

# Wilson loops in $\mathcal{N} = 4$ $SO(N)$ SYM and D-branes in $AdS_5 \times \mathbb{R}P^5$

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**ABSTRACT:** We study the half-BPS circular Wilson loop in  $\mathcal{N} = 4$  super Yang-Mills with orthogonal gauge group. By supersymmetric localization, its expectation value can be computed exactly from a matrix integral over the Lie algebra of  $SO(N)$ . We focus on the large  $N$  limit and present some simple quantitative tests of the duality with type IIB string theory in  $AdS_5 \times \mathbb{R}P^5$ . In particular, we show that the strong coupling limit of the expectation value of the Wilson loop in the spinor representation of the gauge group precisely matches the classical action of the dual string theory object, which is expected to be a D5-brane wrapping a  $\mathbb{R}P^4$  subspace of  $\mathbb{R}P^5$ . We also briefly discuss the large  $N$ , large  $\lambda$  limits of the  $SO(N)$  Wilson loop in the symmetric/antisymmetric representations and their D3/D5-brane duals. Finally, we use the D5-brane description to extract the leading strong coupling behavior of the “bremsstrahlung function” associated to a spinor probe charge, or equivalently the normalization of the two-point function of the displacement operator on the spinor Wilson loop, and obtain agreement with the localization prediction.

**KEYWORDS:** AdS-CFT Correspondence, D-branes, Matrix Models, Wilson, 't Hooft and Polyakov loops

**ARXIV EPRINT:** [2006.10852](https://arxiv.org/abs/2006.10852)

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**1 Introduction**

In addition to being fundamental observables in gauge theory, Wilson loop operators are also versatile probes of the AdS/CFT correspondence [1–3]. They provide a rather direct manifestation of the gauge/string duality: a Wilson loop in the fundamental representation of the gauge group is dual to a fundamental open string worldsheet ending at the boundary of AdS along the curve defining the loop operator [4, 5]. In the  $\mathcal{N} = 4$  SYM theory, it is natural to consider a generalization of the usual Wilson loop, sometimes referred to as the “Maldacena-Wilson” loop, by including couplings to the adjoint scalar fields of the theory.<sup>1</sup> This allows in particular the construction of loop operators that preserve various fractions of the superconformal symmetry of the theory [10, 11]. The most supersymmetric example is the half-BPS Wilson loop, which is defined on a circular (or straight line) contour and couples to one of the six scalars of SYM theory. Its expectation value for circular contours is a non-trivial function of the coupling and representation of the gauge group that can be computed analytically by supersymmetric localization [12–14]. Such exact results can be used to conduct non-trivial tests of the AdS/CFT duality and of the

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<sup>1</sup>On the dual AdS side, these couplings are reflected in the Dirichlet boundary conditions for the string surface along the directions of the “internal” space corresponding to the scalar fields, i.e.  $S^5$  or  $\mathbb{RP}^5$  for  $\mathcal{N} = 4$  SYM. The ordinary Wilson loop with no scalar couplings corresponds instead to Neumann boundary conditions on the internal space directions [6–9]. In this paper, we will only consider the Maldacena-Wilson loop, and will often refer to it simply as the Wilson loop.

detailed structure of the gauge/string dictionary. In particular, one can make contact with the dynamics of various stringy objects in the dual theory: while Wilson loops in the fundamental representation are described by string worldsheets, large representations with “size”<sup>2</sup> of order  $N$  are dual to D-branes that pinch on the boundary contour [15–17], and even larger representations of order  $N^2$  are dual to new supergravity backgrounds [18–20].

Most of the existing work in the literature focuses on the duality between  $\mathcal{N} = 4$  SYM with  $SU(N)$  gauge group and type IIB string theory on  $AdS_5 \times S^5$ . In this note, our aim is to provide some new quantitative tests of the duality in the case of  $\mathcal{N} = 4$  SYM with orthogonal gauge group. The relevant string theory dual was identified in [21]. In flat space, one can obtain a  $SO(N)$  or  $USp(N)$  gauge theory by placing a stack of D3-branes at an orientifold 3-plane. In the near-horizon limit, one gets string theory on the  $AdS_5 \times \mathbb{RP}^5$  orientifold. Here  $\mathbb{RP}^5 = S^5/\mathbb{Z}_2$ , where the  $\mathbb{Z}_2$  action identifies antipodal points on  $S^5$ , yielding the five-dimensional real projective space. The difference between the  $SO(N)$  and  $USp(N)$  gauge theories lies in a choice of “discrete torsion” for the Neveu-Schwarz and Ramond-Ramond  $B$ -fields [21]. While the theories on  $AdS_5 \times S^5$  and  $AdS_5 \times \mathbb{RP}^5$  share several similarities, interesting new features arise in the latter. For example, the genus expansion in string theory includes non-orientable worldsheets, which corresponds to the well-known fact that in  $SO(N)$  and  $USp(N)$  gauge theories the  $1/N$  expansion includes both even and odd powers. Also, the possibility of wrapping branes on the internal  $\mathbb{RP}^5$  gives rise to new objects that do not have counterparts in the  $AdS_5 \times S^5/SU(N)$  duality. For instance, in the  $SO(2k)$  gauge theory one can construct “Pfaffian” local operators  $Pf(\Phi) = \epsilon^{a_1 \dots a_{2k}} \Phi_{a_1 a_2} \dots \Phi_{a_{2k-1} a_{2k}}$  (where each  $\Phi_{ab}$  is any of the six adjoint scalars of the theory, and we are being schematic by omitting the  $R$ -symmetry indices<sup>3</sup>), which are dual to D3-branes wrapping a  $\mathbb{RP}^3 \subset \mathbb{RP}^5$  [21, 22]. Another peculiarity of gauge theory with orthogonal gauge group, which is more relevant for the present paper, is that one can consider Wilson loops in the *spinor representation* of the gauge group. This is the familiar representation of  $SO(N)$  whose generators are the rank-2 antisymmetric products of the Dirac gamma matrices (it has dimension  $2^{N/2}$  for  $N$  even<sup>4</sup> and  $2^{(N-1)/2}$  for  $N$  odd). It does not have a counterpart in the  $SU(N)$  or  $USp(N)$  theories. As argued in [21], the half-BPS Wilson loop in the spinor representation is expected to be dual to a D5-brane wrapping a  $\mathbb{RP}^4 \subset \mathbb{RP}^5$  and occupying an  $AdS_2$  subspace of  $AdS_5$ . In terms of Young tableaux labels, the spinor representation corresponds to  $\vec{m} = [1/2, 1/2, \dots, 1/2]$ , and hence  $\sum_i m_i \sim N$ . Thus, as in the case of large rank (anti)symmetric representations studied in [15–17], it is natural that the dual object should be a D-brane. The fact that Wilson loops in the spinor representation should be dual to a stringy object with tension of order  $N$  (like a D-brane) was also foreshadowed by earlier work [23].

In this paper, we use supersymmetric localization to obtain a simple quantitative test of the correspondence between the spinor Wilson loop and the  $AdS_2 \times \mathbb{RP}^4$  D5-brane.

<sup>2</sup>By “size” we mean essentially  $\sim \sum_i m_i$ , where the  $m_i$ ’s are the Young tableaux labels specifying the representation.

<sup>3</sup>To obtain a chiral primary operator, one should project onto the symmetric traceless representation of  $SO(6)$ .

<sup>4</sup>More precisely, for  $N$  even it splits into two irreducible representations of opposite chirality.

Although this gauge theory observable has already been studied at finite  $N$  and 't Hooft coupling  $\lambda = g_{\text{YM}}^2 N$  [24], here we focus on the large  $N$ , large  $\lambda$  regime and provide a direct confirmation that the expectation value in the gauge theory matches the classical action of the wrapped D5-brane of [21]. We also consider the fundamental Wilson loop (mainly as a check of numerical factors in the gauge/string dictionary and of the consistency of our conventions for the gauge theory and matrix integral), and Wilson loops in the large (anti)symmetric representations. The latter are expected to be dual to D3 and D5-branes with  $\text{AdS}_2 \times S^2$  and  $\text{AdS}_2 \times S^4$  worldvolumes, similarly to the previously studied examples [15–17] in the  $\text{AdS}_5 \times S^5/\text{SU}(N)$  case.

The study of the half-BPS Wilson loop in  $\mathcal{N} = 4$  SYM and its deformations has recently received renewed attention due to its connection to defect CFT. This is because the circular (or straight) half-BPS loop preserves a  $\text{SL}(2, \mathbb{R})$  1d conformal symmetry and hence can be viewed as a conformal defect [25–28] labelled by the choice of representation for the probe particle running around the loop. The case of the fundamental representation of  $\text{SU}(N)$  has been studied extensively both on the gauge theory and string theory sides (see e.g. [26, 27, 29–31]). Likewise, defect CFT correlators on Wilson loops in (anti)symmetric representations and their duality to fluctuations of the D3 and D5-branes were recently studied in detail in [32]. It would be interesting to perform similar defect CFT analyses for  $\text{SO}(N)$  gauge theory and in particular for the spinor representation, which is a unique feature of orthogonal groups. In this paper, we take a first step towards such a study: we will only focus on the transverse fluctuations of the D5-brane within  $\text{AdS}_5$ , which are dual to the displacement operators of the defect CFT. The normalization of their two-point function is given by the so-called “bremsstrahlung function” [33], which can be computed exactly for any representation using supersymmetric localization. Using the D5-brane and applying the  $\text{AdS}_2/\text{dCFT}_1$  correspondence along the lines of [27], we obtain the leading strong coupling prediction for the spinor bremsstrahlung function, finding agreement with the localization result. We leave a more detailed analysis of the D5-brane fluctuations and their defect CFT dual for future work.

The rest of this paper is organized as follows. Section 2 focuses on the calculation of the circular Wilson loop expectation value in the  $\mathcal{N} = 4$   $\text{SO}(N)$  SYM using supersymmetric localization, which reduces it to a matrix integral over the Lie algebra of  $\text{SO}(N)$ . In section 2.1, we compute the expectation value of the Wilson loop in the fundamental representation using the method of orthogonal polynomials, mainly as a check of our matrix model conventions. In section 2.2, we derive the Wigner semicircle law characterizing the distribution of the eigenvalues at the saddle point of the matrix integral at large  $N$ , and use it to derive the large  $N$  limit of the Wilson loop in the spinor representation in section 2.3 and in the large rank (anti)symmetric representations in section 2.4. In section 3, we focus on the dual string theory description. After a brief review of the  $\text{AdS}_5 \times \mathbb{RP}^5$  dual and of the matching between fundamental representation and fundamental string, we show in section 3.2 that the value of the classical action of the  $\text{AdS}_2 \times \mathbb{RP}^4$  D5-brane precisely agrees with the gauge theory calculation in section 2.3. In section 3.3, we describe the D3/D5-branes dual to the Wilson loop in the large rank symmetric and antisymmetric representations. Section 4 is devoted to the two point function of the displacement opera-

tors in the 1d defect CFT defined by the spinor Wilson loop. We calculate the two-point function using both supersymmetric localization and the AdS<sub>2</sub> theory of fluctuations of the wrapped D5-brane, and find agreement. Finally, in section 5, we summarize our results and suggest possible extensions to our work.

## 2 Wilson loops in $\mathcal{N} = 4$ SO( $N$ ) SYM

We begin this section by introducing the SO( $N$ ) half-BPS Maldacena-Wilson loop and the matrix model to which it is equivalent via supersymmetric localization. We then apply the saddle point method to the matrix integral to determine the expectation value of the large  $N$  Wilson loop in various representations, which we will compare to the classical actions of the corresponding strings and D-branes in AdS<sub>5</sub> × ℝP<sup>5</sup> in section 3.

First, we introduce our conventions. We normalize the gauge and scalar fields of  $\mathcal{N} = 4$  SYM so that their kinetic terms take the following standard form:

$$S_{\text{SYM}} = \frac{1}{g_{\text{YM}}^2} \int d^4x \text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu \Phi^I)^2 + \dots \right\}. \tag{2.1}$$

Here  $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu]$  is the gauge field strength,  $A_\mu = A_\mu^a T^a$  is the gauge field,  $\Phi^I = \Phi_a^I T^a$  are the six scalar fields, and  $T^a$  are the generators of the Lie algebra  $\mathfrak{g}$  satisfying  $\text{Tr}(T^a T^b) = C(F) \delta^{ab}$  (the Lie algebra indices  $a, b$  run from 1 to  $\dim(\mathfrak{g})$ ). Here  $C(F)$  corresponds to a choice of normalization for the generators in the fundamental representation. A conventional value is  $C(F) = 1/2$ , but we will keep it arbitrary below (nothing will depend on the choice of  $C(F)$ ). Both the gauge field and the scalar fields transform in the adjoint representation of the gauge group,  $G$ .

The Maldacena-Wilson loop is defined to be the following path-ordered exponential

$$W_R = \text{Tr}_R \mathcal{P} \left[ \oint \left( i A_\mu(x) \dot{x}^\mu + \Theta_I \Phi^I(x) |\dot{x}| \right) dt \right], \tag{2.2}$$

where  $x^\mu(t)$  is a closed path in  $\mathbb{R}^4$ , and  $\Theta^I(t)$  is a unit 6-vector (i.e.,  $\Theta^I \Theta^I = 1$ ), and  $R$  is a choice of representation characterizing the external probe particle propagating along the contour. Our primary focus will be on the spinor representation of  $G = \text{SO}(N)$ , but we will also discuss the fundamental, rank  $k$  symmetric and rank  $k$  antisymmetric representations.

We restrict our attention to the half-BPS Wilson loops, for which  $x^\mu(t)$  is a circle and  $\Theta^I(t)$  is a constant, corresponding to choosing one of the six scalars. The operator in eq. (2.2) then preserves half of the supersymmetry.<sup>5</sup> By supersymmetric localization [14], the expectation value of the half-BPS Wilson loop for a given gauge group  $G$  reduces to an integral over the elements of the Lie algebra  $\mathfrak{g}$  of  $G$ ,

$$\langle W_R \rangle = \frac{1}{Z} \int_{\mathfrak{g}} \mathcal{D}K \text{Tr}_R(e^K) \exp \left( -\frac{2}{g_{\text{YM}}^2} \text{Tr} K^2 \right), \tag{2.3}$$

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<sup>5</sup>The straight Wilson line is also half-BPS, but its expectation value is equal to the representation dimension for any representation and coupling.

where  $K = K^a T^a$  are Lie algebra elements, and  $Z = \int \mathcal{D}K \exp\left(-\frac{2}{g_{\text{YM}}^2} \text{Tr} K^2\right)$  is the partition function of the matrix model. We can check that the normalization of the action in (2.3) agrees with the gauge theory conventions in eq. (2.1) by computing  $\langle W_R \rangle$  perturbatively. Noting that  $\text{Tr} K^2 = K^a K^b \text{Tr}(T^a T^b) = C(F) K^a K^a$ , so that the propagator is  $\langle K^a K^b \rangle = \frac{g_{\text{YM}}^2}{4C(F)} \delta^{ab}$ , one finds

$$\langle W_R \rangle = \dim(R) + \frac{g_{\text{YM}}^2}{8C(F)} C_2(R) \dim(R) + \dots \tag{2.4}$$

where the quadratic Casimir is defined by  $T_R^a T_R^a = C_2(R) \mathbf{1}_{\dim(R) \times \dim(R)}$ . To compare with the perturbative expansion of the Wilson loop in the SYM theory, we expand the exponential in eq. (2.2) and use the gauge and scalar field propagators. The latter follow from the action, eq. (2.1), which in the Feynman gauge gives:

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = \frac{g_{\text{YM}}^2}{8\pi^2 C(F)} \frac{\delta_{\mu\nu} \delta^{ab}}{|x-y|^2}, \quad \langle \Phi^{Ia}(x) \Phi^{Jb}(y) \rangle = \frac{g_{\text{YM}}^2}{8\pi^2 C(F)} \frac{\delta^{ab} \delta^{IJ}}{|x-y|^2}. \tag{2.5}$$

For the circular contour  $x^\mu(t) = (\cos t, \sin t, 0, 0)$  the ‘‘combined’’ gauge field and scalar propagator is a constant

$$-\langle A_\mu^a(x(t)) A_\nu^b(x(s)) \rangle \dot{x}^\mu(t) \dot{x}^\nu(s) + \langle \Theta_I \Phi^{Ia}(x(t)) \Theta_J \Phi^{Jb}(x(s)) \rangle |\dot{x}(t)| |\dot{x}(s)| = \frac{g_{\text{YM}}^2 \delta^{ab}}{16\pi^2 C(F)} \tag{2.6}$$

and one finds

$$\begin{aligned} \langle W_R \rangle &= \dim(R) + \frac{1}{2} \int_0^{2\pi} dt \int_0^{2\pi} ds \text{Tr}_R \{ \langle (iA_\mu \dot{x}^\mu + \Theta_I \Phi^I |\dot{x}|)^2 \rangle \} + O(g_{\text{YM}}^3) \\ &= \dim(R) + \frac{g_{\text{YM}}^2}{8C(F)} C_2(R) \dim(R) + \dots \end{aligned} \tag{2.7}$$

in agreement with eq. (2.4).

