\int_1^t \sum_{k=-\infty}^{\log s^{\delta_N}} 2^{2k}\|P_k\Lambda^\gamma e^{-is\Lambda}\left(e^{is\Delta}f_{\leq s^{1/8}}e^{-is\Delta}\overline{f}\right)\|_{L^1}\,\mathrm{d}s \lesssim 1\,.$$

Converting a factor of $2^{(2-\gamma)k}$ into derivatives $\Lambda^{2-\gamma}$, throwing away the projection P_k, and performing the sum, we see that is suffices to show

$$\int_1^t s^{\gamma\delta_N}\|\Lambda^2 e^{-is\Lambda}\left(e^{is\Delta}f_{\leq s^{1/8}}e^{-is\Delta}\overline{f}\right)\|_{L^1}\,\mathrm{d}s \lesssim 1\,.$$

Since $\|\cdot\|_{L^1} \lesssim \|x\cdot\|_{L^2}^{1/2}\|x^2\cdot\|_{L^2}^{1/2}$, the above estimate will follow from the inequalities

$$\left\||x|\Lambda^2 e^{-is\Lambda}\left(e^{is\Delta}f_{\leq s^{1/8}}e^{-is\Delta}\overline{f}\right)\right\|_{L^2} \lesssim s^{-7/4}\,, \tag{3.14a}$$

$$\left\||x|^2\Lambda^2 e^{-is\Lambda}\left(e^{is\Delta}f_{\leq s^{1/8}}e^{-is\Delta}\overline{f}\right)\right\|_{L^2} \lesssim s^{-1}\,. \tag{3.14b}$$

These two estimates have been already proven in [18] under the same apriori assumptions made in (3.2). Therefore, we omit them and refer the reader to section 6.1 of [18] for a detailed proof.

Decay estimate for $e^{it\Lambda}G_2$. We write

$$e^{it\Lambda}G_2(t,x) = \int_1^t e^{i(t-s)\Lambda}\Lambda^{\gamma-1}\mathcal{F}_\xi^{-1}\left[\int_{\mathbb{R}^3}|\xi|e^{is\tilde{\psi}(\xi,\eta)}\widehat{f_{\geq s^{1/8}}}(\xi-\eta,s)\overline{\widehat{f_{\geq s^{1/8}}}}(\eta,s)\mathrm{d}\eta\right]\,\mathrm{d}s$$

where $\tilde{\psi}(\xi,\eta) = |\xi-\eta|^2 - |\eta|^2 = |\xi|^2 - 2\xi\cdot\eta$. We now want to use (3.5) to integrate by parts in η. By symmetry we can reduce to consider the following term:

$$\int_1^t e^{i(t-s)\Lambda}\frac{1}{s}\,\Lambda^{\gamma-1}\mathcal{F}_\xi^{-1}\left[\int_{\mathbb{R}^3}\frac{\xi}{|\xi|}e^{is\tilde{\psi}(\xi,\eta)}\,\nabla_\eta\widehat{f_{\geq s^{1/8}}}(\xi-\eta,s)\overline{\widehat{f_{\geq s^{1/8}}}}(\eta,s)\mathrm{d}\eta\right]\mathrm{d}s\,. \tag{3.15}$$

The contribution of the time integral between $t-1$ and t can be easily estimated by Sobolev embedding. To estimate the contribution from 1 to $t-1$, we use the linear dispersive estimate for the wave equation (2.15), and our large frequency cutoff convention, to bound it by

$$\begin{aligned}
&\int_1^{t-1}\frac{1}{t-s}\,\frac{1}{s}\sum_{k=-\infty}^{\log s^{\delta_N}}2^{(\gamma+1)k}\left\|P_k\left(e^{is\Delta}xf_{\geq s^{1/8}}\,e^{is\Delta}f_{\geq s^{1/8}}\right)\right\|_{L^1}ds\\
&\lesssim\int_1^{t-1}\frac{1}{t-s}\,\frac{1}{s}\,s^{2\delta_N}\left\|e^{is\Delta}xf_{\geq s^{1/8}}\,e^{is\Delta}f_{\geq s^{1/8}}\right\|_{L^1}ds\\
&\lesssim\int_1^{t-1}\frac{1}{t-s}\,\frac{1}{s}\,s^{2\delta_N}\|xf\|_{L^2}\|f_{\geq s^{1/8}}\|_{L^2}\,ds\lesssim\int_1^{t-1}\frac{1}{t-s}\,\frac{1}{s}\,s^{2\delta_N}\,s^\delta\,\frac{1}{s^{1/8}}s^\delta\,ds\lesssim\frac{1}{t}\,.
\end{aligned}$$

□

4. Estimates for F

Recall that we are making the following a priori assumptions on g and f:

$$\|\Lambda^{1/2}xG\|_{L^2}\leq\varepsilon_1,\qquad\|e^{it\Lambda}x\Lambda G\|_{L^{4/(1+\gamma)}}\leq t^{-1/4+3\gamma/4-2\alpha-3\delta}\varepsilon_1, \tag{4.1}$$

$$\|e^{it\Lambda}G\|_{L^\infty}\leq t^{-1}\varepsilon_1,\qquad\|e^{it\Lambda}\Lambda^{-1}G\|_{L^3}\leq t^{-2\alpha-3\delta}\varepsilon_1, \tag{4.2}$$

$$\|xf\|_{L^\infty}\leq t^\delta\varepsilon_1,\qquad\|x^2f\|_{L^2}\leq t^{1-2\alpha-\delta}\varepsilon_1. \tag{4.3}$$

In this section we want to establish estimates for F defined as

$$\widehat{F}(\xi,t)=\int_0^t\int_{\mathbb{R}^3}e^{is\phi(\xi,\eta)}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\;\mathrm{d}\eta\mathrm{d}s\,. \tag{4.4}$$

Recall that

$$\partial_\xi\phi(\xi,\eta)=-2\eta=-2\frac{\eta}{|\eta|}\left(\frac{\eta}{|\eta|}\cdot\partial_\eta\phi(\xi,\eta)\right)-2\frac{\phi}{|\eta|}\frac{\eta}{|\eta|}\,, \tag{4.5}$$

and note that $\partial^2_{\xi_i}\phi(\xi,\eta)=0$, which in particular leads to $\partial^2_{\xi^i}e^{is\phi(\xi,\eta)}=-4s^2\eta_i^2e^{is\phi(\xi,\eta)}$.

