Supplementary Material to "Series of Abelian and Non-Abelian States in $C > 1$ Fractional Chern Insulators"

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In this Supplementary Material, we provide additional numerical results that might be relevant to a more specialized audience. Our article only showed numerical results for the pyrochlore lattice model [25]. While this supplementary material provides additional evidences for this system, it also gives numerical results about Abelian and non-Abelian states in $C > 1$ fractional Chern insulators (FCI) for the $C = 2$ triangular lattice model [24], the two-orbitals model on triangular lattice [26] with $C = 3$ and the C-orbitals model on a square lattice [26].

PYROCHLORE

In this section, we provide additional evidence for the phase we found on the pyrochlore lattice [25]. For this model, we have used a $(k+1)$-body Hubbard interaction $H_{\text{int},k} = U \sum_i : \rho_i^{k+1} :$ where :: denotes the normal ordering and the sum runs over all the sites. We checked that upon flux insertion the groundstate manifold does not mix with higher energy states. Also, the insertion of one flux restores the original configuration. This can be observed for $N_e = 6$ particles on a $(N_x, N_y) = (3, 4)$ lattice for $C = 3$ with three-body interaction in Fig. 1. We checked that quasiholes states counting obey the same rules that the groundstate. Some energy spectra for quasiholes are shown on Fig. 2 for $C = 2$ with three-body interaction, on Fig. 3 for $C = 3$ with two-body interaction. In both cases, the total observed counting matches the colorful $(k, r)_C$ counting which is identical to the $(k, r + C - 1)_1$ counting.

As discussed in the article, the counting of the groundstate or quasihole degeneracy does not allow to distinguish between a colorless and colorful physics in the absence of an exact mapping [4]. But the particle entanglement spectrum (PES) [37] clearly indicates for $C > 2$ that the phase cannot be understood in terms of a colorless phase. We provide additional PES for $C = 2$ with three-body interaction on Fig. 4 and Fig. 5, and for $C = 3$ with three-body interaction on Fig. 6. As explained on the article, here only the $C = 3$ might display a difference from the spinless counting. Note that the case shown on Fig. 6, the known counting is still given by the $(2, 2)_3$ generalized exclusion principle (which matches the $(2, 4)_1$ counting).

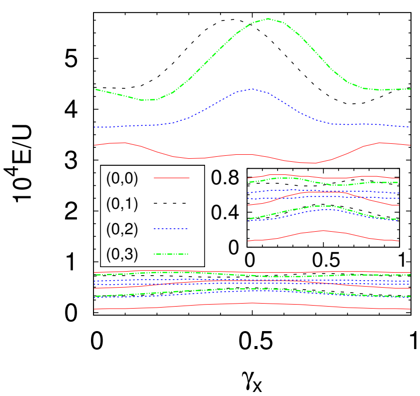

FIG. 1. Evolution of the low-lying states of the pyrochlore lattice model with $C = 3$ and three-body interaction in momentum sectors $(0,0)$, $(0,1)$, $(0,2)$ and $(0,3)$ with $N_e = 6$ bosons on a $(N_x, N_y) = (3, 4)$ lattice upon flux insertion along the x direction. γ_x counts the number of inserted flux quanta. We only show the momentum sector $(K_x, K_y) = (0,0)$, $(0,1)$, $(0,2)$, $(0,3)$ where the almost tenfold degenerate groundstate lies. The inset is a zoom on the low energy part of the spectrum.

$C = 2$ TRIANGULAR LATTICE

We have investigated the $C = 2$ triangular lattice model [24] with three-body interaction, the two-body interaction case have been studied in Ref. [24]. We have used the same parameters as in Ref. [24]. The interaction is given by $H_{\text{int},k} = U \sum_i : \rho_i^3 :$. We find convincing evidence of a sixfold degenerate groundstate at $\nu = \frac{2}{3}$ for even numbers of particles. The energy spectra are shown on Fig. 7 for several system sizes. We have checked that upon flux insertion the groundstate manifold does not mix with higher energy states. Also, the insertion of one flux restores the original configuration. This can be observed for $N_e = 8$ particles on a $(N_x, N_y) = (3, 4)$ lattice in Fig. 8. For the quasiholes and the PES, the results are similar to what we have found for the pyrochlore model, as seen respectively on Fig. 9 and 10.

