Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves

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July 3, 2018

Abstract

The Muskat problem models the evolution of the interface between two different fluids in porous media. The Rayleigh-Taylor condition is natural to reach linear stability of the Muskat problem. We show that the Rayleigh-Taylor condition may hold initially but break down in finite time. As a consequence of the method used, we prove the existence of water waves turning.

1 Introduction

The Muskat problem [26] models the evolution of an interface between two fluids of different characteristics in porous media by means of Darcy's law:

$$\frac{\mu}{\kappa}u = -\nabla p - (0, g\rho), \tag{1}$$

where $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, $u = (u_1(x,t), u_2(x,t))$ is the incompressible velocity (i.e. $\nabla \cdot u = 0$), $p = p(x,t)$ is the pressure, $\mu(x,t)$ is the dynamic viscosity, κ is the permeability of the isotropic medium, $\rho = \rho(x,t)$ is the liquid density, and g is the acceleration due to gravity. More precisely, the interface separates the domains Ω^1 and Ω^2 defined by

$$(\mu, \rho)(x_1, x_2, t) = \begin{cases} (\mu^1, \rho^1), & x \in \Omega^1(t) \\ (\mu^2, \rho^2), & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t), \end{cases}$$

and $\mu^1, \mu^2, \rho^1, \rho^2$ are constants. This physical situation is also related to the evolution of two fluids of different characteristics in a Hele-Shaw cell [22], due to the fact that the laws which model both phenomena are mathematically analogous [31].

This paper is concerned with the case $\mu^1 = \mu^2$ which provides weak solutions of the following transport equation

$$\rho_t + u \cdot \nabla \rho = 0,$$
$$\rho_0 = \rho(x, 0), \qquad x \in \mathbb{R}^2, \tag{2}$$

1

where initially the scalar ρ_0 is given by

$$\rho_0 = \rho(x_1, x_2, 0) = \begin{cases} \rho^1 & \text{in} \quad \Omega^1(0) = \{x_2 > f_0(x_1)\} \\ \rho^2 & \text{in} \quad \Omega^2(0) = \{x_2 < f_0(x_1)\}. \end{cases} \tag{3}$$

Let the free boundary be parametrized by

$$\partial\Omega^j(t) = \{z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) : \alpha \in \mathbb{R}\}$$

where

$$z(\alpha, t) - (\alpha, 0)$$

is 2π-periodic in the space parameter α or, an open contour vanishing at infinity

$$\lim_{\alpha \to \pm\infty} (z(\alpha, t) - (\alpha, 0)) = 0$$

with initial data $z(\alpha, 0) = z_0(\alpha) = (\alpha, f_0(\alpha))$. From Darcy's law, we find that the vorticity is concentrated on the free boundary $z(\alpha, t)$, and is given by a Dirac distribution as follows:

$$\nabla^\perp \cdot u(x, t) = \omega(\alpha, t)\delta(x - z(\alpha, t)),$$

with $\omega(\alpha, t)$ representing the vorticity strength i.e. $\nabla^\perp \cdot u$ is a measure defined by

$$< \nabla^\perp \cdot u, \eta > = \int \omega(\alpha, t)\eta(z(\alpha, t))d\alpha,$$

with $\eta(x)$ a test function.

Then $z(\alpha, t)$ evolves with an incompressible velocity field coming from the Biot-Savart law:

$$u(x, t) = \nabla^\perp \Delta^{-1} \nabla^\perp \cdot u(x, t).$$

As (x, t) approaches a point $z(\alpha, t)$ on the contour the velocity u agrees, modulo tangential terms, with the Birkhoff-Rott integral:

$$BR(z, \omega)(\alpha, t) = \frac{1}{2\pi} PV \int \frac{(z(\alpha, t) - z(\beta, t))^\perp}{|z(\alpha, t) - z(\beta, t)|^2} \omega(\beta, t)d\beta.$$

This yields an appropriate contour dynamics system:

$$z_t(\alpha, t) = BR(z, \omega)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t), \tag{4}$$

where the term c represents the change of parametrization and does not modify the geometric evolution of the curve [24].

The well-posedness is not guaranteed in general, in fact such a result turns out to be false for some initial data. Rayleigh [30] and Saffman-Taylor [31] gave a condition that must be satisfied for the linearized model in order to have a solution locally in time, namely that the normal component of the pressure gradient jump at the interface has to have a distinguished sign. This is known as the Rayleigh-Taylor condition:

$$\sigma(\alpha, t) = -(\nabla p^2(z(\alpha, t), t) - \nabla p^1(z(\alpha, t), t)) \cdot \partial_\alpha^\perp z(\alpha, t) > 0,$$

2

where $\nabla p^j(z(\alpha,t),t)$ denotes the limit gradient of the pressure obtained approaching the boundary in the normal direction inside $\Omega^j(t)$. We call $\sigma(\alpha,t)$ the Rayleigh-Taylor of the solution $z(\alpha,t)$.

Understanding the problem as weak solutions of (1-2) plus the incompressibility of the velocity, we find that the continuity of the pressure $(p^2(z(\alpha,t),t) = p^1(z(\alpha,t),t))$ follows as a mathematical consequence, making unnecessary to impose it as a physical assumption (for more details see [13] and [11]). For the surface tension case, there is a jump discontinuity of the pressure across the interface which is modeled to be equal to the local curvature times the surface tension coefficient:

$$p^2(z(\alpha,t),t) - p^1(z(\alpha,t),t)) = \tau\kappa(\alpha,t).$$

This is known as the Laplace-Young condition, which makes the initial value problem more regular. Then there are no instabilities [18] but fingering phenomena arise [29, 19].

By means of Darcy's law, we can find the following formula for the difference of the gradients of the pressure in the normal direction and the strength of the vorticity:

$$\sigma(\alpha,t) = (\rho^2 - \rho^1)\partial_\alpha z_1(\alpha,t)$$

$$\omega(\alpha,t) = -(\rho^2 - \rho^1)\partial_\alpha z_2(\alpha,t). \tag{5}$$

Above g is taken equal to 1 for the sake of simplicity.

Then, if we choose an appropriate term c in equation (4) (see section 2 below), the dynamics of the interface satisfies

$$z_t(\alpha,t) = \frac{\rho^2 - \rho^1}{2\pi}PV\int \frac{(z_1(\alpha,t) - z_1(\beta,t))}{|z(\alpha,t) - z(\beta,t)|^2}(\partial_\alpha z(\alpha,t) - \partial_\alpha z(\beta,t))d\beta. \tag{6}$$

A wise choice of parametrization of the curve is to have $\partial_\alpha z_1(\alpha,t) = 1$ (for more details see [13]). This yields the denser fluid below the less dense fluid if $\rho^2 > \rho^1$ and therefore the Rayleigh-Taylor condition holds as long as the interface is a graph. This fact has been used in [13] to show local existence in the stable case ($\rho^2 > \rho^1$), together with ill-posedness in the unstable situation ($\rho^2 < \rho^1$). Local existence for the general case ($\mu^1 \neq \mu^2$) is shown in [11], which was also treated in [34, 1].

¿From (6) it is easy to find the evolution equation for the graph:

$$f_t(\alpha,t) = \frac{\rho^2 - \rho^1}{2\pi}PV\int_{\mathbb{R}} \frac{(\alpha - \beta)}{(\alpha - \beta)^2 + (f(\alpha,t) - f(\beta,t))^2}(\partial_\alpha f(\alpha,t) - \partial_\alpha f(\beta,t))d\beta,$$
$$f(\alpha,0) = f_0(\alpha). \tag{7}$$

The above equation can be linearized around the flat solution to find the following nonlocal partial differential equation

$$f_t(x,t) = -\frac{\rho^2 - \rho^1}{2}\Lambda f(x,t),$$
$$f(x,0) = f_0(x), \quad x \in \mathbb{R},$$

3

where the operator Λ is the square root of the Laplacian. This linearization shows the parabolic character of the system.

Furthermore the stable system gives a maximum principle $\|f\|_{L^\infty}(t) \leq \|f\|_{L^\infty}(0)$ [14]; decay rates are obtained for the periodic case:

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty} e^{-Ct},$$

and also for the case on the real line (flat at infinity):

$$\|f\|_{L^\infty}(t) \leq \frac{\|f_0\|_{L^\infty}}{1 + Ct}.$$

There are several results on global existence for small initial data (small compared to 1 in several norms more regular than Lipschitz [9, 35, 32, 13, 19]) taking advantage of the parabolic character of the equation for small initial data. In [8] it is shown in the stable case that global existence for solutions holds if the first derivative of the initial data is smaller than an explicitly computable constant greater than 1/5. Furthermore, if $\|f_0\|_{L^\infty} < \infty$ and $\|\partial_x f_0\|_{L^\infty} < 1$, then there exists a global-in-time solution that satisfies

$$f(x, t) \in C([0, T] \times \mathbb{R}) \cap L^\infty([0, T]; W^{1,\infty}(\mathbb{R})),$$

for each $T > 0$. In particular f is Lipschitz continuous.

Moreover, equation (7) yields an L^2 decay:

$$\|f\|_{L^2}^2(t) + \frac{\rho^2 - \rho^1}{2\pi} \int_0^t ds \int_\mathbb{R} d\alpha \int_\mathbb{R} dx \, \ln\left(1 + \left(\frac{f(x,s) - f(\alpha,s)}{x - \alpha}\right)^2\right) = \|f_0\|_{L^2}^2,$$

which does not imply, for large initial data, a gain of derivatives in the system (see [8]). We will see below that the solutions to the Muskat problem with initial data in H^4 become real analytic immediately despite the weakness of the above decay formula.

The main result we present here is:

Theorem 1.1 *There exists a nonempty open set of initial data in H^4 with Rayleigh-Taylor strictly positive $\sigma > 0$ such that in finite time the Rayleigh-Taylor $\sigma(\alpha, t)$ of the solution of (6) is strictly negative for all α in a nonempty open interval.*

The geometry of this family of initial data is far from trivial: numerical simulations performed in [16] show that there exist initial data with large steepness for which a regularizing effect appears. In fact, as will be explained in Section 2, the first evidence of a change of sign in the Rayleigh-Taylor has been experimentally found in a model with two interfaces.

We proceed as follows:

First, in section 3, we assume initial conditions at time $t = t_0$ that satisfy the Rayleigh-Taylor ($\sigma > 0$) and the arc-chord condition, and for which the boundary z initially belongs to H^4. Let C_1 be the constant in the arc-chord condition, let C_2 be an upper bound for the H^4 norm of the initial data and let c_3 be a lower bound for σ. Then there exists $t_1 > t_0$, with t_1 depending only on C_1, C_2, c_3, such that the Muskat problem has a solution for time $t \in [t_0, t_1]$, satisfying also the arc-chord and Rayleigh-Taylor conditions. Moreover, for $t_0 < t \leq t_1$, the

4

solution $z(\alpha, t)$ is real analytic in a strip $S(t) = \{\alpha + i\zeta : |\zeta| \leq c(t - t_0)\}$, where c depends only on C_1, C_2, c_3.

Our goal in section 4 is to show that the region of analyticity does not collapse to the real axis as long as the Rayleigh-Taylor is greater than or equal to 0. This allows us to reach a regime for which the boundary z develops a vertical tangent.

Section 5 is devoted to showing the existence of a large class of analytic curves for which there exists a point where the tangent vector is vertical and the velocities indicate that the curves are going to turn over and reach the unstable regime for a small time. Plugging these initial data into a Cauchy-Kowalewski theorem indicates that the analytic curves turn over. Therefore the unstable regime is reached.

Finally, in section 6, a perturbative argument allows us to conclude that we can find curves in H^4 close enough to the special class of analytic curves described in Section 5, which satisfy the arc-chord and Rayleigh-Taylor conditions. Then we can show the existence of the curves passing the critical time and actually turning over. Therefore the unstable regime is reached for an entire H^4−neighborhood of initial data.

Remark 1.2 *In a forthcoming paper (see [5]) we will exhibit a particular initial datum for which we will show that once the curve reaches the unstable regime the strip of analyticity collapses in finite time and the solution breaks down. In section 8 we provide a very brief sketch of our proof of breakdown of smoothness for the Muskat equation. These results were announced in [6].*

Remark 1.3 *The same approach can be done for the water waves problem, which shows that, starting with some initial data given by $(\alpha, f_0(\alpha))$, in finite time the interface reaches a regime in which it is no longer a graph. Therefore there exists a time t^* where the solution of the free boundary problem parametrized by $(\alpha, f(\alpha, t))$ satisfies $\|f_\alpha\|_{L^\infty}(t^*) = \infty$ (see section 7). This scenario is known in the literature as wave breaking [7] and there are numerical simulations showing this phenomenon [4].*

Remark 1.4 *We conjecture that a result analogous to Theorem 1.1 holds, in which surface tension is included. We may simply use the same initial data as in Theorem 1.1, and take the coefficient of surface tension to be very small. The solutions are presumably changed only slightly by the surface tension (although we do not have a proof of this plausible assertion). Consequently, we believe that Muskat solutions with small surface tension can turn over.*

A similar remark applies to water waves (see theorem 7.1). There exist initial data for which water waves with surface tension turn over. A rigorous proof may be easily supplied, since local existence (backwards and forward in time) is known for water waves with surface tension (see [3]).

2   The contour equation and numerical simulations

Here we present the evolution equation in terms of the free boundary which is going to be used throughout the paper, and the numerical experiment that motivated the Theorem.

2.1 The equation of motion

By Darcy's law:

$$\nabla^\perp \cdot u = -(\rho^2 - \rho^1)\partial_\alpha z_2(\alpha)\delta(x - z(\alpha)),$$

and Biot-Savart yields

$$z_t(\alpha) = -\frac{(\rho^2 - \rho^1)}{2\pi}PV \int_{\mathbb{R}} \frac{(z(\alpha) - z(\alpha - \beta))^\perp}{|z(\alpha) - z(\alpha - \beta)|^2}\partial_\alpha z_2(\alpha - \beta)d\beta. \tag{8}$$

For the first coordinate above one finds

$$\frac{(\rho^2 - \rho^1)}{2\pi}PV \int_{\mathbb{R}} \frac{(z_2(\alpha) - z_2(\alpha - \beta))}{|z(\alpha) - z(\alpha - \beta)|^2}\partial_\alpha z_2(\alpha - \beta)d\beta$$

$$= -\frac{(\rho^2 - \rho^1)}{2\pi}PV \int_{\mathbb{R}} \frac{(z_1(\alpha) - z_1(\alpha - \beta))}{|z(\alpha) - z(\alpha - \beta)|^2}\partial_\alpha z_1(\alpha - \beta)d\beta$$

using the identity

$$PV \int_{\mathbb{R}} \partial_\beta\big(\ln(|z(\alpha) - z(\alpha - \beta)|^2)\big) d\beta = 0.$$

Therefore

$$z_t(\alpha) = -\frac{(\rho^2 - \rho^1)}{2\pi}PV \int_{\mathbb{R}} \frac{(z_1(\alpha) - z_1(\alpha - \beta))}{|z(\alpha) - z(\alpha - \beta)|^2}\partial_\alpha z(\alpha - \beta)d\beta.$$

Here we point out that in the Biot-Savart law the perpendicular direction appears, but after the above integration by parts, we only see the tangential direction.

