

# A local strengthening of Reed's $\omega$ , $\Delta$ , $\chi$ conjecture for quasi-line graphs

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## Abstract

Reed's  $\omega$ ,  $\Delta$ ,  $\chi$  conjecture proposes that every graph satisfies  $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$ ; it is known to hold for all claw-free graphs. In this paper we consider a local strengthening of this conjecture. We prove the local strengthening for line graphs, then note that previous results immediately tell us that the local strengthening holds for all quasi-line graphs. Our proofs lead to polytime algorithms for constructing colourings that achieve our bounds:  $O(n^2)$  for line graphs and  $O(n^3m^2)$  for quasi-line graphs. For line graphs, this is faster than the best known algorithm for constructing a colouring that achieves the bound of Reed's original conjecture.

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# 1 Introduction

All graphs and multigraphs we consider in this paper are finite. Loops are permitted in multigraphs but not graphs. Given a graph  $G$  with maximum degree  $\Delta(G)$  and clique number  $\omega(G)$ , the chromatic number  $\chi(G)$  is trivially bounded above by  $\Delta(G) + 1$  and below by  $\omega(G)$ . Reed's  $\omega$ ,  $\Delta$ ,  $\chi$  conjecture proposes, roughly speaking, that  $\chi(G)$  falls in the lower half of this range:

**Conjecture 1** (Reed). *For any graph  $G$ ,*

$$\chi(G) \leq \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \rceil.$$

One of the first classes of graphs for which this conjecture was proved is the class of line graphs [7]. Already for line graphs the conjecture is tight, as evidenced by the strong product of  $C_5$  and  $K_\ell$  for any positive  $\ell$ ; this is the line graph of the multigraph constructed by replacing each edge of  $C_5$  by  $\ell$  parallel edges. The proof of Conjecture 1 for line graphs was later extended to quasi-line graphs [5, 6] and later claw-free graphs [5] (we will define these graph classes shortly). In his thesis, King proposed a local strengthening of Reed's conjecture. For a vertex  $v$ , let  $\omega(v)$  denote the size of the largest clique containing  $v$ .

**Conjecture 2** (King [5]). *For any graph  $G$ ,*

$$\chi(G) \leq \max_{v \in V(G)} \lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \rceil.$$

There are several pieces of evidence that lend credence to Conjecture 2. First is the fact that the fractional relaxation holds. This was noted by McDiarmid as an extension of a theorem of Reed [8]; the proof appears explicitly in [5] §2.2:

**Theorem 3** (McDiarmid). *For any graph  $G$ ,*

$$\chi_f(G) \leq \max_{v \in V(G)} \left( \frac{1}{2}(d(v) + 1 + \omega(v)) \right).$$

The second piece of evidence for Conjecture 2 is that the result holds for claw-free graphs with stability number at most three [5]. However, for the remaining classes of claw-free graphs, which are constructed as a generalization of line graphs [3], the conjecture has remained open. In this paper we prove that Conjecture 2 holds for line graphs. We then show that we can extend this result to quasi-line graphs in the same way that Conjecture 1 was extended from line graphs to quasi-line graphs in [6]. Our main result is:

**Theorem 4.** *For any quasi-line graph  $G$ ,*

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil.$$

Furthermore our proofs yield polytime algorithms for constructing a proper colouring achieving the bound of the theorem:  $O(n^2)$  time for a line graph on  $n$  vertices, and  $O(n^3m^2)$  time for a quasi-line graph on  $n$  vertices and  $m$  edges.

Given a multigraph  $G$ , the *line graph of  $G$* , denoted  $L(G)$ , is the graph with vertex set  $V(L(G)) = E(G)$  in which two vertices of  $L(G)$  are adjacent precisely if their corresponding edges in  $H$  share an endpoint. We say that a graph  $G'$  is a *line graph* if for some multigraph  $G$ ,  $L(G)$  is isomorphic to  $G'$ . A graph  $G$  is *quasi-line* if every vertex  $v$  is *bisimplicial*, i.e. the neighbourhood of  $v$  induces the complement of a bipartite graph. A graph  $G$  is *claw-free* if it contains no induced  $K_{1,3}$ . Observe that every line graph is quasi-line and every quasi-line graph is claw-free.

## 2 Proving the local strengthening for line graphs

In order to prove Conjecture 2 for line graphs, we prove an equivalent statement in the setting of edge colourings of multigraphs. Given distinct adjacent vertices  $u$  and  $v$  in a multigraph  $G$ , we let  $\mu_G(uv)$  denote the number of edges between  $u$  and  $v$ . We let  $t_G(uv)$  denote the maximum, over all vertices  $w \notin \{u, v\}$ , of the number of edges with both endpoints in  $\{u, v, w\}$ . That is,

$$t_G(uv) := \max_{w \in N(u) \cap N(v)} (\mu_G(uw) + \mu_G(vw)).$$

We omit the subscripts when the multigraph in question is clear.

Observe that given an edge  $e$  in  $G$  with endpoints  $u$  and  $v$ , the degree of  $uv$  in  $L(G)$  is  $d(u) + d(v) - \mu(uv) - 1$ . And since any clique in  $L(G)$  containing  $e$  comes from the edges incident to  $u$ , the edges incident to  $v$ , or the edges in a triangle containing  $u$  and  $v$ , we can see that  $\omega(v)$  in  $L(G)$  is equal to  $\max\{d(u), d(v), t(uv)\}$ . Therefore we prove the following theorem, which, aside from the algorithmic claim, is equivalent to proving Conjecture 2 for line graphs:

**Theorem 5.** *Let  $G$  be a multigraph on  $m$  edges, and let*

$$\gamma'_l(G) := \max_{uv \in E(G)} \left[ \max \left\{ d(u) + \frac{1}{2}(d(v) - \mu(vu)), d(v) + \frac{1}{2}(d(u) - \mu(uv)), \right. \right. \\ \left. \left. \frac{1}{2}(d(u) + d(v) - \mu_G(uv) + t(uv)) \right\} \right]. \quad (1)$$

*Then  $\chi'(G) \leq \gamma'_l(G)$ , and we can find a  $\gamma'_l(G)$ -edge-colouring of  $G$  in  $O(m^2)$  time.*

The most intuitive approach to achieving this bound on the chromatic index involves assuming that  $G$  is a minimum counterexample, then characterizing  $\gamma'_l(G)$ -edge-colourings of  $G - e$  for an edge  $e$ . We want an algorithmic result, so we will have to be a bit more careful to ensure that we can modify partial  $\gamma'_l(G)$ -edge-colourings efficiently until we find one that we can extend to a complete  $\gamma'_l(G)$ -edge-colouring of  $G$ .