The discussion so far applies to arbitrary gauge group. Let us now specialize to the case of  $\text{SO}(N)$ , which is the main focus of this paper. In this case, the Lie algebra consists of the antisymmetric Hermitian  $N \times N$  matrices  $K_{jk} = K_{kj}^* = -K_{kj}$  and the measure in the matrix model may be written as  $\mathcal{D}K = \prod_{j < k} d(-iK_{jk})$ . Below, we review some of the relevant tools to analyze the large  $N$  limit of the matrix model (see e.g. [34–36] for general discussions of random matrix integral techniques). We will use  $\lambda = g_{\text{YM}}^2 N$  to denote the ’t Hooft coupling in the  $\text{SO}(N)$  gauge theory, which is held fixed as  $N \rightarrow \infty$ .<sup>6</sup>

Antisymmetric Hermitian matrices can be block-diagonalized using orthogonal matrices [37]. Since the traces and the measure in eq. (2.3) are invariant under  $K \rightarrow OKO^T$  if  $OO^T = I$ , conjugation by orthogonal matrices is a gauge symmetry of eq. (2.3) that can be fixed using the Faddeev-Popov procedure. The resulting Faddeev-Popov determinant, denoted  $\Delta^2(\Gamma)$ , can alternatively be viewed as the Jacobian for a change of variables from

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<sup>6</sup>An alternative definition of  $\lambda$  is to let it be  $g_{\text{YM}}^2$  times the rank of  $\text{SO}(N)$ . In that case,  $\lambda = g_{\text{YM}}^2 \frac{N}{2}$  for  $N$  even and  $\lambda = g_{\text{YM}}^2 \frac{N-1}{2}$  for  $N$  odd. We will not adopt this definition in this paper.

$K$  to  $OO^T$ ,  $O$  being orthogonal and  $\Gamma$  being block-diagonal. The resulting integral over the eigenvalues  $\gamma_i$  is slightly different for the even ( $N = 2N'$ ) and odd ( $N = 2N' + 1$ ) cases. One finds (see e.g. [24, 38, 39]):

$$\langle W_R \rangle = \frac{1}{Z'} \int \left( \prod_{i=1}^{N'} d\gamma_i \right) \Delta^2(\Gamma) \text{Tr}_R(e^\Gamma) \exp\left(-\frac{2N}{\lambda} \text{Tr}\Gamma^2\right), \quad (2.8)$$

where

$$\Gamma = \text{diag}(\gamma_1\sigma_y, \gamma_2\sigma_y, \dots, \gamma_{N'}\sigma_y), \quad \Delta^2(\Gamma) = \prod_{1 \leq j < k \leq N'} |\gamma_j^2 - \gamma_k^2|^2 \quad (2.9)$$

if  $N = 2N'$ , and where

$$\Gamma = \text{diag}(\gamma_1\sigma_y, \gamma_2\sigma_y, \dots, \gamma_{N'}\sigma_y, 0), \quad \Delta^2(\Gamma) = \prod_{1 \leq j < k \leq N'} |\gamma_j^2 - \gamma_k^2|^2 \prod_{l=1}^{N'} \gamma_l^2 \quad (2.10)$$

if  $N = 2N' + 1$ . In eqs. (2.9) and (2.10),  $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$  is the second Pauli matrix. In eq. (2.8) the normalization factor  $Z'$  is set equal to the integral expression in eq. (2.8) with  $\text{Tr}_R(e^\Gamma)$  replaced by 1. This ensures that the expectation value of the Wilson loop is equal to the dimension of the representation in the free limit (i.e.,  $\langle W_R \rangle|_{\lambda \rightarrow 0} = \dim R$ ). For later use, we also note that  $\text{Tr}\Gamma^2 = 2 \sum_{i=1}^{N'} \gamma_i^2$  and the eigenvalues of  $e^\Gamma$  are  $\{e^{\pm\gamma_i} | i = 1, \dots, N'\}$  for  $N = 2N'$  and  $\{1\} \cup \{e^{\pm\gamma_i} | i = 1, \dots, N'\}$  for  $N = 2N' + 1$ .

One can in principle use the method of orthogonal polynomials to compute eq. (2.8) exactly for any representation  $R$  and finite  $N$ . The product  $\prod_{i < j} (\gamma_i^2 - \gamma_j^2)$  appearing in  $\Delta(\Gamma)$  is the determinant of the Vandermonde matrix with entries  $V_{ij} = (\gamma_i^2)^{j-1}$ . Using the column addition identity of the determinant, one can write  $|V_{ij}| = |P_j(\gamma_i)|$ , where  $P_n(x)$  is any even monomial in  $x$  of order  $2(n-1)$ . The integral in eq. (2.8) simplifies if we pick the polynomials  $P_n(x)$  to be orthogonal when integrated with the weight  $e^{-\frac{4N}{\lambda}x^2}$  for  $N = 2N'$  and the weight  $e^{-\frac{4N}{\lambda}x^2}$  for  $N = 2N' + 1$ . The orthogonality helps separate the integrals over the different eigenvalues. The right choices for the even and odd cases are  $P_n(x) \propto H_{2n-2}(2\sqrt{N/\lambda}x)$  and  $P_n(x) \propto \frac{1}{x}H_{2n-1}(2\sqrt{N/\lambda}x)$ , respectively, where  $H_n(x)$  is the  $n$ th Hermite polynomial. More details can be found in [24].

In this paper we are mainly interested in the large  $N$  limit, and to analyze higher representations in this limit it will be more convenient to use the saddle point method and corresponding density of eigenvalues. Before doing so, in the next section we will briefly discuss the evaluation of the Wilson loop in the fundamental representation using the orthogonal polynomial method. This will serve as a consistency check of the large  $N$  eigenvalue distribution we will derive in section 2.2.

## 2.1 Fundamental representation

We now apply the method of orthogonal polynomials to determine the expectation value of the Wilson loop in the fundamental representation, which we denote  $R = F$ . In this representation,  $\text{Tr}(e^\Gamma) = 2 \sum_{i=1}^{N'} \cosh(\gamma_i)$  and  $\text{Tr}(e^\Gamma) = 1 + 2 \sum_{i=1}^{N'} \cosh(\gamma_i)$  in the even and



odd cases, respectively. Because  $\text{Tr}(e^\Gamma)$  in eq. (2.8) depends additively on the  $\cosh(\gamma_i)$ , the Wilson loop expectation value reduces to a sum of single integrals involving Hermite polynomials that can be evaluated in terms of associated Laguerre polynomials. Specifically:

$$\begin{aligned} \langle W_F \rangle &= \frac{2}{\sqrt{\pi}} \sum_{i=0}^{N'-1} \frac{1}{(2i)!2^{2i}} \int dx H_{2i}^2(x) \cosh\left(\sqrt{\frac{\lambda}{N}} \frac{x}{2}\right) e^{-x^2} \\ &= 2e^{\frac{\lambda}{16N}} \sum_{i=0}^{N'-1} L_{2i}\left(-\frac{\lambda}{8N}\right) \end{aligned} \quad (2.11)$$

for  $N = 2N'$ , and

$$\begin{aligned} \langle W_F \rangle &= 1 + \frac{2}{\sqrt{\pi}} \sum_{i=0}^{N'-1} \frac{1}{(2i+1)!2^{2i+1}} \int dx H_{2i+1}^2(x) \cosh\left(\sqrt{\frac{\lambda}{N}} \frac{x}{2}\right) e^{-x^2} \\ &= 1 + 2e^{\frac{\lambda}{16N}} \sum_{i=0}^{N'-1} L_{2i+1}\left(-\frac{\lambda}{8N}\right) \end{aligned} \quad (2.12)$$

for  $N = 2N' + 1$ . These expressions agree with those in [24]. Using a contour integral representation of the Laguerre polynomials,<sup>7</sup> it is not difficult to determine the first few terms in the  $1/N$  expansion (with  $\lambda$  fixed) of the circular Wilson loop expectation value directly from eqs. (2.11) and (2.12). For both the even and odd cases,

$$\frac{1}{N} \langle W_F \rangle = \frac{2\sqrt{2}}{\sqrt{\lambda}} I_1\left(\sqrt{\frac{\lambda}{2}}\right) + \frac{1 - I_0\left(\sqrt{\frac{\lambda}{2}}\right)}{2N} + \frac{\lambda I_2\left(\sqrt{\frac{\lambda}{2}}\right)}{96N^2} + O(1/N^3). \quad (2.13)$$

We can compare the small  $g_{\text{YM}}$  expansion of eq. (2.13), with the perturbative expansion in eq. (2.4). When  $R = F$  is the fundamental representation of  $G = \text{SO}(N)$  with  $\dim(R = F) = N$ , the quadratic Casimir is  $C_2(F) = C(F)(N-1)/2$  (see the appendix), so that (2.4) yields  $N^{-1} \langle W_F \rangle = 1 + g_{\text{YM}}^2(N-1)/16 + O(g_{\text{YM}}^4)$ . This agrees with the small  $g_{\text{YM}}$  expansion of eq. (2.13).

## 2.2 Large $N$ saddle point of the $\text{SO}(N)$ matrix model

Computing the large  $N$  behavior of the Wilson loop, eq. (2.8), via the orthogonal polynomial method is more cumbersome for higher representations. In the present section, we use the saddle point method and obtain the density of eigenvalues (given essentially by Wigner's semicircle law) for the  $\text{SO}(N)$  matrix integral. Our analysis is similar to the one in [40].

Consider the large  $N'$  regime of the following  $N'$ -dimensional integral

$$\langle f \rangle = \frac{1}{Z'} \int \left( \prod_{i=1}^{N'} d\gamma_i \right) f(\gamma_1, \dots, \gamma_{N'}) e^{-N'^2 U(\gamma_1, \dots, \gamma_{N'})}, \quad (2.14)$$

<sup>7</sup>One may use  $L_n(x) = (2\pi i)^{-1} \oint dt (t+x)^n t^{-n-1} e^{-t}$ , and expand the sum of Laguerre polynomials as a series in  $1/N$ . This leads to contour integrals of the form  $(2\pi i)^{-1} \oint du e^{-\sqrt{\lambda/8}(u+u^{-1})} u^{-n}$ , which give Bessel functions.



where the normalization  $Z'$  is chosen so that  $\langle 1 \rangle = 1$  and  $U = -\frac{1}{N'^2} \log \left( \Delta^2(\Gamma) e^{-\frac{2N}{\lambda} \text{Tr} \Gamma^2} \right)$  is the logarithm of the measure of eq. (2.8). Let us start with the even  $N$  case, i.e.  $N = 2N'$ . Then we have

$$U(\gamma_1, \dots, \gamma_{N'}) = \frac{8}{\lambda N'} \sum_{i=1}^{N'} \gamma_i^2 - \frac{2}{N'^2} \sum_{1 \leq j < k \leq N'} \log |\gamma_j^2 - \gamma_k^2|. \quad (2.15)$$

We can restrict our attention to functions of the form  $f(\gamma_1, \dots, \gamma_{N'}) = g(\sum_{i=1}^{N'} h(\gamma_i))$ , which is general enough for eq. (2.14) to encompass eq. (2.8) as a special case.

Eq. (2.14) is dominated in the  $N' \rightarrow \infty$  limit by the eigenvalue configurations that minimize the “potential”  $U(\gamma_1, \dots, \gamma_{N'})$ . The invariance of the potential under  $\gamma_i \rightarrow -\gamma_i$  for any  $i = 1, 2, \dots, N'$  means that there are  $2^{N'}$  distinct configurations (or  $2^{N'-1}$  if one of the eigenvalues is zero) that minimize  $U$ . (We do not consider configurations related by permutations of the eigenvalues to be distinct). The degeneracy of the minimum means the minimizing eigenvalues are not specified by a single distribution. By contrast, the distribution of the squares of the eigenvalues, which are also invariant under  $\gamma_i \rightarrow -\gamma_i$ , is the same for all  $\sim 2^{N'}$  configurations. This unique distribution approaches a smooth function,  $\sigma(\gamma^2)$ , in the large  $N'$  limit.

Our task is to find  $\sigma(\gamma^2)$ . The eigenvalues minimizing  $U$  satisfy  $\partial U / \partial \gamma_i = 0$ , which yields:

$$\frac{4\gamma_i}{\lambda} = \frac{\gamma_i}{N'} \sum_{j \neq i} \frac{1}{\gamma_i^2 - \gamma_j^2}, \quad \text{for } i = 1, \dots, N'. \quad (2.16)$$

One must not cancel  $\gamma_i$  from both sides because one of the eigenvalues may be zero (see footnote 8). Instead, we multiply both sides of eq. (2.16) by  $\gamma_i / (z - \gamma_i^2) / z$  and sum over  $i$ . Writing  $\gamma_i^2 / (z - \gamma_i^2) / z = -1/z + 1 / (z - \gamma_i^2)$ , the left hand side yields two terms and, since  $\sum_i \sum_{j \neq i} (\gamma_i^2 - \gamma_j^2)^{-1} = 0$ , the right hand side simplifies. It follows that:

$$\frac{4}{\lambda} \left( -\frac{1}{z} + G(z) \right) = \frac{1}{N'^2} \sum_i \frac{1}{z - \gamma_i^2} \sum_{j \neq i} \frac{1}{\gamma_i^2 - \gamma_j^2}, \quad (2.17)$$

where we have introduced the resolvent,  $G(z)$ :

$$G(z) = \frac{1}{N'} \sum_{i=1}^{N'} \frac{1}{z - \gamma_i^2}. \quad (2.18)$$

The resolvent is analytic on  $\mathbb{C} \setminus [0, a]$ , where  $a \propto \lambda$  is the supremum of the squares of the eigenvalues in the large  $N'$  limit. It satisfies  $\text{Tr}(x - \Gamma)^{-1} = 2xN'G(x^2)$ , which essentially makes  $\langle G(z) \rangle$  a generating function for  $\langle \text{Tr}(\Gamma^n) \rangle$ . Assuming the squares of the eigenvalues at the minimum are characterized by a smooth distribution  $\sigma(\gamma^2)$  in the large  $N'$  limit,  $G(z)$  may be written:

$$G(z) = \int_0^a \frac{\sigma(u) du}{z - u}. \quad (2.19)$$

The residue theorem then implies

$$\sigma(u) = \frac{1}{2\pi i} [G(u + i\epsilon) - G(u - i\epsilon)], \quad (2.20)$$

which means  $G(z)$  determines  $\sigma(u)$  and vice versa.