Adding the tangential term

$$\frac{(\rho^2 - \rho^1)}{2\pi}PV \int_{\mathbb{R}} \frac{(z_1(\alpha) - z_1(\alpha - \beta))}{|z(\alpha) - z(\alpha - \beta)|^2}d\beta\partial_\alpha z(\alpha),$$

we find that the contour equation is given by

$$z_t(\alpha) = \frac{(\rho^2 - \rho^1)}{2\pi}PV \int_{\mathbb{R}} \frac{z_1(\alpha) - z_1(\alpha - \beta)}{|z(\alpha) - z(\alpha - \beta)|^2}(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta))d\beta.$$

For the 2π periodic interface the equation becomes

$$z_t(\alpha) = \frac{(\rho^2 - \rho^1)}{4\pi} \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\alpha - \beta))(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta))}{\cosh(z_2(\alpha) - z_2(\alpha - \beta)) - \cos(z_1(\alpha) - z_1(\alpha - \beta))}d\beta. \tag{9}$$

In order to see (9) we take $z(\alpha) = z_1(\alpha) + iz_2(\alpha)$; it is easy to rewrite (8) as follows;

$$\overline{z}_t(\alpha) = -\frac{(\rho^2 - \rho^1)}{2\pi i}PV \int_{\mathbb{R}} \frac{\partial_\alpha z_2(\beta)}{z(\alpha) - z(\beta)}d\beta.$$

The classical identity

$$\Big(\frac{1}{z} + \sum_{k\geq 1} \frac{z}{z^2 - (2\pi k)^2}\Big) = \frac{1}{2\tan(z/2)}$$

allows us to conclude that

$$z_t(\alpha) = \frac{(\rho^2 - \rho^1)}{4\pi} \int_{\mathbb{T}} \frac{(\sinh(z_2(\alpha) - z_2(\beta)), -\sin(z_1(\alpha) - z_1(\beta)))}{\cosh(z_2(\alpha) - z_2(\beta)) - \cos(z_1(\alpha) - z_1(\beta))} \partial_\alpha z_2(\beta) d\beta,$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Analogously, using the equality

$$\frac{(\rho^2 - \rho^1)}{4\pi} PV \int_{\mathbb{R}} \frac{\sinh(z_2(\alpha) - z_2(\beta))}{\cosh(z_2(\alpha) - z_2(\beta)) - \cos(z_1(\alpha) - z_1(\beta))} \partial_\alpha z_2(\beta) d\beta$$

$$= -\frac{(\rho^2 - \rho^1)}{4\pi} PV \int_{\mathbb{R}} \frac{\sin(z_1(\alpha) - z_1(\beta))}{\cosh(z_2(\alpha) - z_2(\beta)) - \cos(z_1(\alpha) - z_1(\beta))} \partial_\alpha z_1(\beta) d\beta$$

and adding the appropriate tangential term, we obtain equation (9).

2.2   The scenario motivated by the numerics

Our investigations started with the idea that interesting new phenomena may arise if we study three fluids, separated by two interfaces. Careful numerical studies indicated that one of the interfaces may turn over. In attempting to prove analytically the turnover indicated by the numerics, we discovered that a turnover can occur also for a single interface, i.e., for the Muskat problem. This section describes one of our numerical experiments.

Proceeding as in the preceding section, one can derive the equations modeling the evolution of two interfaces separating three fluids with different densities ρ_j ($j = 1, 2, 3$). More precisely, assume that both interfaces can be parametrized by graphs $(\alpha, f(\alpha, t))$ and $(\alpha, g(\alpha, t))$, with f lying above g. These equations read in the periodic case, cf. [16, 15] (this scenario has been recently also considered in [20]),

$$f_t(\alpha, t) = \bar{\rho}_1 \mathcal{I}[f(\cdot, t), f(\cdot, t)] + \bar{\rho}_2 \mathcal{I}[f(\cdot, t), g(\cdot, t)], \qquad f(\alpha, 0) = f_0(\alpha),$$
$$g_t(\alpha, t) = \bar{\rho}_2 \mathcal{I}[g(\cdot, t), g(\cdot, t)] + \bar{\rho}_1 \mathcal{I}[g(\cdot, t), f(\cdot, t)], \qquad g(\alpha, 0) = g_0(\alpha),$$

(10)

where $\bar{\rho}_j = (\rho_{j+1} - \rho_j)/(4\pi)$, $j = 1, 2$, and, for given functions $u(\alpha)$, $v(\alpha)$,

$$\mathcal{I}[u, v] := PV \int_{\mathbb{T}} \frac{(\partial_\alpha u(\alpha) - \partial_\alpha v(\alpha - \beta)) \tan(\beta/2)(1 - \tanh^2((u(\alpha) - v(\alpha - \beta))/2))}{\tan^2(\beta/2) + \tanh^2((u(\alpha) - v(\alpha - \beta))/2)} d\beta.$$

(11)

The first terms $\mathcal{I}[f(\cdot, t), f(\cdot, t)]$ and $\mathcal{I}[g(\cdot, t), g(\cdot, t)]$ in (10) give the velocity of a unique interface. The cross terms $\mathcal{I}[f(\cdot, t), g(\cdot, t)]$ and $\mathcal{I}[g(\cdot, t), f(\cdot, t)]$ take into account the interaction of the two interfaces, and their contribution is getting bigger when the curves are getting closer. This, together with the diffusive behavior reported in [16] for the equation

$$f_t(\alpha, t) = \bar{\rho}_1 \mathcal{I}[f(\cdot, t), f(\cdot, t)], \quad f(\alpha, 0) = f_0(\alpha),$$

(12)

and the mean conservation for f and g, motivate the choice of the following initial data, in the hope that some non regularizing effect arises from the interaction of the two interfaces;

$$f_0(\alpha) = \begin{cases} 0.1 - \sin^3\left(\dfrac{\pi(\alpha - M_1 + r_1)}{2r_1}\right), & \text{if } \alpha \in [M_1 - r_1, M_1 + r_1], \\ 0.1, & \text{otherwise} \end{cases}$$
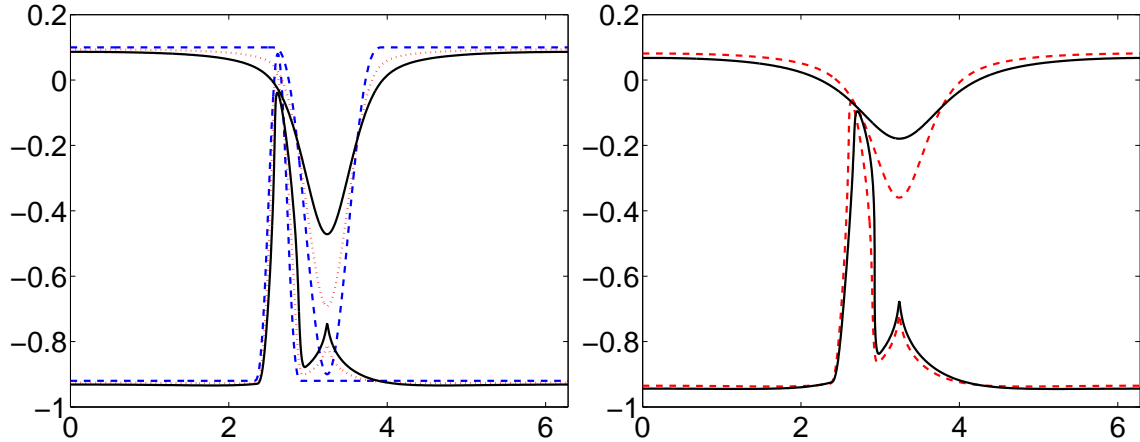
(13)

7

Figure 1: *Left:* Solutions to (10) with initial data (13)-(14) at times $t = 0$ (dashed blue), $t = 3.46 \cdot 10^{-4}$ (red points) and $t = 7.66 \cdot 10^{-4}$ (black). *Right:* Solutions at $t = 1.04 \cdot 10^{-3}$ (dashed red) and $t = 1.84 \cdot 10^{-3}$ (black)

and

$$
g_0(\alpha) = \begin{cases} \sin^3\left(\dfrac{\pi(\alpha - M_2 + r_2)}{2r_2}\right)^3 - 0.92, & \text{if } \alpha \in [M_2 - r_2, M_2 + r_2], \\ g_0(\alpha) = -0.92, & \text{otherwise.} \end{cases}
\tag{14}
$$

The choice of parameters $M_1 = \pi + 0.1$, $r_1 = 0.7$, $M_2 = \pi/1.2$, $r_2 = 0.3$, $\bar{\rho}_1 = 20\pi$ and $\bar{\rho}_2 = \pi/20$, yielded a strong growth of the derivative in the the lower interface as the two curves approach, as shown in Figure 1. Moreover, after introducing a small modification in the lower interface so that the tangent at a certain point becomes actually infinite, and evaluating the normal velocity relative to this point along the modified curve, we obtain the result plotted in Figure 2. This graphic clearly indicates that the velocity field is forcing the interface to turn over.

The numerical approximation of (10) addresses as a main difficulty the absolute lack of knowledge about the behavior of the solutions to (10). Indeed, the goal of our experiments is precisely the search for some singular behavior. The nonlocal terms make the computations expensive and special care has to be taken in order to evaluate the integrands in a neighborhood of $\beta = 0$. For this, we used Taylor expansions locally and computed exactly the principal value. In this situation, adaptivity is strongly indicated, both in space and time, since a good indicator of a singular behavior will be given either by a sudden accumulation of spatial nodes or a sudden reduction of the time steps.

In order to attain the highest resolution in the integration of (10) and compute the solutions shown in Figure 1, cubic spline interpolation of the curves $f(\cdot, t)$ and $g(\cdot, t)$ with periodic boundary conditions was used. This provides a $\mathcal{C}^2$ interpolant of each interface at every time and allows, in particular, the evaluation of the convolution terms at any $\beta \in [0, 2\pi]$. Then, adaptive quadrature can be applied to approximate the integrals and evaluate the derivative at any time. In the experiments reported, adaptive Lobatto quadrature was used,
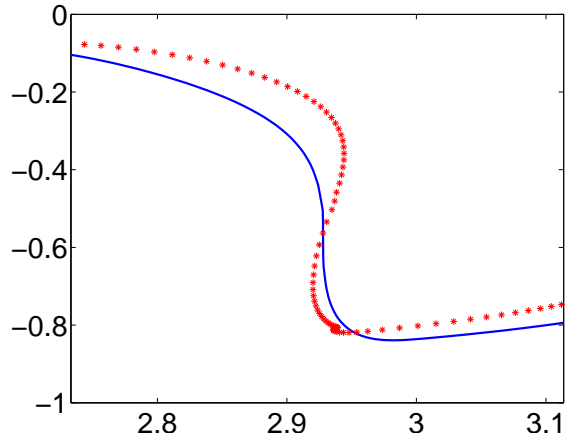
8

Figure 2: Zoom of the interface, modified so that its tangent is vertical at a single point P; and the normal velocity along the curve, minus that at P, scaled by a factor of 100.

by means of the MATLAB routine `quadl`. For the time integration, the embedded Runge–Kutta formula due to Dormand and Prince, DOPRI5(4), was implemented, since the problem was not found to be particularly stiff, see for instance [21]. The time stepping was combined with a spatial node redistribution after every successful step. For the redistribution of the spatial nodes an algorithm following [17] was implemented, with some modifications taking into account that both interfaces are graphs. For several tolerance requirements and different choices of the parameters involved in the full adaptive routine, the integration always failed at a certain critical time, suggesting the explosion of the derivative at a certain point of the lower interface and the lack of validity of (10), once this curve stops being a graph.

   The phenomenon described above and the explicit representations of the maximum of the solutions derived in [14], motivated the search for special initial data which allowed us to understand that this behavior also arises in the one-interface case.

3   Instant Analyticity

Here we show the main estimates that provide local-existence and instant analyticity for a single curve that satisfies initially the arc-chord and Rayleigh-Taylor conditions. We consider the function

$$F(z)(\alpha, \beta) = \frac{\beta^2}{|z(\alpha) - z(\alpha - \beta)|^2}, \quad \alpha, \beta \in \mathbb{R},$$

and in the periodic setting

$$F(z)(\alpha, \beta) = \frac{||\beta||^2}{2(\cosh(z_2(\alpha) - z_2(\alpha - \beta)) - \cos(z_1(\alpha) - z_1(\alpha - \beta)))}, \quad \alpha, \beta \in \mathbb{T},$$

where $||x|| = dist(x, 2\pi\mathbb{Z})$.

   If $F(z) \in L^\infty$ then we say that the curve satisfies the arc-chord condition, and the L^∞ norm of F is called the arc-chord constant.

9

Let us clarify the meaning of the above arc-chord condition. Fix t, and assume that $z(\alpha, t)$ is a smooth function of α. Suppose $F \in L^\infty$. Letting β tend to zero, we conclude that $|\partial_\alpha z(\alpha, t)|$ is bounded below. Since also z is smooth, $|\partial_\alpha z(\alpha, t)|$ is also bounded above. Consequently, the numerator in the fraction defining F is comparable to the square of the arc-length between $z(\alpha, t)$ and $z(\alpha - \beta, t)$. On the other hand, the denominator of that fraction is comparable to the square of the length of the chord joining $z(\alpha, t)$ to $z(\alpha - \beta, t)$. Thus, the boundedness of F expresses the standard arc-chord condition for the curve $z(\cdot, t)$ together with a lower bound for $|\partial_\alpha z(\alpha, t)|$.

Theorem 3.1 *Let $z(\alpha, 0) = z_0(\alpha) \in H^4$, $F(z_0)(\alpha, \beta) \in L^\infty$ and $\partial_\alpha z_1(\alpha, 0) > 0$ (R-T). Then there is a solution of the Muskat problem $z(\alpha, t)$ defined for $0 < t \leq T$ that continues analytically into the strip $S(t) = \{\alpha + i\zeta : |\zeta| < ct\}$ for each t. Here, c and T are determined by upper bounds of the H^4 norm and the arc-chord constant of the initial data and a positive lower bound of $\partial_\alpha z_1(\alpha, 0)$. Moreover, for $0 < t \leq T$, the quantity*

$$\sum_{\pm} \int \left(|z(\alpha \pm ict) - (\alpha + ict, 0)|^2 + |\partial_\alpha^4 z(\alpha \pm ict)|^2 \right) d\alpha$$

is bounded by a constant determined by upper bounds for the H^4 norm and the arc-chord constant of the initial data and a positive lower bound of $\partial_\alpha z_1(\alpha, 0)$. Above $|\cdot|$ is the modulus of a complex number or a vector in $\mathbb{C}^2$.