4.1. Estimate for xF.

In this section we aim to prove the following lemma:

Lemma 4.1. *Let F be defined by (4.4). Under the apriori assumptions (4.1)–(4.3) we have*

$$\|xF\|_{L^2}\lesssim t^\delta\varepsilon_1^2\,. \tag{4.6}$$

Proof. We have that $\nabla_\xi\widehat{F}$ is given by a linear combination of terms of the form

$$\int_0^t\int_{\mathbb{R}^3}e^{is\phi(\xi,\eta)}\nabla_\xi\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\;\mathrm{d}\eta\mathrm{d}s \tag{4.7}$$

$$\int_0^t\int_{\mathbb{R}^3}s\nabla_\xi\phi\,e^{is\phi(\xi,\eta)}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\;\mathrm{d}\eta\mathrm{d}s. \tag{4.8}$$

Using (4.5) to integrate by parts in η and s in equation (4.8), we have the following contributions:

$$\int_0^t \int_{\mathbb{R}^3} e^{is\phi(\xi,\eta)} \nabla_\eta \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s) \, \mathrm{d}\eta \mathrm{d}s \tag{4.9}$$

$$\int_0^t \int_{\mathbb{R}^3} e^{is\phi(\xi,\eta)} \widehat{f}(\xi-\eta,s)\nabla_\eta \widehat{g}(\eta,s) \, \mathrm{d}\eta \mathrm{d}s \tag{4.10}$$

$$\int_{\mathbb{R}^3} te^{it\phi(\xi,\eta)} \widehat{f}(\xi-\eta,t)\frac{1}{|\eta|}\widehat{g}(\eta,t) \, \mathrm{d}\eta \tag{4.11}$$

$$\int_0^t \int_{\mathbb{R}^3} se^{is\phi(\xi,\eta)} \widehat{f}(\xi-\eta,s)\frac{1}{|\eta|}\partial_s \widehat{g}(\eta,s) \, \mathrm{d}\eta \mathrm{d}s \tag{4.12}$$

$$\int_0^t \int_{\mathbb{R}^3} se^{is\phi(\xi,\eta)} \partial_s \widehat{f}(\xi-\eta,s)\frac{1}{|\eta|}\widehat{g}(\eta,s) \, \mathrm{d}\eta \mathrm{d}s. \tag{4.13}$$

The terms (4.7) and (4.9) can be bounded as

$$\|(4.7)\|_{L^2} \lesssim \int_0^t \|xf\|_{L^2} \|e^{is\Lambda}(n_0+G+\Lambda^{-1}n_1)\|_{L^\infty} \, \mathrm{d}s \lesssim \varepsilon_1^2 \int_0^t s^\delta \frac{1}{s} \, ds \lesssim \varepsilon_1^2 t^\delta.$$

For the term (4.10), we first split g into n_0+G and $i\Lambda^{-1}n_1$.

$$\|(4.10)\| \lesssim \int_0^t \|e^{is\Lambda} x(n_0+G)\|_{L^3} \|e^{is\Delta} f\|_{L^6} \, \mathrm{d}s + \int_0^t \|e^{is\Lambda} x\Lambda^{-1} n_1 e^{is\Delta} f\|_{L^2} \, \mathrm{d}s.$$

For the first term, we use the Sobolev embedding and our assumption (1.4):

$$\int_0^t \|x(n_0+G)\|_{L^3} \|e^{is\Delta} f\|_{L^6} \, ds \lesssim \int_0^t \|x(n_0+G)\|_{\dot{H}^{1/2}} \|e^{is\Delta} f\|_{L^6} \, \mathrm{d}s \lesssim \varepsilon_1^2 \int_0^t s^{-1+\delta} \, \mathrm{d}s \lesssim \varepsilon_1^2 t^\delta.$$

For the second term, we commute x and Λ^{-1}, using $x\Lambda^{-1}n_1 = -\frac{\partial}{\Lambda^3}n_1 + \Lambda^{-1}xn_1$, we get:

$$\int_0^t \|e^{is\Lambda} x\Lambda^{-1} n_1 e^{is\Delta} f\|_{L^2} \, \mathrm{d}s \lesssim \int_0^t \|e^{is\Lambda} \Lambda^{-1} x n_1 e^{is\Delta} f\|_{L^2} \, \mathrm{d}s + \int_0^t \|e^{is\Lambda} \frac{\partial}{\Lambda^3} n_1 e^{is\Delta} f\|_{L^2} \, \mathrm{d}s. \tag{4.14}$$

For the first integral on the right-hand side of (4.14) we use Hölder and the dispersive estimates (2.12) and (2.15), to obtain

$$\begin{aligned}\int_1^t \|e^{is\Lambda} \Lambda^{-1} x n_1 e^{is\Delta} f\|_{L^2} \, \mathrm{d}s &\lesssim \int_1^t \|e^{is\Lambda} \Lambda^{-1} x n_1\|_{L^3} \|e^{is\Delta} f\|_{L^6} \, \mathrm{d}s \lesssim \varepsilon_1 \int_1^t s^{-4/3+\delta} \|\Lambda^{-1/3} x n_1\|_{L^{3/2}} \, \mathrm{d}s \\ &\lesssim \varepsilon_1 \int_1^t s^{-4/3+\delta} \|x n_1\|_{L^{9/7}} \, \mathrm{d}s \lesssim \varepsilon_1 \|x^2 n_1\|_{L^2}.\end{aligned}$$