We begin by defining, for a vertex  $v$ , a *fan hinged at  $v$* . Let  $e$  be an edge incident to  $v$ , and let  $v_1, \dots, v_\ell$  be a set of distinct neighbours of  $v$  with  $e$  between  $v$  and  $v_1$ . Let  $c : E \setminus \{e\} \rightarrow \{1, \dots, k\}$  be a proper edge colouring of  $G \setminus \{e\}$  for some fixed  $k$ . Then  $F = (e; c; v; v_1, \dots, v_\ell)$  is a *fan* if for every  $j$  such that  $2 \leq j \leq \ell$ , there exists some  $i$  less than  $j$  such that some edge between  $v$  and  $v_j$  is assigned a colour that does not appear on any edge incident to  $v_i$  (i.e. a colour *missing* at  $v_i$ ). We say that  $F$  is *hinged at  $v$* . If there is no  $u \notin \{v, v_1, \dots, v_\ell\}$  such that  $F' = (e; c; v; v_1, \dots, v_\ell, u)$  is a fan, we say that  $F$  is a *maximal fan*. The *size* of a fan refers to the number of neighbours of the hinge vertex contained in the fan (in this case,  $\ell$ ). These fans generalize Vizing's fans, originally used in the proof of Vizing's theorem [13]. Given a partial  $k$ -edge-colouring of  $G$  and a vertex  $w$ , we say that a colour is *incident to  $w$*  if the colour appears on an edge incident to  $w$ . We use  $\mathcal{C}(w)$  to denote the set of colours incident to  $w$ , and we use  $\bar{\mathcal{C}}(w)$  to denote  $[k] \setminus \mathcal{C}(w)$ .

Fans allow us to modify partial  $k$ -edge-colourings of a graph (specifically those with exactly one uncoloured edge). We will show that if  $k \geq \gamma'_l(G)$ , then either every maximal fan has size 2 or we can easily find a  $k$ -edge-colouring of  $G$ . We first prove that we can construct a  $k$ -edge-colouring of  $G$  from a partial  $k$ -edge-colouring of  $G - e$  whenever we have a fan for which certain sets are not disjoint.

**Lemma 6.** *For some edge  $e$  in a multigraph  $G$  and positive integer  $k$ , let  $c$  be a  $k$ -edge-colouring of  $G - e$ . If there is a fan  $F = (e; c; v; v_1, \dots, v_\ell)$  such that for some  $j$ ,  $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_j) \neq \emptyset$ , then we can find a  $k$ -edge-colouring of  $G$  in  $O(k + m)$  time.*

*Proof.* Let  $j$  be the minimum index for which  $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_j)$  is nonempty. If  $j = 1$  then the result is trivial, since we can extend  $c$  to a proper  $k$ -edge-colouring of  $G$ . Otherwise  $j \geq 2$  and we can find  $j$  in  $O(m)$  time. We define  $e_1$  to be  $e$ . We then construct a function  $f : \{2, \dots, \ell\} \rightarrow \{1, \dots, \ell - 1\}$  such that for each  $i$ , (1)  $f(i) < i$  and (2) there is an edge  $e_i$  between  $v$  and  $v_i$  such that  $c(e_i)$  is missing at  $v_{f(i)}$ . We can find this function in  $O(k + m)$  time by building a list of the earliest  $v_i$  at which each colour is missing, and computing  $f$  for increasing values of  $i$  starting at 2. While doing so we also find the set of edges  $\{e_i\}_{i=2}^\ell$ .

We construct a  $k$ -edge-colouring  $c_j$  of  $G - e_j$  from  $c$  by shifting the colour  $c(e_j)$  from  $e_j$  to  $e_{f(j)}$ , shifting the colour  $c(e_{f(j)})$  from  $e_{f(j)}$  to  $e_{f(f(j))}$ , and so on, until we shift a colour to  $e$ . We now have a  $k$ -edge-colouring  $c_j$  of  $G - e_j$  such that some colour is missing at both  $v$  and  $v_j$ . We can therefore extend  $c_j$  to a proper  $k$ -edge-colouring of  $G$  in  $O(k + m)$  time.  $\square$

**Lemma 7.** *For some edge  $e$  in a multigraph  $G$  and positive integer  $k$ , let  $c$  be a  $k$ -edge-colouring of  $G - e$ . If there is a fan  $F = (e; c; v; v_1, \dots, v_\ell)$  such that for some  $i$  and  $j$  satisfying  $1 \leq i < j \leq \ell$ ,  $\bar{\mathcal{C}}(v_i) \cap \bar{\mathcal{C}}(v_j) \neq \emptyset$ , then we can find  $v_i$  and  $v_j$  in  $O(k + m)$  time, and we can find a  $k$ -edge-colouring of  $G$  in  $O(k + m)$  time.*

*Proof.* We can easily find  $i$  and  $j$  in  $O(k + m)$  time if they exist. Let  $\alpha$  be a colour in  $\bar{\mathcal{C}}(v)$  and let  $\beta$  be a colour in  $\bar{\mathcal{C}}(v_i) \cap \bar{\mathcal{C}}(v_j)$ . Note that by Lemma 6, we can assume  $\alpha \in \mathcal{C}(v_i) \cap \mathcal{C}(v_j)$  and  $\beta \in \mathcal{C}(v)$ .

Let  $G_{\alpha, \beta}$  be the subgraph of  $G$  containing those edges coloured  $\alpha$  or  $\beta$ . Every component of  $G_{\alpha, \beta}$  containing  $v$ ,  $v_i$ , or  $v_j$  is a path on  $\geq 2$  vertices. Thus either  $v_i$  or  $v_j$  is in a component of  $G_{\alpha, \beta}$  not containing  $v$ . Exchanging the colours  $\alpha$  and  $\beta$  on this component leaves us with a  $k$ -edge-colouring of  $G - e$  in which either  $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_i) \neq \emptyset$  or  $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_j) \neq \emptyset$ . This allows us to apply Lemma 6 to find a  $k$ -edge-colouring of  $G$ . We can easily do this work in  $O(m)$  time.  $\square$

The previous two lemmas suggest that we can extend a colouring more easily when we have a large fan, so we now consider how we can extend a fan that is not maximal. Given a fan  $F = (e; c; v; v_1, \dots, v_\ell)$ , we use  $d(F)$  to denote  $d(v) + \sum_{i=1}^{\ell} d(v_i)$ .

**Lemma 8.** *For some edge  $e$  in a multigraph  $G$  and integer  $k \geq \Delta(G)$ , let  $c$  be a  $k$ -edge-colouring of  $G - e$  and let  $F$  be a fan. Then we can extend  $F$  to a maximal fan  $F' = (e; c; v; v_1, v_2, \dots, v_\ell)$  in  $O(k + d(F'))$  time.*

*Proof.* We proceed by setting  $F' = F$  and extending  $F'$  until it is maximal. To this end we maintain two colour sets. The first,  $\mathcal{C}$ , consists of those colours appearing incident to  $v$  but not between  $v$  and another vertex of  $F'$ . The second,  $\bar{\mathcal{C}}_{F'}$ , consists of those colours that are in  $\mathcal{C}$  and are missing at some fan vertex. Clearly  $F'$  is maximal if and only if  $\bar{\mathcal{C}}_{F'} = \emptyset$ . We can perform this initialization in  $O(k + d(F))$  time by counting the number of times each colour in  $\mathcal{C}$  appears incident to a vertex of the fan.