To find  $\sigma(u)$ , we derive a quadratic equation for  $G(z)$  using eq. (2.17). The square of the resolvent can be written:

$$\begin{aligned} G^2(z) &= \frac{1}{N'^2} \sum_{i=1}^{N'} \frac{1}{(z - \gamma_i^2)^2} + \frac{1}{N'^2} \sum_{i=1}^{N'} \sum_{j=1, j \neq i}^{N'} \frac{1}{z - \gamma_i^2} \frac{1}{z - \gamma_j^2} \\ &= -\frac{1}{N'} G'(z) + \frac{2}{N'^2} \sum_{i=1}^{N'} \frac{1}{z - \gamma_i^2} \sum_{j=1, j \neq i}^{N'} \frac{1}{\gamma_i^2 - \gamma_j^2}. \end{aligned} \quad (2.21)$$

Applying eq. (2.17) yields the desired equation:

$$G^2(z) = -\frac{1}{N'} G'(z) + \frac{8}{\lambda} \left[ G(z) - \frac{1}{z} \right]. \quad (2.22)$$

In the large  $N'$  limit, the first term goes to zero and the remaining quadratic equation in  $G(z)$  has the simple solution:

$$G(z) = \frac{4}{\lambda} - \frac{4}{\lambda} \sqrt{1 - \frac{\lambda}{2z}}, \quad (2.23)$$

where we have chosen the root that satisfies  $G(z) \rightarrow 0$  for  $z \rightarrow \infty$ . Eqs. (2.23) and (2.20) together then yield:

$$\sigma(u) = \frac{4}{\pi\lambda} \sqrt{\frac{\lambda}{2u} - 1}, \quad (2.24)$$

with  $a = \frac{\lambda}{2}$ . This result can be put into a more familiar form if we consider the expectation value of a function of the squares of the eigenvalues,  $f(u)$ . It is given by:

$$\langle f \rangle = \int_0^{\frac{\lambda}{2}} f(u) \sigma(u) du = \int_0^{\sqrt{\frac{\lambda}{2}}} f(x^2) \sigma(x^2) 2x dx = \int_{-\sqrt{\frac{\lambda}{2}}}^{\sqrt{\frac{\lambda}{2}}} f(x^2) \sigma(x^2) |x| dx. \quad (2.25)$$

This lets us identify

$$\rho(x) = |x| \sigma(x^2) = \frac{4}{\pi\lambda} \sqrt{\frac{\lambda}{2} - x^2} \quad (2.26)$$

as the distribution characterizing (the even moments of) the eigenvalues at the minimum of  $U$ . This is the well-known Wigner semicircle law with radius  $\sqrt{\lambda/2}$ .

We have focused on the saddle point of  $\text{SO}(N)$  in the even (i.e.,  $N = 2N'$ ) case, but the odd (i.e.,  $N = 2N' + 1$ ) case yields a similar analysis and the same eigenvalue density distribution at large  $N'$ . In particular, the eigenvalue potential energy in eq. (2.15) picks up the additional term  $-\frac{1}{N'^2} \sum_{i=1}^{N'} \log(\gamma_i^2)$  due to the factor  $\prod_i \gamma_i^2$  in the measure in eq. (2.10).

Consequently, no eigenvalue  $\gamma_i$  can sit at the origin in the large  $N'$  limit, and eq. (2.16) is replaced by<sup>8</sup>

$$\frac{4}{\lambda} = \frac{1}{N'} \sum_{j \neq i} \frac{1}{\gamma_i^2 - \gamma_j^2} + \frac{1}{2N'\gamma_i^2}, \quad \text{for } i = 1, \dots, N'. \quad (2.27)$$

Although the details are slightly different, we can use eq. (2.27) to derive a quadratic equation for the resolvent for the odd case just like in the even case, and the result is:

$$G^2(z) = -\frac{1}{N'} \left[ G'(z) + \frac{G(z)}{z} \right] + \frac{8}{\lambda} \left[ G(z) - \frac{1}{z} \right]. \quad (2.28)$$

Unsurprisingly,  $G(z)$  and  $\sigma(u)$  have the same large  $N'$  behavior for the even and odd cases.

Having arrived at the eigenvalue density distribution, eq. (2.26), we can simplify the expectation value in eq. (2.14) for any function of the form  $f = g\left(\frac{1}{N'} \sum_{i=1}^{N'} h(\gamma_i^2)\right)$ . As long as the large  $N'$  behavior of  $f$  is subleading compared to  $e^{-N'^2 U}$ , the leading term in the asymptotic  $1/N'$  expansion is:

$$\left\langle g\left(\frac{1}{N'} \sum_{i=1}^{N'} h(\gamma_i^2)\right) \right\rangle \sim g\left(\int \rho(x) h(x^2) dx\right), \quad N' \rightarrow \infty. \quad (2.29)$$

We must emphasize that the eigenvalue distribution in eq. (2.26) does not correctly characterize the odd moments of the eigenvalues at the saddle points. Therefore, eq. (2.29) cannot be generalized by replacing  $h(\gamma_i^2)$  and  $h(x^2)$  by  $h(\gamma_i)$  and  $h(x)$  for any even or odd function  $h(x)$ .

Derivations of the semicircle distribution of the eigenvalues of the random antisymmetric Hermitian matrix can be found for instance in [38–40]. The derivation in [40] also implements the saddle point method and derives a “loop equation” à la eq. (2.22) for a resolvent closely related to our  $G(z)$  in eq. (2.18). On the other hand, the derivation in [38] proceeds by evaluating  $N' - 1$  of the integrals in eq. (2.14) (excluding, say, the  $\gamma_1$  integral) using the method of orthogonal polynomials. The resulting expression for the distribution of  $\gamma_1$  involves a sum of Hermite polynomials whose leading term in the asymptotic expansion in  $1/N'$  yields the semicircle distribution.

As a check of the semicircle law (in particular of factors of 2), we note that eq. (2.29) implies that the expectation value of the  $SO(N)$  Wilson loop in the fundamental representation is (choosing even  $N$  for simplicity, but the large  $N$  limit in the odd case is identical):

$$\frac{1}{N} \langle W_F \rangle = \left\langle \frac{2}{N} \sum_{i=1}^{N/2} \cosh \gamma_i \right\rangle = \frac{4}{\pi\lambda} \int_{-\sqrt{\lambda/2}}^{\sqrt{\lambda/2}} dx \sqrt{\frac{\lambda}{2} - x^2} \cosh x = \frac{2\sqrt{2}}{\sqrt{\lambda}} I_1 \left( \sqrt{\frac{\lambda}{2}} \right). \quad (2.30)$$

This is in precise agreement with eq. (2.13) at leading order at large  $N$ .

<sup>8</sup>Eqs. (2.16) and (2.27) are more similar than they first appear. One of the eigenvalues in eq. (2.16) (say,  $\gamma_{N'}$ ) is zero, since otherwise we may divide both sides by  $\gamma_i$  and sum over  $i$  to yield  $4N'/\lambda = 0$ , a contradiction. Separating the zero eigenvalue from the others, eq. (2.16) reduces to

$$\frac{4}{\lambda} = \frac{1}{N'} \sum_{j=1, j \neq i}^{N'-1} \frac{1}{\gamma_i^2 - \gamma_j^2} + \frac{1}{N'\gamma_i^2}, \quad \text{for } i = 1, \dots, N' - 1.$$

Up to a factor of 1/2 in the second term on the right hand side, this is the same as eq. (2.27).

### 2.3 Spinor representation

The representation that primarily interests us in the present paper is the spinor representation of  $SO(N)$ , which we denote  $R = sp$ . Using the Weyl character formulas for  $SO(2N')$  and  $SO(2N' + 1)$  (see e.g. [41]), one finds

$$\text{Tr}_{\text{sp}\pm}(e^\Gamma) = 2^{N'-1} \left( \prod_{i=1}^{N'} \cosh\left(\frac{\gamma_i}{2}\right) \pm \prod_{i=1}^{N'} \sinh\left(\frac{\gamma_i}{2}\right) \right) \quad (2.31)$$

when  $N = 2N'$  and

$$\text{Tr}_{\text{sp}}(e^\Gamma) = 2^{N'} \prod_{i=1}^{N'} \cosh\left(\frac{\gamma_i}{2}\right) \quad (2.32)$$

when  $N = 2N' + 1$ . The two choices of sign for even  $N$  correspond to the chiral and antichiral (Weyl spinor) representations of  $SO(2N)$ . One may also consider a non-chiral (Dirac spinor) representation by taking the sum of the two Weyl representations with opposite chirality.

The method of orthogonal polynomials reduces the Wilson loop expectation value in the spinor representation to a determinant of an  $N' \times N'$  matrix of associated Laguerre polynomials:  $\langle W_{\text{sp}} \rangle = \dim(sp) \times |D_{ij}|$ , where

$$\dim(sp) = 2^{N'-1}, \quad D_{ij} = L_{2i-2}^{2j-2i} \left( -\frac{\lambda}{32N} \right) e^{\frac{\lambda}{64N}}, \quad \text{for } N = 2N' \text{ and} \quad (2.33)$$

$$\dim(sp) = 2^{N'}, \quad D_{ij} = L_{2i-1}^{2j-2i} \left( -\frac{\lambda}{32N} \right) e^{\frac{\lambda}{64N}}, \quad \text{for } N = 2N' + 1. \quad (2.34)$$

We cite these results, derived in [24], to illustrate that extracting the large  $N$  limit of the Wilson loop expectation value in the spinor representation directly using the orthogonal polynomial method is more involved than in the case of the fundamental representation. Therefore, we turn to the saddle point method and apply eq. (2.29) to eq. (2.8), given the characters in eqs. (2.31) and (2.32).

Without loss of generality, let us examine the odd case,  $N = 2N' + 1$ . We find:

$$\frac{1}{2^{N'}} \langle W_{\text{sp}} \rangle = \frac{1}{2^{N'} Z'} \int \left( \prod_{i=1}^{N'} d\gamma_i \right) 2^{N'} \prod_{j=1}^{N'} \left( \cosh \frac{\gamma_j}{2} \right) e^{-N'^2 U(\gamma_1, \dots, \gamma_{N'})} \quad (2.35)$$

$$\sim \exp \left( N' \int_{-\sqrt{\lambda/2}}^{\sqrt{\lambda/2}} \rho(x) \log \left( \cosh \left( \frac{x}{2} \right) \right) dx \right). \quad (2.36)$$

To get to the second line, we rewrote  $\prod_{i=1}^{N'} \cosh(\frac{\gamma_i}{2}) = \exp \left( N' \times \frac{1}{N'} \sum_{i=1}^{N'} \log \left( \cosh \left( \frac{\gamma_i}{2} \right) \right) \right)$  and applied eq. (2.29). Substituting the explicit form for  $\rho(x)$  from eq. (2.26), we find that the expectation value of the spinor Wilson loop in the large  $N$  limit is given by

$$\frac{1}{2^{N'}} \langle W_{\text{sp}} \rangle \sim \exp \left[ N' \frac{4}{\pi} \int_0^1 \sqrt{1-u^2} \log \left( \cosh \left( \sqrt{\frac{\lambda}{8}} u \right) \right) du \right] \quad (2.37)$$

This expression is valid at finite  $\lambda$ . We may expand it at small  $\lambda$  using  $\log(\cosh(x)) = x^2/2 - x^4/12 + \dots$  for small  $x$ , from which we get

$$\frac{1}{N} \log \left[ \frac{1}{2^{N'}} \langle W \rangle_{\text{sp}} \right] = \left( \frac{\lambda}{128} - \frac{\lambda^2}{12288} + O(\lambda^3) \right) + O(1/N). \quad (2.38)$$

We can compare this with the leading order result from perturbation theory, eq. (2.4). For  $G = \text{SO}(N)$ , the spinor quadratic Casimir is  $C_2(sp) = C(F) \frac{N(N-1)}{16}$  (see the appendix). Keeping only the leading order terms in  $g_{\text{YM}}$  and  $1/N$ , we thus find from (2.4)  $\langle W \rangle_{\text{sp}} / \dim(sp) = 1 + N\lambda/128(1 + O(1/N)) + O(g_{\text{YM}}^4)$ . This matches the expansion in eq. (2.38) in the regime where  $\lambda \ll 1$ ,  $N \gg 1$  and  $N\lambda \ll 1$  (despite the different order of limits potentially yielding, a priori, different results).

Extracting the leading terms of the expansion of eq. (2.37) at strong coupling involves a little more work. First, we write  $\log \left( \cosh \left( \sqrt{\frac{\lambda}{8}} u \right) \right) = \sqrt{\frac{\lambda}{8}} u - \log 2 + \log \left( 1 + e^{-\sqrt{\frac{\lambda}{2}} u} \right)$ . The integrals over the first two terms can be evaluated explicitly. To expand the integral of the third term in  $1/\sqrt{\lambda}$ , we use  $\log(1 + e^{-\sqrt{\frac{\lambda}{2}} u}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n\sqrt{\frac{\lambda}{2}} u}$ ,  $\sqrt{1-u^2} = 1 - u^2/2 - u^4/8 + \dots$ , and  $\int_0^1 u^m e^{-au} du = \Gamma(m+1) a^{-m-1} + O(e^{-a})$ . Combining these analytic results, we find:

$$\log \left[ \frac{1}{2^{N'}} \langle W \rangle_{\text{sp}} \right] = N \left( \frac{1}{3\pi} \sqrt{\frac{\lambda}{2}} - \frac{1}{2} \log 2 + \frac{\pi}{6} \sqrt{\frac{2}{\lambda}} + O(\lambda^{-3/2}, e^{-\sqrt{\lambda/2}}) \right) + O(N^0). \quad (2.39)$$

We are especially interested in the leading term in the large  $N$ , large  $\lambda$  expansion of the spinor  $\text{SO}(N)$  Wilson loop, which is given by

$$\frac{1}{\dim(sp)} \langle W_{\text{sp}} \rangle \sim \exp \left( \frac{N}{3\pi} \sqrt{\frac{\lambda}{2}} \right). \quad (2.40)$$

We will compare eq. (2.40) to the dual D-brane calculation in section 3.2. Note that, while we specialized to even  $N$  for simplicity in the calculation above, the behavior (2.40) applies to both even and odd  $N$ .