For the second term in (4.14), we have

$$\begin{aligned}\int_0^t \|e^{is\Lambda} \frac{\partial}{\Lambda^3} n_1 e^{is\Delta} f\|_{L^2} \, \mathrm{d}s &\lesssim \int_0^t \|e^{is\Lambda} \frac{\partial}{\Lambda^3} n_1\|_{L^\infty} \|e^{is\Delta} f\|_{L^2} \, \mathrm{d}s \\ &\lesssim \int_0^t s^{-1} \|\frac{\partial^3}{\Lambda^3} n_1\|_{L^1} \|f\|_{L^2} \, \mathrm{d}s \lesssim \varepsilon_1 t^\delta \|n_1\|_{\dot{B}^0_{1,1}}.\end{aligned}$$

To estimate the term (4.11) we first observe that (4.2), and the assumptions (1.3) on the initial data combined with (2.17), give us

$$\|e^{is\Lambda} \Lambda^{-1}(n_0+G)\|_{L^3} \lesssim \varepsilon_1 s^{-2\alpha-3\delta}. \tag{4.15}$$

We then have

$$\begin{aligned}\|(4.11)\|_{L^2} &\lesssim t\|e^{it\Delta} f\|_{L^6} \|e^{it\Lambda} \Lambda^{-1}(n_0+G)\|_{L^3} + t\|f\|_{L^2} \|e^{it\Lambda} \Lambda^{-2} m_0(i\nabla) n_1\|_{L^\infty} \\ &\lesssim t\,\varepsilon_1 t^{-1+\delta}\, \varepsilon_1 t^{-2\alpha+3\delta} + t\,\varepsilon_1 \frac{1}{t} \|n_1\|_{\dot{B}^0_{1,1}} \lesssim \varepsilon_1^2 t^\delta.\end{aligned}$$

Here, and in the remainder of the proof, we denote by m_k a homogeneous multiplier of order k. We also implicitly use the fact that such multipliers operate on homogeneous Besov spaces $\dot{B}^{s+k}_{p,r} \to \dot{B}^{s}_{p,r}$.

For the term (4.12), we observe that $e^{is\Lambda}\partial_s g = \Lambda^\gamma |u|^2$, so that

$$\|(4.12)\|_{L^2} \lesssim \int_1^t s\|e^{is\Delta} f\|_{L^6} \|\Lambda^{-1+\gamma}|u|^2\|_{L^3} \mathrm{d}s \lesssim \varepsilon_1 \int_1^t s s^{-1+\delta} \|u\|^2_{L^{6/(2-\gamma)}} \mathrm{d}s \lesssim \varepsilon_1^2 t^\delta \,,$$

where we use Sobolev's embedding for the second inequality.

Finally, for the term (4.13), we use $e^{is\Delta}\partial_s f = uw$ to estimate

$$\begin{aligned}
\|(4.13)\|_{L^2} &\lesssim \int_1^t s\|e^{is\Delta}\partial_s f\|_{L^6} \|e^{is\Lambda}\Lambda^{-1}(n_0+G)\|_{L^3} + s\|\partial_s f\|_{L^2} \|e^{is\Lambda}\Lambda^{-2} m_0(i\nabla) n_1\|_{L^\infty} \,\mathrm{d}s \\
&\lesssim \int_1^t s\|u\|_{L^6}\|w\|_{L^\infty} s^{-2\alpha+3\delta} + s\|u\|_{L^\infty}\|w\|_{L^2}\frac{1}{s}\|n_1\|_{\dot{B}^0_{1,1}} \,\mathrm{d}s \\
&\lesssim \varepsilon_1^2 \int_1^t s s^{-1+\delta} s^{-1} s^{-2\alpha+3\delta} + s s^{-1-\alpha} s^{-1} \,\mathrm{d}s \lesssim \varepsilon_1^2 t^\delta \,,
\end{aligned}$$

having used $\alpha > 3\delta/2$. □

4.2. **Estimate for $x^2 F$.** In this last section we want to estimate $\|x^2 F\|_{L^2}$ and show

Proposition 4.2. *Under the apriori assumptions* (4.1)–(4.3) *we have*

$$\|x^2 F\|_{L^2} \lesssim \varepsilon_1^2 t^{1-2\alpha-\delta} \,.$$

Fix i and differentiate twice with respect to ξ_i in (4.4), generating three terms:

$$\widehat{F}_1 = -4\int_0^t \int e^{is\phi(\xi,\eta)} s^2 \eta_i^2 \widehat{f}(\xi-\eta)\widehat{g}(\eta,s) \,\mathrm{d}\eta\mathrm{d}s \tag{4.16}$$

$$\widehat{F}_2 = -4\int_0^t \int e^{is\phi(\xi,\eta)} is\eta_i \partial_{\xi_i} \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s) \,\mathrm{d}\eta\mathrm{d}s \tag{4.17}$$

$$\widehat{F}_3 = \int_0^t \int e^{is\phi(\xi,\eta)} \partial^2_{\xi_i} \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s) \,\mathrm{d}\eta\mathrm{d}s. \tag{4.18}$$

4.2.1. *Estimate for F_1.* As a consequence of the second equality in (4.5), we have

$$\eta_i^2 = \frac{\eta_i^2}{|\eta|}\frac{\eta}{|\eta|}\cdot\partial_\eta\phi + \frac{\phi}{|\eta|}\frac{\eta_i^2}{|\eta|}. \tag{4.19}$$

Plugging this into (4.16) we obtain two terms (we omit the constant factor):

$$\int_0^t \int s^2 e^{is\phi(\xi,\eta)} \frac{\eta_i^2}{|\eta|}\frac{\eta}{|\eta|}\cdot\partial_\eta\phi\, \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s) \,\mathrm{d}\eta\mathrm{d}s \,, \tag{4.20}$$

$$\int_0^t \int s^2 e^{is\phi(\xi,\eta)} \frac{\eta_i^2}{|\eta|}\frac{\phi}{|\eta|} \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s) \,\mathrm{d}\eta\mathrm{d}s \,. \tag{4.21}$$