Now suppose we have  $F' = (e; c; v; v_1, v_2, \dots, v_\ell)$ , along with sets  $\mathcal{C}$  and  $\bar{\mathcal{C}}_{F'}$ , which we may assume is not empty. Take an edge incident to  $v$  with a colour in  $\bar{\mathcal{C}}_{F'}$ ; call its other endpoint  $v_{\ell+1}$ . We now update  $\mathcal{C}$  by removing all colours appearing between  $v$  and  $v_{\ell+1}$ . We update  $\bar{\mathcal{C}}_{F'}$  by removing all colours appearing between  $v$  and  $v_{\ell+1}$ , and adding all colours in  $\mathcal{C} \cap \bar{\mathcal{C}}(v_{\ell+1})$ . Set  $F' = (e; c; v; v_1, v_2, \dots, v_{\ell+1})$ . We can perform this update in  $d(v_{\ell+1})$  time; the lemma follows.  $\square$

We can now prove that if  $k \geq \gamma'_l(G)$  and we have a maximal fan of size 1 or at least 3, we can find a  $k$ -edge-colouring of  $G$  in  $O(k + m)$  time.

**Lemma 9.** *For some edge  $e$  in a multigraph  $G$  and positive integer  $k \geq \gamma'_l(G)$ , let  $c$  be a  $k$ -edge-colouring of  $G - e$  and let  $F = (e; c; v; v_1)$  be a fan. If  $F$  is a maximal fan we can find a  $k$ -edge-colouring of  $G$  in  $O(k + m)$  time.*

*Proof.* If  $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_1)$  is nonempty, then we can easily extend the colouring of  $G - e$  to a  $k$ -edge-colouring of  $G$ . So assume  $\bar{\mathcal{C}}(v) \cap \bar{\mathcal{C}}(v_1)$  is empty. Since  $k \geq \gamma'_\ell(G) \geq 1$ ,  $\bar{\mathcal{C}}(v_1)$  is nonempty. Therefore there is a colour in  $\bar{\mathcal{C}}(v_1)$  appearing on an edge incident to  $v$  whose other endpoint, call it  $v_2$ , is not  $v_1$ . Thus  $(e; c; v; v_1, v_2)$  is a fan, contradicting the maximality of  $F$ .  $\square$

**Lemma 10.** *For some edge  $e$  in a multigraph  $G$  and positive integer  $k \geq \gamma'_\ell(G)$ , let  $c$  be a  $k$ -edge-colouring of  $G - e$  and let  $F = (e; c; v; v_1, v_2, \dots, v_\ell)$  be a maximal fan with  $\ell \geq 3$ . Then we can find a  $k$ -edge-colouring of  $G$  in  $O(k + m)$  time.*

*Proof.* Let  $v_0$  denote  $v$  for ease of notation. If the sets  $\bar{\mathcal{C}}(v_0), \bar{\mathcal{C}}(v_1), \dots, \bar{\mathcal{C}}(v_\ell)$  are not all pairwise disjoint, then using Lemma 6 or Lemma 7 we can find a  $k$ -edge-colouring of  $G$  in  $O(m)$  time. We can easily determine whether or not these sets are pairwise disjoint in  $O(k + m)$  time. Now assume they are all pairwise disjoint; we will exhibit a contradiction, which is enough to prove the lemma.

The number of missing colours at  $v_i$ , i.e.  $|\bar{\mathcal{C}}(v_i)|$ , is  $k - d(v_i)$  if  $2 \leq i \leq \ell$ , and  $k - d(v_i) + 1$  if  $i \in \{0, 1\}$ . Since  $F$  is maximal, any edge with one endpoint  $v_0$  and the other endpoint outside  $\{v_0, \dots, v_\ell\}$  must have a colour not appearing in  $\cup_{i=0}^\ell \bar{\mathcal{C}}(v_i)$ . Therefore

$$\left( \sum_{i=0}^{\ell} k - d(v_i) \right) + 2 + \left( d(v_0) - \sum_{i=1}^{\ell} \mu(v_0 v_i) \right) \leq k. \quad (2)$$

Thus

$$\ell k + 2 - \sum_{i=1}^{\ell} \mu(v_0 v_i) \leq \sum_{i=1}^{\ell} d(v_i). \quad (3)$$

But since  $k \geq \gamma'_\ell(G)$ , (1) tells us that for all  $i \in [\ell]$ ,

$$d(v_i) + \frac{1}{2}(d(v_0) - \mu(v_0 v_i)) \leq k \quad (4)$$

Thus substituting for  $k$  tells us

$$\sum_{i=1}^{\ell} \frac{d(v_0) + 2d(v_i) - \mu(v_0 v_i)}{2} + 2 - \sum_{i=1}^{\ell} \mu(v_0 v_i) \leq \sum_{i=1}^{\ell} d(v_i).$$

So

$$\begin{aligned} 2 + \frac{1}{2}\ell d(v_0) - \frac{3}{2} \sum_{i=1}^{\ell} \mu(v_0 v_i) &\leq 0 \\ 2 + \frac{1}{2}\ell d(v_0) &\leq \frac{3}{2} \sum_{i=1}^{\ell} \mu(v_0 v_i) \\ \frac{\ell}{2} d(v_0) &< \frac{3}{2} d(v_0). \end{aligned}$$

This is a contradiction, since  $\ell \geq 3$ . □

We are now ready to prove the main lemma of this section.

**Lemma 11.** *For some edge  $e_0$  in a multigraph  $G$  and positive integer  $k \geq \gamma'_l(G)$ , let  $c_0$  be a  $k$ -edge-colouring of  $G - e_0$ . Then we can find a  $k$ -edge-colouring of  $G$  in  $O(k + m)$  time.*

As we will show, this lemma easily implies Theorem 5. We approach this lemma by constructing a sequence of overlapping fans of size two until we can apply a previous lemma. If we cannot do this, then our sequence results in a cycle in  $G$  and a set of partial  $k$ -edge-colourings of  $G$  with a very specific structure that leads us to a contradiction.

*Proof.* We postpone algorithmic considerations until the end of the proof.

Let  $v_0$  and  $v_1$  be the endpoints of  $e_0$ , and let  $F_0 = (e_0; c_0; v_1; v_0, u_1, \dots, u_\ell)$  be a maximal fan. If  $|\{u_1, \dots, u_\ell\}| \neq 1$  then we can apply Lemma 9 or Lemma 10. More generally, if at any time we find a fan of size three or more we can finish by applying Lemma 10. So assume  $\{u_1, \dots, u_\ell\}$  is a single vertex; call it  $v_2$ .

Let  $\bar{C}_0$  denote the set of colours missing at  $v_0$  in the partial colouring  $c_0$ , and take some colour  $\alpha_0 \in \bar{C}_0$ . Note that if  $\alpha_0$  does not appear on an edge between  $v_1$  and  $v_2$  then we can find a fan  $(e_0; c_0; v_1; v_0, v_2, u)$  of size 3 and apply Lemma 10 to complete the colouring. So we can assume that  $\alpha_0$  does appear on an edge between  $v_1$  and  $v_2$ .