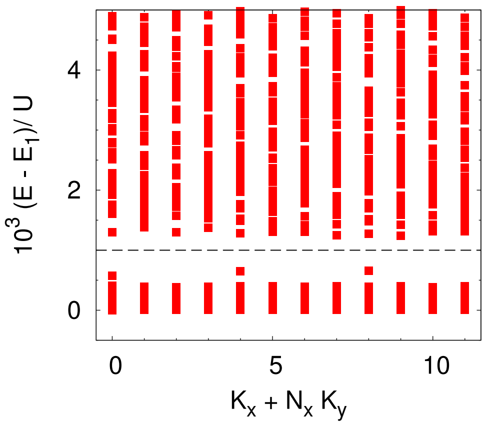

FIG. 2. Low energy spectra on the pyrochlore $C = 3$ model with three-body interaction for the $N_e = 6$ bosons on a $(N_x, N_y) = (4, 3)$ pyrochlore lattice (three sites added compared to the $\nu = 2/3$ groundstate). The energies are shifted by E_1, the lowest energy for each system size. The number of states below the gap (materialized by a dashed line) — is equal to 676 and is in agreement with the $(2, 3)_1$ counting.

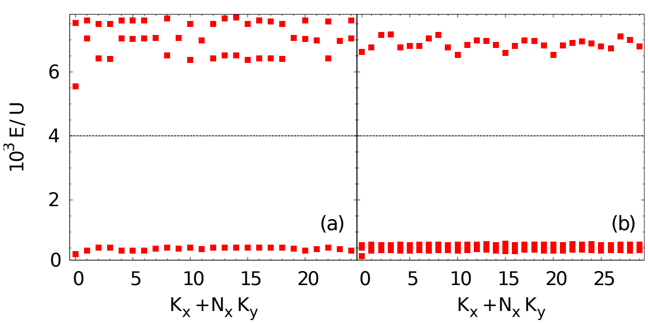

FIG. 3. Low energy spectra for the $N_e = 6$ bosons on a $(N_x, N_y) = (5, 5)$ pyrochlore lattice with $C = 3$ and two-body interaction (one site added compared to the $\nu = 1/4$ groundstate) (a) and for the $N_e = 7$ bosons on a $(N_x, N_y) = (5, 6)$ pyrochlore lattice (two sites added compared to the $\nu = 1/4$ groundstate) (b). The number of states below the gap (materialized by a dashed line) — respectively 25 (a) and 120 (b) — is in agreement with the $(1, 4)_1$ counting.

$C = 3$ TRIANGULAR LATTICE

On the 2-orbital triangular lattice [26] with $C = 3$, we have found a clear signature of a strongly correlated topological phase at $\nu = \frac{1}{4}$ with two-body interaction. We have use the model of ref. [26] with the optimized parameters $t_2/t_1 = 0.28$ and $t_3/t_1 = -0.22$. The interaction is slightly different from the one we have considered previously in order to make contact with the physical system. It consists of an isotropic interaction $H_{\text{int},k} = U \sum_i : (\sum_j \rho_{i,j})^2 :$ where the first sum runs over all the sites whereas the second sum runs over the different orbitals on the same site. On Fig. 11, we pro-