Integration by parts in s – the term (4.21). Note that

$$\phi e^{is\phi(\xi,\eta)} = -i\partial_s e^{is\phi(\xi,\eta)},$$

and so (4.21) may be rewritten as a sum of the terms

$$\widehat{F}_{11} = -\int i m_0(\eta) t^2 e^{is\phi(\xi,\eta)} \widehat{f}(\xi-\eta,t)\widehat{g}(\eta,t)\mathrm{d}\eta \qquad (4.22)$$

$$+ i\int_0^t \int m_0(\eta) s^2\, e^{is\phi(\xi,\eta)} \partial_s \left(\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\right) \mathrm{d}\eta\mathrm{d}s,$$

$$\widehat{F}_{13} = 2i\int_0^t \int m_0(\eta) s\, e^{is\phi(\xi,\eta)} \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\, \mathrm{d}\eta\mathrm{d}s\,. \qquad (4.23)$$

Integration by parts in η *– the term* (4.20). Write out (4.20) as a sum of terms

$$\int_0^t \int s^2 \frac{\eta_i^2}{|\eta|^2}\eta_j \partial_{\eta_j}\phi\, e^{is\phi(\xi,\eta)} \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\, \mathrm{d}\eta\mathrm{d}s.$$

Fix one summand. Since $s\partial_{\eta_j}\phi e^{is\phi} = \partial_{\eta_j} e^{is\phi}$, integration by parts yields

$$-\int_0^t \int s\partial_{\eta_j}\left(\frac{\eta_i^2\eta_j}{|\eta|^2}\right) e^{is\phi(\xi,\eta)} \widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\, \mathrm{d}\eta\mathrm{d}s \qquad (4.24)$$

$$-\int_0^t \int s\frac{\eta_i^2\eta_j}{|\eta|^2} e^{is\phi(\xi,\eta)} \partial_{\eta_j}\left(\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\right) \mathrm{d}\eta\mathrm{d}s\,. \qquad (4.25)$$

The first term is analogous to $\widehat{F}_{13}$ in (4.23). The quantity (4.25) gives two contributions:

$$\int_0^t \int m_1(\eta) s e^{is\phi(\xi,\eta)} \partial_{\eta_j}\widehat{f}(\xi-\eta)\widehat{g}(\eta)\, \mathrm{d}\eta\mathrm{d}s \qquad (4.26)$$

$$\int_0^t \int m_1(\eta) s e^{is\phi(\xi,\eta)} \widehat{f}(\xi-\eta)\partial_{\eta_j}\widehat{g}(\eta)\, \mathrm{d}\eta\mathrm{d}s\,, \qquad (4.27)$$

where the symbol $m_1(\eta)$ denotes a multiplier which is homogenous of order 1. The first term above is analogous to the term F_2 in (4.17) and will be estimated later. We denote the second term by $\widehat{F}_{12}$. We now proceed to estimate the terms F_{11}, F_{12} and F_{13}, defined respectively in (4.22), (4.27) and (4.23).

Estimate for F_{11}. This term was defined in (4.22). The first term

$$\int m_0(\eta) t^2 e^{is\phi(\xi,\eta)} \widehat{f}(\xi-\eta,t)\widehat{g}(\eta,t)\mathrm{d}\eta$$

can be dealt with by an $L^6 - L^3$ estimate: we pair the zero-th order multiplier with $g = n_0 + G + \Lambda^{-1}n_1$ to find a bound of

$$t^2\|e^{it\Delta}f\|_{L^6}\|e^{is\Lambda}(n_0+G)\|_{L^3} + t^2\|xf\|_{L^2}\|e^{it\Lambda}\Lambda^{-1}n_1\|_\infty$$
$$\lesssim \varepsilon_1 t\|xf\|_{L^2}t^{-1/3} + \varepsilon_1^2 t^\delta \lesssim \varepsilon_1^2 t^{2/3+\delta}.$$

This is acceptable since $2/3+\delta \le 1-2\alpha-\delta$, see (3.4). For the second term in (4.22), we first expand the derivative

$$\partial_s\left(\widehat{f}(\xi-\eta)\widehat{g}(\eta)\right) = \partial_s\widehat{f}(\xi-\eta)\widehat{g}(\eta) + \widehat{f}(\xi-\eta)\partial_s\widehat{g}(\eta).$$

Using $\partial_s f = e^{is\Delta}un$ and $\partial_s g = e^{is\Lambda}\Lambda^\gamma|u|^2$, we obtain a bound of the form

$$\int_0^t s^2\|un\|_{L^6}\|n\|_{L^3}\,\mathrm{d}s + \int_0^t s^2 s^\delta\|u\|_{L^6}\||u|^2\|_{L^3}\,\mathrm{d}s\,.$$

Using the a priori assumptions (4.1), this is bounded by

$$\varepsilon_1^2\int_0^t s s^{-1+\delta}s^{-1/3}\,\mathrm{d}s + \varepsilon_1^2\int_0^t s^{2+\delta}(s^{-1+\delta})^3\,\mathrm{d}s \lesssim \varepsilon_1^2\int_0^t s^{\delta-1/3}\,\mathrm{d}s \lesssim \varepsilon_1^2 t^{2/3+\delta}\,.$$