Let  $e_1$  denote the edge between  $v_1$  and  $v_2$  given colour  $\alpha_0$  in  $c_0$ . We construct a new colouring  $c_1$  of  $G - e_1$  from  $c_0$  by uncolouring  $e_1$  and assigning  $e_0$  colour  $\alpha_0$ . Let  $\bar{C}_1$  denote the set of colours missing at  $v_1$  in the colouring  $c_1$ . Now let  $F_1 = (e_1; c_1; v_2; v_1, v_3)$  be a maximal fan. As with  $F_0$ , we can assume that  $F_1$  exists and is indeed maximal. The vertex  $v_3$  may or may not be the same as  $v_0$ .

Let  $\alpha_1 \in \bar{C}_1$  be a colour in  $\bar{C}_1$ . Just as  $\alpha_0$  appears between  $v_1$  and  $v_2$  in  $c_0$ , we can see that  $\alpha_1$  appears between  $v_2$  and  $v_3$ . Now let  $e_2$  be the edge between  $v_2$  and  $v_3$  having colour  $\alpha_1$  in  $c_1$ . We construct a colouring  $c_2$  of  $G - e_2$  from  $c_1$  by uncolouring  $e_2$  and assigning  $e_1$  colour  $\alpha_1$ .

We continue to construct a sequence of fans  $F_i = (e_i, c_i; v_{i+1}; v_i, v_{i+2})$  for  $i = 0, 1, 2, \dots$  in this way, maintaining the property that  $\alpha_{i+2} = \alpha_i$ . This is possible because when we construct  $c_{i+1}$  from  $c_i$ , we make  $\alpha_i$  available at  $v_{i+2}$ , so the set  $\bar{C}_{i+2}$  (the set of colours missing at  $v_{i+2}$  in the colouring  $c_{i+2}$ ) always contains  $\alpha_i$ . We continue constructing our sequence of fans until we reach some  $j$  for which  $v_j \in \{v_i\}_{i=0}^{j-1}$ , which will inevitably happen if we never find a fan of size 3 or greater. We claim that  $v_j = v_0$  and  $j$  is odd. To see this, consider the original edge-colouring of  $G - e_0$  and note that for  $1 \leq i \leq j - 1$ ,  $\alpha_0$  appears on an edge between  $v_i$  and  $v_{i+1}$  precisely if  $i$  is odd, and  $\alpha_1$  appears on an edge between  $v_i$  and  $v_{i+1}$  precisely if  $i$  is even. Thus since the edges of colour  $\alpha_0$  form a matching, and so do the edges of colour  $\alpha_1$ , we indeed have  $v_j = v_0$

and  $j$  odd. Furthermore  $F_0 = F_j$ . Let  $C$  denote the cycle  $v_0, v_1, \dots, v_{j-1}$ . In each colouring,  $\alpha_0$  and  $\alpha_1$  both appear  $(j-1)/2$  times on  $C$ , in a near-perfect matching. Let  $H$  be the sub-multigraph of  $G$  consisting of those edges between  $v_i$  and  $v_{i+1}$  for  $0 \leq i \leq j-1$  (with indices modulo  $j$ ). Let  $A$  be the set of colours missing on at least one vertex of  $C$ , and let  $H_A$  be the sub-multigraph of  $H$  consisting of  $e_0$  and those edges receiving a colour in  $A$  in  $c_0$  (and therefore in any  $c_i$ ).

Suppose  $j = 3$ . If some colour is missing on two vertices of  $C$  in  $c_0, c_1$ , or  $c_2$ , we can easily find a  $k$ -edge-colouring of  $G$  since any two vertices of  $C$  are the endpoints of  $e_0, e_1$ , or  $e_2$ . We know that every colour in  $\bar{C}_0$  appears between  $v_1$  and  $v_2$ , and every colour in  $\bar{C}_1$  appears between  $v_2$  and  $v_3 = v_0$ . Therefore  $|E(H_A)| = |A| + 1$ . Our construction tells us that every colour in  $\bar{C}_0$  appears between  $v_1$  and  $v_2$ , and every colour in  $\bar{C}_1$  appears between  $v_2$  and  $v_3 = v_0$ . Therefore

$$\begin{aligned}
2\gamma'_l(G) &\geq d_G(v_0) + d_G(v_1) + t_G(v_0v_1) - \mu_G(v_0v_1) \\
&= d_{H_A}(v_0) + d_{H_A}(v_1) + 2(k - |A|) + t_G(v_0v_1) - \mu_G(v_0v_1) \\
&\geq d_{H_A}(v_0) + d_{H_A}(v_1) + 2(k - |A|) + t_{H_A}(v_0v_1) - \mu_{H_A}(v_0v_1) \\
&\geq 2|E(H_A)| + 2(k - |A|) \\
&> 2|A| + 2(k - |A|) = 2k
\end{aligned}$$

This is a contradiction since  $k \geq \gamma'_l(G)$ . We can therefore assume that  $j \geq 5$ .

Let  $\beta$  be a colour in  $A \setminus \{\alpha_0, \alpha_1\}$ . If  $\beta$  is missing at two consecutive vertices  $v_i$  and  $v_{i+1}$  then we can easily extend  $c_i$  to a  $k$ -edge-colouring of  $G$ . Bearing in mind that each  $F_i$  is a maximal fan, we claim that if  $\beta$  is not missing at two consecutive vertices then either we can easily  $k$ -edge-colour  $G$ , or the number of edges coloured  $\beta$  in  $H_A$  is at least twice the number of vertices at which  $\beta$  is missing in any  $c_i$ .

To prove this claim, first assume without loss of generality that  $\beta \in \bar{C}_0$ . Since  $\beta$  is not missing at  $v_1$ ,  $\beta$  appears on an edge between  $v_1$  and  $v_2$  for the same reason that  $\alpha_0$  does. Likewise, since  $\beta$  is not missing at  $v_{j-1}$ ,  $\beta$  appears on an edge between  $v_{j-1}$  and  $v_{j-2}$ . Finally, suppose  $\beta$  appears between  $v_1$  and  $v_2$ , and is missing at  $v_3$  in  $c_0$ . Then let  $e_\beta$  be the edge between  $v_1$  and  $v_2$  with colour  $\beta$  in  $c_0$ . We construct a colouring  $c'_0$  from  $c_0$  by giving  $e_2$  colour  $\beta$  and giving  $e_\beta$  colour  $\alpha_1$  (i.e. we swap the colours of  $e_\beta$  and  $e_2$ ). Thus  $c'_0$  is a  $k$ -edge-colouring of  $G - e_0$  in which  $\beta$  is missing at both  $v_0$  and  $v_1$ . We can therefore extend  $G - e_0$  to a  $k$ -edge-colouring of  $G$ . Thus if  $\beta$  is missing at  $v_3$  or  $v_{j-3}$  we can easily  $k$ -edge-colour  $G$ . We therefore have at least two edges of  $H_A$  coloured  $\beta$  for every vertex of  $C$  at which  $\beta$  is missing, and we do not double-count edges. This proves the claim, and the analogous claim for any colour in  $A$  also holds.