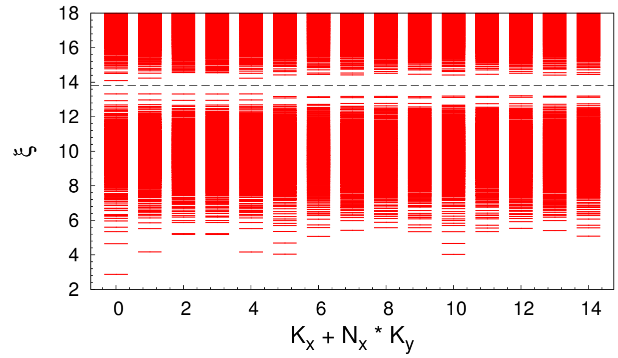

FIG. 4. PES for low energy groundstate manifold on the pyrochlore lattice with $C = 2$ and three-body interaction for $N_e = 10$ bosons and $N_A = 5$ on a $(N_x, N_y) = (5, 3)$ lattice. The number of states below the dotted line is 4278. This is equal to the $(2, 3)_1$ counting.

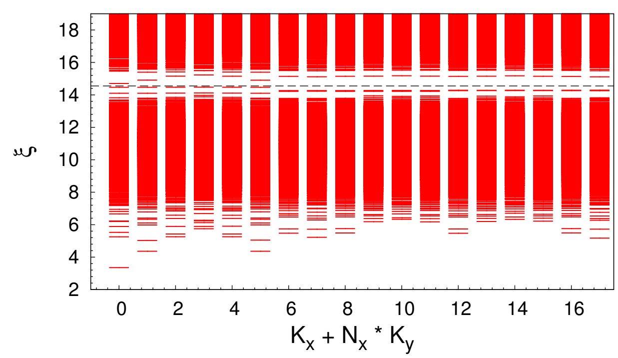

FIG. 5. PES for low energy groundstate manifold on the pyrochlore lattice with $C = 2$ and three-body interaction for $N_e = 12$ bosons and $N_A = 5$ on a $(N_x, N_y) = (6, 3)$ lattice. The number of states below the dotted line is 12870. This is equal to the $(2, 3)_1$ counting.

vide the energy spectrum for the groundstate at $\nu = \frac{1}{4}$ up to $N_e = 9$.

We present the PES in Fig. 12 for the groundstate manifold with $N_e = 7$ (upper panel), $N_e = 8$ (middle panel) and $N_e = 9$ (lower panel). For $N_e = 7$, the PES total counting is given by the $(1, 4)_1$ principle. The counting is identical to the pyrochlore case for $N_e = 9$. For $N_e = 8$ on the pyrochlore model, no clear entanglement gap was observed for $N_A = 4$. Here the clear entanglement gap allows to deduce the counting. This latest is lower than the one of the generalized Pauli exclusion principle $(1, 4)_1$, clearly showing that this phase cannot be associated to a Laughlin-like phase.

Note that for $N_e = 9$, the ratio gap over spread is equal to 45370 whereas for the pyrochlore model this ratio is equal to 58. This makes the $C = 3$ triangular lattice the best model we have studied. As observed in Fig. 12, the entanglement gaps are also bigger for this model than for

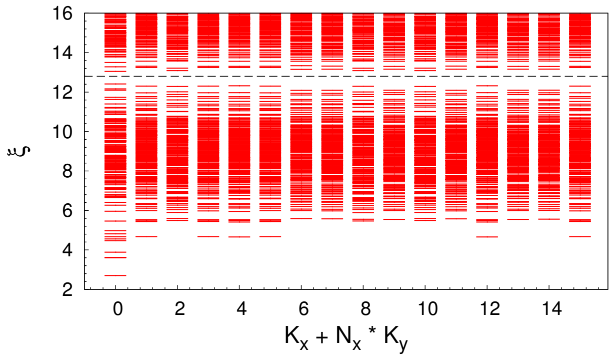

FIG. 6. PES for low energy groundstate manifold on the pyrochlore lattice with $C = 3$ and three-body interaction for $N_e = 8$ bosons and $N_A = 4$ on a $(N_x, N_y) = (4, 4)$ lattice. The number of states below the dotted line is 1956. This is equal to the number of states given by the $(2, 2)_3$ counting.