**Defect fermion description.** Before moving on to discuss the Wilson loop in the large rank (anti)symmetric representations, we point out a description of the spinor Wilson loop in terms of auxiliary “defect” fermions living on the circle. This is similar to the known description of the  $\text{SU}(N)$  Wilson loops in the symmetric and antisymmetric representations [17, 42]. In the case of the spinor representation, the defect theory consists of real (Majorana) fermions  $\chi^i$ ,  $i = 1, \dots, N$ , transforming in the fundamental representation of  $\text{SO}(N)$  and coupled to  $iA + \Phi^6$  (we choose  $\Phi^6$  to be the scalar that couples to the half-BPS Wilson loop). The worldline action is given by:

$$S_{\text{defect}} = \frac{i}{2} \int_0^1 dt \left[ \chi^i \dot{\chi}^i + \chi^i \Gamma_{ij} \chi^j \right], \quad (2.41)$$

where we write  $\Gamma = iA + \Phi^6$  for short-hand and, by suitable gauge-fixing, we may take  $\Gamma$  to be constant on the Wilson loop. Consider the even  $N$  case first. Upon canonical

quantization, the fermion operators satisfy the Clifford algebra  $\{\chi^i, \chi^j\} = \delta^{ij}$  and their Hilbert space is the  $2^{\frac{N}{2}}$ -dimensional space of the spinor representation. We can explicitly check that the contribution of the defect action, eq. (2.41), to the partition function yields the characters of the spinor representation. Using an orthogonal transformation on the  $\chi^i$  fields, we may write  $\Gamma_{ij} = \text{diag}(\gamma_1\sigma_y, \gamma_2\sigma_y, \dots, \gamma_{N/2}\sigma_y)$ , and therefore:

$$\begin{aligned} Z_{\text{defect}} &= \int \prod_{i=1}^N D\chi_i e^{-S_{\text{defect}}} \\ &= \prod_{i=1}^{N/2} \int D\chi_{2i-1} D\chi_{2i} \exp\left(\frac{1}{2} \int_0^1 dt (\chi^{2i-1} \chi^{2i}) \begin{pmatrix} i \frac{d}{dt} & \gamma_i \\ -\gamma_i & i \frac{d}{dt} \end{pmatrix} \begin{pmatrix} \chi_{2i-1} \\ \chi_{2i} \end{pmatrix}\right) \\ &= \prod_{i=1}^{N/2} \text{Det}^{\frac{1}{2}} \begin{pmatrix} i \frac{d}{dt} & \gamma_i \\ -\gamma_i & i \frac{d}{dt} \end{pmatrix} = \prod_{i=1}^{N/2} \prod_{n=0}^{\infty} (2n+1)^2 \pi^2 \left(1 + \frac{\gamma_i^2}{(2n+1)^2 \pi^2}\right). \end{aligned} \quad (2.42)$$

The product over odd integers,  $2n+1$ , appears because the functional determinant is taken over antiperiodic functions on the unit interval,  $f_n(t) \propto e^{i(2n+1)\pi t}$ . The functional determinant can be zeta-function regularized to yield:

$$Z_{\text{defect}} = 2^{\frac{N}{2}} \prod_{i=1}^{\frac{N}{2}} \cosh\left(\frac{\gamma_i}{2}\right). \quad (2.43)$$

This is precisely the sum of the characters of the two chiral representations of  $\text{SO}(N)$  for even  $N$ , given in eq. (2.31). If one wanted to extract the chiral characters separately, one could compute the functional determinant over suitable linear combinations of periodic and antiperiodic boundary conditions on the circle.<sup>9</sup>

The case of odd  $N$  yields eq. (2.42) with one additional path integral over  $\chi^N$  with action  $S = \frac{i}{2} \int dt \chi^N \dot{\chi}^N$ . By suitable normalization of the path integral, the additional factor should be set to unity because the one-dimensional Clifford algebra  $\{\chi, \chi\} = 1$  defines a trivial spinorial representation of  $\text{SO}(N)$ . It follows that the defect theory also reproduces the character of  $\text{SO}(N)$  for odd  $N$ , given in eq. (2.32).

The defect theory defining the spinor Wilson loop differs from the defect theory defining the (anti)symmetric Wilson loops [17] in that the fermions are real and there is no additional gauge field on the worldline, which is necessary in the (anti)symmetric to fix the rank of the representation. The absence of the worldline gauge field should be reflected on the boundary conditions for the worldvolume gauge field on the dual D5-brane (the gauge field vanishes on the classical solution discussed in section 3.2 below).

## 2.4 Large rank (anti)symmetric representation

We next examine the Wilson loop in the antisymmetric and symmetric representations with large rank,  $k \sim N$ , which we denote  $A_k$  and  $S_k$ , respectively. The generating functions for

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<sup>9</sup>This is because the path integral with periodic boundary conditions computes the trace with an extra insertion of  $(-1)^{\mathcal{F}}$ , where  $\mathcal{F}$  is the fermion number. In terms of the Dirac matrices representation,  $(-1)^{\mathcal{F}}$  is equivalent to the chirality matrix, and hence  $(1 \pm (-1)^{\mathcal{F}})/2$  projects onto the chiral and antichiral Weyl spinor representations.

the characters of the antisymmetric and symmetric representations of  $SO(N)$  take simple forms and can be used to determine the corresponding Wilson loops using the saddle point approximation in the usual 't Hooft limit with, additionally,  $f = \frac{k}{N}$  held fixed. The analysis is nearly identical to the one for  $SU(N)$  carried out in [43].

We recall from [43] that, if  $H$  is a Hermitian matrix with eigenvalues  $h_i$ , then the trace of  $e^H$  in the rank  $k$  antisymmetric representation of  $SU(N)$  is given by  $\text{tr}_{A_k}[e^H] = \frac{1}{2\pi i} \oint dt \frac{F_A(t)}{t^{N-k+1}}$ , where the antisymmetric generating function is  $F_A(t) = \det(1 + te^H) = \prod_{i=1}^N (1 + te^{h_i})$ . Since antisymmetric Hermitian matrices are a subset of Hermitian matrices for which the eigenvalues come in pairs (e.g.,  $h_i = -h_{i+N/2}$  for  $i = 1, \dots, N/2$  for the even  $N$  case and  $h_i = -h_{i+(N-1)/2}$  for  $i = 1, \dots, (N-1)/2$  and  $h_N = 0$  for the odd  $N$  case), the formula for the characters of  $SO(N)$  in the rank  $k$  antisymmetric representation is essentially the same:

$$\text{Tr}_{A_k}[e^K] = \frac{1}{2\pi i} \oint dt \frac{F_A(t)}{t^{N-k+1}} = \frac{1}{2\pi i} \oint dt \frac{F_A(t)}{t^{k+1}}, \tag{2.44}$$

where

$$F_A(t) = \det(1 + te^K) = \prod_{i=1}^{N'} (1 + te^{k_i})(1 + te^{-k_i}) \begin{cases} 1 + t & N \text{ is odd} \\ 1 & N \text{ is even} \end{cases}. \tag{2.45}$$

Note that  $F_A(t)$  is a symmetric polynomial (i.e.,  $F_A(t) = t^N F_A(t^{-1})$ ), which justifies the second equality in eq. (2.44).

Similarly, the character of  $SU(N)$  in the rank  $k$  symmetric representation is  $\text{tr}_{S_k}[e^H] = \frac{1}{2\pi i} \oint dt \frac{F_S(t)}{t^{k+1}}$ , where the symmetric generating function is  $F_S(t) = \det(1 - te^H)^{-1} = \prod_{i=1}^N (1 - te^{h_i})^{-1}$  [43]. We can again straightforwardly obtain the formula for the characters of  $SO(N)$  in the rank  $k$  symmetric representation from the character formula for  $SU(N)$ , with one modification (i.e., an extra factor of  $1 - t^2$  in the contour integral) related to the fact that the irreducible symmetric representations of  $SO(N)$  must be traceless. Hence, the formula for the characters is:

$$\text{Tr}_{S_k}[e^H] = \frac{1}{2\pi i} \oint dt \frac{(1 - t^2)F_S(t)}{t^{k+1}}, \tag{2.46}$$

where

$$F_S(t) = \frac{1}{\det(1 - te^K)} = \prod_{i=1}^{N'} \frac{1}{(1 - te^{k_i})(1 - te^{-k_i})} \begin{cases} (1 - t)^{-1} & N \text{ is odd} \\ 1 & N \text{ is even} \end{cases}. \tag{2.47}$$

One can check eqs. (2.44)–(2.47) by comparing with the character formulas for  $SO(N)$  [41].

Let us use the generating functions and the saddle point analysis to determine the Wilson loop expectation value at large  $N$  in the antisymmetric and symmetric representations. For simplicity, let us work in the even  $N$  case. We first note that the two generating functions can be written compactly as:

$$F_{A,S}(t) = \exp \left( \pm \sum_{i=1}^{\frac{N}{2}} \left( \log(1 \pm te^{k_i}) + \log(1 \pm te^{-k_i}) \right) \right), \tag{2.48}$$



where the upper sign corresponds to  $A$  and the lower sign to  $S$ . Since both are dominated by  $e^{-N^2U}$  at large  $N$ ,  $U$  being the eigenvalue potential in eq. (2.15), the expectation values of  $F_A$  and  $F_S$  at large  $N$  are both determined by the Wigner semicircle law. Thus, we find that the large  $N$  expectation value of the Wilson loop in the antisymmetric or symmetric representations of rank  $k = fN$  with  $f$  fixed as  $N \rightarrow \infty$  is given by:

$$\langle W \rangle_{A_k, S_k} = \frac{1}{2\pi i} \oint dt \frac{(1-t^2)^{\frac{1\mp 1}{2}}}{t^{k+1}} \exp \left( \pm \frac{4N}{\pi\lambda} \int_{-\sqrt{\frac{\lambda}{2}}}^{\sqrt{\frac{\lambda}{2}}} dx \sqrt{\frac{\lambda}{2} - x^2} \log(1 \pm te^x) \right). \quad (2.49)$$

By expanding this result at small  $\lambda$  and evaluating the resulting integrals,<sup>10</sup> one may check that this is in agreement with (2.4), using the values of the quadratic Casimir collected in appendix.

Note that the contour integrals in (2.49) are the same as the analogous ones appearing in [43] with the replacement  $\lambda \rightarrow \lambda/2$ .<sup>11</sup> Consequently, to obtain the strong coupling behavior, we can simply borrow their results. Then, for the antisymmetric representation in the  $\lambda \gg 1$  regime, one finds:<sup>12</sup>

$$\langle W_{A_k} \rangle \sim \exp \left[ \frac{2N}{3\pi} \sqrt{\frac{\lambda}{2}} \sin^3 \theta_k \right], \quad (2.50)$$

where  $\theta_k$  satisfies:

$$\pi f = \theta_k - \sin \theta_k \cos \theta_k. \quad (2.51)$$

For the symmetric representation, the saddle point that dominates in the large  $\lambda$  limit with  $\kappa = \frac{f}{4} \sqrt{\frac{\lambda}{2}}$  fixed yields the result:

$$\langle W_{S_k} \rangle \sim \exp \left[ 2N(\kappa \sqrt{1 + \kappa^2} + \sinh^{-1} \kappa) \right]. \quad (2.52)$$

We will compare these results below to D3 and D5-branes in  $\text{AdS}_5 \times \mathbb{RP}^5$ .

### 3 Dual string theory in $\text{AdS}_5 \times \mathbb{RP}^5$

As explained in [21], the holographic dual to  $\mathcal{N} = 4$  SYM with  $\text{SO}(N)$  or  $\text{USp}(N)$  gauge group is type IIB string theory on the  $\text{AdS}_5 \times \mathbb{RP}^5$  orientifold. Here  $\mathbb{RP}^5 = S^5/\mathbb{Z}_2$ , where

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<sup>10</sup>The contour integrals may be evaluated using  $\oint \frac{dt}{2\pi i} \frac{(1+t)^b}{t^{a+1}} = \binom{b}{a}$  and  $\oint \frac{dt}{2\pi i} \frac{1}{(1-t)^b t^{a+1}} = \binom{b+a-1}{a}$ . Note also that  $\dim(A_k) = \binom{N}{k}$  and  $\dim(S_k) = \binom{N+k-1}{k} - \binom{N+k-3}{k-2}$ .

<sup>11</sup>A small difference between the result in (2.49) and those in [43] is the measure factor  $1 - t^2$  which implements tracelessness of the symmetric representation. However, the contribution of the  $t^2$  subtraction term is subleading at leading order at large  $N$ .

<sup>12</sup>It should be noted that when  $k = \frac{N}{2}$  and  $\theta_k = \frac{\pi}{2}$  the antisymmetric Wilson loop expectation value given in eq. (2.50) is the square of the spinor Wilson loop expectation value given in eq. (2.40). This reflects the fact that the tensor product of two spinor representations can be decomposed into a sum of antisymmetric representations that is dominated at large  $N$  by the largest representation,  $k = \frac{N}{2}$ . We thank the referee for bringing this to our attention. An analogous statement in terms of the dual D5 branes is given in footnote 20.

the  $\mathbb{Z}_2$  quotient acts by identification of antipodal points (the orientifolding procedure implies that as the string goes around a non-contractible cycle, its orientation is reversed). This background arises as the near-horizon limit of a stack of D3-branes placed at an orientifold plane in flat space. The difference between the orthogonal and symplectic groups corresponds to a choice of “discrete torsion” for the Neveu-Schwarz and Ramond-Ramond 2-forms  $B_{NS}$  and  $B_{RR}$ . Essentially, this corresponds to a discrete choice for the holonomy on a  $\mathbb{RP}^2 \subset \mathbb{RP}^5$ ,  $\exp(i \int_{\mathbb{RP}^2} B_{NS}) = e^{2\pi i \theta_{NS}} = \pm 1$  and  $\exp(i \int_{\mathbb{RP}^2} B_{RR}) = e^{2\pi i \theta_{RR}} = \pm 1$ . The choices  $(\theta_{NS}, \theta_{RR}) = (0, 0)$  and  $(\theta_{NS}, \theta_{RR}) = (0, 1/2)$  correspond respectively to the  $SO(2N')$  and  $SO(2N' + 1)$  gauge theory, while the two choices with non-zero  $\theta_{NS}$  are dual to the gauge theory with symplectic group. The discrete torsion will not play an explicit role in the calculations below, but it is important to note that the D5-brane dual to the spinor Wilson loop, which wraps a  $\mathbb{RP}^4 \subset \mathbb{RP}^5$ , is only possible when  $\theta_{NS} = 0$  [21]. This is in line with the fact that there are no spinor representations for  $USp(N)$  group.

Before moving on to compare the Wilson loop results to strings and branes in  $AdS_5 \times \mathbb{RP}^5$ , let us first briefly review the relation between gauge theory and string theory parameters. The relevant part of the type IIB supergravity Lagrangian that involves the metric and self-dual RR 5-form is (we work in Euclidean signature throughout)

$$S = -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{g} \left( \mathcal{R} - \frac{1}{4 \cdot 5!} (F_5)^2 + \dots \right), \tag{3.1}$$

where  $2\kappa_{10}^2 = (2\pi)^7 g_s^2 \alpha'^4$ . In the more familiar case of the duality with the  $SU(N)$  gauge theory, the  $AdS_5 \times S^5$  background is given by

$$ds^2 = L^2 \left( ds_{AdS_5}^2 + ds_{S^5}^2 \right) \tag{3.2}$$

where  $ds_{AdS_5}^2$  and  $ds_{S^5}^2$  are the metrics on unit radius  $AdS_5$  and  $S^5$ . The 5-form field strength is given by  $F_5 = \frac{4}{L}(\omega_5 + \tilde{\omega}_5)$ , where  $\omega_5$  and  $\tilde{\omega}_5$  are volume forms on  $AdS_5$  and  $S^5$  with unit radius. The radius is fixed by the condition that the flux of the 5-form is  $\frac{1}{2\kappa_{10}^2 T_3} \int_{S^5} F_5 = N$ , corresponding to  $N$  D3-branes (here  $T_3 = (2\pi)^{-3} (\alpha')^{-2} g_s^{-1}$  is the D3-brane tension). Using  $\text{vol}(S^5) = \pi^3 L^5$ , this gives the relation  $L^4 = 4\pi g_s \alpha'^2 N$ . The string coupling  $g_s$  is related to the  $SU(N)$  SYM coupling normalized as in (2.1) by  $g_{YM}^2 = 4\pi g_s$ .