Estimate for F_{12}. The term F_{12} was defined in (4.27) as a term of the form
$$\widehat{F}_{12} = \int_0^t \int m_1(\eta) s\, e^{is\phi(\xi,\eta)} \widehat{f}(\xi-\eta)\partial_\eta \widehat{g}(\eta)\,\mathrm{d}\eta\mathrm{d}s.$$
Up to a commutator term resulting from $\eta\partial_\eta\widehat{g}(\eta) = \partial_\eta(\eta\widehat{g}(\eta)) - \widehat{g}(\eta)$, we can apply Hölder's inequality with exponents $p = 4/(1+\gamma)$ and $p' = 4/(1-\gamma)$ to find a bound of the type
$$\int_0^t s\|m_0(\nabla)e^{is\Lambda}x\Lambda g\|_{L^{4/(1+\gamma)}}\|e^{is\Delta}f\|_{L^{4/(1-\gamma)}}\,\mathrm{d}s.$$
By (4.1),
$$\|m_0(\nabla)e^{is\Lambda}x\Lambda(n_0+G)\|_{L^{4/(1+\gamma)}} \lesssim \varepsilon_1 s^{-1/4+3\gamma/4-2\alpha-3\delta}$$
and, by interpolating the L^2 and L^6 bounds, we have
$$\|e^{is\Delta}f\|_{L^{4/(1-\gamma)}} \lesssim \varepsilon_1 s^{-(3/4+3\gamma/4)}.$$

On the other hand, we have
$$\begin{aligned}\|e^{is\Lambda}xn_1\|_{L^{4/(1+\gamma)}} &\lesssim t^{-1/2+\gamma/2}\|\Lambda^{1-\gamma}(xn_1)\|_{L^{4/(3-\gamma)}}\\ &\lesssim t^{-1/2+\gamma/2}(\|x\Lambda n_1\|_{L^{12/(9+\gamma)}} + \|n_1\|_{L^{12/(9+\gamma)}})\\ &\lesssim t^{-1/2+\gamma/2}\|\langle x\rangle^2\Lambda n_1\|_{L^2}.\end{aligned}$$
Combining these, we obtain the result.

The term F_{13}. This was defined in (4.23). Taking $L^6 - L^3$ we have a bound of the form
$$\int_0^t s\|u\|_{L^6}\|n\|_{L^3}\,\mathrm{d}s + \int_0^t s\|u\|_{L^6}\|e^{is\Lambda}\Lambda^{-1}n_1\|_{L^3}\,\mathrm{d}s.$$
For the first term in the preceding, we have the bound
$$\int_0^t s^{\delta-1/3}\,\mathrm{d}s \lesssim t^{2/3+\delta}.$$
For the second term, we use the dispersive estimate and Sobolev embedding to obtain the bound $t^{2/3}\|n_1\|_{L^{9/7}}$, which is more than what is needed.

4.2.2. *Estimate for F_2.* This was defined in (4.17) and is of the form
$$\int_0^t \int e^{is\phi(\xi,\eta)} is\eta_i\partial_{\xi_i}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s.$$
We use (4.5) to obtain the two terms
$$-\int_0^t \int e^{is\phi(\xi,\eta)} is\frac{\eta_i}{|\eta|}\frac{\eta}{|\eta|}\cdot\partial_\eta\phi\,\partial_{\eta_i}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s \tag{4.28}$$
$$-\int_0^t \int e^{is\phi(\xi,\eta)} is\frac{\eta_i}{|\eta|}\frac{\phi(\xi,\eta)}{|\eta|}\partial_{\eta_i}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s\,. \tag{4.29}$$
We integrate by parts in (4.28), obtaining terms
$$\int_0^t \partial_{\eta_j}\left(\frac{\eta_i\eta_j}{|\eta|^2}\right)e^{is\phi(\xi,\eta)}\partial_{\eta_i}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s \tag{4.30}$$
$$\int_0^t \frac{\eta_i\eta_j}{|\eta|^2}e^{is\phi(\xi,\eta)}\partial^2_{\eta_i\eta_j}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s \tag{4.31}$$
$$\int_0^t \frac{\eta_i\eta_j}{|\eta|^2}e^{is\phi(\xi,\eta)}\partial_{\eta_i}\widehat{f}(\xi-\eta)\partial_{\eta_j}\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s\,. \tag{4.32}$$

Notice that the multiplier in (4.30) is of order -1 while in (4.31) and (4.32) it is of order zero. In particular, (4.31) is similar to (4.18) and can be estimated as in (4.41) below, so we skip it.

For (4.30), we use the Mikhlin multiplier Theorem[10] and the second assumption in (4.2) and (4.3), to find the bound

$$\begin{aligned}\|(4.30)\|_{L^2} &\lesssim \int_0^t \|e^{is\Delta}xf\|_{L^6}\|e^{is\Lambda}m_{-1}(\nabla)(n_0+G)\|_{L^3}\,\mathrm{d}s + \int_0^t \|xf\|_{L^2}\|e^{is\Lambda}\Lambda^{-2}n_1\|_{L^\infty}\,\mathrm{d}s \\ &\lesssim \int_0^t s^{-1}\|x^2f\|_{L^2}\|e^{is\Lambda}\Lambda^{-1}(n_0+G)\|_{L^3}\,\mathrm{d}s + \varepsilon_1^2 t^\delta \\ &\lesssim \varepsilon_1^2\int_0^t s^{-1}s^{1-2\alpha-\delta}s^{-2\alpha-3\delta}\,\mathrm{d}s + \varepsilon_1^2 t^\delta \lesssim \varepsilon_1^2 t^{1-2\alpha-3\delta}\,.\end{aligned}$$