Now we have

$$\sum_{i=0}^{j-1} \mu_{H_A}(v_i v_{i+1}) = |E(H_A)| > 2 \sum_{i=0}^{j-1} (k - d_G(v_i)). \quad (5)$$



Therefore taking indices modulo  $j$ , we have

$$\sum_{i=0}^{j-1} (d_G(v_i) + \frac{1}{2}\mu_{H_A}(v_{i+1}v_{i+2})) > jk. \quad (6)$$

Therefore there exists some index  $i$  for which

$$d_G(v_i) + \frac{1}{2}\mu_{H_A}(v_{i+1}v_{i+2}) > k. \quad (7)$$

Therefore

$$k \geq d_G(v_i) + \frac{1}{2}\mu_G(v_{i+1}v_{i+2}) > k. \quad (8)$$

This is a contradiction, so we can indeed find a  $k$ -edge-colouring of  $G$ . It remains to prove that we can do so in  $O(k+m)$  time.

Given the colouring  $c_i$ , we can construct the fan  $F_i = (e_i, c_i; v_{i+1}; v_i, v_{i+2})$  and determine whether or not it is maximal in  $O(k+d(F_i))$  time. If it is not maximal, we can complete the  $k$ -edge-colouring of  $G$  in  $O(m)$  time; this will happen at most once throughout the entire process. Therefore we will either complete the colouring or construct our cycle of fans  $F_0, \dots, F_{j-1}$  in  $O(\sum_{i=0}^{j-1}(k+d(F_i)))$  time. This is not the desired bound, so suppose there is an index  $i$  for which  $k > d(F_i)$ . In this case we certainly have two intersecting sets of available colours in  $F_i$ , so we can apply Lemma 6 or 7 when we arrive at  $F_i$ , and find the  $k$ -edge-colouring of  $G$  in  $O(k+m)$  time. If no such  $i$  exists, then  $jk = O(\sum_{i=0}^{j-1}(d(F_i))) = O(m)$ , and we indeed complete the construction of all fans in  $O(k+m)$  time.

Since each  $F_i$  is a maximal fan, in  $c_0$  there must be some colour  $\beta \notin \{\alpha_0, \alpha_1\}$  missing at two consecutive vertices  $v_i$  and  $v_{i+1}$ , otherwise we reach a contradiction. We can find this  $\beta$  and  $i$  by going around the cycle of fans and comparing  $\bar{C}_i$  and  $\bar{C}_{i+1}$ , and since this is trivial if  $|\bar{C}_i| + |\bar{C}_{i+1}| > d(v_i) + d(v_{i+1})$  we can find  $\beta$  and  $i$  in  $O(k+m)$  time, after which it is easy to construct the  $k$ -edge-colouring of  $G$  from  $c_i$ .  $\square$

We now complete the proof of Theorem 5.

*Proof of Theorem 5.* Order the edges of  $G$   $e_1, \dots, e_m$  arbitrarily and let  $k = \gamma'_l(G)$ , which we can easily compute in  $O(nm)$  time. For  $i = 0, \dots, m$ , let  $G_i$  denote the subgraph of  $G$  on edges  $\{e_j \mid j \leq i\}$ . Since  $G_0$  is empty it is vacuously  $k$ -edge-coloured. Given a  $k$ -edge-colouring of  $G_i$ , we can find a  $k$ -edge-colouring of  $G_{i+1}$  in  $O(k+m)$  time by applying Lemma 11. Since  $k = \gamma'_l(G) = O(m)$ , the theorem follows.  $\square$

This gives us the following result for line graphs, since for any multigraph  $G$  we have  $|V(L(G))| = |E(G)|$ :

**Theorem 12.** *Given a line graph  $G$  on  $n$  vertices, we can find a proper colouring of  $G$  using  $\gamma_l(G)$  colours in  $O(n^2)$  time.*

This is faster than the algorithm of King, Reed, and Vetta [7] for  $\gamma(G)$ -colouring line graphs, which is given an improved complexity bound of  $O(n^{5/2})$  in [5], §4.2.3.

### 3 Extending the result to quasi-line graphs

We now leave the setting of edge colourings of multigraphs and consider vertex colourings of simple graphs. As mentioned in the introduction, we can extend Conjecture 2 from line graphs to quasi-line graphs using the same approach that King and Reed used to extend Conjecture 1 from line graphs to quasi-line graphs in [6]. We do not require the full power of Chudnovsky and Seymour’s structure theorem for quasi-line graphs [2]. Instead, we use a simpler decomposition theorem from [3].

#### 3.1 The structure of quasi-line graphs

We wish to describe the structure of quasi-line graphs. If a quasi-line graph does not contain a certain type of homogeneous pair of cliques, then it is either a circular interval graph or built as a generalization of a line graph – where in a line graph we would replace each edge with a vertex, we now replace each edge with a linear interval graph. We now describe this structure more formally.

##### 3.1.1 Linear and circular interval graphs

A *linear interval graph* is a graph  $G = (V, E)$  with a *linear interval representation*, which is a point on the real line for each vertex and a set of intervals, such that vertices  $u$  and  $v$  are adjacent in  $G$  precisely if there is an interval containing both corresponding points on the real line. If  $X$  and  $Y$  are specified cliques in  $G$  consisting of the  $|X|$  leftmost and  $|Y|$  rightmost vertices (with respect to the real line) of  $G$  respectively, we say that  $X$  and  $Y$  are *end-cliques* of  $G$ . These cliques may be empty.

Accordingly, a *circular interval graph* is a graph with a *circular interval representation*, i.e.  $|V|$  points on the unit circle and a set of intervals (arcs) on the unit circle such that two vertices of  $G$  are adjacent precisely if some arc contains both corresponding points. Circular interval graphs are the first of two fundamental types of quasi-line graph. Deng, Hell, and Huang proved that we can identify and find a representation of a circular or linear interval graph in  $O(m)$  time [4].

##### 3.1.2 Compositions of linear interval strips

We now describe the second fundamental type of quasi-line graph.

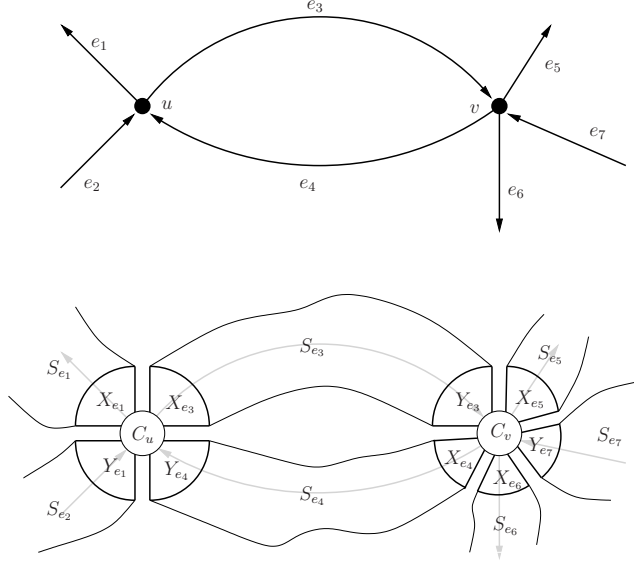


Figure 1: We compose a set of strips  $\{(S_e, X_e, Y_e) \mid e \in E(H)\}$  by joining them together on their end-cliques. A hub clique  $C_u$  will arise for each vertex  $u \in V(H)$ .