Similarly, the  $AdS_5 \times \mathbb{RP}^5$  background relevant to  $SO(N)$  and  $USp(N)$  gauge theory takes the form

$$ds^2 = L^2 \left( ds_{AdS_5}^2 + ds_{\mathbb{RP}^5}^2 \right) \tag{3.3}$$

where the metric on  $\mathbb{RP}^5$  is locally the same as  $S^5$ , but due to the  $\mathbb{Z}_2$  quotient  $\text{vol}(\mathbb{RP}^5) = \frac{1}{2} \text{vol}(S^5)$ . The 5-form field strength still takes the form  $F_5 = \frac{4}{L}(\omega_5 + \tilde{\omega}_5)$ , but now the flux quantization condition for  $SO(2N')$ ,  $SO(2N' + 1)$  and  $USp(2N')$  requires

$$\frac{1}{2\kappa_{10}^2 T_3} \int_{\mathbb{RP}^5} F_5 = N'. \tag{3.4}$$

This gives  $L^4 = 8\pi g_s \alpha'^2 N'$ . This relation is valid to leading order at large  $N$ . Taking into account also the contribution to the RR charge due to the orientifold three-plane,<sup>13</sup> the

<sup>13</sup>The charge of the orientifold three-plane for the  $SO(2N')$ ,  $SO(2N' + 1)$  and  $USp(2N')$  cases is respectively  $-1/4$ ,  $+1/4$ ,  $+1/4$ , see e.g. table 1 in [44]. Adding this contribution to the right-hand side of (3.4), one can write the result for the radius in terms of  $N$  (rather than  $N'$ ) as in (3.5).

formula for the radius can be written for  $SO(N)$  and  $USp(N)$  as [45]

$$L^4 = 8\pi g_s \alpha'^2 \left( \frac{N}{2} \mp \frac{1}{4} \right). \tag{3.5}$$

In eq. (3.5) above and in eq. (3.9) below, the upper (lower) sign gives the result for  $SO(N)$  ( $USp(N)$ ). The mapping between string coupling and SYM coupling normalized as in (2.1) is given by

$$g_{\text{YM}}^2 = 8\pi g_s. \tag{3.6}$$

Note the factor of 2 difference compared to the analogous relation in the  $SU(N)$  case.<sup>14</sup>

As a check of the dictionary reviewed above, let us briefly recall the matching of the conformal anomalies [45, 47, 48]. A quick way to extract the  $a$ -anomaly coefficient from the holographic dual is to evaluate the on-shell action on the background solution, using the hyperbolic ball coordinates on  $AdS_5$  where the boundary is a sphere  $S^4$ . The conformal anomaly then manifests itself in the fact that the volume of  $AdS_{d+1}$  with even  $d$  is logarithmically divergent (see e.g. [49, 50])

$$\text{vol}(AdS_{d+1}) = \frac{2(-\pi)^{d/2}}{\Gamma(1 + \frac{d}{2})} \log R \tag{3.7}$$

where  $R$  may be identified with the radius of the boundary sphere. Integrating over the  $\mathbb{R}P^5$  directions yields the action

$$S = -\frac{L^5}{2\kappa_{10}^2} \text{vol}(\mathbb{R}P^5) \int_{AdS_5} d^5x \sqrt{g} \left( \mathcal{R}_5 + \frac{12}{L^2} \right) \tag{3.8}$$

where  $\text{vol}(\mathbb{R}P^5) = \pi^3/2$  is the volume of  $\mathbb{R}P^5$  with unit radius. Using (3.5) and (3.7) this yields [45]

$$S = \left( \frac{N(N \mp 1)}{2} + \frac{1}{8} \right) \log R. \tag{3.9}$$

The holographic prediction for the  $a$ -anomaly coefficient (normalized so that  $a = 1/90$  for a free massless scalar) is then  $a = \frac{N(N \mp 1)}{2} + O(N^0)$ , in agreement, up to order  $N$ , with the anomaly of the dual  $SO(N)$  and  $USp(N)$  SYM theories.<sup>15</sup> Note that the term of order  $N^0$  is not expected to match from this classical calculation, because there can be corrections of order  $N^0$  coming from supergravity one-loop effects. It would be interesting to compute them, along the lines of what was done in [51] for the  $SU(N)$  case, and check whether they agree with the exact gauge theory answer,  $a = \frac{N(N \mp 1)}{2}$ .

In the remainder of this section, we will focus exclusively on the  $SO(N)$  case, and compare the localization prediction for the Wilson loop with the dual strings and D-branes in  $AdS_5 \times \mathbb{R}P^5$ . We will work in the leading large  $N$  limit, where the finite shift in (3.5) will not play a role.

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<sup>14</sup>For an explanation of this factor of 2 difference between  $U(N)$  and  $SO(N)$  gauge theory on D-branes, see for instance section 13.3 of [46].

<sup>15</sup>The  $a$ -anomaly in  $\mathcal{N} = 4$  SYM with group  $G$  is given by  $a = \frac{1}{90} (6 + 4 \cdot \frac{11}{2} + 62) \dim(G) = \dim(G)$ .

### 3.1 Fundamental string

As is well-known, the dual to the Wilson loop in the fundamental representation is a string worldsheet corresponding to a minimal surface ending on the Wilson loop contour on the boundary [4]. The fundamental string action is

$$S_{F1} = T_{F1} \int d^2\sigma \sqrt{h}, \quad T_{F1} = \frac{1}{2\pi\alpha'} \quad (3.10)$$

where  $h = \det h_{\alpha\beta}$  and  $h_{\alpha\beta}$  is the induced metric.

For a Wilson loop with constant scalar coupling, the string surface is pointlike in the internal  $\mathbb{RP}^5$  space and extends entirely within  $\text{AdS}_5$ . In the case of the 1/2-BPS Wilson loop supported on a circle or infinite straight line, the minimal surface is well known [52], and corresponds to an  $\text{AdS}_2 \subset \text{AdS}_5$ . Specifically, using the Poincaré coordinates in  $\text{AdS}_5$

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + dx_\mu dx^\mu \right) \quad (3.11)$$

the minimal surface dual to a circular Wilson loop of radius  $a$  in the  $(x^1, x^2)$ -plane may be written as

$$x^1 = \frac{a \cos \tau}{\cosh \sigma}, \quad x^2 = \frac{a \sin \tau}{\cosh \sigma}, \quad z = a \tanh \sigma \quad (3.12)$$

where  $0 < \sigma < \infty$  and  $0 < \tau < 2\pi$  are the worldsheet coordinates. This is a “hemisphere”  $z^2 + (x^1)^2 + (x^2)^2 = a^2$  embedded in  $\text{AdS}_5$ . The induced worldsheet metric is

$$ds_2^2 = \frac{L^2}{\sinh^2 \sigma} \left( d\tau^2 + d\sigma^2 \right) \quad (3.13)$$

which is the hyperbolic disk metric of  $\text{AdS}_2$ .<sup>16</sup> To compute the regularized action on this solution, one may either introduce a cutoff at  $z = \epsilon$  and add a boundary term to remove the linear divergence proportional to the perimeter [4, 52], or directly use the well-known formula for the regularized volume of the hyperbolic disk (with unit radius)

$$\text{vol}(\text{AdS}_2) = -2\pi. \quad (3.14)$$

Making use of this result, we get

$$S_{F1} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h} = \frac{L^2}{2\pi\alpha'} \text{vol}(\text{AdS}_2). \quad (3.15)$$

Using  $L^4 = 8\pi g_s \alpha'^2 \frac{N}{2}$  for  $\text{SO}(N)$  as reviewed above (we drop the finite shift in (3.5) here and everywhere below), we find

$$S_{F1} = -\sqrt{4\pi g_s N} = -\sqrt{\frac{\lambda}{2}}, \quad (3.16)$$

where in the last step we used the relation (3.6) and  $\lambda = g_{\text{YM}}^2 N$ . The strong coupling behavior of the fundamental Wilson loop is then

$$\frac{1}{N} \langle W_F \rangle \sim e^{-S_{F1}} = e^{\sqrt{\frac{\lambda}{2}}}. \quad (3.17)$$

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<sup>16</sup>Similarly, for the 1/2-BPS Wilson loop on the infinite straight line, the string solution is  $z = \sigma$ ,  $x^1 = \tau$ , with  $0 < \sigma < \infty$ ,  $-\infty < \tau < \infty$ . This gives the Poincaré half-plane.

Comparing with the asymptotic expansion in  $1/\lambda$  of the  $O(N^0)$  term in eq. (2.13),  $\frac{1}{N} \langle W_F \rangle \sim \sqrt{\frac{2}{\pi}} \left(\frac{2}{\lambda}\right)^{\frac{3}{4}} e^{\sqrt{\frac{\lambda}{2}}(1 + O(1/\sqrt{\lambda}))}$ , we see that eq. (3.17) reproduces the leading large  $\lambda$  behavior of the large  $N$  limit of the fundamental Wilson loop in  $\mathcal{N} = 4$  SO( $N$ ) SYM. Reproducing the subleading order contribution,  $\sqrt{\frac{2}{\pi}} \left(\frac{2}{\lambda}\right)^{\frac{3}{4}}$ , is more subtle and requires taking into account the semiclassical fluctuations about the minimal surface in the string disk partition function as well as the proper normalization of the string path integral. There has been much effort [53–61] to implement this precise matching for the circular Wilson loop in SU( $N$ ) SYM, starting with [13], which attributed the factor  $\lambda^{-3/4}$  to the contributions of the three SL(2,  $\mathbb{R}$ ) ghost zero modes on the disk. This problem was revisited recently in [62], where it was observed that, up to numerical factors, the subleading correction in the circular Wilson loop in both SU( $N$ ) SYM and U( $N$ ) $_k \times$  U( $N$ ) $_{-k}$  ABJM theory, when not normalized by the dimension of the representation, takes the universal form  $\sim \sqrt{T}/g_s$ . The factor of  $\sqrt{T}$  was argued to arise from the special structure of the UV divergences of the one-loop determinants for the quantum fluctuations, while the factor of  $1/g_s$  is the standard one for the contribution of the disk topology to the genus expansion of the string path integral.<sup>17</sup> The same universal factor applies for the string dual to the fundamental Wilson loop in SO( $N$ ) SYM, as we can easily see from eqs. (3.6) and (3.10).

It would also be interesting to try to reproduce the  $1/N$  corrections given in (2.13). In particular, the term of order  $1/N$ , which is absent in the SU( $N$ ) theory, should come from a worldsheet with a crosscap.

### 3.2 The D5-brane dual to the spinor Wilson loop

Let us now turn to the case of main interest in this paper — namely, the Wilson loop in the spinor representation of SO( $N$ ). As argued in [21], the dual object is a D5-brane wrapping a  $\mathbb{R}P^4$  “equator” in  $\mathbb{R}P^5$  (see figure 1a), and having AdS<sub>2</sub> induced geometry within AdS<sub>5</sub>, like the fundamental string of the previous section. As mentioned above, such wrapping is only possible when the NS discrete torsion vanishes,  $\theta_{NS} = 0$ . Unlike the D-branes dual to the (anti)symmetric representations [15, 17, 63], on this D5-brane worldvolume there is no U(1) gauge field turned on. The classical DBI action evaluated on the solution is then simply given by the volume of the D5-brane, which has AdS<sub>2</sub>  $\times$   $\mathbb{R}P^4$  induced geometry. The on-shell action is then

$$S_{D5} = T_5 \int d^6\sigma \sqrt{h} = T_5 L^6 \text{vol}(\text{AdS}_2) \text{vol}(\mathbb{R}P^4), \tag{3.18}$$

where  $\text{vol}(\text{AdS}_2) = -2\pi$  and  $\text{vol}(\mathbb{R}P^4) = \frac{4}{3}\pi^2$  are the volumes of AdS<sub>2</sub> and  $\mathbb{R}P^4$  with unit radius. Here  $T_5$  is the D5-brane tension, which reads<sup>18</sup>

$$T_5 = \frac{1}{(2\pi)^5 \alpha'^3 g_s} = \frac{N}{8\pi^4 L^6} \sqrt{\frac{\lambda}{2}} \tag{3.19}$$

<sup>17</sup>At higher genus, the universal factor proposed in [62] generalizes to  $(\sqrt{T}/g_s)^\chi$ , where  $\chi$  is the Euler number.

<sup>18</sup>In general, for a D $_p$ -brane, the tension is  $T_p = (2\pi)^{-p} (\alpha')^{-(p+1)/2} g_s^{-1}$ , see e.g. [64].

where we used (3.5) and (3.6). Hence we find

$$S_{D5} = -\frac{N}{3\pi} \sqrt{\frac{\lambda}{2}}, \tag{3.20}$$

or, in terms of the spinor Wilson loop expectation value

$$\frac{1}{2^{N/2}} \langle W_{\text{sp}} \rangle \sim e^{-S_{D5}} = \exp\left(\frac{N}{3\pi} \sqrt{\frac{\lambda}{2}}\right). \tag{3.21}$$

This precisely matches the  $\mathcal{N} = 4$  SYM result predicted by localization, eq. (2.40).

By comparing eqs. (3.20) and (3.16), we see that the actions of the D5-brane dual to the circular Wilson loop in the spinor representation and the fundamental string dual to the circular Wilson loop in the fundamental representation are related by

$$S_{D5} = \frac{N}{3\pi} S_{F1}. \tag{3.22}$$

This relationship in fact holds for any Wilson loop that is  $SO(5)$  invariant (i.e., for which  $\Theta^I$  in eq. (2.2) is a point in  $\mathbb{RP}^5$ ), regardless of the shape of the contour in  $\mathbb{R}^4$ . The reason is simple. If the minimal fundamental string surface in  $AdS_5$  is  $\Sigma$  and  $A(\Sigma)$  is its area, then  $S_{F1} = T_{F1} A(\Sigma)$ . On the other hand, the volume of the minimal wrapped D5-brane dual to the spinor Wilson loop, which has induced geometry  $\Sigma \times \mathbb{RP}^4$ , is  $A(\Sigma) \frac{4\pi^2 L^4}{3}$ . Given  $S_{D5} = T_5 A(\Sigma) \frac{4\pi^2 L^4}{3}$  and  $T_5 L^6 = \frac{N}{4\pi^3} T_{F1} L^2$  (see eq. (3.19)), then eq. (3.22) follows. This argument is similar to the one in [65], which established an analogous universal relationship between the fundamental Wilson loop and the rank  $k$  antisymmetric Wilson loop. The argument for the spinor Wilson loop is even simpler since the wrapped D5-brane dual to the spinor Wilson loop carries zero electric field and is governed simply by the Nambu-Goto action. One example of a useful consequence of eq. (3.22), is that, since we know the quark-antiquark potential at strong coupling takes the form  $V_F(R) = -2\pi\gamma T_{F1} L^2 / R = -\gamma \sqrt{\frac{\lambda}{2}} \frac{1}{R}$  where  $\gamma$  is a numerical factor that can be obtained from the relevant minimal surface [4],<sup>19</sup> and  $R$  the distance between the quarks, then the potential between probe charges in the spinor representation must be  $V_{\text{sp}} = -2\pi\gamma \frac{N}{3\pi} \frac{T_{F1} L^2}{R} = -\gamma \frac{N \sqrt{\frac{\lambda}{2}}}{3\pi} \frac{1}{R}$ .