Proof: For the proof we consider the contour $z \in H^4$ with $z - (\alpha, 0)$ periodic and $\partial_\alpha z_1(\alpha, 0) > 0$. In the case of the real line similar arguments hold. The Muskat equation reads

$$z_t(\alpha) = \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\alpha - \beta))(\partial_\alpha z(\alpha) - \partial_\alpha z(\alpha - \beta))}{\cosh(z_2(\alpha) - z_2(\alpha - \beta)) - \cos(z_1(\alpha) - z_1(\alpha - \beta))} d\beta, \tag{15}$$

where we suppose $\partial_\alpha z_1(\alpha, 0) > 0$. We also take $\rho^2 - \rho^1 = 4\pi$ since we are studying the case $\rho^2 > \rho^1$. For the complex extension one finds

$$z_t(\alpha + i\zeta) = \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha + i\zeta) - z_1(\alpha + i\zeta - \beta))(\partial_\alpha z(\alpha + i\zeta) - \partial_\alpha z(\alpha + i\zeta - \beta))}{\cosh(z_2(\alpha + i\zeta) - z_2(\alpha + i\zeta - \beta)) - \cos(z_1(\alpha + i\zeta) - z_1(\alpha + i\zeta - \beta))} d\beta. \tag{16}$$

We will use energy estimates. Consider

$$S(t) = \{\alpha + i\zeta \in \mathbb{C} : \alpha \in \mathbb{T}, |\zeta| < ct\},$$

for c given below[1],

$$\|z\|_{L^2(S)}^2(t) = \sum_{\pm} \int_{\mathbb{T}} |z(\alpha \pm ict, t) - (\alpha \pm ict, 0)|^2 d\alpha,$$

$$\|z\|_{H^k(S)}^2(t) = \|z\|_{L^2(S)}^2(t) + \sum_{\pm} \int_{\mathbb{T}} |\partial_\alpha^k z(\alpha \pm ict, t)|^2 d\alpha,$$

---

[1]At the end of the proof we can take any $c < \min_\alpha(\partial_\alpha z_1(\alpha, 0)/|\partial_\alpha z(\alpha, 0)|^2)$.

where $k \geq 2$ as an integer, and

$$F(z)(\alpha + i\zeta, \beta) = \frac{||\beta||^2}{2(\cosh(z_2(\alpha + i\zeta) - z_2(\alpha + i\zeta - \beta)) - \cos(z_1(\alpha + i\zeta) - z_1(\alpha + i\zeta - \beta)))}, \tag{17}$$

with norm

$$\|F(z)\|_{L^\infty(S)}(t) = \sup_{\alpha + i\zeta \in S(t), \beta \in \mathbb{T}} |F(z)(\alpha + i\zeta, \beta)|.$$

Next, we define as follows:

$$\|z\|_S^2(t) = \|z\|_{H^4(S)}^2(t) + \|F(z)\|_{L^\infty(S)}(t).$$

We shall analyze the evolution of $\|z\|_{H^4(S)}(t)$.

Before starting the energy estimates, we mention an idea used previously e.g. in the proof of (6.3) in [11]. Suppose $A(\alpha, \beta)$ is a $C^1(\mathbb{T})$ function, and suppose $f(\alpha)$ belongs to $L^2(\mathbb{T})$. To estimate

$$\int_{-\pi}^{\pi} A(\alpha, \alpha - \beta)\frac{1}{2}\cot(\frac{\beta}{2})f(\alpha - \beta)d\beta \tag{18}$$

we break up this integral as the sum of

$$A(\alpha, \alpha)\int_{-\pi}^{\pi} \frac{1}{2}\cot(\frac{\beta}{2})f(\alpha - \beta)d\beta \tag{19}$$

and

$$\int_{-\pi}^{\pi} \left\{[A(\alpha, \alpha - \beta) - A(\alpha, \alpha)]\frac{1}{2}\cot(\frac{\beta}{2})\right\}f(\alpha - \beta)d\beta. \tag{20}$$

The integral in (19) is simply the Hilbert transform of f and the quantity in curly brackets in (19) is bounded. This idea will be used repeatedly, with $A(\alpha, \beta)$ arising from derivatives $\partial_\alpha^k z(\alpha, t)$ up to order 2, and with $f(\alpha) = \partial_\alpha^4 z_\mu(\alpha, t)$ ($\mu = 1, 2$). Whenever we use this scheme, we will simply say that "a Hilbert transform arises". For similar simple ideas used below, we refer the reader to the term J_1 in pg. 485 in [11].

Then, using above scheme, for the low order terms in derivatives, it is easy to find that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |z(\alpha \pm ict), t) - (\alpha \pm ict, 0)|^2 d\alpha \leq C(\|z\|_S(t) + 1)^k. \tag{21}$$

In (21) and in several of the estimates, k denotes a enough large universal constant.

Next, we check that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z(\alpha \pm ict, t)|^2 d\alpha = \sum_{j=1,2} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z_j(\alpha \pm ict, t)|^2 d\alpha$$

where

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z_j(\alpha \pm ict, t)|^2 d\alpha = \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z_j(\alpha \pm ict, t)}(\partial_t(\partial_\alpha^4 z_j)(\alpha \pm ict, t) \pm ic\partial_\alpha^5 z_j(\alpha \pm ict, t))d\alpha. \tag{22}$$

11

In order to simplify the exposition we write $z(\alpha, t) = z(\alpha)$ for a fixed t, we treat both coordinates at the same time, we write $(x_1, x_2) \cdot (x_3, x_4) = x_1 x_3 + x_2 x_4$ for $x_j \in \mathbb{C}$, $j = 1, ..., 4$, we denote $\alpha \pm ict = \gamma$, and we define

$$Q(\gamma, \beta) = \cosh(z_2(\gamma) - z_2(\gamma - \beta)) - \cos(z_1(\gamma) - z_1(\gamma - \beta)).$$

Then we split the right hand side of (22) by writing

$$I_1 = \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 z_t(\gamma) d\alpha,$$

and

$$I_2 = \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot ic\partial_\alpha^5 z(\gamma) d\alpha.$$

In I_1 we will find the R-T and use it to absorb I_2. We will decompose I_1 in order to find the terms of at least fourth order. In order to estimate the lower order terms, we refer the reader to the paper [11] (see, e.g., Lemma 6.1). We have $I_1 = J_1 + J_2 + J_3 +$ l.o.t., where

$$\|\text{l.o.t}\|_{L^2(\mathbb{T})} \leq C(\|z\|_S + 1)^k,$$

and J_1, J_2, J_3 are defined as follows:

$$J_1 = \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \left(\int_{\mathbb{T}} A(\gamma, \beta) \frac{\partial_\alpha^4 z_1(\gamma) - \partial_\alpha^4 z_1(\gamma - \beta)}{Q(\gamma, \beta)} (\partial_\alpha z(\gamma) - \partial_\alpha z(\gamma - \beta)) d\beta \right) d\alpha,$$

where $A(\gamma, \beta) = \cos(z_1(\gamma) - z_1(\gamma - \beta))$,

$$J_2 = -\Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \left(\int_{\mathbb{T}} \frac{\sin(z_1(\gamma) - z_1(\gamma - \beta))}{(Q(\gamma, \beta))^2} (\partial_\alpha z(\gamma) - \partial_\alpha z(\gamma - \beta)) B(\gamma, \beta) d\beta \right) d\alpha$$

where

$$B(\gamma, \beta) = (\sin(z_1(\gamma) - z_1(\gamma - \beta)), \sinh(z_2(\gamma) - z_2(\gamma - \beta))) \cdot (\partial_\alpha^4 z(\gamma) - \partial_\alpha^4 z(\gamma - \beta)),$$

and

$$J_3 = \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \left(\int_{\mathbb{T}} \frac{\sin(z_1(\gamma) - z_1(\gamma - \beta))}{Q(\gamma, \beta)} (\partial_\alpha^5 z(\gamma) - \partial_\alpha^5 z(\gamma - \beta)) d\beta \right) d\alpha.$$

We split further $J_1 = K_1 + K_2$ where

$$K_1 = \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^4 z_1(\gamma) \left(PV \int_{\mathbb{T}} \frac{A(\gamma, \beta)}{Q(\gamma, \beta)} (\partial_\alpha z(\gamma) - \partial_\alpha z(\gamma - \beta)) d\beta \right) d\alpha,$$

$$K_2 = -\Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \left(PV \int_{\mathbb{T}} \frac{A(\gamma, \beta)}{Q(\gamma, \beta)} (\partial_\alpha z(\gamma) - \partial_\alpha z(\gamma - \beta)) \partial_\alpha^4 z_1(\gamma - \beta) d\beta \right) d\alpha.$$

Taking into account the complex extension of the arc-chord condition, it is easy to deal with K_1 to obtain

$$K_1 \leq (\|z\|_S(t) + 1)^k.$$

12

In K_2 it is possible to find a "Hilbert transform" applied to $\partial_\alpha^4 z_1$ as in (18), and therefore an analogous estimate follows. We are done with J_1. For J_2 we obtain similarly

$$J_2 \leq (\|z\|_S(t) + 1)^k.$$

Next, we split $J_3 = K_3 + K_4$ where

$$K_3 = \Re \int_\mathbb{T} \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha^5 z(\gamma) \left(PV \int_\mathbb{T} \frac{\sin(z_1(\gamma) - z_1(\gamma - \beta))}{Q(\gamma, \beta)} d\beta \right) d\alpha,$$

$$K_4 = -\Re \int_\mathbb{T} \overline{\partial_\alpha^4 z(\gamma)} \cdot \left(PV \int_\mathbb{T} \frac{\sin(z_1(\gamma) - z_1(\gamma - \beta))}{Q(\gamma, \beta)} \partial_\alpha^5 z(\gamma - \beta) d\beta \right) d\alpha.$$

We have to be careful, because K_3 for real curves is harmless, but for complex curves we need to use the dissipative term to cancel out a dangerous term. We denote

$$f(\gamma) = PV \int_\mathbb{T} \frac{\sin(z_1(\gamma) - z_1(\gamma - \beta))}{Q(\gamma, \beta)} d\beta \tag{23}$$

and therefore $K_3 = L_1 + L_2$ where

$$L_1 = \int_\mathbb{T} \Re(f)(\Re(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z) + \Im(\partial_\alpha^4 z)\Im(\partial_\alpha^5 z))d\alpha,$$

$$L_2 = \int_\mathbb{T} \Im(f)(-\Re(\partial_\alpha^4 z)\Im(\partial_\alpha^5 z) + \Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z))d\alpha.$$

An easy integration by parts allows us to get

$$L_1 = -\frac{1}{2} \int_\mathbb{T} \Re(\partial_\alpha f)|\partial_\alpha^4 z|^2 d\alpha \leq C(\|z\|_S(t) + 1)^k.$$

For L_2 we find

$$L_2 = \int_\mathbb{T} \Im(\partial_\alpha f)\Re(\partial_\alpha^4 z)\Im(\partial_\alpha^4 z)d\alpha + 2\int_\mathbb{T} \Im(f)\Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z))d\alpha.$$

The first term on the right is easy to dominate by $C(\|z\|_S + 1)^k$. We denote the second one by M_1. We claim that

$$M_1 \leq C(\|z\|_S(t) + 1)^k + K\|\Im(f)\|_{H^2(S)}\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(S)}^2, \tag{24}$$

for $K > 0$ universal constant. To see this, we rewrite

$$M_1 = -2\int_\mathbb{T} \Im(f)\Im(\partial_\alpha^4 z)\Re(\Lambda(H(\partial_\alpha^4 z)))d\alpha$$

which yields

$$M_1 = -2\int_\mathbb{T} \Lambda^{1/2}(\Im(f)\Im(\partial_\alpha^4 z))\Re(\Lambda^{1/2}(H(\partial_\alpha^4 z)))d\alpha$$

13

and therefore

$$
\begin{aligned}
M_1 \leq & 2\|\Lambda^{1/2}(\Im(f)\Im(\partial_\alpha^4 z))\|_{L^2(S)}\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(S)} \\
\leq & C\|\Im(f)\|_{H^2(S)}(\|\partial_\alpha^4 z\|_{L^2(S)} + \|\Lambda^{1/2}(\partial_\alpha^4 z)\|_{L^2(S)})\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(S)} \\
\leq & C(\|z\|_S(t) + 1)^k + K\|\Im(f)\|_{H^2(S)}\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(S)}^2.
\end{aligned}
$$

Finally we find that

$$
K_3 \leq C(\|z\|_S(t) + 1)^k + K\|\Im(f)\|_{H^2(S)}\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(S)}^2. \tag{25}
$$

We will use the thickness of the strip to control the unbounded term above.

For K_4 we decompose further: $K_4 = L_3 + L_4 + L_5 + L_6$ where

$$
L_3 = -\Re \int_{-\pi}^\pi \overline{\partial_\alpha^4 z(\gamma)} \cdot \int_{-\pi}^\pi \frac{\beta^2}{Q(\gamma,\beta)} \frac{1}{\beta} \Big(\frac{\sin(z_1(\gamma) - z_1(\gamma-\beta))}{\beta} - \partial_\alpha z_1(\gamma) \Big) \partial_\alpha^5 z(\gamma-\beta) d\beta d\alpha,
$$

$$
L_4 = -\Re \int_{-\pi}^\pi \overline{\partial_\alpha^4 z(\gamma)} \cdot \partial_\alpha z_1(\gamma) \int_{-\pi}^\pi \Big(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{2}{|\partial_\alpha z(\gamma)|^2} \Big) \frac{1}{\beta} \partial_\alpha^5 z(\gamma-\beta) d\beta d\alpha,
$$

$$
L_5 = -\Re \int_{-\pi}^\pi \overline{\partial_\alpha^4 z(\gamma)} \cdot \frac{\partial_\alpha z_1(\gamma)}{|\partial_\alpha z(\gamma)|^2} \int_{-\pi}^\pi \Big(\frac{2}{\beta} - \frac{1}{\tan(\beta/2)} \Big) \partial_\alpha^5 z(\gamma-\beta) d\beta d\alpha,
$$

$$
L_6 = -\Re \int_{-\pi}^\pi \overline{\partial_\alpha^4 z(\gamma)} \cdot \frac{\partial_\alpha z_1(\gamma)}{|\partial_\alpha z(\gamma)|^2} \Lambda(\partial_\alpha^4 z)(\gamma) d\alpha.
$$

Inside L_3, L_4 and L_5 we can integrate by parts and therefore

$$
L_3 + L_4 + L_5 \leq C(\|z\|_S(t) + 1)^k.
$$

In L_6 we use the splitting $L_6 = M_2 + M_3$ where

$$
M_2 = \int_{\mathbb{T}} \Im\Big(\frac{\partial_\alpha z_1}{|\partial_\alpha z|^2} \Big)(-\Re(\partial_\alpha^4 z) \cdot \Im(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z) \cdot \Re(\Lambda(\partial_\alpha^4 z))) d\alpha,
$$

$$
M_3 = -\int_{\mathbb{T}} \Re\Big(\frac{\partial_\alpha z_1}{|\partial_\alpha z|^2} \Big)(\Re(\partial_\alpha^4 z) \cdot \Re(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z) \cdot \Im(\Lambda(\partial_\alpha^4 z))) d\alpha.
$$

