## Jet-like tunneling from a trapped vortex

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We analyze the tunneling of vortex states from elliptically shaped traps. Using the hydrodynamic representation of the Gross-Pitaevskii (Nonlinear Schrdinger) equation, we derive analytically and demonstrate numerically a novel type of quantum fluid flow: a jet-like singularity formed by the interaction between the vortex and the nonhomogenous field. For strongly elongated traps, the ellipticity overwhelms the circular rotation, resulting in the ejection of field in narrow, well-defined directions. These jets can also be understood as a formation of caustics since they correspond to a convergence of trajectories starting from the top of the potential barrier and meeting at a certain point on the exit line. They will appear in any coherent wave system with angular momentum and non-circular symmetry, such as superfluids, Bose-Einstein condensates, and light.

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Topological charges, such as vortices, are fundamental to the dynamics of coherent fields<sup>1,2</sup>. They appear in laser systems, carry charge in superconductors, characterize turbulence in quantum fluids, and hold potential for quantum memory<sup>3</sup>. To date, the main focus in vortex dynamics has been on transport, so that the charges could move and interact. (see e.g. Ref. 4) However, it is often desirable, and sometimes necessary, to confine and trap vortex structures. This is a basic problem in trapping theory, yet it has received very little attention. Here, we consider the dynamics of vortex decay in a potential and show that asymmetry in the potential can lead to the development of jets during wave tunneling. These formations concentrate wave density in the form of caustics and represent a new type of coherent structure for wave transport.

The emphasis on vorticity implies that phase dynamics will be important to the tunneling process. Even in the context of simple wavefunctions, without angular momentum, phase can have profound effects. Examples include the recent prediction of "blips" in the outgoing matter through a trap<sup>5–8</sup> and the development of dispersive shock waves<sup>9–12</sup>, e.g. when tunneling through a barrier.<sup>13</sup> These latter structures are traveling waves with oscillating phase that are finding increasing importance in fluids<sup>14,15</sup>, optics<sup>2,11</sup>, and Bose-Einstein condensates<sup>16,17</sup>. In spatially inhomogeneous potentials, such as the elliptical wells typical of BEC experiments<sup>18</sup>, both shock waves and blips can go unstable and generate vortices. Here, we consider the simplest case of a circular vortex trapped in an elliptical well and examine the competition of symmetry during wavefunction tunneling.

Tunneling problems are usually discussed within the framework of the WKB approximation, which looks for a solution of the Schrödinger equation (Nonlinear



FIG. 1: A cross-section of the potential  $V_{trap}(x, y)$  of an elliptical trap with cylindrically symmetric internal potential well, which holds a vortex state shown symbolically. The potential  $V_{well}(x, y)$ , which tends to a constant value for large x and/or y (shown by dashed-doted lines) is used for the preparation of the initial state.  $V_0$  is the difference between the top rim of the barrier and the chemical potential level  $\mu$  whose crossing with the barrier determines the exit line. The insert shows a view from above with the circular top rim, a vortex inside it, and the elliptic exit line outside.

Schrödinger (NLS) or Gross-Pitaevskii (GP) equation in our case)

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V_{trap}(\mathbf{r})\psi + \lambda|\psi|^2\psi \qquad(1)$$

as an expansion over  $\hbar$ . In the case of the NLS equation describing propagation of a classical coherent electromagnetic wave  $\hbar = 1$ ,  $\psi$  is the amplitude of the electric component, the propagation distance z plays part of time, the part of mass m is played by the wave vector of the light emitted by the laser, and the potential V is created by changing the linear refraction index of the medium. The appearance of singularities near the turn-

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ing points is typical for WKB approach. These singularities make generalization to 2d and higher dimensions a formidable technical problem (see, e.g. Ref. 19 and references therein) since then the points become singular lines or surfaces. (Nevertheless calculations can be quite straightforward in the cylindrical symmetry case<sup>20</sup> even from the vortex state.) The most promising way to deal with this problem is to use the hydrodynamic approach, which was applied to the tunneling dynamics in Refs. 5,21 [See also Refs. 9–12]. Two hydrodynamic equations

 $\frac{\partial}{\partial t}\rho + \nabla(\rho \mathbf{v}) = 0$ 

(2)

and

$$\frac{\partial}{\partial t}\mathbf{v} + \frac{1}{2}\nabla\mathbf{v}^2 = -\frac{1}{m}\nabla\left(V_{trap}(\mathbf{r}) - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} + \lambda\rho\right)$$
(3)

follow directly from the GP equation (1) for the density  $\rho = |\Psi|^2$  and the velocity  $m\mathbf{v} = -\hbar\nabla\varphi$  fields determined by the phase  $\varphi$  of the wave function  $\Psi$ . The principal advantage of this approach is the absence of any singular behavior of the wave function near the turning points. For example, the temporal tunneling dynamics was deduced analytically<sup>5,21</sup> from these equations accounting for the role of the interaction (nonlinearity) in 1d systems.