A *linear interval strip*  $(S, X, Y)$  is a linear interval graph  $S$  with specified end-cliques  $X$  and  $Y$ . We compose a set of strips as follows. We begin with an underlying directed multigraph  $H$ , possibly with loops, and for every edge  $e$  of  $H$  we take a linear interval strip  $(S_e, X_e, Y_e)$ . For  $v \in V(H)$  we define the *hub clique*  $C_v$  as

$$C_v = \left( \bigcup \{X_e \mid e \text{ is an edge out of } v\} \right) \cup \left( \bigcup \{Y_e \mid e \text{ is an edge into } v\} \right).$$

We construct  $G$  from the disjoint union of  $\{S_e \mid e \in E(H)\}$  by making each  $C_v$  a clique;  $G$  is then a *composition of linear interval strips* (see Figure 1). Let  $G_h$  denote the subgraph of  $G$  induced on the union of all hub cliques. That is,

$$G_h = G[\bigcup_{v \in V(H)} C_v] = G[\bigcup_{e \in E(H)} (X_e \cup Y_e)].$$

Compositions of linear interval strips generalize line graphs: note that if each  $S_e$  satisfies  $|S_e| = |X_e| = |Y_e| = 1$  then  $G = G_h = L(H)$ .

### 3.1.3 Homogeneous pairs of cliques

A pair of disjoint nonempty cliques  $(A, B)$  in a graph is a *homogeneous pair of cliques* if  $|A| + |B| \geq 3$ , every vertex outside  $A \cup B$  is adjacent to either all or none of  $A$ , and

every vertex outside  $A \cup B$  is adjacent to either all or none of  $B$ . Furthermore  $(A, B)$  is *nonlinear* if  $G$  contains an induced  $C_4$  in  $A \cup B$  (this condition is equivalent to insisting that the subgraph of  $G$  induced by  $A \cup B$  is a linear interval graph).

### 3.1.4 The structure theorem

Chudnovsky and Seymour's structure theorem for quasi-line graphs [3] tells us that any quasi-line graph not containing a clique cutset is made from the building blocks we just described.

**Theorem 13.** *Any quasi-line graph containing no clique cutset and no nonlinear homogeneous pair of cliques is either a circular interval graph or a composition of linear interval strips.*

To prove Theorem 4, we first explain how to deal with circular interval graphs and nonlinear homogeneous pairs of cliques, then move on to considering how to decompose a composition of linear interval strips.

## 3.2 Circular interval graphs

We can easily prove Conjecture 2 for circular interval graphs by combining previously known results. Niessen and Kind proved that every circular interval graph  $G$  satisfies  $\chi(G) = \lceil \chi_f(G) \rceil$  [9], so Theorem 3 immediately implies that Conjecture 2 holds for circular interval graphs. Furthermore Shih and Hsu [11] proved that we can optimally colour circular interval graphs in  $O(n^{3/2})$  time, which gives us the following result:

**Lemma 14.** *Given a circular interval graph  $G$  on  $n$  vertices, we can  $\gamma_l(G)$ -colour  $G$  in  $O(n^{3/2})$  time.*

## 3.3 Nonlinear homogeneous pairs of cliques

There are many lemmas of varying generality that tell us we can easily deal with nonlinear homogeneous pairs of cliques; we use the version used by King and Reed [6] in their proof of Conjecture 1 for quasi-line graphs:

**Lemma 15.** *Let  $G$  be a quasi-line graph on  $n$  vertices containing a nonlinear homogeneous pair of cliques  $(A, B)$ . In  $O(n^{5/2})$  time we can find a proper subgraph  $G'$  of  $G$  such that  $G'$  is quasi-line,  $\chi(G') = \chi(G)$ , and given a  $k$ -colouring of  $G'$  we can find a  $k$ -colouring of  $G$  in  $O(n^{5/2})$  time.*

It follows immediately that no minimum counterexample to Theorem 4 contains a nonlinear homogeneous pair of cliques.

### 3.4 Decomposing: Clique cutsets

Decomposing graphs on clique cutsets for the purpose of finding vertex colourings is straightforward and well understood.

For any monotone bound on the chromatic number for a hereditary class of graphs, no minimum counterexample can contain a clique cutset, since we can simply “paste together” two partial colourings on a clique cutset. Tarjan [12] gave an  $O(nm)$ -time algorithm for constructing a clique cutset decomposition tree of any graph, and noted that given  $k$ -colourings of the leaves of this decomposition tree, we can construct a  $k$ -colouring of the original graph in  $O(n^2)$  time. Therefore if we can  $\gamma_l(G)$ -colour any quasi-line graph containing no clique cutset in  $O(f(n, m))$  time for some function  $f$ , we can  $\gamma_l(G)$ -colour any quasi-line graph in  $O(f(n, m) + nm)$  time.

If the multigraph  $H$  contains a loop or a vertex of degree 1, then as long as  $G$  is not a clique, it will contain a clique cutset.

### 3.5 Decomposing: Canonical interval 2-joins

A canonical interval 2-join is a composition by which a linear interval graph is attached to another graph. Canonical interval 2-joins arise from compositions of strips, and can be viewed as a local decomposition rather than one that requires knowledge of a graph’s global structure as a composition of strips.

Given four cliques  $X_1, Y_1, X_2,$  and  $Y_2$ , we say that  $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$  is an *interval 2-join* if it satisfies the following:

- $V(G)$  can be partitioned into nonempty  $V_1$  and  $V_2$  with  $X_1 \cup Y_1 \subseteq V_1$  and  $X_2 \cup Y_2 \subseteq V_2$  such that for  $v_1 \in V_1$  and  $v_2 \in V_2$ ,  $v_1 v_2$  is an edge precisely if  $\{v_1, v_2\}$  is in  $X_1 \cup X_2$  or  $Y_1 \cup Y_2$ .
- $G|V_2$  is a linear interval graph with end-cliques  $X_2$  and  $Y_2$ .

If we also have  $X_2$  and  $Y_2$  disjoint, then we say  $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$  is a *canonical interval 2-join*. The following decomposition theorem is a straightforward consequence of the structure theorem for quasi-line graphs:

**Theorem 16.** *Let  $G$  be a quasi-line graph containing no nonlinear homogeneous pair of cliques. Then one of the following holds.*

- $G$  is a line graph
- $G$  is a circular interval graph
- $G$  contains a clique cutset
- $G$  admits a canonical interval 2-join.

Therefore to prove Theorem 4 it only remains to prove that a minimum counterexample cannot contain a canonical interval 2-join. Before doing so we must give some notation and definitions.

We actually need to bound a refinement of  $\gamma_l(G)$ . Given a canonical interval 2-join  $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$  in  $G$  with an appropriate partitioning  $V_1$  and  $V_2$ , let  $G_1$  denote  $G|V_1$ , let  $G_2$  denote  $G|V_2$  and let  $H_2$  denote  $G|(V_2 \cup X_1 \cup Y_1)$ . For  $v \in H_2$  we define  $\omega'(v)$  as the size of the largest clique in  $H_2$  containing  $v$  and not intersecting both  $X_1 \setminus Y_1$  and  $Y_1 \setminus X_1$ , and we define  $\gamma_l^j(H_2)$  as  $\max_{v \in H_2} [d_G(v) + 1 + \omega'(v)]$  (here the superscript  $j$  denotes *join*). Observe that  $\gamma_l^j(H_2) \leq \gamma_l(G)$ . If  $v \in X_1 \cup Y_1$ , then  $\omega'(v)$  is  $|X_1| + |X_2|$ ,  $|Y_1| + |Y_2|$ , or  $|X_1 \cap Y_1| + \omega(G|(X_2 \cup Y_2))$ .