In M_2 it is easy to find a commutator formula:

$$
M_2 = \int_{\mathbb{T}} [-\Lambda\big(\Im(\frac{\partial_\alpha z_1}{|\partial_\alpha z|^2})\Re(\partial_\alpha^4 z)\big) + \Im(\frac{\partial_\alpha z_1}{|\partial_\alpha z|^2})\Re(\Lambda(\partial_\alpha^4 z))] \cdot \Im(\partial_\alpha^4 z) d\alpha,
$$

and the appropriate estimate follows. We find that $M_2 \leq C(\|z\|_S + 1)^k$. For M_3 we write $M_3 = N_1 + N_2$ where

$$
N_1 = -\int_{\mathbb{T}} [\Re(\frac{\partial_\alpha z_1}{|\partial_\alpha z|^2}) - m(t)](\Re(\partial_\alpha^4 z) \cdot \Re(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z) \cdot \Im(\Lambda(\partial_\alpha^4 z))) d\alpha,
$$

$$
N_2 = -m(t)\|\Lambda^{1/2}(\partial_\alpha^4 z)\|_{L^2(S)}^2,
$$

14

where
$$m(t) = \min_\gamma \Re(\frac{\partial_\alpha z_1(\gamma)}{|\partial_\alpha z(\gamma)|^2}).$$

We use the pointwise estimate [10]

$$2g\Lambda(g) - \Lambda(g^2) \geq 0. \tag{26}$$

Therefore

$$N_1 \leq \frac{1}{2}\|\Lambda(\Re(\frac{\partial_\alpha z_1}{|\partial_\alpha z|^2}))\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2 \leq C(\|z\|_S(t) + 1)^k$$

as long as

$$\Re(\frac{\partial_\alpha z_1(\gamma)}{|\partial_\alpha z(\gamma)|^2}) > 0.$$

Remember that initially $\Re(\frac{\partial_\alpha z_1(\gamma)}{|\partial_\alpha z(\gamma)|^2})$ is greater than zero (R-T). We will prove that it is going to keep like that for a short time. For I_2 we find as before

$$I_2 = c\int_{\mathbb{T}} (\Im(\partial_\alpha^4 z)(\gamma) \cdot \Re(\partial_\alpha^5 z)(\gamma) - \Re(\partial_\alpha^4 z)(\gamma) \cdot \Im(\partial_\alpha^5 z)(\gamma))d\alpha \leq c\|\Lambda^{1/2}(\partial_\alpha^4 z)\|_{L^2(S)}^2.$$

Finally

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z(\alpha \pm ict)|^2 d\alpha \leq C(\|z\|_S(t) + 1)^k + (c + K\|\Im(f)\|_{H^2(S)}(t) - m(t))\|\Lambda^{1/2}(\partial_\alpha^4 z)\|_{L^2(S)}^2(t).$$

Note that $\|\Im(f)\|_{H^2(S)}(0) = 0$. If $c - m(0) < 0$, we will show that

$$c + K\|\Im(f)\|_{H^2(S)}(t) - m(t) < 0$$

for short time. It yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z(\alpha \pm ict)|^2 d\alpha \leq C(\|z\|_S(t) + 1)^k,$$

as long as $c + K\|\Im(f)\|_{H^2(S)}(t) - m(t) < 0$. Using Sobolev estimates, we proceed as in section 8 in [11] to show that

$$\frac{d}{dt}\|F(z)\|_{L^\infty(S)} \leq C(\|z\|_S(t) + 1)^k.$$

From the two inequalities above and (21) it is easy to obtain a priori energy estimates that depend upon the negativity of $c + K\|\Im(f)\|_{H^2(S)}(t) - m(t)$. We get bona fide energy estimates as follows. We denote

$$\|z\|_{RT}^2(t) = \|z\|_S^2(t) + 1/(m(t) - c - K\|\Im(f)\|_{H^2(S)}(t)).$$

At this point, it is easy to find that

$$-\frac{d}{dt}\|\Im(f)\|_{H^2(S)}(t) \leq C(\|z\|_S(t) + 1)^k$$

15

using (23), and therefore (see section 9 in [11] for more details)

$$\frac{d}{dt}\|z\|_{RT}(t) \leq C(\|z\|_{RT}(t) + 1)^k.$$

It follows that

$$\|z\|_{RT}(t) \leq \frac{\|z\|_{RT}(0) + 1}{(1 - C(\|z\|_{RT}(0) + 1)^k t)^{1/k}} - 1,$$

providing the a priori estimate with C and k universal constants.

We approximate the problem as follows

$$z_t^\varepsilon(\alpha, t) = \phi_\varepsilon * \int \frac{\sin(\phi_\varepsilon * z_1^\varepsilon(\alpha) - \phi_\varepsilon * z_1^\varepsilon(\beta))(\partial_\alpha(\phi_\varepsilon * z^\varepsilon)(\alpha) - \partial_\beta(\phi_\varepsilon * z^\varepsilon)(\beta))}{\cosh(z_2^\varepsilon(\alpha) - z_2^\varepsilon(\beta)) - \cos(z_1^\varepsilon(\alpha) - z_1^\varepsilon(\beta))} d\beta$$
$$z^\varepsilon(\alpha, 0) = \phi_\varepsilon * z_0(\alpha),$$

where $\phi_\varepsilon(x) = \phi(\alpha/\varepsilon)/\varepsilon$, ϕ is the heat kernel and $\varepsilon > 0$. Picard's theorem yields the existence of a solution $z^\varepsilon(\alpha, t)$ in $C\left([0, T^\varepsilon); H^4\right)$ which is analytic in the whole space for z_0 satisfying the arc-chord condition and ε small enough. Using the same techniques we have developed above we obtain a bound for $z^\varepsilon(\alpha, t)$ in H^4 in the strip $S(t)$ for a small enough T which is independent of ε. We need arc-chord, R-T, $z_0 \in H^4$ and $c - m(0) < 0$. Then we can pass to the limit.

4   Getting all the way to breakdown of Rayleigh-Taylor

This section is devoted to proving the following theorem.

Theorem 4.1 *Let $z(\alpha, 0) = z^0(\alpha)$ be an analytic curve in the strip*

$$S = \{\alpha + i\zeta \in \mathbb{C} \ : \ |\zeta| < h(0)\},$$

with $h(0) > 0$ and satisfying:

- *The arc-chord condition, $F(z^0)(\alpha + i\zeta, \beta) \in L^\infty(S \times \mathbb{R})$*

- *The Rayleigh-Taylor condition, $\partial_\alpha z_1^0(\alpha) > 0$.*

- *The curve $z^0(\alpha)$ is real for real α.*

- *The functions $z_1^0(\alpha) - \alpha$ and $z_2^0(\alpha)$ are periodic with period 2π.*

- *The functions $z_1^0(\alpha) - \alpha$ and $z_2^0(\alpha)$ belong to $H^4(\partial S)$.*

Then there exist a time T and a solution of the Muskat problem $z(\alpha, t)$ defined for $0 < t \leq T$ that continues analytically into some complex strip for each fixed $t \in [0, T]$. Here T is either a small constant depending only on $\|z^0\|_S$ or it is the first time a vertical tangent appears, whichever occurs first.

Thus our Muskat solution is analytic as long as $\partial_\alpha z_1(\alpha, t) \geq 0$.

We will use the following:

16

Lemma 4.2 *Let $\varphi(\alpha \pm i\zeta) = \sum_{k=-N}^{N} A_k e^{ik\alpha \mp k\zeta}$. Then, for $\zeta > 0$, we have*

$$\frac{\partial}{\partial \zeta} \sum_{\pm} \int_{\mathbb{T}} |\varphi(\alpha \pm i\zeta)|^2 d\alpha \geq \frac{1}{10} \sum_{\pm} \int_{\mathbb{T}} \Lambda \varphi(\alpha \pm i\zeta) \overline{\varphi(\alpha \pm i\zeta)} d\alpha - 10 \int_{\mathbb{T}} \Lambda \varphi(\alpha) \overline{\varphi}(\alpha) d\alpha, \quad (27)$$

where $\Lambda \varphi(\alpha \pm i\zeta) = \sum_{k=-N}^{N} |k| A_k e^{ik\alpha} e^{\mp k\zeta}$.

Proof: First we shall compute the left hand side in the frequency space:

$$\sum_{\pm} \int_{\mathbb{T}} |\varphi(\alpha \pm i\zeta)|^2 d\alpha = 4\pi \sum_{k=-N}^{N} |A_k|^2 \cosh(2|k|\zeta).$$

On the other hand we have that

$$\sum_{\pm} \int_{\mathbb{T}} \Lambda \varphi(\alpha \pm i\zeta) \overline{\varphi(\alpha \pm i\zeta)} d\alpha = 4\pi \sum_{k=-N}^{N} |k| |A_k|^2 \cosh(2|k|\zeta),$$

while

$$\int_{\mathbb{T}} \Lambda \varphi(\alpha) \overline{\varphi(\alpha)} d\alpha = 2\pi \sum_{k=-N}^{N} |k| |A_k|^2.$$

Differentiating in ζ we obtain

$$\frac{\partial}{\partial \zeta} \int_{\mathbb{T}} |\varphi(\alpha \pm i\zeta)|^2 d\alpha = 8\pi \sum_{k=-N}^{N} |k| |A_k|^2 \sinh(2|k|\zeta).$$

The lemma holds since $\sinh(\zeta) \geq \cosh(\zeta) - 1$ for any $\zeta > 0$.

Corollary 4.3 *Let $\varphi(\alpha \pm i\zeta, t) = \sum_{k=-N}^{N} A_k(t) e^{ik\alpha} e^{\mp k\zeta}$ and $h(t) > 0$ be a decreasing function of t. Then*

$$\frac{\partial}{\partial t} \sum_{\pm} \int_{\mathbb{T}} |\varphi(\alpha \pm ih(t))|^2 d\alpha \leq \frac{h'(t)}{10} \sum_{\pm} \int_{\mathbb{T}} \Lambda \varphi(\alpha \pm ih(t)) \overline{\varphi(\alpha \pm ih(t))} d\alpha$$

$$-10h'(t) \int_{\mathbb{T}} \Lambda \varphi(\alpha) \overline{\varphi}(\alpha) d\alpha + 2\Re \sum_{\pm} \int_{\mathbb{T}} \varphi_t(\alpha \pm ih(t)) \overline{\varphi(\alpha \pm ih(t))} d\alpha.$$

This corollary allows us to prove Theorem 4.1.

Proof (Theorem 4.1): The norms $\|z\|_{H^k(S)}$ and $\|z\|_S$ are defined as before using the new strip $S(t)$ defined by

$$S(t) = \{\alpha + i\zeta \in \mathbb{C} : |\zeta| < h(t)\},$$

where $h(t)$ is a positive decreasing function of t.

17

We use the Galerkin approximation of equation (15), i.e.

$$\partial_t z^{[N]}(\zeta, t) = \Pi_N[J[z^{[N]}]](\zeta, t),$$

where $\zeta \in \overline{S}(t)$, Π_N will be specified below, and

$$J[z](\alpha, t) = \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\beta))(\partial_\alpha z(\alpha) - \partial_\alpha z(\beta))}{\cosh(z_2(\alpha) - z_2(\beta)) - \cos(z_1(\alpha) - z_1(\beta))} d\beta.$$

We impose the initial condition

$$z^{[N]}(\alpha, 0) = z^{[N]}(\alpha).$$

Here, for a large enough positive integer N, we define $z^{[N]}(\alpha, 0)$ from $z^0(\alpha)$ by using the projection

$$\Pi_N : \sum_{-\infty}^{\infty} A_k e^{ik\alpha} \mapsto \sum_{-N}^{N} A_k e^{ik\alpha}.$$

We define $z^{[N]}(\alpha)$ by stipulating that

$$z_1^{[N]}(\alpha) - \alpha = \Pi_N[z_1^0(\alpha) - \alpha]$$

and

$$z_2^{[N]}(\alpha) = \Pi_N[z_2^0(\alpha)].$$

For N large enough, the functions $z^{[N]}(\alpha, 0)$ satisfy the arc-chord and Rayleigh-Taylor condition.

We shall consider the evolution of the most singular quantity

$$\sum_{\pm} \int_{\mathbb{T}} |\partial_\alpha^4 z^{[N]}(\alpha \pm ih_N(t), t)|^2 d\alpha,$$

where $h_N(t)$ is a smooth positive decreasing function on t, with $h_N(0) = h(0)$, which will be given below. Also we denote

$$S_N(t) = \{\alpha + i\zeta \in \mathbb{C} : |\zeta| < h_N(t)\}.$$

¿From now on, we will drop the dependency on N from $z^{[N]}$ and $h_N(t)$ in our notation. We will return to the previous notation in the discussion below at the end of the section. Taking the derivative with respect to t yields

$$\frac{d}{dt} \int_{\alpha \in \mathbb{T}} \left|\partial_\alpha^4 z_\mu(\alpha \pm ih(t), t)\right|^2 d\alpha$$

$$= 2\Re \int_{\alpha \in \mathbb{T}} \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t), t)} \left\{\partial_t \partial_\alpha^4 z_\mu(\alpha \pm ih(t), t) + ih'(t)\partial_\alpha^5 z_\mu(\alpha \pm ih(t), t)\right\} d\alpha$$

$$= 2\Re \int_{\alpha \in \mathbb{T}} \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t), t)} \left\{\partial_\alpha^4 \Pi_N[J_\mu[z]](\alpha \pm ih(t), t) + ih'(t)\partial_\alpha^5 z_\mu(\alpha \pm ih(t), t)\right\} d\alpha$$

18

$$= 2\Re \int_{\alpha \in \mathbb{T}} \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t), t)} \left\{ \Pi_N[\partial_\alpha^4 J_\mu[z]](\alpha \pm ih(t), t) + ih'(t)\partial_\alpha^5 z_\mu(\alpha \pm ih(t), t) \right\} d\alpha$$

$$= 2\Re \int_{\alpha \in \mathbb{T}} \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t), t)} \left\{ \partial_\alpha^4 J_\mu[z](\alpha \pm ih(t), t) + ih'(t)\partial_\alpha^5 z_\mu(\alpha \pm ih(t), t) \right\} d\alpha,$$

since $\partial_\alpha^4 z_\mu(\alpha \pm ih(t), t)$ is a trigonometric polynomial in the range of Π_N. Here $\mu = 1, 2$.