Here we consider wave tunneling from a twodimensional trap. We first consider an irrotational initial state in order to present our approach and for the sake of comparison. We then generalize it to the rotational case with a vortex initial state. We will show that in the case of nonzero angular momentum, jets appear in which matter/intensity exit the trap along certain preferential directions.

In the 1d case<sup>21</sup> (see also a more detailed derivation in Ref. 5), the adiabatic approximation in the Euler equation (3) yields the velocity at the exit point in the form  $v(x_{ex}) = \sqrt{2V_0/m}$ , where  $V_0$  is the energy difference between the top of the barrier and the chemical potential in the trap. The same definition holds for the 2d case (see Fig. 1). In this higher dimension, there is an exit curve where the chemical potential  $\mu$  crosses the trap potential. Integrating the exit flux  $\rho(\mathbf{r})\mathbf{v}_{ex}(\mathbf{r}) \cdot \hat{n}$  over this closed curve ( $\hat{n}$  is the unit vector normal to this curve), gives the total exit flux

$$\frac{dN(t)}{dt} = -\oint \rho(\mathbf{r})(\mathbf{v}_{ex}(\mathbf{r}) \cdot \hat{n})d\mathbf{l}.$$
(4)

The 2d tunneling problem<sup>5</sup> is mapped onto the classical motion of a fluid droplet (tracer) falling down from the top of the difference potential  $\Delta V(x, y) = V_{trap}(x, y) - V_{well}(x, y)$  to the observation points (x, y) on the exit line. Here,  $V_{well}(x, y)$  is the potential well used to prepare the initial state and  $V_{trap}(x, y)$  is the actual potential of the trap from which the tunneling takes place (see Fig. 1). In the case of an irrotational flow, the exit velocity vector is found from the equation  $m\ddot{\mathbf{r}} = -\nabla\Delta V(\mathbf{r})$ . Considering the example of the elliptic trap shown in Fig. 1,

$$\Delta V(x,y) = -\frac{m\omega^2}{2}(y^2 + \epsilon^2 x^2) \tag{5}$$

with the aspect ratio  $\epsilon \leq 1$  we get  $\mathbf{v}_{ex}(x, y) = (\epsilon \omega x, \omega y)$ .

Adiabatically slow varying density<sup>5,21</sup> in the trap may be described as  $\rho(N, \mathbf{r}, t) = N(t)\rho_0 \exp(-\frac{2}{\hbar}\sqrt{2mV_0}|\mathbf{r}|)$ and Eq. (4) becomes

$$\frac{dN}{dt} = -NI_{irrot} \tag{6}$$

where  $I_{irrot}$  is the integral escape rate. Using the polar coordinates,  $(r, \theta)$ , the differential escape rate reads

$$\frac{dI_{irrot}}{d\theta} = \frac{2V_0}{m\omega} \rho_0 \frac{\sin^2 \theta + \epsilon^3 \cos^2 \theta}{\chi^2(\varepsilon, \theta)} e^{-\frac{4\Delta u}{\sqrt{\chi(\varepsilon, \theta)}}}$$
(7)

where  $\Delta u = V_0/(\hbar\omega)$  and  $\chi(\varepsilon,\theta) = \sin^2 \theta + \epsilon^2 \cos^2 \theta$ . This equation is obtained by using the simple connection  $\epsilon \tan \beta = \tan \theta$  between the polar angle  $\theta$  and the angle  $\beta$  of the exit velocity direction (see  $v_{ex}$  after Eq. 5).

Assuming certain dependence (say, linear) of the interaction on the number of particles N remaining within the trap, its dependence on time for various values of the parameters of the system can be found from Eq. (6). Qualitatively these dependencies are rather close to those considered in detail in Ref. [<sup>21</sup>], discussing tunneling from one dimensional traps, although the numerical values may differ.