### 3.3 The D3/D5-branes dual to large rank (anti)symmetric Wilson loops

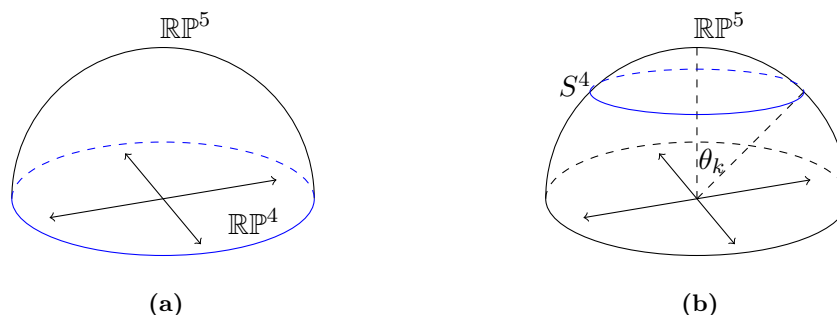
In section 2.4, we found that the  $SO(N)$  Wilson loops in the large rank symmetric and antisymmetric representations, in the supergravity regime, were equal to the corresponding  $SU(N)$  Wilson loops up to the replacement  $\lambda \rightarrow \lambda/2$ . Consequently, we expect that the D3 and D5-branes dual to the  $SO(N)$  Wilson loops are closely related to the corresponding branes dual to the  $SU(N)$  Wilson loops, which were identified in [15–17, 43].

To review the basic facts about those D-brane solutions, it is convenient to use the following  $AdS_2 \times S^2$  slicing of the  $AdS_5$  metric (see e.g. [32] for more details on the form of the brane solutions and their actions using these coordinates)

$$ds_{AdS_5}^2 = L^2 \left( du^2 + \cosh^2 u ds_{AdS_2}^2 + \sinh^2 u d\Omega_2^2 \right). \tag{3.23}$$

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<sup>19</sup>The explicit value is  $\gamma = 4\pi^2 \sqrt{2} / \Gamma\left(\frac{1}{4}\right)^4$ .



**Figure 1.** The real projective space  $\mathbb{RP}^5$  can be visualized as the upper hemisphere of  $S^5$  with antipodal points on the equator identified. Since the neutral D5-brane dual to the spinor Wilson loop wraps an  $\mathbb{RP}^4$  submanifold of  $\mathbb{RP}^5$ , we can think of it as wrapping the equator, as is indicated by the blue curve in (a). Meanwhile, the D5-brane dual to the antisymmetric Wilson loop wraps an  $S^4$  submanifold of  $\mathbb{RP}^5$  at polar angle  $\theta_k$ , as is indicated by the blue curve in (b).

The D3-brane of [15] extends along the  $\text{AdS}_2$  and  $S^2$  directions within  $\text{AdS}_5$ , with  $u = u_k$  constant, and it is point-like in the internal space  $S^5$  (or, in our case,  $\mathbb{RP}^5$ ). In addition, there is a non-zero worldvolume gauge field given by

$$F = i \cosh u_k \omega_2 \quad (3.24)$$

where  $\omega_2$  denotes the volume form on  $\text{AdS}_2$ . The parameter  $u_k$  is fixed by requiring that the fundamental string charge is equal to  $k$ , the rank of the dual representation. This yields the condition, written in terms of D3-brane tension and radius of  $\text{AdS}_5$

$$k = 2\pi\alpha' T_3 4\pi L^2 \sinh u_k = \frac{L^2}{\pi\alpha' g_s} \sinh u_k \quad (3.25)$$

We have written the result in this (perhaps unfamiliar-looking) way so that it can be easily applied to both the  $\text{AdS}_5 \times S^5$  and  $\text{AdS}_5 \times \mathbb{RP}^5$  cases, using the appropriate value of the  $\text{AdS}_5$  radius. For the former case, using  $L^4 = 4\pi g_s \alpha'^2 N = g_{\text{YM}}^2 N \alpha'^2$ , (3.25) yields

$$\sinh u_k = \frac{f\sqrt{\lambda}}{4}, \quad f = \frac{k}{N}, \quad (3.26)$$

while for the  $\text{AdS}_5 \times \mathbb{RP}^5$  dual to  $\text{SO}(N)$  gauge group, using (3.5) and (3.6), we get

$$\sinh u_k = \frac{f\sqrt{\lambda/2}}{4}, \quad f = \frac{k}{N}. \quad (3.27)$$

The D3-brane on-shell action, obtained by summing the DBI and Wess-Zumino contributions, and supplemented by a boundary term implementing a Legendre transform on the gauge field, yields [15] (see [32] for the calculation in the coordinates used here)

$$S_{D3} = -4\pi^2 L^4 T_3 (u_k + \sinh u_k \cosh u_k) = -2N (u_k + \sinh u_k \cosh u_k) \quad (3.28)$$

In the  $\text{SO}(N)$  case, using (3.27), we see that this agrees with (2.52), i.e. it is the same as the  $\text{SU}(N)$  case, up to the replacement  $\frac{f\sqrt{\lambda}}{4N} \rightarrow \frac{f\sqrt{\lambda/2}}{4N}$ .



The  $\text{AdS}_2 \times S^4$  D5-brane [16, 66] dual to the antisymmetric Wilson loop occupies the  $\text{AdS}_2$  in the metric (3.23) with  $u = 0$ , and it wraps an  $S^4 \subset S^5$  at a constant polar angle  $\theta = \theta_k$ . Similarly to the D3-brane, there is a worldvolume gauge field with field strength components along  $\text{AdS}_2$ , given explicitly by

$$F = i \cos \theta_k \omega_2. \tag{3.29}$$

We can have the same solution in the  $\text{AdS}_5 \times \mathbb{RP}^5$  case. We may think of  $\mathbb{RP}^5$  as the upper hemisphere of  $S^5$  with antipodal points identified on the equator. The brane solution then wraps an  $S^4$  “latitude” at angle  $\theta = \theta_k$  on the upper hemisphere, as shown in figure 1b. The requirement that the flux of the worldvolume gauge field is equal to  $k$  yields the condition

$$k = 8\pi^3 \alpha' T_5 L^4 (\theta_k - \sin \theta_k \cos \theta_k) = \frac{L^4}{4\pi^2 g_s \alpha'^2} (\theta_k - \sin \theta_k \cos \theta_k). \tag{3.30}$$

In terms of gauge theory parameters, this can be written, both for the  $\text{SU}(N)$  and  $\text{SO}(N)$  cases, as

$$\theta_k - \sin \theta_k \cos \theta_k = \frac{\pi k}{N}. \tag{3.31}$$

Note that in the  $\text{SO}(N)$  case we can restrict  $k \leq N'$ , where  $N'$  is the rank of  $\text{SO}(N)$ , which is  $N' = N/2$  for  $N$  even and  $N' = (N - 1)/2$  for  $N$  odd. Then we see that it is consistent to restrict  $\theta_k$  to the upper hemisphere.<sup>20</sup> The total action of the D5-brane, including the relevant boundary terms, yields [16]

$$S_{\text{D5}} = -\frac{16\pi^3}{3} T_5 L^6 \sin^3 \theta_k. \tag{3.32}$$

For the  $\text{AdS}_5 \times \mathbb{RP}^5$  dual to the  $\text{SO}(N)$  theory, we then have, using (3.19):

$$S_{\text{D5}} = -\frac{2N}{3\pi} \sqrt{\frac{\lambda}{2}} \sin^3 \theta_k, \tag{3.33}$$

which matches the expectation value of the  $\text{SO}(N)$  Wilson loop in the antisymmetric representation given in eq. (2.50). Again, the difference with the  $\text{SU}(N)$  theory result to this order is simply the replacement  $\sqrt{\lambda} \rightarrow \sqrt{\lambda/2}$ , in agreement with the localization analysis.

#### 4 The bremsstrahlung function for the spinor defect CFT

The 1/2-BPS Wilson loop defines an interesting example of a conformal defect labelled by a choice of representation of the gauge group. Using AdS/CFT, we can extract strong

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<sup>20</sup>The equatorial case  $\theta_k = \frac{\pi}{2}$ , corresponding to  $k = \frac{N}{2}$  for even  $N$ , is a special case because antipodal points on the equator are identified under the quotient  $S^5/\mathbb{Z}_2 \cong \mathbb{RP}^5$ . The fact that the volume of  $\mathbb{RP}^4$  is half that of  $S^4$  indicates that the D5-brane dual to the antisymmetric Wilson loop corresponding to  $k = \frac{N}{2}$  is doubly wrapped and is therefore a double cover of the D5-brane dual to the spinor Wilson loop. (Note from eq. (3.29) that  $F = 0$  in this case). Alternatively, the singly-wrapped D5-brane should be dual to just one of the two irreducible, self-dual representations into which the  $k = N/2$  antisymmetric representation may be decomposed.

coupling predictions for the defect CFT<sub>1</sub> defined by the Wilson loop in the spinor representation by studying fluctuations of the dual D5-brane and their correlation functions. For the D3/D5-brane dual to the Wilson loops in the (anti)symmetric representations of SU(N), this analysis was carried out in detail recently in [32] both on the string and gauge theory sides. Our analysis in the present paper will be minimal: we restrict our attention to the two-point correlation functions of the displacement operators characterizing transverse displacements of the Wilson line, which are dual to transverse fluctuations of the D5-brane within AdS<sub>5</sub>. For simplicity we will carry out the computation for the straight Wilson line, but we can easily map correlation functions on the straight Wilson line defect to correlation functions on the circular Wilson loop defect, when necessary. Our analysis of the fluctuating D5-brane is similar to the one in [27] for the AdS<sub>2</sub> fundamental string. We leave a more detailed analysis along the lines of [32] to future work.

Defect correlation functions of operators  $\mathcal{O}_1, \dots, \mathcal{O}_n$  living on the 1/2-BPS spinor Wilson loop are defined to be:

$$\langle\langle \mathcal{O}_1(t_1) \dots \mathcal{O}_n(t_n) \rangle\rangle := \langle \text{Tr}_{\text{sp}} \mathcal{P} \left[ \mathcal{O}_1(t_1) \dots \mathcal{O}_n(t_n) e^{\int dt (iA_\mu \dot{x}^\mu + \Phi^6 |\dot{x}|)} \right] \rangle. \quad (4.1)$$

Certain operators have natural interpretations in terms of deformations of the contour  $x^\mu(t)$  in  $\mathbb{R}^4$  and of the scalar coupling  $\Theta^I(t)$ , defining the Wilson loop (2.2). Here we focus on the displacement operators  $\mathbb{D}_i(t)$  for the straight Wilson line, which generates transverse deformations of the curve in  $\mathbb{R}^4$ . They can be defined in terms of the variation of the Wilson loop in response to a variation in the  $\mathbb{R}^4$  contour [33, 67]:

$$\delta W = \mathcal{P} \int dt \delta x^i(t) \mathbb{D}_i(t) W. \quad (4.2)$$

Here,  $i = 1, 2, 3$  denotes the three directions transverse to the unperturbed straight line contour  $x^\mu(t) = (x^0(t), \dots, x^3(t)) = (t, 0, 0, 0)$ . By taking a variational derivative of eq. (2.2) with respect to  $\delta x^j$ , one finds  $\mathbb{D}_i := iF_{ti} + D_i\Phi^6$ . Every defect CFT has displacement operators, whose scaling dimension is protected and equal to  $\Delta_{\mathbb{D}} = 2$  for a one-dimensional defect [67].

On the string theory side, the fluctuations of the D5-brane dual to the spinor Wilson loop can be described by a field theory on  $\text{AdS}_2 \times \mathbb{RP}^4$ . Turning off the fluctuations of the U(1) gauge field (which is zero on the solution),<sup>21</sup> the bosonic part of the D5-brane action is given by the Nambu-Goto-like action

$$S = \frac{T_5 L^6}{2} \int d^2\sigma d^4\tau \sqrt{\det_{\alpha\beta} \left[ \frac{\partial_\alpha x^r \partial_\beta x^r + \partial_\alpha z \partial_\beta z}{z^2} + \frac{\partial_\alpha y^a \partial_\beta y^a}{\left(1 + \frac{1}{4}y^2\right)^2} \right]}, \quad (4.3)$$

where  $\sigma^\alpha = (\sigma^1, \sigma^2, \tau^1, \dots, \tau^4)$  are world-sheet coordinates on the brane,  $x^r$ ,  $r = 0, \dots, 3$  are spacetime coordinates on the boundary of AdS<sub>5</sub> and  $y^a$ ,  $a = 1, \dots, 5$ , are stereographic spacetime coordinates on  $\mathbb{RP}^5$ .

<sup>21</sup>The gauge field fluctuations do not affect the leading order calculation of the displacement two-point function that we perform below, but they may affect four and higher point correlation functions, even if we restrict to the displacement sector for external states, as seen in the D5-brane calculations in [32].

Since here we will be interested only in the behavior of the displacement operators in the Wilson loop, which should be dual to the transverse fluctuations within AdS<sub>5</sub>, we ignore the fluctuations of the D5-brane in  $\mathbb{RP}^5$ . Second, we pick coordinates on AdS<sub>5</sub> that make the AdS<sub>2</sub> geometry of the D5-brane manifest, as in [27]. The AdS<sub>5</sub> metric in such coordinates takes the form

$$ds_{\text{AdS}_5}^2 = \frac{\left(1 + \frac{1}{4}x^2\right)}{\left(1 - \frac{1}{4}x^2\right)^2} ds_2^2 + \frac{dx^i dx^i}{\left(1 - \frac{1}{4}x^2\right)^2}, \quad ds_2^2 = \frac{1}{z^2} \left(dx_0^2 + dz^2\right) \quad (4.4)$$

Here  $x^i$ ,  $i = 1, 2, 3$ , are the coordinates characterizing transverse displacements of the AdS<sub>2</sub> part of the D5-brane. Finally, working in the static gauge (i.e.,  $\sigma^1 = x^0$ ,  $\sigma^2 = z$ , and the  $\tau^1, \dots, \tau^4$  are some parametrization of the static  $\mathbb{RP}^4$  submanifold) and assuming the  $x^i$  are constant on  $\mathbb{RP}^4$  (i.e., considering only the lowest Kaluza-Klein modes in the dimensional reduction from AdS<sub>2</sub>  $\times$   $\mathbb{RP}^4$  to AdS<sub>2</sub>), we find that the effective Nambu-Goto action describing the transverse fluctuations of the D5-brane is:

$$S = T_5 L^6 \text{vol}(\mathbb{RP}^4) \int d^2\sigma \sqrt{\det_{\mu\nu} \left( \frac{\left(1 + \frac{1}{4}x^2\right)}{\left(1 - \frac{1}{4}x^2\right)^2} g_{\mu\nu} + \frac{\partial_\mu x^i \partial_\nu x^i}{\left(1 - \frac{1}{4}x^2\right)^2} \right)}, \quad (4.5)$$

where  $\mu, \nu = 1, 2$  are the AdS<sub>2</sub> “worldsheet” coordinate indices and  $g_{\mu\nu} = \delta_{\mu\nu}/(\sigma^2)^2$ . We can expand the action in small  $x^i$ , and to quadratic order we find:

$$S = T_5 L^6 \text{vol}(\mathbb{RP}^4) \int d^2\sigma \sqrt{g} \left( 1 + \frac{1}{2} \partial_\mu x^i \partial^\mu x^i + x^i x^i + O(x^4) \right). \quad (4.6)$$

Eq. (4.6) tells us that the  $x^i$  fields are scalars in AdS<sub>2</sub> with mass  $m^2 = 2$ , corresponding to  $\Delta = 2$  using the familiar AdS/CFT dictionary  $m^2 = \Delta(\Delta - 1)$ . The  $x^i$  are dual to the  $\mathbb{D}_i$  displacement operators, which have the right scaling dimension, the same SO(3) symmetry corresponding to rotations about the straight Wilson line, and the same interpretation in terms of transverse deformations of the Wilson line/surface.