Using the above corollary we have that

$$\frac{d}{dt} \sum_\pm \int_{\alpha \in \mathbb{T}} \left| \partial_\alpha^4 z_\mu(\alpha \pm ih(t), t) \right|^2 d\alpha \le \frac{h'(t)}{10} \sum_\pm \int_\mathbb{T} \Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha$$

$$-10h'(t) \int_\mathbb{T} \Lambda(\partial_\alpha^4 z_\mu)(\alpha) \cdot \overline{\partial_\alpha^4 z_\mu}(\alpha) d\alpha + 2 \sum_\pm \Re \int_\mathbb{T} \partial_\alpha^4 J_\mu[z](\alpha, t)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha.$$

We shall study in detail the most singular term in $\partial^4 J[z](\alpha, t)$, i.e.

$$
\begin{aligned}
\partial_\alpha^4 J[z](\alpha \pm ih(t), t) &= \int_{-\pi}^\pi \frac{\sin(z_1(\alpha \pm ih(t), t) - z_1(\beta, t))(\partial_\alpha^5 z(\alpha \pm ih(t), t) - \partial_\beta^5 z(\beta, t))}{\cosh(z_2(\alpha \pm ih(t), t) - z_2(\beta, t)) - \cos(z_1(\alpha \pm ih(t), t) - z_1(\beta, t))} d\beta \\
&+ \text{l.o.t} \equiv X + \text{l.o.t.},
\end{aligned}
$$

where $\|l.o.t\|_{L^2(\mathbb{T})} \le C(\|z\|_S(t) + 1)^k$ (see [11] and our previous discussion of (18)). We split X in to the following terms

$$
\begin{aligned}
X &= \int_{-\pi}^\pi K(\alpha \pm ih(t), \beta)(\partial_\alpha^5 z(\alpha \pm ih(t), t) - \partial_\beta^5 z(\beta, t)) d\beta \\
&+ \sigma(\alpha \pm ih(t), t) \int_{-\pi}^\pi \cot\left(\frac{\alpha \pm ih(t) - \beta}{2}\right) (\partial_\alpha^5 z(\alpha \pm ih(t), t) - \partial_\beta^5 z(\beta, t)) d\beta \\
&\equiv X_1 + X_2,
\end{aligned}
$$

where

$$
\begin{aligned}
K(\alpha, \beta) =& \frac{\sin(z_1(\alpha, t) - z_1(\beta, t))}{\cosh(z_2(\alpha, t) - z_2(\beta, t)) - \cos(z_1(\alpha, t) - z_1(\beta, t))} \\
&- \frac{\partial_\alpha z_1(\alpha, t)}{(\partial_\alpha z_2(\alpha, t))^2 + (\partial_\alpha z_1(\alpha, t))^2} \cot\left(\frac{\alpha - \beta}{2}\right)
\end{aligned}
$$

and

$$\tilde{\sigma}(\alpha, t) = \frac{\partial_\alpha z_1(\alpha, t)}{(\partial_\alpha z_1(\alpha, t))^2 + (\partial_\alpha z_2(\alpha, t))^2}.$$

Let us denote

$$\Gamma_\pm(t) = \{\zeta \in \mathbb{C} : \zeta = \alpha \pm ih(t), \alpha \in \mathbb{T}\}.$$

Since $K(\alpha, \beta)$ is a holomorphic function in α and β, with $\alpha, \beta \in S(t)$, for fixed t we have that

$$X_1 = \int_{\pi}^{\pi} K(\alpha \pm ih(t), \beta) \partial_\alpha^5 z(\alpha \pm ih(t), t) d\beta$$
$$- \int_{\pi}^{\pi} K(\alpha \pm ih(t), \beta) \partial_\alpha^5 z(\beta, t) d\beta$$
$$\equiv X_{11} + X_{12},$$

and integration by parts shows that the term X_{12} satisfies $||X_{12}||_{L^2(\mathbb{T})} \leq C(||z||_S + 1)^k$. In addition, we can write X_{11} as follows

$$X_{11} = \int_{w \in \Gamma_\pm(t)} K(\alpha \pm ih(t), w) \partial^5 z(\alpha \pm ih(t), t) dw$$
$$= P.V. \int_{w \in \Gamma_\pm(t)} \frac{\sin(z_1(\alpha \pm ih(t), t) - z_1(w, t)) \partial^5 z(\alpha \pm ih(t), t)}{\cosh(z_2(\alpha \pm ih(t), t) - z_2(w, t)) - \cos(z_1(\alpha \pm ih(t), t) - z_1(w, t))} dw$$
$$- \partial^5 z(\alpha \pm ih(t), t) \sigma(\alpha \pm ih(t), t) P.V. \int_{w \in \Gamma_\pm(t)} \cot\left(\frac{\alpha \pm ih(t) - w}{2}\right) dw$$
$$= P.V. \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha \pm ih(t), t) - z_1(\beta \pm ih(t), t)) \partial^5 z(\alpha \pm ih(t), t)}{\cosh(z_2(\alpha \pm ih(t), t) - z_2(\beta \pm ih(t), t)) - \cos(z_1(\alpha \pm ih(t), t) - z_1(\beta \pm ih(t), t))} d\beta,$$

As before we call

$$f(\alpha \pm ih(t), t)$$
$$= P.V. \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha \pm ih(t), t) - z_1(\beta \pm ih(t), t))}{\cosh(z_2(\alpha \pm ih(t), t) - z_2(\beta \pm ih(t), t)) - \cos(z_1(\alpha \pm ih(t), t) - z_1(\beta \pm ih(t), t))} d\beta$$
$$= P.V. \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha \pm ih(t), t) - z_1(\alpha \pm ih(t) - \beta, t))}{\cosh(z_2(\alpha \pm ih(t), t) - z_2(\alpha \pm ih(t) - \beta, t)) - \cos(z_1(\alpha \pm ih(t), t) - z_1(\alpha \pm ih(t) - \beta, t))} d\beta.$$

Thus
$$X_{11} = \partial^5 z(\alpha \pm ih(t), t) f(\alpha \pm ih(t), t).$$

Also we can write X_2 in the following way;

$$X_2 = \tilde{\sigma}(\alpha \pm ih(t), t) \int_{-\pi}^{\pi} \cot\left(\frac{\alpha \pm ih(t) - \beta}{2}\right) (\partial_\alpha^5 z(\alpha \pm ih(t), t) - \partial_\beta^5 z(\beta, t)) d\beta$$
$$= \tilde{\sigma}(\alpha \pm ih(t), t) \int_{w \in \Gamma_\pm(t)} \cot\left(\frac{\alpha \pm ih(t) - w}{2}\right) (\partial_\alpha^5 z(\alpha \pm ih(t), t) - \partial_\beta^5 z(w, t)) dw$$

$$= \tilde{\sigma}(\alpha \pm ih(t), t) P.V. \int_{w \in \Gamma_\pm(t)} \cot\left(\frac{\alpha \pm ih(t) - w}{2}\right) \partial_\alpha^5 z(\alpha \pm ih(t), t) dw$$
$$- \tilde{\sigma}(\alpha \pm ih(t), t) P.V. \int_{w \in \Gamma_\pm(t)} \cot\left(\frac{\alpha \pm ih(t) - w}{2}\right) \partial_\alpha^5 z(w, t) dw$$

20

$$= - \tilde{\sigma}(\alpha \pm ih(t), t) P.V. \int_{-\pi}^{\pi} \cot\left(\frac{\alpha - \beta}{2}\right) \partial_{\alpha}^5 z(\beta \pm ih(t), t) d\beta$$

$$= - \tilde{\sigma}(\alpha \pm ih(t), t) P.V. \int_{-\pi}^{\pi} \frac{1}{2} \csc^2\left(\frac{\alpha - \beta}{2}\right) (\partial_{\alpha}^4 z(\alpha \pm ih(t), t) - \partial_{\alpha}^4 z(\beta \pm ih(t), t)) d\beta$$

and finally

$$X_2 = - 2\pi \tilde{\sigma}(\alpha \pm ih(t), t)(\Lambda \partial_{\alpha}^4 z)(\alpha \pm ih(t), t).$$

Then we find two dangerous terms

$$I_1 = 2\Re \int_{\mathbb{T}} f(\alpha \pm ih(t), t) \overline{(\partial_{\alpha}^4 z_{\mu})(\alpha \pm ih(t))} \cdot (\partial_{\alpha}^5 z_{\mu})(\alpha \pm ih(t)) d\alpha$$

and

$$I_2 = -4\pi \Re \int_{\mathbb{T}} \tilde{\sigma}(\alpha \pm ih(t), t) \Lambda(\partial_{\alpha}^4 z_{\mu})(\alpha \pm ih(t)) \cdot \overline{\partial_{\alpha}^4 z_{\mu}(\alpha \pm ih(t))} d\alpha.$$

The rest can be bounded by $C(\|z\|_S + 1)^k(t)$ as in the previous section. In order to bound I_1 and I_2 we use the following commutator estimate:

$$||\Lambda^{\frac{1}{2}}(fg) - f\Lambda^{\frac{1}{2}}g||_{L^2(\mathbb{T})} \leq C||\Lambda^{1+\varepsilon}f||_{L^2(\mathbb{T})}||g||_{L^2(\mathbb{T})}, \tag{28}$$

for $f(\alpha) = \sum_{-N}^{N} f_k e^{ikx}$ and $g(\alpha) = \sum_{-N}^{N} g_k e^{ikx}$, where $\varepsilon > 0$ and C does not depend on N. The proof of (28) will be left to the reader.

First we estimate I_1. We denote $\gamma = \alpha + ih(t)$.

$$I_1 = 2\Re \int_{-\pi}^{\pi} f(\gamma, t) \overline{\partial_{\alpha}^4 z_{\mu}(\gamma, t)} \partial_{\alpha}^5 z_{\mu}(\gamma, t) d\alpha$$

$$= 2 \int_{-\pi}^{\pi} \Re(f(\gamma)) \left\{ \Re(\partial_{\alpha}^4 z_{\mu}(\gamma, t)) \partial_{\alpha}(\Re(\partial_{\alpha}^4 z_{\mu}(\gamma, t))) + \Im(\partial_{\alpha}^4 z_{\mu}(\gamma, t)) \partial_{\alpha}(\Im(\partial_{\alpha}^4 z_{\mu}(\gamma, t))) \right\} d\alpha$$

$$- 2 \int_{-\pi}^{\pi} \Im(f(\gamma)) \left\{ \Re(\partial_{\alpha}^4 z_{\mu}(\gamma, t)) \partial_{\alpha}(\Im(\partial_{\alpha}^4 z_{\mu}(\gamma, t))) + \Im(\partial_{\alpha}^4 z_{\mu}(\gamma, t)) \partial_{\alpha}(\Re(\partial_{\alpha}^4 z_{\mu}(\gamma, t))) \right\} d\alpha$$

$$\equiv I_{11} + I_{12}.$$

Integrating by parts we have that $||I_{11}||_{L^2(\mathbb{T})} \leq C(\|z\|_S + 1)^k$. In order to estimate I_{12} we note that $f(\gamma, t)$ is real for real γ. Then

$$\Im(f(\alpha \pm ih(t), t)) = h(t)\tilde{f}_{\pm}(\alpha, t),$$

where

$$||\tilde{f}_{\pm}||_{H^2(\mathbb{T})} \leq C(\|z\|_S(t) + 1)^k.$$

Then we can write

$$\int_{-\pi}^{\pi} \Im(f(\gamma))\Re(\partial_\alpha^4 z_\mu(\gamma,t))\partial_\alpha(\Im(\partial_\alpha^4 z_\mu(\gamma,t)))d\alpha$$

$$= h(t)\int_{-\pi}^{\pi} \tilde{f}_\pm(\alpha,t)\Re(\partial_\alpha^4 z_\mu(\gamma,t))\partial_\alpha(\Im(\partial_\alpha^4 z_\mu(\gamma,t)))d\alpha$$

$$= -h(t)\int_{-\pi}^{\pi} \tilde{f}_\pm(\alpha,t)\Re(\partial_\alpha^4 z_\mu(\gamma,t))\Lambda H(\Im(\partial_\alpha^4 z_\mu(\gamma,t)))d\alpha$$

$$= -h(t)\int_{-\pi}^{\pi} \Lambda^{\frac{1}{2}}(\tilde{f}_\pm(\alpha,t)\Re(\partial_\alpha^4 z_\mu)(\gamma,t))\Lambda^{\frac{1}{2}}H(\Im(\partial_\alpha^4 z_\mu(\gamma,t)))d\alpha$$

$$= -h(t)\int_{-\pi}^{\pi}\left\{\Lambda^{\frac{1}{2}}(\tilde{f}_\pm(\alpha,t)\Re(\partial_\alpha^4 z_\mu)(\gamma,t)) - \tilde{f}_\pm(\alpha)\Lambda^{\frac{1}{2}}\Re(\partial_\alpha^4 z_\mu)\right\}\Lambda^{\frac{1}{2}}H(\Im(\partial_\alpha^4 z_\mu(\gamma,t)))d\alpha$$

$$- h(t)\int_{-\pi}^{\pi}\tilde{f}_\pm(\alpha)\Lambda^{\frac{1}{2}}\Re(\partial_\alpha^4 z_\mu)\Lambda^{\frac{1}{2}}H(\Im(\partial_\alpha^4 z_\mu(\gamma,t)))d\alpha$$

$$\leq h(t)\|\Lambda^{\frac{1}{2}}(\tilde{f}_\pm(\cdot,t))\Re(\partial_\alpha^4 z_\mu(\cdot\pm ih(t),t)) - \tilde{f}_\pm(\cdot,t)\Lambda^{\frac{1}{2}}\Re(\partial_\alpha^4 z_\mu(\cdot\pm ih(t)))\|_{L^2(\mathbb{T})}$$

$$\times \|\Lambda^{\frac{1}{2}}H(\Im(\partial_\alpha^4 z_\mu(\cdot\pm ih(t),t)))\|_{L^2(\mathbb{T})}$$

$$+ h(t)\|\tilde{f}_\pm\|_{L^\infty(\mathbb{T})}\|\Lambda^{\frac{1}{2}}\Re(\partial_\alpha^4 z_\mu(\cdot\pm ih(t)))\|_{L^2(\mathbb{T})}\|\Lambda^{\frac{1}{2}}H\Im(\partial_\alpha^4 z_\mu(\cdot\pm ih(t)))\|_{L^2(\mathbb{T})}.$$

Using the estimate (28) yields

$$\int_{-\pi}^{\pi} \Im(f(\gamma))\Re(\partial_\alpha^4 z_\mu(\gamma,t))\partial_\alpha(\Im(\partial_\alpha^4 z_\mu(\gamma,t)))d\alpha$$

$$\leq h(t)\|\Lambda^{1+\varepsilon}\tilde{f}_\pm\|_{L^2(\mathbb{T})}\|\Re(\partial_\alpha^4 z_\mu(\cdot\pm ih(t),t))\|_{L^2(\mathbb{T})}\|\Lambda^{\frac{1}{2}}(\Im(\partial_\alpha^4 z_\mu(\cdot\pm ih(t),t)))\|_{L^2(\mathbb{T})}$$

$$+ h(t)\|\tilde{f}_\pm\|_{L^\infty(\mathbb{T})}\|\Lambda^{\frac{1}{2}}\Re(\partial_\alpha^4 z_\mu(\cdot\pm ih(t)))\|_{L^2(\mathbb{T})}\|\Lambda^{\frac{1}{2}}\Im(\partial_\alpha^4 z_\mu(\cdot\pm ih(t)))\|_{L^2(\mathbb{T})}$$

$$\leq Ch(t)(\|z\|_S + 1)^k + Ch(t)(\|z\|_S + 1)^k\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z_\mu(\cdot\pm ih(t),t)\|_{L^2(\mathbb{T})}^2$$

$$= Ch(t)(\|z\|_S + 1)^k + Ch(t)(\|z\|_S + 1)^k\int_{-\pi}^{\pi}\overline{\partial_\alpha^4 z_\mu(\gamma,t)}\Lambda\partial_\alpha^4 z_\mu(\gamma,t)d\alpha.$$

Now I_1 is equal to the integral to the left, plus a similar integral that can be bounded in a similar way.