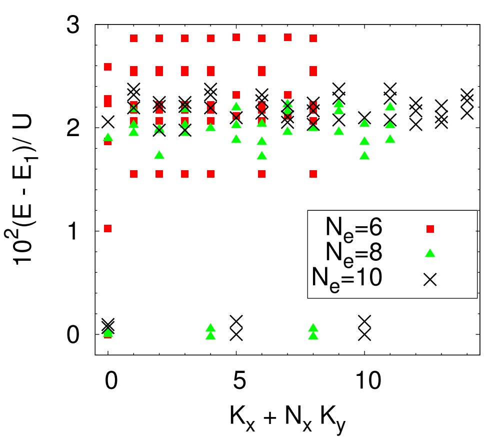

FIG. 7. Low energy spectra on the triangular lattice with $C = 2$ and three-body interaction for $N_e = 6, 8, 10$ bosons at $\nu = \frac{2}{3}$ on a $(N_x, N_y) = (N_e/2, 3)$ lattice. We observe an almost sixfold degenerate groundstate only for an even number of particles.

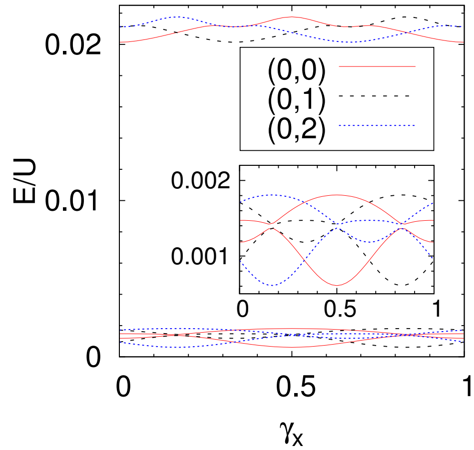

FIG. 8. Evolution of the low-lying states of the triangular lattice model with $C = 2$ and three-body interaction in momentum sectors $(0, 0)$, $(0, 1)$ and $(0, 2)$ with $N_e = 8$ bosons on a $(N_x, N_y) = (4, 3)$ lattice upon flux insertion along the x direction. γ_x counts the number of inserted flux quanta. We only show the momentum sector $(K_x, K_y) = (0, 0)$, $(0, 1)$, $(0, 2)$ where the almost sixfold degenerate groundstate lies.

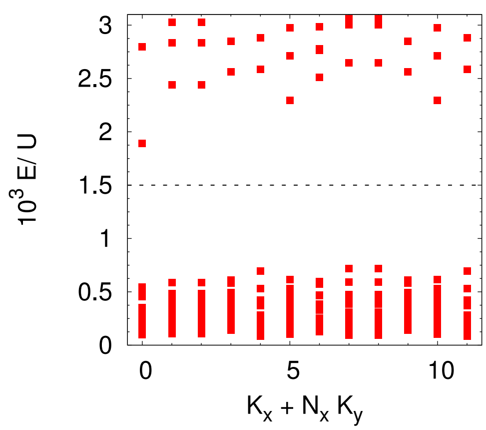

FIG. 9. Low energy spectra for $N_e = 7$ bosons on a $(N_x, N_y) = (3, 4)$ $C = 2$ triangular lattice with three-body interaction. The number of states below the gap (materialized by a dashed line) is 144, in agreement with the $(2, 3)_1$ counting.

the pyrochlore model.

4-ORBITALS MODEL WITH $C = 4$

On the C-orbitals model on a square lattice [26], we find convincing evidence for $C = 4$ at $\nu = \frac{1}{5}$ with two-body interaction. We have used the parameters that optimize the band flatness as given in Ref. [26]. For this model, we have used an isotropic interaction $H_{\text{int},1} = U \sum_i : (\sum_j \rho_{i,j})^2 :$ where the first sum runs over all the sites whereas the second sum runs over the different orbitals on the same site. On Fig. 13, we show the energy spectrum for the groundstate for $N_e = 6, 7$ and 8 bosons. For $N_e = 8$, the ratio gap over spread is equal to 15 whereas for the pyrochlore model this ratio is equal to 6.