A 2d configuration allows the consideration of a rotational initial state, i.e. one with nonzero vorticity  $\tilde{\boldsymbol{\omega}} = \boldsymbol{\nabla} \times \mathbf{v}$ , which is not possible in 1d. Since the velocity field is  $m\mathbf{v} = -\hbar\boldsymbol{\nabla}\varphi$ , a finite vorticity in Eq.(3) appears if the phase  $\varphi$  is singular along some lines in 3d space or points in 2d space. As the wave function is single valued, the equation  $m \oint \mathbf{v}(\mathbf{r}) d\mathbf{l} = 2\pi\hbar\nu$  with an integer  $\nu$  holds for integration over any closed path. The phase  $\varphi = -\frac{2\pi\nu}{m} \arctan \frac{y}{x}$  corresponds to a vortex around the line x = y = 0 with the velocity field  $\mathbf{v}_{rot}(\mathbf{r}) = \nu \frac{\hbar}{m} \hat{n}_z \delta(x) \delta(y)$ . Eq. (3) requires that the density field vanishes as  $\rho(\varrho) \propto \varrho^{2|\nu|}$  at  $\varrho \to 0$  and the quantum potential blows up as  $\varrho^{-2}$ .

The total velocity field  $\mathbf{v}(\mathbf{r}) = \mathbf{v}_{rot}(\mathbf{r}) + \mathbf{v}_{pot}(\mathbf{r})$  is a sum of the rotational and irrotational velocity fields with  $\nabla \times$  $\mathbf{v}_{pot} = 0$ . Then we may carry out the same program as above, i.e. we prepare the initial state in a potential well  $V_{well}(x, y)$  with a vorticity characterized by an integer quantum number  $\nu$ . Then we change the potential into  $V_{trap}(x, y)$  allowing the wave function to tunnel through the barrier and apply the adiabatic approximation.<sup>5,21</sup> As a result, the Euler equation (3) takes the form

$$m\frac{\partial \mathbf{v}_{pot}(\mathbf{r},t)}{\partial t} + \frac{m}{2}\boldsymbol{\nabla}[\mathbf{v}_{pot}(\mathbf{r},t) + \mathbf{v}_{rot}(\mathbf{r})]^2 = -\boldsymbol{\nabla}\left[\Delta V(\mathbf{r}) - \frac{\hbar^2 \nu^2}{2m} \frac{1}{\varrho^2}\right]$$
(8)

with the centrifugal potential in the right hand side. Now the Cole - Hopf transformation<sup>5</sup> allows one to map the tunneling problem on the classical motion of a fluid tracer described by the equation

$$m\dot{\mathbf{v}}_{rot} + m\dot{\mathbf{v}}_{pot} = m\mathbf{v} \times \widetilde{\boldsymbol{\omega}} - \boldsymbol{\nabla} \left[ \Delta V(\mathbf{r}) - \frac{\hbar^2 \nu^2}{2m\varrho^2} \right].$$
 (9)

The "Lorentz force" in (9) is zero everywhere except for the line x = y = 0 and does not play a role.

For  $\mathbf{v}_{pot} = 0$  and  $\Delta V(\mathbf{r}) = 0$ , one gets  $\mathbf{v}_{rot} = \Omega \cdot \hat{n}_z \times \mathbf{r}_{\Omega}(t)$  with  $\mathbf{r}_{rot}(t) = \rho (\cos \Omega t, \sin \Omega t, 0)$  and  $\Omega = \frac{\hbar \nu}{m \rho^2}$ , which corresponds to the tracer making a circular rotation with the velocity  $v = \hbar \nu / m \rho$ . We again use the elliptic potential (5) and consider the simplest vortex with  $\nu = 1$ ; then the escape rate is calculated similarly to the rotationless case. We are interested in the angular dependence of the differential escape rate

$$\frac{dI_{rot}}{d\beta} = \frac{2V_0}{m\omega} \sqrt{1 + \frac{4\Delta u^2}{\chi(\varepsilon,\theta)}}.$$

$$\frac{\epsilon\cos\beta\cos\theta + \sin\beta\sin\theta}{\sqrt{\chi(\varepsilon,\theta)}} \left|\frac{d\theta}{d\beta}\right| e^{-\frac{4\Delta u}{\sqrt{\chi(\varepsilon,\theta)}}}$$
(10)

Contrary to the irrotational case (7), there is not now a simple relation between the polar coordinate  $\theta$  and the angle  $\beta$  of the escape direction. Therefore, we have to use the more general equation (10) in which the dependence  $\beta(\theta)$  is found by solving equation of motion (9) numerically.