Thus we obtain that

$$\sum_\pm I_1 \leq C(\|z\|_S + 1)^k + Ch(t)(\|z\|_S + 1)^k\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(S)}^2. \tag{29}$$

By assumption the R-T $\tilde{\sigma}$ is bigger than zero for real values. In order to avoid problems with the imaginary part we may write

$$\frac{\partial_\alpha z_1(\alpha\pm ih(t),t)}{(\partial_\alpha z_1(\alpha\pm ih(t)))^2 + (\partial_\alpha z_2(\alpha\pm ih(t)))^2} = \frac{\partial_\alpha z_1(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2} + h(t)g_\pm(\alpha,t).$$

where

$$\|g_\pm\|_{H^2(\mathbb{T})} \leq C(\|z\|_S + 1)^k.$$

One finds,

$$I_2 = -2\Re \int_{\mathbb{T}} \frac{\partial_\alpha z_1(\alpha)}{|\partial_\alpha z(\alpha)|^2} \Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha$$

$$-h(t)2\Re \int_{\mathbb{T}} g_\pm(\alpha, t)\Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha.$$

The first term above can be treated as in section 3 taking advantage of the inequality (26). Here we just need $\partial_\alpha z_1(\alpha) \geq 0$. The second term can be treated using the inequality (28) as with the term I_1. We find that

$$\sum_\pm I_2 \leq C(\|z\|_S + 1)^k + Ch(t)\|g_\pm\|_{H^2(S)}\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(S)}^2,$$

and therefore

$$\sum_\pm I_2 \leq C(\|z\|_S + 1)^k + Ch(t)(\|z\|_S + 1)^k\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(S)}^2. \qquad (30)$$

Using (29) and (30) we have that

$$\frac{d}{dt}\sum_\pm \int_{\mathbb{T}} |\partial_\alpha^4 z_\mu(\alpha \pm ih(t))|^2 d\alpha \leq C(\|z\|_S(t) + 1)^k - 10h'(t)\int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_\mu)(\alpha) \cdot \overline{\partial_\alpha^4 z_\mu}(\alpha)d\alpha$$

$$+(C(\|z\|_S(t) + 1)^k h(t) + \frac{1}{10}h'(t))\int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha.$$

Choosing

$$h(t) = h(0)\exp(-10C\int_0^t (\|z\|_S + 1)^k(r)dr)$$

we eliminate the most dangerous term. The other term in the expression above involves with a function on the real line and it is easily controlled. Indeed

$$\int_{\mathbb{T}} \Lambda\partial_\alpha^4 z_\mu(\alpha) \cdot \partial_\alpha^4 z_\mu(\alpha) \leq \frac{C}{h(t)}\sum_\pm \int_{\mathbb{T}} |\partial_\alpha^4 z_\mu(\alpha \pm ih(t))|^2 d\alpha,$$

as one sees by examining the Fourier expansion of $\partial_\alpha^4 z_\mu(\alpha, t)$.

Thus

$$\left|10h'(t)\int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_\mu)(\alpha) \cdot \partial_\alpha^4 z_\mu(\alpha)d\alpha\right| \leq C\frac{|h'(t)|}{h(t)}\|z\|_S^2 \leq C(\|z\|_S + 1)^{k+2}.$$

And we obtain finally

$$\frac{d}{dt}\sum_\pm \int_{\mathbb{T}} |\partial_\alpha^4 z(\alpha \pm ih(t))|^2 d\alpha \leq C(\|z\|_S(t) + 1)^{k+2}.$$

23

Recovering the dependency on N in our notation we have that

$$\frac{d}{dt} \sum_{\pm} \int_{\mathbb{T}} |\partial_\alpha^4 z^{[N]}(\alpha \pm i h_N(t))|^2 d\alpha \leq C(\|z^{[N]}\|_{S_N}(t) + 1)^{k+2}. \tag{31}$$

As in the previous section, we can obtain a bound of the evolution of the arc-chord condition that depends on $C(\|z^{[N]}\|_{S_N}(t) + 1)^{k+2}$.

This estimate is true whenever $t \in [0, T_N]$, where T_N is the maximal time of existence of the solution $z^{[N]}$. In addition inequality (31) shows that we can extend these solutions in $H^4(S)$ up to a small enough time T independent of N and depending on the initial data.

The above calculation shows that the strip may shrink but does not collapse as long as $\partial_\alpha z_1(\alpha, t) \geq 0$.

5 From an analytic curve in the stable regime to an analytic curve in the unstable regime

In this section we show that there exist some initial data which are analytic curves satisfying the arc-chord and R-T conditions such that the solution of the Muskat problem reaches the unstable regime. In order to do it we will prove the local existence of solutions for analytic initial data without assuming the R-T condition. Then we will construct some suitable initial data for our purpose.

Theorem 5.1 *Let z_0 be an analytic curve satisfying the arc-chord condition. Then there exists an analytic solution for the Muskat problem in some interval $[-T, T]$ for a small enough $T > 0$.*

Remark 5.2 *Notice that in theorem (5.1) there is no assumption on the R-T condition. The proof we use here is analogous to the one in [33] based on Cauchy-Kowalewski theorems [27, 28] (for an application to the Euler equation see [2]). Here we cannot parametrize the curve as a graph, so we have to change the argument substantially in the proof in order to deal with the arc-chord condition.*

Proof: We use the same notation as before. Let $\{X_r\}_{r>0}$ be a scale of Banach spaces given by $\mathbb{R}^2-$valued real functions f that can be extended into the complex strip $S_r = \{\alpha + i\zeta \in \mathbb{C} : |\zeta| < r\}$ such that the norm

$$\|f\|_r^2 = \sum_{\pm} \int_{\mathbb{T}} |f(\alpha \pm ir) - (\alpha \pm ir, 0)|^2 d\alpha + \int_{\mathbb{T}} |\partial_\alpha^4 f(\alpha \pm ir)|^2 d\alpha,$$

is finite and $f(\alpha) - (\alpha, 0)$ is $2\pi-$periodic.

Let $z^0(\alpha)$ be a curve satisfying the arc-chord condition and $z^0(\alpha) \in X_{r_0}$ for some $r_0 > 0$. Then, we will show that there exist a time $T > 0$ and $0 < r < r_0$ so that there is a unique solution to (16) in $C([0, T]; X_r)$.

It is easy to check that $X_r \subset X_{r'}$ for $r' \leq r$ due to the fact that $\|f\|_{r'} \leq \|f\|_r$. A simple application of the Cauchy formula gives

$$\|\partial_\alpha f\|_{r'} \leq \frac{C}{r - r'} \|f\|_r, \tag{32}$$

for $r' < r$. Next, we write equation (16) as follows:

$$z_t(\alpha + i\zeta, t) = G(z(\alpha + i\zeta, t)),$$

with

$$G(z(\alpha+i\zeta,t)) = \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha + i\zeta) - z_1(\alpha + i\zeta - \beta))(\partial_\alpha z(\alpha + i\zeta) - \partial_\alpha z(\alpha + i\zeta - \beta))}{\cosh(z_2(\alpha + i\zeta) - z_2(\alpha + i\zeta - \beta)) - \cos(z_1(\alpha + i\zeta) - z_1(\alpha + i\zeta - \beta))} d\beta.$$

We take $0 \leq r' < r$ and we introduce the open set O in S_r given by

$$O = \{z, \omega \in X_r : \|z\|_r < R, \quad \|F(z)\|_{L^\infty(S_r)} < R^2\}, \tag{33}$$

with $F(z)(\alpha + i\zeta, \beta, t)$ given by (17). Then the function G for $G : O \to X_{r'}$ is a continuous mapping. In addition, there is a constant C_R (depending on R only) such that

$$\|G(z)\|_{r'} \leq \frac{C_R}{r - r'} \|z\|_r, \tag{34}$$

$$\|G(z^2) - G(z^1)\|_{r'} \leq \frac{C_R}{r - r'} \|z^2 - z^1\|_r, \tag{35}$$

and

$$\sup_{\alpha+i\zeta \in S_r, \beta \in \mathbb{T}} |G(z)(\alpha + i\zeta) - G(z)(\alpha + i\zeta - \beta)| \leq C_R|\beta|, \tag{36}$$

for $z, z^j \in O$. The above inequalities can be proved by estimating as in previous sections. Then they yield the proof of theorem 5.1. The argument is analogous to [27] and [28]. We have to deal with the arc-chord condition so we will point out the main differences. For initial data $z^0 \in X_{r_0}$ satisfying arc-chord, we can find a $0 < r_0' < r_0$ and a constant R_0 such that $\|z^0\|_{r_0'} < R_0$ and

$$2\frac{\cosh(z_2^0(\alpha + i\zeta) - z_2^0(\alpha + i\zeta - \beta)) - \cos(z_1^0(\alpha + i\zeta) - z_1^0(\alpha + i\zeta - \beta))}{||\beta||^2} > \frac{1}{R_0^2}, \tag{37}$$

for $\alpha + i\zeta \in S_{r_0'}$. We take $0 < r < r_0'$ and $R_0 < R$ to define the open set O as in (33). Therefore we can use the classical method of successive approximations:

$$z^{n+1}(t) = z^0 + \int_0^t G(z^n(s)) ds, \tag{38}$$

for $G : O \to X_{r'}$ and $0 < r' < r$. We assume by induction that

$$\|z^k\|_r(t) < R, \qquad \text{and} \qquad \|F(z^k)\|_{L^\infty(S_r)}(t) < R$$

25

for $k \le n$ and $0 < t < T$ with $T = \min(T_A, T_{CK})$ and T_{CK} the time obtained in the proofs in [27] and [28], and T_A determined below. Now, we will check that $\|F(z^{n+1})\|_{L^\infty(S_r)}(t) < R$ for suitable T_A. The rest of the proof follows in the same way as in [27], [28].

Definitions (38) and (17) easily imply that

$$|(F(z^{n+1})(\alpha + i\zeta, \beta, t))^{-1}| \ge |(F(z^0)(\alpha + i\zeta, \beta, t))^{-1}| - C_R(t^2 + t) \ge \frac{1}{R_0^2} - C_R(t^2 + t).$$

To see this, we just use the formulas for $\cos(a + b)$ and $\cosh(a + b)$, and bounds for the functions $\frac{\cosh(x)-1}{x^2}$, $\frac{1-\cos(x)}{x^2}$, $\frac{\sinh(x)}{x}$, $\frac{\sin(x)}{x}$, for bounded x. Therefore, taking

$$0 < T_A < \min\left\{1, \sqrt{\left(\frac{1}{R_0^2} - \frac{1}{R^2}\right)\frac{1}{2C_R}}\right\},$$

we obtain $\|F(z^{n+1})\|_{L^\infty(S_r)}(t) < R$. This completes the proof of theorem 5.1.

The next step will be the construction of analytic initial data such that

$$a.\ \partial_\alpha z_1(\alpha) > 0 \text{ if } \alpha \ne 0. \qquad b.\ \partial_\alpha z_1(0) = 0.$$

$$c.\ \partial_\alpha z_2(0) > 0. \qquad d.\ \partial_\alpha v_1(0) < 0.$$

Also $z_1(\alpha) - \alpha$ and $z_2(\alpha)$ are $2\pi-$periodic.

Here $v_\mu(\alpha, t)$, with $\mu = 1, 2$, are the velocities given by

$$v_\mu(\alpha, t) = \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\beta))}{\cosh(z_2(\alpha) - z_2(\beta)) - \cos(z_1(\alpha) - z_1(\beta))}(\partial_\alpha z_\mu(\alpha) - \partial_\alpha z_\mu(\beta))d\beta.$$
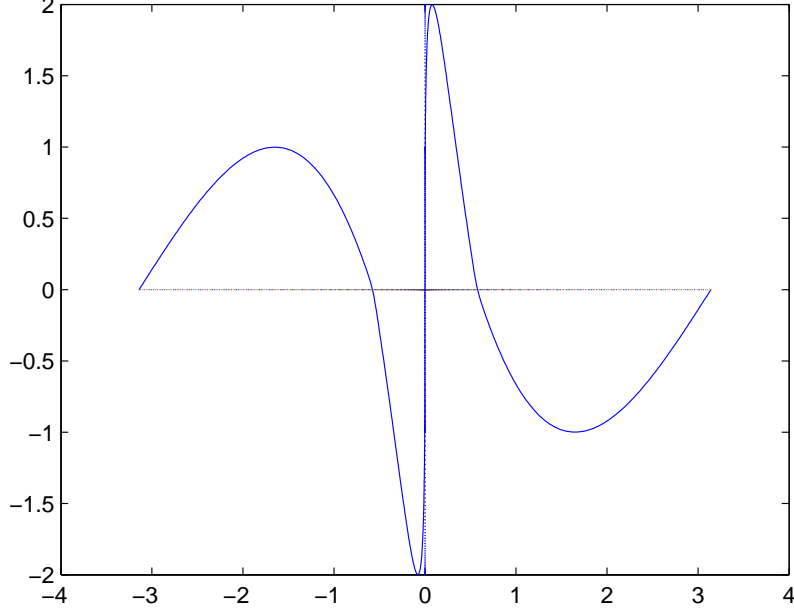
Notice that in this situation the graph $f : \mathbb{R} \to \mathbb{R}$ defined by the equation $z_2(\alpha) = f(z_1(\alpha))$, has a vertical tangent at the point $z(0)$. See the figure below for an example. We shall prove the following lemma:

Lemma 5.3 *There exists a curve $z(\alpha) = (z_1(\alpha), z_2(\alpha))$ with the following properties:*

1. *$z_1(\alpha) - \alpha$ and $z_2(\alpha)$ are analytic $2\pi-$periodic functions and $z(\alpha)$ satisfies the arc-chord condition,*

2. *$z(\alpha)$ is odd and*

3. *$\partial_\alpha z_1(\alpha) > 0$ if $\alpha \ne 0$, $\partial_\alpha z_1(0) = 0$ and $\partial_\alpha z_2(0) > 0$,*

   such that

$$\begin{aligned}(\partial_\alpha v_1)(0) &= \left(\partial_\alpha \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\beta))}{\cosh(z_2(\alpha) - z_2(\beta)) - \cos(z_1(\alpha) - z_1(\beta))}(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\beta))d\beta\right)\bigg|_{\alpha=0} \\ &< 0.\end{aligned} \tag{39}$$

Proof: We shall assume that $z(\alpha)$ is a smooth curve satisfying the properties 2 and 3. Differentiating the expression for the horizontal component of the velocity, it is easy to obtain

$$(\partial_\alpha v_1)(\alpha) = \partial_\alpha \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\alpha - \beta))}{\cosh(z_2(\alpha) - z_2(\alpha - \beta)) - \cos(z_1(\alpha) - z_1(\alpha - \beta))}(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha - \beta))d\beta$$

$$= \int_{-\pi}^{\pi} \frac{\cos(z_1(\alpha) - z_1(\alpha - \beta))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha - \beta))^2}{\cosh(z_2(\alpha) - z_2(\alpha - \beta)) - \cos(z_1(\alpha) - z_1(\alpha - \beta))}d\beta$$

$$+ \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\alpha - \beta))(\partial_\alpha^2 z_1(\alpha) - \partial_\alpha^2 z_1(\alpha - \beta))}{\cosh(z_2(\alpha) - z_2(\alpha - \beta)) - \cos(z_1(\alpha) - z_1(\alpha - \beta))}d\beta$$

$$- \int_{-\pi}^{\pi} \sin((z_1(\alpha) - z_1(\alpha - \beta)))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha - \beta))$$

$$\times \frac{\sinh(z_2(\alpha) - z_2(\alpha - \beta))(\partial_\alpha z_2(\alpha) - \partial_\alpha z_2(\alpha - \beta))}{(\cosh(z_2(\alpha) - z_2(\alpha - \beta)) - \cos(z_1(\alpha) - z_1(\alpha - \beta)))^2}d\beta$$

$$- \int_{-\pi}^{\pi} \sin((z_1(\alpha) - z_1(\alpha - \beta)))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha - \beta))$$

$$\times \frac{\sin(z_2(\alpha) - z_2(\alpha - \beta))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha - \beta))}{(\cosh(z_2(\alpha) - z_2(\alpha - \beta)) - \cos(z_1(\alpha) - z_1(\alpha - \beta)))^2}d\beta.$$