The PES of the groundstate manifold for $N_e = 8$ and $N_A = 4$ is shown on Fig. 14. Interestingly, this counting does not match any of the known counting: in one hand, it is lower than the $(1, 2)_3$, either complete or reduced by the configurations than involves more than three particles with the same color. On the other hand, the counting is higher than the PES counting of the corresponding $SU(4)$ Halperin state. This example, that one should be

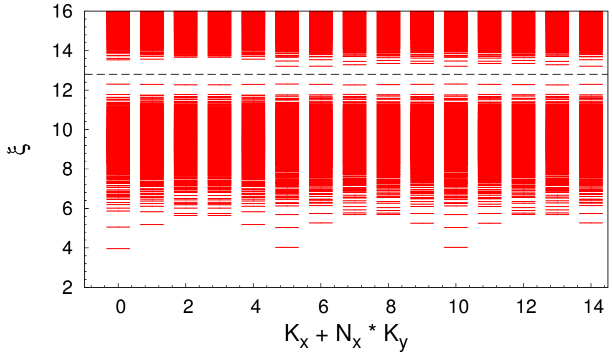

FIG. 10. PES for low energy groundstate manifold on the triangular lattice with $C = 2$ and three-body interaction for $N_e = 10$ bosons and $N_A = 5$ on a $(N_x, N_y) = (5, 3)$ lattice. The energies are shifted by E_1, the lowest energy for each system size. The number of states below the dotted line is 4278. This is equal to the $(2, 3)_1$ counting.

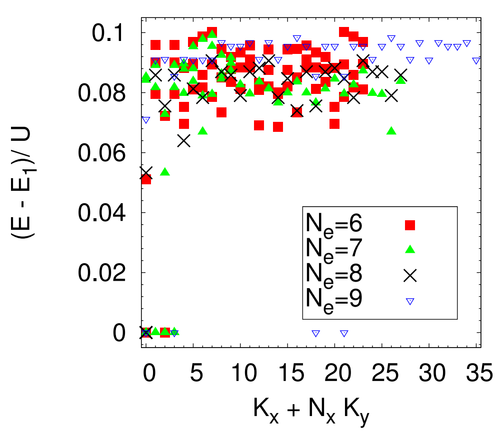

FIG. 11. Low energy spectra on the 2-orbitals on triangular lattice with $C = 3$ and two-body interaction for $N_e = 6, 7, 8$ bosons at $\nu = \frac{1}{4}$ on a $(N_x, N_y) = (N_e, 4)$ lattice. The energies are shifted by E_1, the lowest energy for each system size. As expected, we observe an almost fourfold degenerate groundstate. Note that for $N_e = 8$, the four lowest energy states in the $(K_x, K_y) = (0, 0)$ are so close in energy that it is impossible to distinguish them.