Thus, by an AdS<sub>2</sub>/dCFT<sub>1</sub> correspondence [27] between the defect CFT on the spinor Wilson loop and the fluctuations of the dual D5-brane, we expect:

$$\langle \langle \mathbb{D}_i(t_1) \mathbb{D}_j(t_2) \rangle \rangle_{\text{CFT}_1} = \langle x^i(t_1) x^j(t_2) \rangle_{\text{AdS}_2}. \quad (4.7)$$

We can use eq. (4.7) to find  $\langle \langle \mathbb{D}_i(t_1) \mathbb{D}_j(t_2) \rangle \rangle_{\text{CFT}_1}$  at strong coupling. From the action in eq. (4.6), the boundary two-point function in AdS<sub>d+1</sub> can be readily calculated. In particular, the two-point function for a scalar field governed by the action  $S = \frac{T}{2} \int d^d z dz_0 \sqrt{g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2)$ , with  $m^2 = \Delta(\Delta - d)$  and  $g_{\mu\nu} = \frac{1}{z_0^2} \delta_{\mu\nu}$ , is [68]:<sup>22</sup>

$$\langle O_\Delta(x) O_\Delta(y) \rangle_{\text{AdS}_{d+1}} = T \frac{(2\Delta - d) \Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - \frac{d}{2})} \frac{1}{|x - y|^{2\Delta}}. \quad (4.8)$$

<sup>22</sup>Notice that the factor of  $T$  appears in the numerator rather than the denominator of the two point function, the opposite of what one would find for 2-point functions in ordinary field theory. This follows from the definition of the AdS correlation functions as variational derivatives of the exponential of the action with respect to the fields at the boundary.

For the case of the  $x^i$  fields, with  $d = 1$  and  $\Delta = 2$  and taking into account the factor of  $T_5 \text{vol}(\mathbb{RP}^4)$  in the normalization of the action, this yields:

$$\langle x^i(t_1)x^j(t_2) \rangle = \frac{6T_5 L^6 \text{vol}(\mathbb{RP}^4)}{\pi} \frac{\delta_{ij}}{|t_1 - t_2|^4} = \frac{N\sqrt{\lambda}}{\sqrt{2}\pi^3} \frac{\delta_{ij}}{|t_1 - t_2|^4}. \quad (4.9)$$

Because there is a canonical choice for the normalization of the  $x^i$  fields according to eq. (4.3) and for the  $\mathbb{D}_i(t)$  operators according to eq. (4.2), the coefficient of the two-point function in eq. (4.9) is a meaningful quantity that we can compare to physical properties of the boundary CFT. In particular, we can relate the normalization of the two-point function of the  $x^i$  fields to the bremsstrahlung function of the Wilson loop in the spinor representation, by using the results derived in [33]. Section 4 of [33] argues for precise relationships between three observables in the defect conformal field theory associated with a Wilson line: the derivative of the anomalous dimension of a cusp in the Wilson line evaluated at zero angle, the normalization of the two-point function of the displacement operators, and the energy radiated by an accelerating quark.

Most relevant to our present purpose, the two-point function of the displacement operators  $\mathbb{D}_i$  on a straight Wilson loop is related to the bremsstrahlung function,  $B(\lambda)$ , by:<sup>23</sup>

$$\langle \langle \mathbb{D}_i(t_1)\mathbb{D}_j(t_2) \rangle \rangle = \frac{12B(\lambda)\delta_{ij}}{t_{12}^4}. \quad (4.10)$$

From eq. (4.7), eq. (4.9), and eq. (4.10), it follows that the leading contribution to the spinor Wilson loop bremsstrahlung function at strong coupling is:

$$B_{\text{sp}}(\lambda) = \frac{N\sqrt{\lambda}}{12\sqrt{2}\pi^3} + O\left(\frac{1}{\sqrt{\lambda}}\right). \quad (4.11)$$

We can also calculate the bremsstrahlung function directly in the defect CFT using supersymmetric localization. The argument is the same as for the case of the fundamental Wilson loop [27, 33]. First, one notes that the bremsstrahlung function can also be extracted from the two-point function of the scalar operators  $\Phi^a$ ,  $a = 1, \dots, 5$ , i.e. those that do not appear in the Wilson loop (2.2). This is because these operators belong to the same supermultiplet of  $\text{OSp}(4^*|4)$  as the displacement. Their two-point function is given by

$$\langle \langle \Phi^a(t_1)\Phi^b(t_2) \rangle \rangle = \frac{2B(\lambda)\delta_{ab}}{t_{12}^2}. \quad (4.12)$$

In particular, the dimensions of the  $\Phi^a$  fields are protected and equal to 1, and the normalizations of the two different two-point functions are related by a factor of 6. These properties follow from supersymmetry and are independent of the gauge group and representation.

The normalization constant in (4.12) can be extracted via localization by considering certain “twisted” combinations of the scalars  $\Phi^a$  that have position independent correlation

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<sup>23</sup>The bremsstrahlung function is also related to the anomalous dimension of a cusp of angle  $\phi$  by the equation  $B = \frac{1}{2}\partial_\phi^2 \Gamma_{\text{cusp}}(\phi)|_{\phi=0}$ , and yields the energy of a moving quark moving at low velocities by  $\Delta E = 2\pi B \int dt(\dot{v})^2$ .

functions on the circle, see [24, 27, 28, 31, 33] for more details. The end result is that, for any representation  $R$ , one obtains the exact result

$$B_R(\lambda) = \frac{\lambda}{2\pi^2} \frac{\partial}{\partial \lambda} \log \langle W_R \rangle. \tag{4.13}$$

For the spinor Wilson loop, using eq. (2.37), this yields

$$\begin{aligned} B_{\text{sp}}(\lambda) &= \frac{\lambda}{2\pi^2} \frac{\partial}{\partial \lambda} \frac{2N}{\pi} \int_0^1 \sqrt{1-u^2} \log \left( \cosh \left( \frac{1}{2} \sqrt{\frac{\lambda}{2}} u \right) \right) du \\ &= \frac{N\sqrt{\lambda}}{4\sqrt{2}\pi^3} \int_0^1 \sqrt{1-u^2} u \tanh \left( \frac{1}{2} \sqrt{\frac{\lambda}{2}} u \right) du. \end{aligned} \tag{4.14}$$

In the large  $\lambda$  regime, this reduces to  $B_{\text{sp}}(\lambda) = \frac{N\sqrt{\lambda}}{12\sqrt{2}\pi^3} + O\left(\frac{1}{\sqrt{\lambda}}\right)$ , in precise agreement with eq. (4.11). Localization may be similarly applied to obtain predictions for higher-point functions of the protected scalar operators. It would be interesting to match them with the correlators obtained from the D5-brane.

## 5 Conclusions

In this paper, we studied the half-BPS circular Wilson loop in the  $\mathcal{N} = 4$  SYM theory with orthogonal gauge group. Using supersymmetric localization, we derived exact results for its expectation value, focusing on the large  $N$  limit and on the fundamental, (anti)symmetric and spinor representations. In the latter case we obtained a simple quantitative test of the proposal of [21] by matching the classical action of the  $\text{AdS}_2 \times \mathbb{RP}^4$  D5-brane with the strong coupling limit of the spinor Wilson loop expectation value.

There are many natural extensions of the calculations we have discussed. One could try to study in detail the spectrum of fluctuations of the D5-brane dual to the spinor Wilson loop, along the lines of earlier work [69–71] done for the D3/D5-branes dual to (anti)symmetric representations. Knowing the spectrum of excitations of the D5-brane would allow one to compute the one-loop correction to the Wilson loop expectation value, as in [55, 72], which would be very interesting to match with the localization prediction (for this, one would need to extend the analysis done in section 2.3 to order  $N^0$ ). From the point of view of defect CFT, it would also be interesting to match the spectrum of excitations of the brane to the spectrum of defect operator insertions on the Wilson loop. Using localization and techniques similar to the ones developed recently in [32], one should be able to obtain exact results for a subsector of defect correlation functions and match them with the D5-brane dynamics. While the analysis should be somewhat similar to the one done in [32] for the antisymmetric representation (dual to the  $\text{AdS}_2 \times S^4$  D5-brane), there should be important differences. For instance, in the spinor Wilson loop case there should be no “Legendre transform” with respect to the gauge field, which is related in the case of (anti)symmetric representation to fixing the rank  $k$  of the representation. It could also be worthwhile to see whether the defect spectrum and correlation functions on the spinor Wilson loop may be controlled by some underlying integrability. In the case of the

antisymmetric Wilson loop in the  $SU(N)$  theory, some steps along these lines were taken in [73]. The case of the spinor representation should be similar, and perhaps simpler: in some sense, the spinor representation is more “fundamental” than the rank- $k$  antisymmetric one because it does not arise from the tensor product of  $k$  fundamental representations.

Another extension natural from the defect CFT point of view would be to study “bulk-defect” observables, i.e. correlation functions of the Wilson loop, with or without defect insertions, and local operators inserted away from the Wilson loop. Some such calculations were done in [74] for the (anti)symmetric representations of  $SU(N)$ , and it would be nice to extend them to the  $SO(N)$  case and the spinor representation. The “bulk-defect” correlation functions play an important role in the bootstrap approach [28] to defect CFT, and are also accessible by supersymmetric localization when restricted to a special protected subsector.

Finally, it may also be interesting to study other representations of the orthogonal gauge group, like the spin- $s$  tensor-spinor representations  $[s, 1/2, \dots, 1/2]$  (with  $s$  a half-integer), and see whether they have a natural D-brane description on the string theory side. The analysis of larger representations of size  $\sim N^2$  (that may also include large spinorial representations), would also be of interest: they should be dual to “bubbling geometries” with  $AdS_5 \times \mathbb{RP}^5$  asymptotics, similarly to the  $AdS_5 \times S^5$  case studied in [18–20],<sup>24</sup> but they may involve new features that are worth investigating.

## Acknowledgments

We thank Shota Komatsu for useful discussions and collaboration on related topics. This work was supported in part by the US NSF under Grants No. PHY-1620542 and PHY-1914860.

## A Quadratic Casimirs of $SO(N)$

The quadratic Casimir of  $SO(N)$  appears in the leading term in the Feynman diagram expansion of the Wilson loop, given in eq. (2.4). Presently, we collect the expressions for the quadratic Casimirs in the fundamental, the symmetric, the antisymmetric, and the spinor representations.

As in the main text, let  $T_R^a$  be the generators of  $SO(N)$  in the representation  $R$  of dimension  $\dim(R)$ . The quadratic Casimir is defined by  $T_R^a T_R^a = C_2(R) \mathbf{1}_{\dim(R) \times \dim(R)}$ . Once we fix the normalization of the generators in the fundamental representation by  $\text{Tr}(T_F^a T_F^b) = C(F) \delta^{ab}$ ,  $C_2(R)$  is fixed in terms of  $C(F)$ , the Dynkin labels  $a_i$  of  $R$  and the inverse Cartan matrix  $G_{ij}$  by the equation (see eq. 27 in [76]):

$$C_2(R) = \frac{C(F)}{2} \sum_{i=1}^{N'} \sum_{j=1}^{N'} (a_i + 2) G_{ij} a_j. \tag{A.1}$$

Here,  $N'$  is the rank of  $SO(N)$  when  $N = 2N'$  or  $N = 2N' + 1$ .

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<sup>24</sup>The bubbling geometries dual to large chiral primary local operators in the case of  $SO(N)/USp(N)$  gauge theory was studied in [75] and further discussed in [24].

To determine  $C_2(R)$  for the four representations, we need the inverse Cartan matrix for  $SO(N)$  and the Dynkin labels for each representation. The inverse Cartan matrices for  $SO(2N')$  and  $SO(2N'+1)$  are given in table 3 of [63]. Furthermore, the Dynkin labels generically can be related to the Young tableau indices  $m_i$ ,  $i = 1, 2, \dots, N'$ , which specify the number of boxes in the  $i$ th row of the representation's Young tableau. Section 1.58 of [77] tells us that when  $N = 2N'$ , then  $a_i = m_i - m_{i+1}$  for  $1 \leq i < N'$  and  $a_{N'} = m_{N'-1} + m_{N'}$ ; and when  $N = 2N'+1$ , then  $a_i = m_i - m_{i+1}$  for  $1 \leq i < N'$  and  $a_{N'} = 2m_{N'}$ . Using the explicit forms of  $G_{ij}$  and the relations between the  $a_i$  and  $m_i$  for both even and odd  $N$ , we can rewrite eq. (A.1) in a more useable form:

$$C_2(R) = \frac{C(F)}{2} \left( Nr + \sum_{i=1}^{N'} m_i(m_i - 2i) \right), \tag{A.2}$$

where  $r = \sum_i m_i$ .

Finally, we consider the four specific representations. For the fundamental representation,  $m_1 = 1$ ,  $m_i = 0$  for  $2 \leq i \leq N'$ . For the rank  $k$  symmetric representation,  $m_1 = k$ ,  $m_i = 0$  for  $2 \leq i \leq N'$ . For the rank  $k$  antisymmetric representation,  $m_i = 1$  for  $1 \leq i \leq k$  and  $m_i = 0$  for  $k < i \leq N'$ . Finally, for the spinor representation,  $m_i = 1/2$  for  $1 \leq i \leq N'$ . It follows that the non-zero Dynkin labels are  $a_1 = 1$  for the fundamental,  $a_1 = k$  for the symmetric,  $a_k = 1$  for the antisymmetric, and  $a_{N'} = 1$  for the spinor. Using either eq. (A.1) or eq. (A.2), we arrive at the following expressions:

$$C_2(F) = C(F) \frac{N-1}{2}, \tag{A.3}$$

$$C_2(S_k) = C(F) \frac{k(k+N-2)}{2}, \tag{A.4}$$

$$C_2(A_k) = C(F) \frac{(N-k)k}{2}, \tag{A.5}$$

$$C_2(sp) = C(F) \frac{N(N-1)}{16}. \tag{A.6}$$

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## References

- [1] J.M. Maldacena, *The large  $N$  limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [4] J.M. Maldacena, *Wilson loops in large  $N$  field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [[hep-th/9803002](#)] [[INSPIRE](#)].