The following lemma is due to King and Reed and first appeared in [5]; we include the proof for the sake of completeness.

**Lemma 17.** *Let  $G$  be a graph on  $n$  vertices and suppose  $G$  admits a canonical interval 2-join  $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$ . Then given a proper  $l$ -colouring of  $G_1$  for any  $l \geq \gamma_l^j(H_2)$ , we can find a proper  $l$ -colouring of  $G$  in  $O(nm)$  time.*

Since  $\gamma_l^j(H_2) \leq \gamma_l(G)$ , this lemma implies that no minimum counterexample to Theorem 4 contains a canonical interval 2-join.

It is easy to see that a minimum counterexample cannot contain a simplicial vertex (i.e. a vertex whose neighbourhood is a clique). Therefore in a canonical interval 2-join  $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$  in a minimum counterexample, all four cliques  $X_1$ ,  $Y_1$ ,  $X_2$ , and  $Y_2$  must be nonempty.

*Proof.* We proceed by induction on  $l$ , observing that the case  $l = 1$  is trivial. We begin by modifying the colouring so that the number  $k$  of colours used in both  $X_1$  and  $Y_1$  in the  $l$ -colouring of  $G_1$  is maximal. That is, if a vertex  $v \in X_1$  gets a colour that is not seen in  $Y_1$ , then every colour appearing in  $Y_1$  appears in  $N(v)$ . This can be done in  $O(n^2)$  time. If  $l$  exceeds  $\gamma_l^j(H_2)$  we can just remove a colour class in  $G_1$  and apply induction on what remains. Thus we can assume that  $l = \gamma_l^j(H_2)$  and so if we apply induction we must remove a stable set whose removal lowers both  $l$  and  $\gamma_l^j(H_2)$ .

We use case analysis; when considering a case we may assume no previous case applies. In some cases we extend the colouring of  $G_1$  to an  $l$ -colouring of  $G$  in one step. In other cases we remove a colour class in  $G_1$  together with vertices in  $G_2$  such that everything we remove is a stable set, and when we remove it we reduce  $\gamma_l^j(v)$  for every  $v \in H_2$ ; after doing this we apply induction on  $l$ . Notice that if  $X_1 \cap Y_1 \neq \emptyset$  and there are edges between  $X_2$  and  $Y_2$  we may have a large clique in  $H_2$  which contains some but not all of  $X_1$  and some but not all of  $Y_1$ ; this is not necessarily obvious but we deal with it in every applicable case.

Case 1.  $Y_1 \subseteq X_1$ .

$H_2$  is a circular interval graph and  $X_1$  is a clique cutset. We can  $\gamma_l(H_2)$ -colour  $H_2$  in  $O(n^{3/2})$  time using Lemma 14. By permuting the colour classes we can ensure that this colouring agrees with the colouring of  $G_1$ . In this case  $\gamma_l(H_2) \leq \gamma_l^j(H_2) \leq l$  so we are done. By symmetry, this covers the case in which  $X_1 \subseteq Y_1$ .

Case 2.  $k = 0$  and  $l > |X_1| + |Y_1|$ .

Here  $X_1$  and  $Y_1$  are disjoint. Take a stable set  $S$  greedily from left to right in  $G_2$ . By this we mean that we start with  $S = \{v_1\}$ , the leftmost vertex of  $X_2$ , and we move along the vertices of  $G_2$  in linear order, adding a vertex to  $S$  whenever doing so will leave  $S$  a stable set. So  $S$  hits  $X_2$ . If it hits  $Y_2$ , remove  $S$  along with a colour class in  $G_1$  not intersecting  $X_1 \cup Y_1$ ; these vertices together make a stable set. If  $v \in G_2$  it is easy to see that  $\gamma_l^j(v)$  will drop: every remaining vertex in  $G_2$  either loses two neighbours or is in  $Y_2$ , in which case  $S$  intersects every maximal clique containing  $v$ . If  $v \in X_1 \cup Y_1$  then since  $X_1$  and  $Y_1$  are disjoint,  $\omega'(v)$  is either  $|X_1| + |X_2|$  or  $|Y_1| + |Y_2|$ ; in either case  $\omega'(v)$ , and therefore  $\gamma_l^j(v)$ , drops when  $S$  and the colour class are removed. Therefore  $\gamma_l^j(H_2)$  drops, and we can proceed by induction.

If  $S$  does not hit  $Y_2$  we remove  $S$  along with a colour class from  $G_1$  that hits  $Y_1$  (and therefore not  $X_1$ ). Since  $S \cap Y_2 = \emptyset$  the vertices together make a stable set. Using the same argument as before we can see that removing these vertices drops both  $l$  and  $\gamma_l^j(H_2)$ , so we can proceed by induction.

Case 3.  $k = 0$  and  $l = |X_1| + |Y_1|$ .

Again,  $X_1$  and  $Y_1$  are disjoint. By maximality of  $k$ , every vertex in  $X_1 \cup Y_1$  has at least  $l - 1$  neighbours in  $G_1$ . Since  $l = |X_1| + |Y_1|$  we know that  $\omega'(X_1) \leq |X_1| + |Y_1| - |X_2|$  and  $\omega'(Y_1) \leq |X_1| + |Y_1| - |Y_2|$ . Thus  $|Y_1| \geq 2|X_2|$  and similarly  $|X_1| \geq 2|Y_2|$ . Assume without loss of generality that  $|Y_2| \leq |X_2|$ .

We first attempt to  $l$ -colour  $H_2 - Y_1$ , which we denote by  $H_3$ , such that every colour in  $Y_2$  appears in  $X_1$  – this is clearly sufficient to prove the lemma since we can permute the colour classes and paste this colouring onto the colouring of  $G_1$  to get a proper  $l$ -colouring of  $G$ . If  $\omega(H_3) \leq l - |Y_2|$  then this is easy: we can  $\omega(H_3)$ -colour the vertices of  $H_3$ , then use  $|Y_2|$  new colours to recolour  $Y_2$  and  $|Y_2|$  vertices of  $X_1$ . This is possible since  $Y_2$  and  $X_1$  have no edges between them.

Define  $b$  as  $l - \omega(H_3)$ ; we can assume that  $b < |Y_2|$ . We want an  $\omega(H_3)$ -colouring of  $H_3$  such that at most  $b$  colours appear in  $Y_2$  but not  $X_1$ . There is some clique  $C = \{v_i, \dots, v_{i+\omega(H_3)-1}\}$  in  $H_3$ ; this clique does not intersect  $X_1$  because  $|X_1 \cup X_2| \leq l - \frac{1}{2}|Y_1| \leq l - |Y_2| < l - b$ . Denote by  $v_j$  the leftmost neighbour of  $v_i$ . Since  $\gamma_l^j(v_i) \leq l$ , it is clear that  $v_i$  has at most  $2b$  neighbours outside  $C$ ,

and since  $b < |Y_2| \leq \frac{1}{2}|X_1|$  we can be assured that  $v_i \notin X_2$ . Since  $\omega(H_3) > |Y_2|$ ,  $v_i \notin Y_2$ .