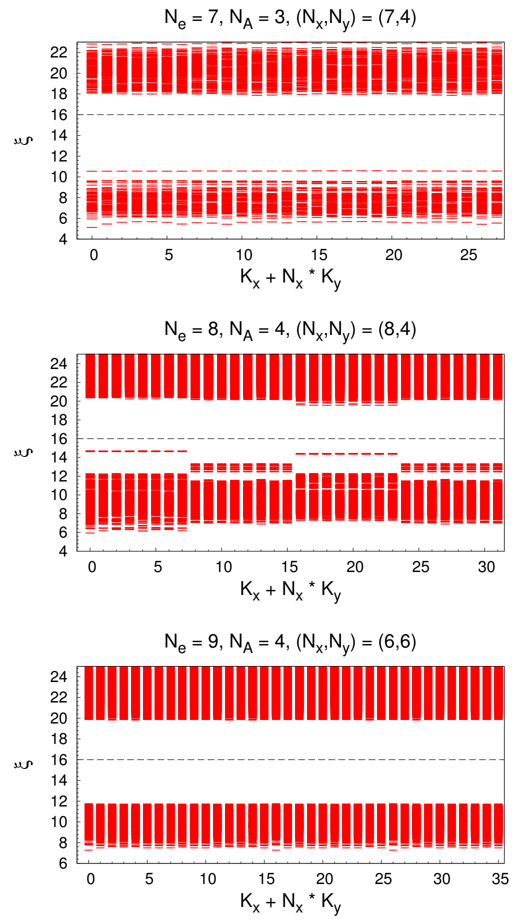

FIG. 12. Upper panel: PES for low energy groundstate manifold on the 2-orbitals on triangular lattice with $C = 3$ and two-body interaction for $N_e = 7$ bosons and $N_A = 3$ on a $(N_x, N_y) = (7, 4)$ lattice. The number of states below the dotted line is 1428 which is equal to the $(1, 4)_1$ counting. Middle panel: PES for low energy groundstate manifold on the 2-orbitals on triangular lattice with $C = 3$ and two-body interaction for $N_e = 8$ bosons and $N_A = 4$ on a $(N_x, N_y) = (8, 4)$ lattice. The number of states below the dotted line is 7112. The $(1, 4)_1$ counting gives 20 more states per sector. Lower panel: PES for low energy groundstate manifold on the 2-orbitals on triangular lattice with $C = 3$ and two-body interaction for $N_e = 9$ bosons and $N_A = 4$ on a $(N_x, N_y) = (6, 6)$ lattice. The number of states below the dotted line is 14364.

in principle related directly to the $SU(4)$ Halperin state but does not fall into any simple explanation, might be crucial to test any prediction on the effect of dislocation on the state counting.

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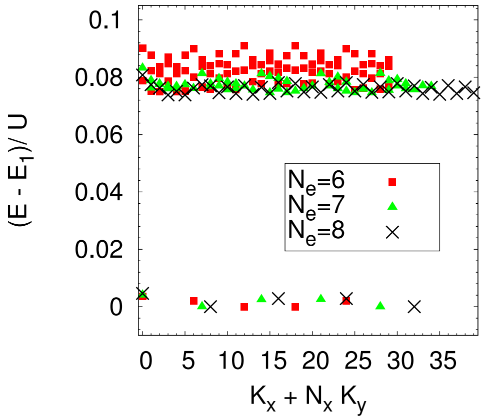

FIG. 13. Low energy spectra on the C-orbitals with $C = 4$ and two-body interaction for $N_e = 6, 7, 8$ bosons at $\nu = \frac{1}{5}$ on a $(N_x, N_y) = (N_e, 5)$ lattice. The energies are shifted by E_1, the lowest energy for each system size. We observe an almost fivefold degenerate groundstate.

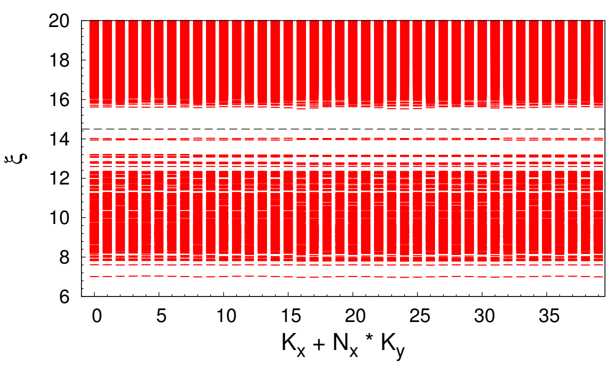

FIG. 14. PES for low energy groundstate manifold on the C-orbitals lattice with $C = 4$ and two-body interaction for $N_e = 8$ bosons and $N_A = 4$ on a $(N_x, N_y) = (8, 5)$ lattice. The number of states below the dotted line is 11410. The total $(1, 5)_1$ counting gives 17710 states. The counting of Halperin state PES is 8960 states. The number of configuration after suppressing the root configurations that involve more than 2 particles of the same color is 15210.