- [5] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large  $N$  gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J. C* **22** (2001) 379 [[hep-th/9803001](#)] [[INSPIRE](#)].
- [6] L.F. Alday and J. Maldacena, *Comments on gluon scattering amplitudes via AdS/CFT*, *JHEP* **11** (2007) 068 [[arXiv:0710.1060](#)] [[INSPIRE](#)].
- [7] J. Polchinski and J. Sully, *Wilson Loop Renormalization Group Flows*, *JHEP* **10** (2011) 059 [[arXiv:1104.5077](#)] [[INSPIRE](#)].
- [8] M. Beccaria, S. Giombi and A. Tseytlin, *Non-supersymmetric Wilson loop in  $\mathcal{N} = 4$  SYM and defect 1d CFT*, *JHEP* **03** (2018) 131 [[arXiv:1712.06874](#)] [[INSPIRE](#)].
- [9] M. Beccaria, S. Giombi and A.A. Tseytlin, *Correlators on non-supersymmetric Wilson line in  $\mathcal{N} = 4$  SYM and  $AdS_2/CFT_1$* , *JHEP* **05** (2019) 122 [[arXiv:1903.04365](#)] [[INSPIRE](#)].
- [10] K. Zarembo, *Supersymmetric Wilson loops*, *Nucl. Phys. B* **643** (2002) 157 [[hep-th/0205160](#)] [[INSPIRE](#)].
- [11] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, *Supersymmetric Wilson loops on  $S^3$* , *JHEP* **05** (2008) 017 [[arXiv:0711.3226](#)] [[INSPIRE](#)].
- [12] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in  $N = 4$  supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **582** (2000) 155 [[hep-th/0003055](#)] [[INSPIRE](#)].
- [13] N. Drukker and D.J. Gross, *An exact prediction of  $N = 4$  SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [[hep-th/0010274](#)] [[INSPIRE](#)].
- [14] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [[arXiv:0712.2824](#)] [[INSPIRE](#)].
- [15] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010 [[hep-th/0501109](#)] [[INSPIRE](#)].
- [16] S. Yamaguchi, *Wilson loops of anti-symmetric representation and D5-branes*, *JHEP* **05** (2006) 037 [[hep-th/0603208](#)] [[INSPIRE](#)].
- [17] J. Gomis and F. Passerini, *Holographic Wilson Loops*, *JHEP* **08** (2006) 074 [[hep-th/0604007](#)] [[INSPIRE](#)].
- [18] S. Yamaguchi, *Bubbling geometries for half BPS Wilson lines*, *Int. J. Mod. Phys. A* **22** (2007) 1353 [[hep-th/0601089](#)] [[INSPIRE](#)].
- [19] O. Lunin, *On gravitational description of Wilson lines*, *JHEP* **06** (2006) 026 [[hep-th/0604133](#)] [[INSPIRE](#)].
- [20] E. D'Hoker, J. Estes and M. Gutperle, *Gravity duals of half-BPS Wilson loops*, *JHEP* **06** (2007) 063 [[arXiv:0705.1004](#)] [[INSPIRE](#)].
- [21] E. Witten, *Baryons and branes in anti-de Sitter space*, *JHEP* **07** (1998) 006 [[hep-th/9805112](#)] [[INSPIRE](#)].
- [22] O. Aharony, Y.E. Antebi, M. Berkooz and R. Fishman, *'Holey sheets': Pfaffians and subdeterminants as D-brane operators in large  $N$  gauge theories*, *JHEP* **12** (2002) 069 [[hep-th/0211152](#)] [[INSPIRE](#)].
- [23] E. Witten, *Current Algebra, Baryons, and Quark Confinement*, *Nucl. Phys. B* **223** (1983) 433 [[INSPIRE](#)].
- [24] B. Fiol, B. Garolera and G. Torrents, *Exact probes of orientifolds*, *JHEP* **09** (2014) 169 [[arXiv:1406.5129](#)] [[INSPIRE](#)].

- [25] N. Drukker and S. Kawamoto, *Small deformations of supersymmetric Wilson loops and open spin-chains*, *JHEP* **07** (2006) 024 [[hep-th/0604124](#)] [[INSPIRE](#)].
- [26] M. Cooke, A. Dekel and N. Drukker, *The Wilson loop CFT: Insertion dimensions and structure constants from wavy lines*, *J. Phys. A* **50** (2017) 335401 [[arXiv:1703.03812](#)] [[INSPIRE](#)].
- [27] S. Giombi, R. Roiban and A.A. Tseytlin, *Half-BPS Wilson loop and  $AdS_2/CFT_1$* , *Nucl. Phys. B* **922** (2017) 499 [[arXiv:1706.00756](#)] [[INSPIRE](#)].
- [28] P. Liendo, C. Meneghelli and V. Mitev, *Bootstrapping the half-BPS line defect*, *JHEP* **10** (2018) 077 [[arXiv:1806.01862](#)] [[INSPIRE](#)].
- [29] M. Kim, N. Kiryu, S. Komatsu and T. Nishimura, *Structure Constants of Defect Changing Operators on the 1/2 BPS Wilson Loop*, *JHEP* **12** (2017) 055 [[arXiv:1710.07325](#)] [[INSPIRE](#)].
- [30] N. Kiryu and S. Komatsu, *Correlation Functions on the Half-BPS Wilson Loop: Perturbation and Hexagonalization*, *JHEP* **02** (2019) 090 [[arXiv:1812.04593](#)] [[INSPIRE](#)].
- [31] S. Giombi and S. Komatsu, *Exact Correlators on the Wilson Loop in  $\mathcal{N} = 4$  SYM: Localization, Defect CFT, and Integrability*, *JHEP* **05** (2018) 109 [Erratum *ibid.* **11** (2018) 123] [[arXiv:1802.05201](#)] [[INSPIRE](#)].
- [32] S. Giombi, J. Jiang and S. Komatsu, *Giant Wilson loops and  $AdS_2/dCFT_1$* , *JHEP* **11** (2020) 064 [[arXiv:2005.08890](#)] [[INSPIRE](#)].
- [33] D. Correa, J. Henn, J. Maldacena and A. Sever, *An exact formula for the radiation of a moving quark in  $N = 4$  super Yang-Mills*, *JHEP* **06** (2012) 048 [[arXiv:1202.4455](#)] [[INSPIRE](#)].
- [34] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar Diagrams*, *Commun. Math. Phys.* **59** (1978) 35 [[INSPIRE](#)].
- [35] D. Bessis, C. Itzykson and J.B. Zuber, *Quantum field theory techniques in graphical enumeration*, *Adv. Appl. Math.* **1** (1980) 109 [[INSPIRE](#)].
- [36] M. Mariño, *Les Houches lectures on matrix models and topological strings*, (2004) [[hep-th/0410165](#)] [[INSPIRE](#)].
- [37] B. Zumino, *Normal Forms of Complex Matrices*, *J. Math. Phys.* **3** (1962) 1055.
- [38] M.L. Mehta and N. Rosenzweig, *Distribution laws for the roots of a random antisymmetric hermitian matrix*, *Nucl. Phys. A* **109** (1968) 449 [[INSPIRE](#)].
- [39] M.L. Mehta and A. Pandey, *On Some Gaussian Ensembles of Hermitian Matrices*, *J. Phys. A* **16** (1983) 2655 [[INSPIRE](#)].
- [40] S.K. Ashok, R. Corrado, N. Halmagyi, K.D. Kennaway and C. Romelsberger, *Unoriented strings, loop equations, and  $N = 1$  superpotentials from matrix models*, *Phys. Rev. D* **67** (2003) 086004 [[hep-th/0211291](#)] [[INSPIRE](#)].
- [41] A.B. Balantekin and P. Cassak, *Character expansions for the orthogonal and symplectic groups*, *J. Math. Phys.* **43** (2002) 604 [[hep-th/0108130](#)] [[INSPIRE](#)].
- [42] C. Hoyos, *A defect action for Wilson loops*, *JHEP* **07** (2018) 045 [[arXiv:1803.09809](#)] [[INSPIRE](#)].
- [43] S.A. Hartnoll and S.P. Kumar, *Higher rank Wilson loops from a matrix model*, *JHEP* **08** (2006) 026 [[hep-th/0605027](#)] [[INSPIRE](#)].

- [44] O. Bergman, E.G. Gimon and S. Sugimoto, *Orientifolds, RR torsion, and k-theory*, *JHEP* **05** (2001) 047 [[hep-th/0103183](#)] [[INSPIRE](#)].
- [45] M. Blau, K.S. Narain and E. Gava, *On subleading contributions to the AdS /CFT trace anomaly*, *JHEP* **09** (1999) 018 [[hep-th/9904179](#)] [[INSPIRE](#)].
- [46] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2007).
- [47] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)] [[INSPIRE](#)].
- [48] V. Balasubramanian and P. Kraus, *A stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [[hep-th/9902121](#)] [[INSPIRE](#)].
- [49] D.E. Diaz and H. Dorn, *Partition functions and double-trace deformations in AdS/CFT*, *JHEP* **05** (2007) 046 [[hep-th/0702163](#)] [[INSPIRE](#)].
- [50] H. Casini, M. Huerta and R.C. Myers, *Towards a derivation of holographic entanglement entropy*, *JHEP* **05** (2011) 036 [[arXiv:1102.0440](#)] [[INSPIRE](#)].
- [51] M. Beccaria and A.A. Tseytlin, *Higher spins in AdS<sub>5</sub> at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT*, *JHEP* **11** (2014) 114 [[arXiv:1410.3273](#)] [[INSPIRE](#)].
- [52] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, *Phys. Rev. D* **60** (1999) 125006 [[hep-th/9904191](#)] [[INSPIRE](#)].
- [53] M. Kruczenski and A. Tirziu, *Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling*, *JHEP* **05** (2008) 064 [[arXiv:0803.0315](#)] [[INSPIRE](#)].
- [54] C. Kristjansen and Y. Makeenko, *More about One-Loop Effective Action of Open Superstring in AdS<sub>5</sub> × S<sup>5</sup>*, *JHEP* **09** (2012) 053 [[arXiv:1206.5660](#)] [[INSPIRE](#)].
- [55] E.I. Buchbinder and A.A. Tseytlin, *1/N correction in the D3-brane description of a circular Wilson loop at strong coupling*, *Phys. Rev. D* **89** (2014) 126008 [[arXiv:1404.4952](#)] [[INSPIRE](#)].
- [56] V. Forini, V. Giangreco M. Puletti, L. Griguolo, D. Seminara and E. Vescovi, *Precision calculation of 1/4-BPS Wilson loops in AdS<sub>5</sub> × S<sup>5</sup>*, *JHEP* **02** (2016) 105 [[arXiv:1512.00841](#)] [[INSPIRE](#)].
- [57] A. Faraggi, L.A. Pando Zayas, G.A. Silva and D. Trancanelli, *Toward precision holography with supersymmetric Wilson loops*, *JHEP* **04** (2016) 053 [[arXiv:1601.04708](#)] [[INSPIRE](#)].
- [58] V. Forini, A.A. Tseytlin and E. Vescovi, *Perturbative computation of string one-loop corrections to Wilson loop minimal surfaces in AdS<sub>5</sub> × S<sup>5</sup>*, *JHEP* **03** (2017) 003 [[arXiv:1702.02164](#)] [[INSPIRE](#)].
- [59] A. Cagnazzo, D. Medina-Rincon and K. Zarembo, *String corrections to circular Wilson loop and anomalies*, *JHEP* **02** (2018) 120 [[arXiv:1712.07730](#)] [[INSPIRE](#)].
- [60] D. Medina-Rincon, A.A. Tseytlin and K. Zarembo, *Precision matching of circular Wilson loops and strings in AdS<sub>5</sub> × S<sup>5</sup>*, *JHEP* **05** (2018) 199 [[arXiv:1804.08925](#)] [[INSPIRE](#)].
- [61] L. Botao and D. Medina-Rincon, *On precision holography for the circular Wilson loop in AdS<sub>5</sub> × S<sup>5</sup>*, *Phys. Lett. B* **810** (2020) 135789 [[arXiv:2007.15760](#)] [[INSPIRE](#)].
- [62] S. Giombi and A.A. Tseytlin, *Strong coupling expansion of circular Wilson loops and string theories in AdS<sub>5</sub> × S<sup>5</sup> and AdS<sub>4</sub> × CP<sup>3</sup>*, *JHEP* **10** (2020) 130 [[arXiv:2007.08512](#)] [[INSPIRE](#)].

- [63] N. Yamatsu, *Finite-Dimensional Lie Algebras and Their Representations for Unified Model Building*, [arXiv:1511.08771](#) [[INSPIRE](#)].
- [64] C.V. Johnson, *D-brane primer*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 99): Strings, Branes, and Gravity*, (2000), pp. 129–350, DOI [[hep-th/0007170](#)] [[INSPIRE](#)].
- [65] S.A. Hartnoll, *Two universal results for Wilson loops at strong coupling*, *Phys. Rev. D* **74** (2006) 066006 [[hep-th/0606178](#)] [[INSPIRE](#)].
- [66] J.M. Camino, A. Paredes and A.V. Ramallo, *Stable wrapped branes*, *JHEP* **05** (2001) 011 [[hep-th/0104082](#)] [[INSPIRE](#)].
- [67] M. Billò, V. Gonçalves, E. Lauria and M. Meineri, *Defects in conformal field theory*, *JHEP* **04** (2016) 091 [[arXiv:1601.02883](#)] [[INSPIRE](#)].
- [68] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT(d)/AdS(d+1) correspondence*, *Nucl. Phys. B* **546** (1999) 96 [[hep-th/9804058](#)] [[INSPIRE](#)].
- [69] S. Harrison, S. Kachru and G. Torroba, *A maximally supersymmetric Kondo model*, *Class. Quant. Grav.* **29** (2012) 194005 [[arXiv:1110.5325](#)] [[INSPIRE](#)].
- [70] A. Faraggi and L.A. Pando Zayas, *The Spectrum of Excitations of Holographic Wilson Loops*, *JHEP* **05** (2011) 018 [[arXiv:1101.5145](#)] [[INSPIRE](#)].
- [71] A. Faraggi, W. Mueck and L.A. Pando Zayas, *One-loop Effective Action of the Holographic Antisymmetric Wilson Loop*, *Phys. Rev. D* **85** (2012) 106015 [[arXiv:1112.5028](#)] [[INSPIRE](#)].
- [72] A. Faraggi, J.T. Liu, L.A. Pando Zayas and G. Zhang, *One-loop structure of higher rank Wilson loops in AdS/CFT*, *Phys. Lett. B* **740** (2015) 218 [[arXiv:1409.3187](#)] [[INSPIRE](#)].
- [73] D.H. Correa and F.I. Schaposnik Massolo, *D5-brane boundary reflection factors*, *JHEP* **05** (2013) 095 [[arXiv:1301.3412](#)] [[INSPIRE](#)].
- [74] S. Giombi, R. Ricci and D. Trancanelli, *Operator product expansion of higher rank Wilson loops from D-branes and matrix models*, *JHEP* **10** (2006) 045 [[hep-th/0608077](#)] [[INSPIRE](#)].
- [75] S. Mukhi and M. Smedback, *Bubbling orientifolds*, *JHEP* **08** (2005) 005 [[hep-th/0506059](#)] [[INSPIRE](#)].
- [76] T. van Ritbergen, A.N. Schellekens and J.A.M. Vermaseren, *Group theory factors for Feynman diagrams*, *Int. J. Mod. Phys. A* **14** (1999) 41 [[hep-ph/9802376](#)] [[INSPIRE](#)].
- [77] L. Frappat, A. Sciarrino and P. Sorba, *Dictionary on Lie algebras and superalgebras*, vol. 10, Academic Press San Diego, CA, U.S.A. (2000).