The angular dependence (10) of the escape rate from a vortex state is the principal result of this paper. It shows that the rate is determined largely by the function  $\beta(\theta)$ , which is not necessarily monotonous. Fig. 2 shows that at small enough aspect ratio  $\epsilon$  and moderate barrier height  $V_0$ , the escape direction  $\beta(\theta)$  obtains a maximum value  $\beta_{max}$  on the exit curve at an angle  $\theta_{max}$ . At this point, the derivative  $\frac{d\theta}{d\beta}$  blows up, indicating that an ensemble of streamlines originating from a spread of  $\theta$  angles around  $\theta_{max}$  collapses together. This bunching results in the formation of a jet in the  $\beta_{max}$  direction.

For small eccentricity and moderate vorticity  $\nu$ , the exit flow is distorted only weakly. For large enough eccentricity and low barrier heights, or high enough vorticity, this distortion becomes strong and jets appear. An example of well-developed jets are shown in Fig. 3, obtained when the shape of the trap strongly deviates from the circular one ( $\epsilon = 0.1$ ).

The jets are caused by the interplay between the elliptical shape of the trap and the spherical symmetry of the vortex. If the equivalent tracer motion were dominated by the circular motion only, then we could have a strange situation in which the tracer could have left the elongated trap, traveled along its circular orbit, and then tried to re-enter the trap. Interestingly, this scenario is prevented by the *irrotational* part of the velocity



FIG. 2: Exit angle  $\beta$  as a function of polar angle  $\theta$  for the barrier hight  $\Delta u$  varying from 0.5 to 4 for a highly eccentric trap (aspect ratio  $\epsilon = 0.1$ ). All the curves pass maxima indicating the jets coming out of the trap.



FIG. 3: Normalized polar graphs of the differential escape rate  $\frac{dI}{d\beta}$  as a function of  $\beta$  for  $\epsilon = 0.1$ .

field, whose contribution leads to a frustration point and the formation of the jets. The meeting of many different tracer trajectories in the vicinity of  $\theta_{max}$  on the exit line gives rise to the formation of caustics. The resulting interference is similar to the tunneling dynamics considered in Ref. [19]. As shown in Fig. 3, the variation in tunneling speeds gives rise to wave steepening and a new type of angular shock.

In principle, the exponentially weak tunneling from an elliptically shaped trap violates the spherical symmetry of the problem and may cause a decay of the vortex state. In practice, however, the time span of the decay can be rather long (esp. if the vortex can be stabilized by a strong enough interaction)<sup>22</sup>, and observation of the above dynamics should be possible. Currently, the best candidate systems to observe jet-like tunneling are cold atoms in elliptically trapped Bose-Einstein condensates and coherent light confined in elongated optical waveguides.



FIG. 4: Numerical calculation of the evolution of a light propagating in an elliptically shaped semitransparent waveguide for the vorticity  $\nu = 1$  (left) for  $\Delta n/n_0 = 5.0 \cdot 10^{-5}$  written in PMMA and  $\nu = 8$  for  $\Delta n/n_0 = 1.0 \cdot 10^{-4}$  (right) induced in SBN. Formation of jets with the propagation of light at a distance 7.5 mm is clearly seen.

Here, we numerically demonstrate angular caustic formation in the optical case. The cylindrical waveguide required for an experimental realization can be written inside a medium such as a glass or polymethyl methacrylate (PMMA) using femtosecond laser pulses. As a realistic example, we consider a waveguide with a transverse refractive index profile (potential well barrier) shown in Fig. 1. The inner side of the well is circular while the outer edge is an ellipse with a semi-major axis of  $40\mu m$ (corresponding to an eccentricity  $\sqrt{1-\epsilon^2} = 0.9$ ). Simulations were carried out for the 2+1d system by solving the NLS equation (1) using a split-step beam propagation code. Results are shown in Fig. 4. Even with a small refractive index difference (potential barrier height) of  $\Delta n/n_0 = 5.0 \cdot 10^{-5}$ , jet-like tunneling occurs for a singlycharged vortex  $\nu = 1$  (see Fig. 4, left panel). For comparison, we also give similar results for light propagating in an optically-induced waveguide in SBN (Strontium Barium Niobate). In this case, the writing beam diffracts,

so the waveguide diameter is limited. Even with a  $50\mu$ m inner radius, the potential still diffracts. This weaker potential means that we cannot see the jet form if the input vortex is only singly-charged. Shown is the output field of a charge  $\nu = 8$  vortex after propagating 8mm in the crystal (see Fig. 4, right panel).

In summary, by using the hydrodynamic formulation of the nonlinear Schrödinger equation, we were able to carry out an analysis of tunneling from a vortex field trapped in an asymmetric potential well. Interference between the rotational motion of the field with a strongly asymmetric tunneling flow created angular caustics, resulting in jet-like radiation patterns. Analytic results were verified with numerical simulation.

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