We now colour  $H_3$  from left to right, modulo  $\omega(H_3)$ . If at most  $b$  colours appear in  $Y_2$  but not  $X_1$  then we are done, otherwise we will “roll back” the colouring, starting at  $v_i$ . That is, for every  $p \geq i$ , we modify the colouring of  $H_3$  by giving  $v_p$  the colour after the one that it currently has, modulo  $\omega(H_3)$ . Since  $v_i$  has at most  $2b$  neighbours behind it, we can roll back the colouring at least  $\omega(H_3) - 2b - 1$  times for a total of  $\omega(H_3) - 2b$  proper colourings of  $H_3$ .

Since  $v_i \notin Y_2$  the colours on  $Y_2$  will appear in order modulo  $\omega(H_3)$ . Thus there are  $\omega(H_3)$  possible sets of colours appearing on  $Y_2$ , and in  $2b + 1$  of them there are at most  $b$  colours appearing in  $Y_2$  but not  $X_1$ . It follows that as we roll back the colouring of  $H_3$  we will find an acceptable colouring.

Henceforth we will assume that  $|X_1| \geq |Y_1|$ .

Case 4.  $0 < k < |X_1|$ .

Take a stable set  $S$  in  $G_2 - X_2$  greedily from left to right. If  $S$  hits  $Y_2$ , we remove  $S$  from  $G$ , along with a colour class from  $G_1$  intersecting  $X_1$  but not  $Y_1$ . Otherwise, we remove  $S$  along with a colour class from  $G_1$  intersecting both  $X_1$  and  $Y_1$ . In either case it is a simple matter to confirm that  $\gamma_l^j(v)$  drops for every  $v \in H_2$  as we did in Case 2. We proceed by induction.

Case 5.  $k = |Y_1| = |X_1| = 1$ .

In this case  $|X_1| = k = 1$ . If  $G_2$  is not connected then  $X_1$  and  $Y_1$  are both clique cutsets and we can proceed as in Case 1. If  $G_2$  is connected and contains an  $l$ -clique, then there is some  $v \in V_2$  of degree at least  $l$  in the  $l$ -clique. Thus  $\gamma_l^j(H_2) > l$ , contradicting our assumption that  $l \geq \gamma_l^j(H_2)$ . So  $\omega(G_2) < l$ . We can  $\omega(G_2)$ -colour  $G_2$  in linear time using only colours not appearing in  $X_1 \cup Y_1$ , thus extending the  $l$ -colouring of  $G_1$  to a proper  $l$ -colouring of  $G$ .

Case 6.  $k = |Y_1| = |X_1| > 1$ .

Suppose that  $k$  is not minimal. That is, suppose there is a vertex  $v \in X_1 \cup Y_1$  whose closed neighbourhood does not contain all  $l$  colours in the colouring of  $G_1$ . Then we can change the colour of  $v$  and apply Case 4. So assume  $k$  is minimal.

Therefore every vertex in  $X_1$  has degree at least  $l + |X_2| - 1$ . Since  $X_1 \cup X_2$  is a clique,  $\gamma_l^j(H_2) \geq l \geq \frac{1}{2}(l + |X_2| + |X_1| + |X_2|)$ , so  $2|X_2| \leq l - k$ . Similarly,  $2|Y_2| \leq l - k$ , so  $|X_2| + |Y_2| \leq l - k$ . Since there are  $l - k$  colours not appearing in  $X_1 \cup Y_1$ , we can  $\omega(G_2)$ -colour  $G_2$ , then permute the colour classes so that no colour appears in both  $X_1 \cup Y_1$  and  $X_2 \cup Y_2$ . Thus we can extend the  $l$ -colouring of  $G_1$  to an  $l$ -colouring of  $G$ .



These cases cover every possibility, so we need only prove that the colouring can be found in  $O(nm)$  time. If  $k$  has been maximized and we apply induction,  $k$  will stay maximized: every vertex in  $X_1 \cup Y_1$  will have every remaining colour in its closed neighbourhood except possibly if we recolour a vertex in Case 6. In this case the overlap in what remains is  $k - 1$ , which is the most possible since we remove a vertex from  $X_1$  or  $Y_1$ , each of which has size  $k$ . Hence we only need to maximize  $k$  once. We can determine which case applies in  $O(m)$  time, and it is not hard to confirm that whenever we extend the colouring in one step our work can be done in  $O(nm)$  time. When we apply induction, i.e. in Cases 2, 4, and possibly 6, all our work can be done in  $O(m)$  time. Since  $l < n$  it follows that the entire  $l$ -colouring can be completed in  $O(nm)$  time.  $\square$

### 3.6 Putting the pieces together

We can now prove an algorithmic version of Theorem 4.

**Theorem 18.** *Let  $G$  be a quasi-line graph on  $n$  vertices and  $m$  edges. Then we can find a proper colouring of  $G$  using  $\gamma(G)$  colours in  $O(n^3m^2)$  time.*

*Proof.* We proceed by induction on  $n$ . As already explained, we need only consider graphs containing no clique cutsets since  $n^3m^2 \geq nm$ . We begin by applying Lemma 15 at most  $m$  times in order to find a quasi-line subgraph  $G'$  of  $G$  such that  $\chi(G) = \chi(G')$ , and given a  $k$ -colouring of  $G'$ , we can find a  $k$ -colouring of  $G$  in  $O(n^2m^2)$  time. We must now colour  $G'$ .

If  $G'$  is a circular interval graph we can determine this and  $\gamma_l(G)$ -colour it in  $O(n^{3/2})$  time. If  $G'$  is a line graph we can determine this in  $O(m)$  time using an algorithm of Roussopoulos [10], then  $\gamma_l(G)$ -colour it in  $O(n^2)$  time. Otherwise,  $G'$  must admit a canonical interval 2-join. In this case Lemma 6.18 in [5], due to King and Reed, tells us that we can find such a decomposition in  $O(n^2m)$  time.

This canonical interval 2-join  $((V_1, X_1, Y_1), (V_2, X_2, Y_2))$  leaves us to colour the induced subgraph  $G_1$  of  $G'$ , which has at most  $n - 1$  vertices and is quasi-line. Given a  $\gamma_l(G)$ -colouring of  $G_1$  we can  $\gamma_l(G)$ -colour  $G'$  in  $O(nm)$  time, then reconstruct the  $\gamma_l(G)$ -colouring of  $G$  in  $O(n^2m^2)$  time. The induction step takes  $O(n^2m^2)$  time and reduces the number of vertices, so the total running time of the algorithm is  $O(n^3m^2)$ .  $\square$

**Remark:** With some care and using more sophisticated results on decomposing quasi-line graphs (see [1]), we believe it should be possible to reduce the running time of the entire algorithm to  $O(m^2)$ .

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