Evaluating this expression at $\alpha = 0$ we have that

$$(\partial_\alpha v_1)(0) = \int_{-\pi}^{\pi} \frac{\cos(z_1(\beta))(\partial_\alpha z_1(\beta))^2 + \sin(z_1(\beta))\partial_\alpha^2 z_1(\beta)}{\cosh(z_2(\beta)) - \cos(z_1(\beta))}d\beta$$

27

$$-\int_{-\pi}^{\pi} \sin(z_1(\beta))\partial_\alpha z_1(\beta)\frac{\sin(z_1(\beta))\partial_\alpha z_1(\beta) - \sinh(z_2(\beta))(\partial_\alpha z_2(0) - \partial_\alpha z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2}d\beta.$$

Integration by parts yields

$$\int_{-\pi}^{\pi} \frac{\sin(z_1(\beta))\partial_\alpha^2 z_1(\beta)}{\cosh(z_2(\beta)) - \cos(z_1(\beta))}d\beta$$

$$= -\int_{-\pi}^{\pi} \cos(z_1(\beta))\frac{(\partial_\alpha z_1(\beta))^2}{\cosh(z_2(\beta)) - \cos(z_1(\beta))}d\beta$$

$$+\int_{-\pi}^{\pi} \sin(z_1(\beta))\partial_\alpha z_1(\beta)\frac{\sin(z_1(\beta))\partial_\alpha z_1(\beta) + \sinh(z_2(\beta))\partial_\alpha z_2(\beta)}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2}d\beta.$$

The above integrals converge because z_1 and z_2 satisfy the properties 2 and 3. Therefore we obtain that

$$(\partial_\alpha v_1)(0) = \partial_\alpha z_2(0)\int_{-\pi}^{\pi} \frac{\sin(z_1(\beta))\sinh(z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2}\partial_\alpha z_1(\beta)d\beta$$

$$= 2\partial_\alpha z_2(0)\int_0^{\pi} \frac{\sin(z_1(\beta))\sinh(z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2}\partial_\alpha z_1(\beta)d\beta \tag{40}$$

¿From the expression (40) we can control the sign of $(\partial_\alpha v_1)(0)$. In order to clarify the proof we shall take

$$z_1(\beta) = -\sin(\beta) + \beta.$$

We construct the function $z_2(\beta)$ in the following way:

Let β_1 and β_2 be real numbers satisfying $0 < \beta_1 < \beta_2 < \pi$, and let $z^*(\beta)$ be a smooth function on $[-\pi, \pi]$, with the following properties,

a. $z^*(\beta)$ is odd.   b. $(\partial_\beta z^*)(0) > 0$.

c. $z^*(\beta) > 0$ if $\beta \in (0, \beta_1)$.   d. $z^*(\beta) < 0$ if $\beta \in (\beta_1, \beta_2]$

e. $z^*(\beta) \leq 0$ if $\beta \in [\beta_2, \pi]$.

For a positive real number b to be fixed later, we define a piecewise smooth function $\tilde{z}(\beta)$ on $[-\pi, \pi]$, by setting

$$\tilde{z}(\beta) = bz^*(\beta) \quad \text{if } |\beta| \leq \beta_1,$$
$$\tilde{z}(\beta) = z^*(\beta) \quad \text{if } \beta_1 < |\beta| < \pi.$$

Then

$$\int_{\beta_1}^{\pi} \frac{\sin(z_1(\beta))\sinh(\tilde{z}(\beta))}{(\cosh(\tilde{z}(\beta)) - \cos(z_1(\beta)))^2}\partial_\alpha z_1(\beta)d\beta$$

is negative and independent of b, while

$$\int_0^{\beta_1} \frac{\sin(z_1(\beta))\sinh(\tilde{z}(\beta))}{(\cosh(\tilde{z}(\beta)) - \cos(z_1(\beta)))^2}\partial_\alpha z_1(\beta)d\beta$$

28

tends to zero as $b \to \infty$.

Therefore, we can fix b large enough so that

$$\int_0^\pi \frac{\sin(z_1(\beta)) \sinh(\tilde{z}(\beta))}{(\cosh(\tilde{z}(\beta)) - \cos(z_1(\beta)))^2} \partial_\alpha z_1(\beta) d\beta < 0.$$

It is now easy to approximate $\tilde{z}(\beta)$ in $L^2[-\pi, \pi]$ by an odd, real-analytic $2\pi-$periodic function such that

$$\int_0^\pi \frac{\sin(z_1(\beta)) \sinh(z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2} \partial_\alpha z_1(\beta) d\beta < 0,$$

and $\partial_\alpha z_2(0) > 0$.

The conclusions of lemma (5.3) follow, thanks to (40).

Theorem (5.1) and lemma (5.3) allow us to show the breakdown of the R-T condition.

Theorem 5.4 *Let z_0 a curve satisfying the requirements of lemma (5.3). Then there exists an analytic solution of the Muskat problem satisfying the arc-chord condition in some interval $[-T, T]$ such that for small enough $T > 0$ we have that:*

1. $\partial_\alpha z_1(\alpha, -t) > 0 \ \forall \alpha$ and

2. $\partial_\alpha z_1(0, t) < 0$

for all $t \in (0, T]$. In addition $\partial_\alpha z_2(0, t) > 0$ in $[-T, T]$.

Proof: We use theorem (5.1) to obtain the existence and from lemma (5.3) we have that

$$(\partial_t \partial_\alpha z_1)(0, 0) < 0.$$

Remark 5.5 *For $t \in [-T, 0]$, our solution satisfies*

$$\min_\alpha \partial_\alpha z_1(\alpha, t) > c|t|.$$

This follows easily, since $\partial_\alpha z_1(\alpha, 0)$ has a non-degenerate minimum at $\alpha = 0$, and $\partial_t \partial_\alpha z_1(0, 0) < 0$.

6 From a curve in H^4 in the stable regime to an analytic curve in the unstable regime

Finally we show that there exists an open set of initial data in the H^4 topology satisfying the arc-chord and R-T conditions such that the solution for the Muskat problem reaches the unstable regime. This section is devoted to proving theorem (1.1).

Proof of theorem (1.1): The idea is simply to take a small H^4-neighborhood of the initial data of an analytic solution. Let z_0 be a curve as in lemma (5.3). Let $z(\alpha, t)$ with $t \in [-T, T]$ for some $T > 0$, the solution for the equation (6) given by theorem 5.1. We consider the curve

$w_\delta^\varepsilon(\alpha) = ((w_\delta^\varepsilon(\alpha)_1, (w_\delta^\varepsilon(\alpha))_2)$ which is a small perturbation in $H^4(\mathbb{T})$ of the curve $z(\alpha, t)$ at time $t = -\delta$, with $0 < \delta < T$, i. e.

$$\|w_\delta^\varepsilon(\cdot) - z(\cdot, -\delta)\|_{H^4} = \|\eta_\delta^\varepsilon\|_{H^4} \leq \varepsilon.$$

Also, $w_\delta^\varepsilon(\alpha)$ satisfies the R-T condition

$$\sigma_{w_\delta^\varepsilon}(\alpha) \equiv (\rho^2 - \rho^1)\partial_\alpha(w_\delta^\varepsilon)_1(\alpha) > 0,$$

if $0 < \delta \leq \delta_0$ and $0 < \varepsilon \leq \varepsilon(\delta)$. From now on, we take ε and δ to satisfy this condition. Also, we may take $\varepsilon(\delta) < \varepsilon_0$.

Since $z(\alpha, 0) = z_0(\alpha)$ is a smooth curve satisfying the arc-chord condition we can assume that there exist $\varepsilon_0 > 0$ and $0 < \delta_0 < T$ such that

$$\sup_{0 < \varepsilon \leq \varepsilon_0, \, 0 < \delta \leq \delta_0} \|w_\delta^\varepsilon(\cdot)\|_{H^4(\mathbb{T})} \leq C(z_0, \varepsilon_0, \delta_0), \tag{41}$$

and

$$\sup_{0 < \varepsilon \leq \varepsilon_0, \, 0 < \delta \leq \delta_0} \|F(w_\delta^\varepsilon)\|_{L^\infty(\mathbb{T})} \leq C(z_0, \varepsilon_0, \delta_0). \tag{42}$$

Now, let the curve $w^\varepsilon(\alpha, t)$ be the solution to the equation

$$\partial_t w^\varepsilon(\alpha, t) = \int \frac{\sin((w^\varepsilon)_1(\alpha, t) - (w^\varepsilon)_1(\beta, t))}{\cosh((w^\varepsilon)_2(\alpha, t) - (w^\varepsilon)_2(\beta, t)) - \cos((w^\varepsilon)_1(\alpha, t) - (w^\varepsilon)_1(\beta, t))}$$
$$\times (\partial_\alpha w^\varepsilon(\alpha, t) - \partial_\alpha w^\varepsilon(\beta, t))d\beta$$
$$w^\varepsilon(\alpha, -\delta) = w_\delta^\varepsilon(\alpha).$$

¿From theorems 3.1 and 4.1 and the inequalities (41) and (42) we see that we can choose ε_0 and δ small enough in such a way that $w^\varepsilon(\alpha, t)$ is well defined, for all $0 < \varepsilon < \varepsilon_0$, in $t \in [-\delta, 0]$ unless $w^\varepsilon(\alpha, t)$ loses the R-T condition. That means there exist some point α_0 and some time $t_0 \in [-\delta, 0]$ satisfying $\sigma_{w^\varepsilon}(\alpha_0, t_0) < 0$. Also, for small enough ε_0 and fixed δ we have that

$$\sigma_{w^\varepsilon}(\alpha) \geq a > 0, \tag{43}$$

where a is a real number independent of ε. The numbers ε_0 and δ are fixed for the rest of the proof.

If there exist times t such that there exists some point α_0 with $\sigma_{w^\varepsilon}(\alpha_0, t) = 0$, we denote the first of these times to be $T^\varepsilon \in (-\delta, \infty)$. Also we set $\tilde{T}^\varepsilon = \min\{T^\varepsilon, 0\}$ and $I^\varepsilon = [-\delta, \tilde{T}^\varepsilon]$. Due to (43) we have that

$$\inf_{0 < \varepsilon < \varepsilon(\delta)} \tilde{T}^\varepsilon > t_b > -\delta,$$

for some number t_b.

¿From the proof of theorems 3.1 and 4.1 we know that there exists a function $h(t)$, given by the expression

$$h(t) = \begin{cases} c_a(t + \delta) & -\delta \leq t \leq t_a \\ c_a(t_a + \delta)e^{-C_a(t - t_a)} & t > t_a, \end{cases}$$

30

where t_a (small enough), c_a and C_a are constants which only depend on the constant $C(z_0, \varepsilon_0, \delta)$ (see (41) and (42)), such that $w^\varepsilon(\alpha, t)$ is an analytic function in the strip

$$S(t) = \{\zeta \in \mathbb{C} : |\Im(\zeta)| < h(t)\},$$

and also

$$(\|w^\varepsilon(\cdot, t)\|_S + 1)^k \leq C_a,$$

for some large enough k and $t \in [t_a, \tilde{T}^\varepsilon]$ (notice that the constants t_a, c_a and C_a do not depend on ε).

In this situation we claim the following:

$$\frac{d}{dt} \int_{-\pi}^{\pi} |\partial_\alpha^4 (w^\varepsilon(\alpha \pm ih(t), t) - z(\alpha \pm ih(t), t))|^2 d\alpha$$

$$\leq C(\|\partial_\alpha^4 (w^\varepsilon(\cdot \pm ih(t), t) - z(\cdot \pm ih(t), t))\|_{L^2(\mathbb{T})}^2 + \|w^\varepsilon(\cdot + ih(t), t) - z(\cdot + ih(t), t)\|_{L^2(\mathbb{T})}^2) \quad (44)$$

for $t \in I^\varepsilon$ and where C is a constant just depending on $C(z_0, \varepsilon_0, \delta)$.

We will prove this inequality at the end of the section. Let us assume that (44) holds.

We notice that we can always choose either a subsequence $\{\varepsilon_n\}_1^\infty$ with $\varepsilon_n \to 0$ when $n \to \infty$ such that $T^{\varepsilon_n} < 0$ $\forall n$ or a subsequence $\{\varepsilon_m\}_1^\infty$ with $\varepsilon_m \to 0$ when $m \to \infty$ such that $T^{\varepsilon_m} \geq 0$ $\forall m$ (the case in which there exist only a finite number of times T^ε can be treated as this last case). We deal with these two cases, I and II, separately:

I. $T^\varepsilon < 0$ for all ε. From inequality (44) we can take ε small enough such that

$$w^\varepsilon(\alpha, T^\varepsilon) - z(\alpha, T^\varepsilon)$$

has norm $\leq C\varepsilon$ in $H^4(S(T^\varepsilon))$.

Note that

$$0 = \min_\alpha \partial_\alpha(w^\varepsilon)_1(\alpha, T^\varepsilon) \geq -C\varepsilon + \min_\alpha \partial_\alpha z_1^0 \geq -C\varepsilon + c|T^\varepsilon|$$

by the remark at the end of section 5. Thus, $|T^\varepsilon| < C\varepsilon$.

Then

$$z(\alpha, T^\varepsilon) - z_0(\alpha)$$

has norm $\leq C\varepsilon$ in $H^4(S(0))$ and therefore

$$|(\partial_\alpha(v_{w^\varepsilon})_1)(\alpha_0, T^\varepsilon) - (\partial_\alpha(v_{z_0})_1)(0)| \leq C\varepsilon,$$

and we can conclude that

$$(\partial_\alpha(v_{w^\varepsilon})_1)(\alpha_0, T^\varepsilon) < 0.$$

Here we recall that

$$(v_{w^\varepsilon})_1 = \partial_t w^\varepsilon(\alpha, t) = \int \frac{\sin((w^\varepsilon)_1(\alpha, t) - (w^\varepsilon)_1(\beta, t))(\partial_\alpha w^\varepsilon(\alpha, t) - \partial_\alpha w^\varepsilon(\beta, t))d\beta}{\cosh((w^\varepsilon)_2(\alpha, t) - (w^\varepsilon)_2(\beta, t)) - \cos((w^\varepsilon)_1(\alpha, t) - (w^\varepsilon)_1(\beta, t))}$$

Applying the same argument as in section 5 to the curve $w^\varepsilon(\alpha, T^\varepsilon)$ we finish the proof of theorem 1.1 in the case $T^\varepsilon < 0$ for all ε.