The term (4.32) can be estimated similarly using Sobolev's embedding and (4.1):

$$\begin{aligned}\|(4.32)\|_{L^2} &\lesssim \int_0^t \|e^{is\Delta}xf\|_{L^6}\|e^{is\Lambda}x(n_0+G)\|_{L^3}\,\mathrm{d}s + \int_0^t \|e^{is\Delta}xf\|_{L^6}\|e^{is\Lambda}\Lambda^{-1}xn_1\|_{L^3}\,\mathrm{d}s \\ &\quad + \int_0^t \|e^{is\Delta}xf\|_{L^2}\|e^{is\Lambda}\frac{\partial}{\Lambda^3}n_1\|_{L^\infty}\,\mathrm{d}s \\ &\lesssim \int_0^t s^{-1}\|x^2f\|_{L^2}\|x(n_0+G)\|_{\dot H^{\frac12}}\,\mathrm{d}s + \int_0^t s^{-1}\|x^2f\|_{L^2}\|\Lambda^{-1/3}xn_1\|_{L^{3/2}}\,\mathrm{d}s \\ &\quad + \int_0^t s^{-1}\|\frac{\partial^3}{\Lambda^3}n_1\|_{L^1}\|xf\|_{L^2}\,\mathrm{d}s \\ &\lesssim \varepsilon_1^2\int_0^t s^{-1}s^{1-2\alpha-\delta}\,\mathrm{d}s + \varepsilon_1\int_0^t s^{-2\alpha-\delta}s^{-1/3}\|xn_1\|_{L^{9/7}}\,\mathrm{d}s + \varepsilon_1^2 t^\delta \\ &\lesssim \varepsilon_1^2 t^{1-2\alpha-\delta}.\end{aligned}$$

We now integrate by parts in s in the term (4.29) to find the terms

$$-\int_0^t\int e^{is\phi(\xi,\eta)}s\frac{\eta_i}{|\eta|^2}\partial_s\partial_{\eta_i}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s \qquad (\equiv \widehat{F}_{21}) \tag{4.33}$$

$$-\int_0^t\int e^{is\phi(\xi,\eta)}s\frac{\eta_i}{|\eta|^2}\partial_{\eta_i}\widehat{f}(\xi-\eta,s)\partial_s\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s \qquad (\equiv \widehat{F}_{22}) \tag{4.34}$$

$$\int e^{it\phi(\xi,\eta)}t\frac{\eta_i}{|\eta|^2}\partial_{\eta_i}\widehat{f}(\xi-\eta,t)\widehat{g}(\eta,t)\mathrm{d}\eta \qquad (\equiv \widehat{F}_{23}) \tag{4.35}$$

$$-\int_0^t\int e^{is\phi(\xi,\eta)}\frac{\eta_i}{|\eta|^2}\partial_{\eta_i}\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}s\,. \qquad (\equiv \widehat{F}_{24}) \tag{4.36}$$

We estimate these four terms below.

Estimate for F_{21}. We first compute the ∂_s derivative:

$$\partial_s\widehat{f}(\xi-\eta,s) = i\int e^{is\phi(\xi-\eta,\tau)}\widehat{f}(\xi-\eta-\tau,s)\widehat{g}(\tau,s)\,\mathrm{d}\tau.$$

The derivative ∂_{η_j} now generates two terms:

$$2i\int_0^t\int\int e^{is\phi(\xi,\eta)}s^2\frac{\eta_i}{|\eta|^2}e^{is\phi(\xi-\eta,\tau)}\widehat{f}(\xi-\eta-\tau,s)\tau_j\widehat{g}(\tau,s)\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}\tau\mathrm{d}s \tag{4.37}$$

$$\int_0^t\int\int e^{is\phi(\xi,\eta)}s\frac{\eta_i}{|\eta|^2}e^{is\phi(\xi-\eta,\tau)}\partial_{\eta_j}\widehat{f}(\xi-\eta-\tau,s)\widehat{g}(\tau,s)\widehat{g}(\eta,s)\,\mathrm{d}\eta\mathrm{d}\tau\mathrm{d}s. \tag{4.38}$$

[10] $|\eta|m_{-1}(\eta)$ is bounded on L^p, $1<p<\infty$.

For the first term we pair the multiplier with $\widehat{g}(\eta)$ and use $L^6 - L^3$ to obtain

$$\|(4.37)\|_{L^2} \lesssim \int_0^t s^2 \|u\partial n\|_{L^6} \|e^{is\Lambda}\Lambda^{-1}(G+n_0)\|_{L^3}\,\mathrm{d}s + \int_0^t s^2\|u\partial n\|_{L^2}\|e^{is\Lambda}\Lambda^{-2}n_1\|_{L^\infty}\,\mathrm{d}s. \tag{4.39}$$

Estimating $\|e^{is\Lambda}\Lambda^{-1}(G+n_0)\|_{L^3}$ by (4.15), we can bound the first of the two expressions above by

$$\int_0^t s^2\|u\|_{L^6}\|\partial n\|_{L^\infty}\|e^{is\Lambda}\Lambda^{-1}(G+n_0)\|_{L^3}\mathrm{d}s \lesssim \varepsilon_1^2 \int_0^t s^{2\delta}s^{-2\alpha-3\delta}\mathrm{d}s \lesssim \varepsilon_1^2 t^{1-2\alpha-\delta}.$$

For the second integral in (4.39) we have

$$\begin{aligned}\int_0^t s^2\|u\partial n\|_{L^2}\|e^{is\Lambda}\Lambda^{-2}n_1\|_{L^\infty}\,\mathrm{d}s &\lesssim \int_0^t s\|u\|_{L^6}\|\partial n\|_{L^3}\|n_1\|_{\dot{B}^0_{1,1}}\,\mathrm{d}s \\ &\lesssim \int_0^t \varepsilon_1^3 s^{-1/3+2\delta}\,\mathrm{d}s \lesssim \varepsilon_1^3 t^{2/3+2\delta}.\end{aligned}$$

Moving on to (4.38), we reproduce the same pairing as before and once again use $L^6 - L^3$ and $L^2 - L^\infty$ estimates to find a bound

$$\int_0^t s\|e^{is\Delta}(xf)\,n\|_{L^6}\|e^{is\Lambda}\Lambda^{-1}(n_0+G)\|_{L^3}\mathrm{d}s + \int_0^t s\|e^{is\Delta}(xf)\,n\|_{L^2}\|e^{is\Lambda}\Lambda^{-2}n_1\|_{L^\infty}\,\mathrm{d}s. \tag{4.40}$$

Observe that

$$\|e^{is\Delta}xf\|_{L^6} \lesssim s^{-1}\|x^2 f\|_{L^2} \lesssim \varepsilon_1 s^{-2\alpha-\delta}\,.$$

Using this, together with $\|n\|_{L^\infty} \le \varepsilon_1 s^{-1}$ and (4.2), we obtain the desired bound of $\varepsilon_1^2 t^{1-2\alpha-\delta}$ for the first term in (4.40). For the second term, we can obtain a bound of $\varepsilon_1^3 t^\delta$ by using

$$\|e^{is\Delta}(xf)\,n\|_{L^2} \le \|xf\|_{L^2}\|n\|_{L^\infty} \lesssim \varepsilon_1^2 s^{-1+\delta}\,,$$

combined with the the linear dipersive estimate (2.17) and the assumption on n_1 in (1.4).

Estimate for F_{22}. This term is defined in (4.34). We pair the multiplier with $\partial_s g = e^{is\Lambda}\Lambda^\gamma|u|^2$, use an $L^6 - L^3$ estimate, Sobolev's embedding and (4.3), to find the bound

$$\begin{aligned}\|(4.34)\|_{L^2} &\lesssim \int_0^t s\|e^{is\Delta}xf\|_{L^6}\|\Lambda^{-1+\gamma}|u|^2\|_{L^3}\,\mathrm{d}s \lesssim \varepsilon_1\int_0^t s^{1-2\alpha-\delta}\||u|^2\|_{L^{3/(2+\gamma)}}\,\mathrm{d}s \\ &\lesssim \varepsilon_1^2\int_0^t s^{1-2\alpha-\delta}\|u\|_{L^2}\|u\|_{L^{6/(1-2\gamma)}}\,\mathrm{d}s \lesssim \varepsilon_1^2 t^{1-2\alpha-\delta}\,.\end{aligned}$$

Estimate for F_{23}. This term is defined in (4.35). We can use $L^6 - L^3$ and $L^\infty - L^2$ as before to get

$$\begin{aligned}\|(4.35)\|_{L^2} &\lesssim t\|e^{it\Delta}xf\|_{L^6}\|\Lambda^{-1}e^{is\Lambda}(n_0+G)\|_{L^3} \\ &\quad + t\|xf\|_{L^2}\|e^{it\Lambda}\Lambda^{-2}n_1\|_{L^\infty} \lesssim \varepsilon_1^2 t^{1-4\alpha-5\delta}\end{aligned}$$

Estimate for F_{24}. This was defined in (4.36). It is dealt with by $L^6 - L^3$ and $L^\infty - L^2$, in an identical fashion to the previous case.

4.2.3. *Estimate for F_3.* This was defined in (4.18). It can be dealt with by using an $L^2 - L^\infty$ estimate leading to

$$\begin{aligned}\|((4.18)\|_{L^2} &\lesssim \int_0^t \|x^2 f\|_{L^2}\|e^{is\Lambda}(n_0+G) + e^{is\Lambda}\Lambda^{-1}n_1\|_{L^\infty}\,\mathrm{d}s \\ &\lesssim \int_0^t \|x^2 f\|_{L^2}\|n\|_{L^\infty}\,\mathrm{d}s + \int_0^t \|x^2 f\|_{L^2}\|\Lambda n_1\|_{\dot{B}^0_{1,1}}\,\mathrm{d}s \lesssim \varepsilon_1^2 t^{1-2\alpha-\delta}\,.\end{aligned} \tag{4.41}$$

At this point all terms are accounted for. □

References

[1] Bourgain, J. and Colliander, J. On the Well-posedness of the Zakharov system. *IMRN* 11 (1996), 515-546.

[2] Bejenaru, I., Herr, S., Holmer, J., and Tataru, D. On the 2D Zakharov system with L^2-Schrödinger data. *Nonlinearity* 22 (2009), no. 5, 1063-1089.

[3] Bejenaru, I. and Herr, S. Convolutions of singular measures and applications to the Zakharov system. *J. Funct. Anal.* 261 (2011), no. 2, 478-506.

[4] Ghidaglia, J-M. and Saut, J-C. On the initial value problem for the Davey-Stewartson systems. *Nonlinearity*, 3 (1990), 475-506.

[5] Ghidaglia, J-M. and Saut, J-C. On the Zakharov-Schulman equations. *in Nonlinear dispersive waves,* L. Debnath Ed. World Scientific (1992) 83-97.

[6] Germain, P., Masmoudi, N. and Shatah, J. Global solutions for 3D quadratic Schrödinger equations. *Int. Math. Res. Not. IMRN*, 3 (2009), 414-432.

[7] Germain P., Masmoudi N. and Shatah, J. Global solutions for the gravity surface water waves equation in dimension 3. *Ann. of Math.*, 175 (2012), no. 2, 691-754.

[8] Ginibre, J., Tsutsumi, Y. and Velo, G. On the Cauchy problem for the Zakharov system. *J. Funct. Anal.* 151 (1997), no. 2, 384-436.

[9] Ginibre, J. and Velo, G. Scattering theory for the Zakharov system. *Hokkaido Math. J.* 35 (2006), no. 4, 865-892.

[10] Ginibre, J. and Velo, G. Long range scattering and modified wave operators for the Wave Schrödinger system. *Ann. Henri Poincaré* 3 (2002), 537-612.

[11] Ginibre, J. and Velo, G. Long range scattering and modified wave operators for the Wave Schrödinger system II. *Ann. Henri Poincaré* 4 (2003), 973-999.

[12] Ginibre, J. and Velo, G. Long range scattering and modified wave operators for the Wave Schrödinger system III. *Dyn. PDE 2* (2005), 101-125.

[13] Ginibre, J. and Velo, G. Long range scattering for the wave-Schrödinger system revisited. *J. Differential Equations* 252 (2012), no. 2, 1642-1667.