II. $T^\varepsilon \geq 0$ for all ε. Then we can apply a Cauchy-Kowalewski theorem to the initial data

$$w^\varepsilon(\alpha, 0) - z(\alpha, 0)$$

satisfying

$$||w^\varepsilon - z||_{S(0)} \leq C\varepsilon.$$

For $t > 0$ small enough, $z(\alpha, t)$ is in the unstable regime. We achieve the conclusion of theorem 1.1 by continuity with respect to the initial data.

The rest of the section is devoted to proving inequality (44). We shall denote $\gamma = \alpha + ih(t)$ and $d(\gamma, t) = \partial_\alpha^4(w(\gamma, t) - z(\gamma, t))$ (we omit the superscript ε in the notation) and we recall that $w(\alpha, t)$ and $z(\alpha, t)$ are real for real α (therefore we obtain similar similar estimates for $\gamma = \alpha - ih(t)$). In order to prove inequality (44) we have to compute the following quantity

$$\frac{d}{dt}\int_{-\pi}^{\pi}|d(\gamma, t)|^2 d\alpha \quad = 2\Re\left\{\int_{-\pi}^{\pi}\overline{d(\gamma, t)}d_t(\gamma, t)d\alpha\right\} + 2\Re\left\{ih'(t)\int_{-\pi}^{\pi}\overline{d(\gamma, t)}\partial_\alpha d(\gamma, t)d\alpha\right\}.$$

Again we treat in detail the most singular term in $d_t(\gamma, t)$. Recall $K(\alpha, \beta)$ from section 3 and write K_w and K_z for corresponding expressions arising from z and w. Then we have that

$$d_t(\gamma, t) = \int_{-\pi}^{\pi} K_w(\gamma, \gamma - \beta)\partial_\alpha^5(w(\gamma, t) - w(\gamma - \beta, t)d\beta$$

$$- \int_{-\pi}^{\pi} K_z(\gamma, \gamma - \beta)\partial_\alpha^5(z(\gamma, t) - z(\gamma - \beta, t))d\beta + \text{l.o.t}(\alpha, t),$$

where

$$2\Re\left\{\int_{-\pi}^{\pi}\overline{d(\gamma, t)}\text{l.o.t}(\alpha)d\alpha\right\} \leq C(||d(\cdot + ih(t), t)||^2_{L^2(\mathbb{T})} + ||w(\cdot + ih(t), t) - z(\cdot + ih(t), t)||^2_{L^2(\mathbb{T})}).$$

Here C is a constant which just depends on ε_0 and δ.

We can write

$$\int_{-\pi}^{\pi} K_w(\gamma, \gamma - \beta)\partial_\alpha^5(w(\gamma, t) - w(\gamma - \beta, t))d\beta$$

$$- \int_{-\pi}^{\pi} K_z(\gamma, \gamma - \beta)\partial_\alpha^5(z(\gamma, t) - z(\gamma - \beta, t))d\beta$$

$$= \int_{-\pi}^{\pi} K_w(\gamma, \gamma - \beta)\partial_\alpha^5((w(\gamma, t) - z(\gamma, t)) - (w(\gamma - \beta, t) - z(\gamma - \beta, t)))d\beta$$

$$+ \int_{-\pi}^{\pi} \{K_z(\gamma, \gamma - \beta) - K_w(\gamma, \gamma - \beta)\}\partial_\alpha^5(z(\gamma, t) - z(\gamma - \beta, t))d\beta$$

$$= \int_{-\pi}^{\pi} K_w(\gamma, \gamma - \beta)\partial_\alpha(d(\gamma, t) - d(\gamma - \beta, t))d\beta$$

$$+ \int_{-\pi}^{\pi} \{K_z(\gamma, \gamma - \beta) - K_w(\gamma, \gamma - \beta)\}\partial_\alpha^5(z(\gamma, t) - z(\gamma - \beta, t))d\beta$$

$$\equiv X_1(\alpha, t) + X_2(\alpha, t).$$

32

Therefore

$$\frac{d}{dt}\int_{-\pi}^{\pi}|d(\gamma,t)|^2 d\alpha \leq C||d(\cdot+ih(t),t)||_{L^2(\mathbb{T})}^2$$

$$+ 2\Re\left\{\int_{-\pi}^{\pi}\overline{d(\gamma,t)}X_1(\alpha,t)d\alpha\right\} + 2\Re\left\{\int_{-\pi}^{\pi}\overline{d(\gamma,t)}X_2(\alpha,t)d\alpha\right\}$$

$$+ 2\Re\left\{ih'(t)\int_{-\pi}^{\pi}\overline{d(\gamma,t)}\partial_\alpha d(\gamma,t)d\alpha\right\}.$$

Following the computations in section 3 when $t \in [-\delta, t_a]$ and those in section 4 when $t \in [t_a, \tilde{T}^\varepsilon]$ we have that

$$\frac{d}{dt}\int_{-\pi}^{\pi}|d(\gamma,t)|^2 d\alpha \leq C||d(\cdot+ih(t),t)||_{L^2(\mathbb{T})}^2 + 2\Re\left\{\int_{-\pi}^{\pi}\overline{d(\gamma,t)}X_2(\alpha,t)d\alpha\right\}.$$

In addition

$$\left|\frac{\sin(w^1(\gamma)-w^1(\gamma-\beta))}{\cosh(w^2(\gamma)-w^2(\gamma-\beta))-\cos(w^1(\gamma)-w^1(\gamma-\beta))}\right.$$
$$\left.-\frac{\sin(z^1(\gamma)-z^1(\gamma-\beta))}{\cosh(z^2(\gamma)-z^2(\gamma-\beta))-\cos(z^1(\gamma)-z^1(\gamma-\beta))}\right|$$
$$=\left|\left\{\left[K_w(\gamma,\gamma-\beta)-\frac{\partial_\alpha w^1(\gamma)}{(\partial_\alpha w^1(\gamma))^2+(\partial_\alpha w^1(\gamma))^2}\cot\left(\frac{\beta}{2}\right)\right]\right.\right.$$
$$\left.-\left[K_z(\gamma,\gamma-\beta)-\frac{\partial_\alpha z^1(\gamma)}{(\partial_\alpha z^1(\gamma))^2+(\partial_\alpha z^1(\gamma))^2}\cot\left(\frac{\beta}{2}\right)\right]\right\}$$
$$\left.+\left\{\frac{\partial_\alpha w^1(\gamma)}{(\partial_\alpha w^1(\gamma))^2+(\partial_\alpha w^1(\gamma))^2}-\frac{\partial_\alpha z^1(\gamma)}{(\partial_\alpha z^1(\gamma))^2+(\partial_\alpha z^1(\gamma))^2}\right\}\cot\left(\frac{\beta}{2}\right)\right|$$
$$\leq(||d(\cdot+ih(t),t)||_{L^2(\mathbb{T})}+||w(\cdot+ih(t),t)-z(\cdot+ih(t),t)||_{L^2(\mathbb{T})})\left\{C+C\left|\cot\left(\frac{\beta}{2}\right)\right|\right\}.$$

Also,

$$|\partial_\alpha^5 z(\alpha\pm ih(t),t)-\partial_\alpha^5 z(\alpha\pm ih(t)-\beta,t)| \leq C\left|\tan\left(\frac{\beta}{2}\right)\right|$$

since z is the analytic unperturbed solution. Therefore

$$2\Re\left\{\int_{-\pi}^{\pi}\overline{d(\gamma,t)}X_2(\alpha,t)d\alpha\right\} \leq C(||d(\cdot+ih(t),t)||_{L^2(\mathbb{T})}^2+||w(\cdot+ih(t),t)-z(\cdot+ih(t),t)||_{L^2(\mathbb{T})}^2).$$

We are done.

7   Turning water waves

Let us consider an incompressible irrotational flow satisfying the Euler equations

$$\rho(v_t+v\cdot\nabla v)=-\nabla p-g\rho(0,1), \tag{45}$$

33

where ρ satisfies (2,3) and $\rho^1 = 0$. This system of equations provides the motion of the interface for the water wave problem (see [3, 25] and references therein), whose contour equation is given by

$$z_t(\alpha, t) = BR(z, \omega)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t), \tag{46}$$

and

$$\begin{aligned}
\omega_t(\alpha, t) = &-2\partial_t BR(z, \omega)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - \partial_\alpha(\frac{|\omega|^2}{4|\partial_\alpha z|^2})(\alpha, t) + \partial_\alpha(c\omega)(\alpha, t) \\
&+ 2c(\alpha, t)\partial_\alpha BR(z, \omega)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) - 2g\partial_\alpha z_2(\alpha, t).
\end{aligned} \tag{47}$$

The values of $z(\alpha, t)$ and $w(\alpha, t)$ are given at an initial time t_0: $z(\alpha, t_0) = z^0(\alpha)$ and $w(\alpha, t_0) = w^0(\alpha)$. For more details see [12].

As an application of section 5, we can consider initial data given by a graph $(\alpha, f_0(\alpha))$ and show that in finite time the interface evolution reaches a regime where the contour only can be parametrized as $z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t))$, for $\alpha \in \mathbb{R}$, with $\partial_\alpha z_1(\alpha, t) < 0$ for $\alpha \in I$, a non-empty interval. This implies that there exists a time t^* where the solution of the free boundary problem reparametrized by $(\alpha, f(\alpha, t))$ satisfies $\|f_\alpha\|_{L^\infty}(t^*) = \infty$.

Theorem 7.1 *There exists a non-empty open set of initial data $z^0(\alpha) = (\alpha, f_0(\alpha))$ and $w^0(\alpha)$, with $f_0 \in H^5$ and $w^0 \in H^4$, such that in finite time t^* the solution of the water wave problem (46,47) given by $(\alpha, f(\alpha, t))$ satisfies $\|f_\alpha\|_{L^\infty}(t^*) = \infty$. The solution can be continued for $t > t^*$ as $z(\alpha, t)$ with $\partial_\alpha z_1(\alpha, t) < 0$ for $\alpha \in I$, a non-empty interval.*

Proof: Let us consider a curve $z^*(\alpha) \in H^5$ satisfying 1., 2. and 3. of Lemma 5.3. We point out that analyticity is not required here. In order to find a velocity with property (39) we pick for water waves $\omega(\alpha, t^*) = -\partial_\alpha z_2^*(\alpha)$ and a suitable $z(\alpha, t^*) = z^*(\alpha)$ as an initial datum. Notice that the tangential term does not affect the evolution. Then, with the appropriate $c(\alpha, t)$, we can apply the local existence result in [12]: There exists a solution of the water wave problem with $z(\alpha, t) \in C([t^* - \delta, t^* + \delta]; H^5)$, $\omega(\alpha, t) \in C([t^* - \delta, t^* + \delta]; H^4)$ and $\delta > 0$ small enough. The initial data promised by theorem 7.1 are any sufficiently small perturbations of $z(\alpha, t)$ and $w(\alpha, t)$ at time $t = t^* - \delta$.

8   Breakdown of Smoothness

In [5] we will exhibit a solution $z(\alpha, t)$ of the Muskat equation, with the following properties.

1. At time t_0, the interface is real-analytic and satisfies the arc-chord and Rayleigh-Taylor conditions.

2. At time $t_1 > t_0$, the interface turns over.

3. At time $t_2 > t_1$, the interface no longer belongs to C^4, although it is real-analytic for all times $t \in [t_0, t_2)$.

In this section we provide a brief sketch of our proof of the existence of such a Muskat solution.

Our Muskat solution $z(\alpha, t)$ will be a small perturbation of a Muskat solution $z^{00}(\alpha, t)$, with the following properties.

4. $z^{00}(\alpha, t)$ is real analytic in α, for $|\Im\alpha| < \varepsilon^{00}$ and $|\tau| \leq \tau^{00}$.

5. For $t \in [-\tau^{00}, 0)$, $z^{00}(\alpha, t)$ satisfies the Rayleigh-Taylor and arc-chord conditions.

6. For $t = 0$, the curve $z^{00}(\alpha, t)$ has a vertical tangent at $\alpha = 0$.

7. For $t \in (0, \tau^{00}]$, the curve $z^{00}(\alpha, t)$ fails to satisfy the Rayleigh-Taylor condition.

This paper constructs Muskat solutions z^{00} satisfying 4., 5., 6. and 7. Our problem is to pass from z^{00} to a nearby Muskat solution z satisfying 1., 2. and 3. The idea is as follows.

So far, we have studied the analytic continuation of Muskat solutions to a time-varying strip

$$S(t) = \{|\Im\alpha| \leq h(t)\},$$

in the complex plane. In our forthcoming paper [5], we will study the analytic continuation of a Muskat solution to a carefully chosen time-varying domain of the form

$$\Omega(t) = \{|\Im\alpha| \leq h(\Re\alpha, t)\}, \tag{48}$$

defined for $t \in [-\tau^{10}, \tau]$. Here, τ is a small enough positive number.

For $t \in [-\tau^{10}, \tau]$, we will work with the space $H^4(\Omega(t))$, consisting of all analytic functions $F : \Omega(t) \mapsto \mathbb{C}^2$ whose derivatives up to order 4 belong to $L^2(\partial\Omega(t))$.

We will pick our time-varying domain $\Omega(t)$ in (48) so that $h(x, t) > 0$ for all $(x, t) \in \mathbb{R}/2\pi\mathbb{Z} \times [-\tau^{10}, \tau)$ and $h(x, \tau) > 0$ for all $x \in \mathbb{R}/2\pi\mathbb{Z} \setminus \{0\}$, but $h(0, \tau) = 0$. Thus, the domain $\Omega(t)$ has 'thickness' zero at the origin. Consequently, $H^4(\Omega(\tau))$ is not contained in $C^4(\mathbb{R}/2\pi\mathbb{Z})$.

We will also take $\tau < \tau^{00}$ and $h(x, t) < \varepsilon^{00}$, so that the Muskat solution $z^{00}(\alpha, t)$ continues analytically to $\Omega(t)$, for each $t \in [-\tau^{10}, \tau]$.

We can therefore pick an 'initial' curve $z^0(\alpha)$, such that

8. $z^0(\alpha) - z^{00}(\alpha, \tau)$ belongs to $H^4(\Omega(\tau))$ and has small norm, yet

9. $z^0(\alpha)$ does not belong to $C^4(\mathbb{R}/2\pi\mathbb{Z})$.

We solve the Muskat problem backwards in time, with the 'initial' condition

10. $z(\alpha, \tau) = z^0(\alpha)$.

By a more elaborate version of the analytic continuation arguments used in this paper, we find that our Muskat solution exists and continues analytically into $\Omega(t)$, for all $t \in [t_*, \tau]$ (for a suitable time t_*); moreover,

11. $z(\alpha, t) - z^{00}(\alpha, t)$ has small norm in $H^4(\Omega(\tau))$, for all $t \in [t_*, \tau]$.

Here, either

12. $t_* = -\tau^{10}$ or

13. a modified Rayleigh-Taylor condition, adapted to the time-varying domain, fails at time t_*.

We can rule out 13., thanks to 11., together with our understanding of $z^{00}(t)$ and $\Omega(t)$.

Thus, we obtain a Muskat solution $z(\alpha, t)$, satisfying 9., 10., 11. and 12. Properties 1., 2. and 3. of $z(\alpha, t)$ now follow easily.

Acknowledgements

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