[14] Guo, Z. and Nakanishi, K. Small energy scattering for the Zakharov system with radial symmetry. *arXiv:1203.3959*, 2012.

[15] Guo, Z., Nakanishi, K. and Wang, S. Global dynamics below the ground state energy for the Zakharov system in the 3D radial case. *Advances in Math.* 238 (2013), 412441

[16] Guo, Z., Lee, S., Nakanishi, K. and Wang, C. Generalized Strichartz estimates and scattering for 3D Zakharov system. to appear *to Comm. Math. Phys.*.

[17] Gustafson, S., Nakanishi, K. and Tsai, T. Scattering for the Gross-Pitaevsky equation in 3 dimensions. *Commun. Contemp. Math.* 11 (2009), no. 4, 657-707.

[18] Hani, Z., Pusateri, F. and Shatah, J. Scattering for the Zakharov system in three dimensions. *Comm. Math. Phys.* 322 (2013), no. 3, 731-753.

[19] Hayashi, N. and Naumkin, P. Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations. *American Journal of Mathematics*, 120 (1998), 369-389.

[20] Hayashi, N. and Naumkin, P. Large time behavior of solutions for the modified Korteweg-de Vries equation. *Int. Math. Res. Not.*, (1999), no. 8, 395-418.

[21] Hayashi, N. and Naumkin, P. Large time asymptotics of solutions to the generalized Benjamin-Ono equation. *Trans. Amer. Math. Soc.*, 351 (1999), no.1, 109-130.

[22] Ionescu, A. and Pausader, B. The Euler-Poisson system in 2D: global stability of the constant equilibrium solution. *Int. Math. Res. Not.* (2013), 761-826 .

[23] Ionescu, A. and Pausader, B. Global solutions of quasilinear systems of Klein-Gordon equations in 3D. *J. Eur. Math. Soc.*, to appear. arXiv:1208.2661.

[24] Ionescu, A. and Pusateri, F. Nonlinear fractional Schrödinger equations in one dimension. *arXiv:1209.4943*, (2012).

[25] Ionescu, A. and Pusateri, F. Global existence of solution for the gravity water waves system in 2d. *Invent. Math.*, to appear. arXiv:1303.5357.

[26] Kato, J. and Pusateri, F. A new proof of long range scattering for critical nonlinear Schrödinger equations. *Diff. Int. Equations*, 24 (2011), no. 9-10, 923-940.

[27] Kenig, C. E., Ponce, G. and Vega, L. On the Zakharov and Zakharov-Schulman systems. *J. Funct. Anal.* 127 (1995), no. 1, 204-234.

[28] Kenig, C. E., Wang, W. Existence of local smooth solution for a generalized Zakharov system. *J. Fourier Anal. Appl.* 4 (1998), no. 4-5, 469-490.

[29] Klainerman, S. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38 (1985), no. 3, 321-332.

[30] Klainerman, S. *The null condition and global existence for systems of wave equations.* Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 293–326. Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986.

[31] Masmoudi, N. and Nakanishi, K. Energy convergence for singular limits of Zakharov type systems. *Invent. Math.* 172 (2008), no. 3, 535-583..

[32] Musher, S. L., Rubenchik, A. M. and Zakharov, V. E. Hamiltonian approach to the description of nonlinear plasma phenomena. *Phys. Rep.* 129 (1985), no. 5, 285-366.

[33] Ozawa, T. and Tsutsumi, Y. Global existence and asymptotic behavior of solutions for the Zakharov equations in three space dimensions. *Adv. Math. Sci. Appl.* 3 (1993/94), Special Issue, 301-334.

[34] Pusateri, F. and Shatah, J. Space-time resonances and the null condition for (first order) systems of wave equations. arXiv:1109.5662v2. *Comm. Pure and Appl. Math.*, to appear.

[35] Schulman, I. and Zakharov, V.E. Integrability of nonlinear systems and perturbation theory. *What is integrability?* (1991) Spring Ser. Nonlinear Dyn., Springer, Berlin, 185-250.

[36] Shatah, J. Normal forms and quadratic nonlinear Klein-Gordon equations. *Comm. Pure Appl. Math.*, 38(5):685-696, 1985.

[37] Shatah, J. and Struwe, M. Geometric wave equations, volume 2 of *Courant Lecture Notes in Mathematics.* New York University, Courant Institute of Mathematical Sciences, New York, 1998.

[38] Shimomura, A. Scattering theory for Zakharov equations in three space dimensions with large data. *Commun. Contemp. Math*, 6 (2004), 881-899.

[39] Shimomura, A. Modified wave operators for the coupled wave-Schrödinger equations in three space dimensions. *Discrete Contin. Dyn. Syst.*, 9 (2003), no. 6, 1571-1586.

[40] Shimomura, A. Modified wave operators for Maxwell-Schrödinger equations in three-dimensions space. *Ann. Henri Poincaré* 4 (2003) 661-683.

[41] Sulem, C. and Sulem, P.L. The nonlinear Schrödinger equation. Self-focussing and wave collapse. *Applied Mathematical Sciences*, 139. Springer-Verlag, New York, 1999.

[42] Texier, B. Derivation of the Zakharov equations. *Arch. Ration. Mech. Anal.* 184 (2007), no. 1, 121-183.

[43] Tsutsumi, Y. Global existence and asymptotic behavior of solutions for the Maxwell-Schrödinger equations in three space dimensions. *Comm. Math. Phys.*, 151 (1993), no. 3, 543-576.

[44] Zakharov, V.E. Collapse of Langmuir waves. *Zh. Eksp. Teor. Fiz.* 62, 1745-1751 (1972) [*Sov. Phys. JETP* 35, 908-914 (1972)].

Princeton University
E-mail address: tdbeck@math.princeton.edu

Princeton University
E-mail address: fabiop@math.princeton.edu

Princeton University
E-mail address: psosoe@math.princeton.edu

Princeton University
E-mail address: pakwong@math.princeton.edu