EFFECTIVE LIMITING ABSORPTION PRINCIPLES, AND APPLICATIONS

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ABSTRACT. The limiting absorption principle asserts that if H is a suitable Schrödinger operator, and f lives in a suitable weighted L^2 space (namely $H^{0,1/2+\sigma}$ for some $\sigma > 0$), then the functions $R(\lambda + i\varepsilon)f := (H - \lambda - i\varepsilon)^{-1}f$ converge in a another weighted L^2 space $H^{0,-1/2-\sigma}$ to the unique solution u of the Helmholtz equation $(H - \lambda)u = f$ which obeys the Sommerfeld outgoing radiation condition. In this paper, we investigate more quantitative (or effective) versions of this principle, for the Schrödinger operator on asymptotically conic manifolds with short-range potentials, and in particular consider estimates of the form

$$\|R(\lambda + i\varepsilon)f\|_{H^{0,-1/2-\sigma}} \leq C(\lambda, H)\|f\|_{H^{0,1/2+\sigma}}.$$

We are particularly interested in the exact nature of the dependence of the constants $C(\lambda, H)$ on both λ and H. It turns out that the answer to this question is quite subtle, with distinctions being made between low energies $\lambda \ll 1$, medium energies $\lambda \sim 1$, and large energies $\lambda \gg 1$, and there is also a non-trivial distinction between "qualitative" estimates on a single operator H (possibly obeying some spectral condition such as non-resonance, or a geometric condition such as non-trapping), and "quantitative" estimates (which hold uniformly for all operators H in a certain class). Using elementary methods (integration by parts and ODE techniques), we give some sharp answers to these questions. As applications of these estimates, we present a global-in-time local smoothing estimate and pointwise decay estimates for the associated time-dependent Schrödinger equation, as well as an integrated local energy decay estimate and pointwise decay estimates for solutions of the corresponding wave equation, under some additional assumptions on the operator H.

1. INTRODUCTION

1.1. The limiting absorption principle for the free Schrödinger operator.

The *limiting absorption principle* for the free Schrödinger operator $H_0 := -\Delta$ on the Euclidean space $\mathbf{R}^n$ describes the behavior of the resolvents $R_0(\lambda \pm i\varepsilon) := (H_0 - (\lambda \pm i\varepsilon))^{-1}$ in the limit $\varepsilon \to 0$ in the weighted Sobolev spaces $H^{s,m}(\mathbf{R}^n)$, defined for $s = 0, 1, 2$ and $m \in \mathbf{R}$ by

$$\|f\|_{H^{s,m}(\mathbf{R}^n)} := \sum_{j=0}^{s} \|\langle x \rangle^m |\nabla^j f|\|_{L^2(\mathbf{R}^n)},$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Proposition 1.2 (Limiting absorption principle for H_0)**.** *Let $\lambda > 0$, $n \geq 3$, $\epsilon, \sigma > 0$, and $\pm$ be a sign. Then for any $f \in H^{0,1/2+\sigma}(\mathbf{R}^n)$ the problem*

$$(H_0 - (\lambda \pm i\varepsilon))u = f$$

has the unique solution $u_{\pm\varepsilon} = R_0(\lambda \pm i\varepsilon)f \in H^2(\mathbf{R}^n)$ and we have the estimate[1]

$$\text{(1)} \qquad \|R_0(\lambda \pm i\varepsilon)f\|_{H^{0,-1/2-\sigma}(\mathbf{R}^n)} \leq C\lambda^{-1/2}\|f\|_{H^{0,1/2+\sigma}(\mathbf{R}^n)}$$

for all $\varepsilon > 0$, and we have the variant estimate

$$\text{(2)} \qquad \|R_0(\lambda \pm i\varepsilon)f\|_{H^{2,-1/2-\sigma}(\mathbf{R}^n)} \leq C(1+\lambda)\lambda^{-1/2}\|f\|_{H^{0,1/2+\sigma}(\mathbf{R}^n)}.$$

Furthermore, as $\varepsilon \to 0$,

$$u_{\pm\varepsilon} = R_0(\lambda \pm i\varepsilon)f \to u_{\pm} = R_0(\lambda \pm i0)f \quad in \ H^{2,-1/2-\sigma}(\mathbf{R}^n).$$

The function $u_{\pm}$ is the unique solution in $H^{2,-1/2-\sigma}(\mathbf{R}^n)$ to the Helmholtz equation $(H_0 - \lambda)u = f$ which obeys the Sommerfeld radiation condition

$$(\partial_r \mp i\sqrt{\lambda})u_{\pm} \in H^{0,-1/2+\sigma'}(\mathbf{R}^n) \ for \ all \ \sigma' < \sigma,$$

where $\partial_r := \frac{x}{|x|} \cdot \nabla_x$ is the radial derivative. Moreover,

$$(\partial_r \mp i\sqrt{\lambda \pm i\varepsilon})u_{\pm\varepsilon} \to (\partial_r \mp i\sqrt{\lambda})u_{\pm} \quad in \ H^{0,-1/2+\sigma'}(\mathbf{R}^n).$$

Finally, if $\sigma < 1$, we may replace (1) with the estimate

$$\text{(3)} \qquad \|R_0(\lambda \pm i\varepsilon)f\|_{H^{0,-3/2+\sigma}(\mathbf{R}^n)} \leq C(1+\lambda)^{-1/2}\|f\|_{H^{0,1/2+\sigma}(\mathbf{R}^n)}.$$

A similar version, with the factor of $(1+\lambda)^{-1/2}$ instead of $\lambda^{-1/2}$, holds in place of (2).

The result is well-known for the free Schrödinger Hamiltonian $H_0 = -\Delta$. It was established for a more general class of differential operators with *constant* coefficients by Agmon [1]. We will reprove this result in the course of this paper (see in particular Section 6.1 for an elementary argument). The sharpness of the above estimates can be easily verified from the behavior of the kernel

$$R_0(\lambda \pm i\varepsilon)(x,y) \sim C\frac{e^{i\sqrt{\lambda \pm i\varepsilon}\,|x-y|}}{|x-y|^{n-2}}$$

of the resolvent $R_0(\lambda \pm i\varepsilon)$ for $|x-y| \geq C\lambda^{-1/2}$ and the bound $R_0(\lambda \pm i\varepsilon)(x,y) \leq C|x-y|^{2-n}$ for $|x-y| \leq C\lambda^{-1/2}$.

The limiting absorption principle can be viewed as a quantitative formulation of the fact that the operator H_0 has no embedded eigenvalues or resonances in its absolutely continuous spectrum $[0,+\infty)$. As is well known, it is also closely connected to the *limiting amplitude principle* for the wave equation, and the *local smoothing estimate* for the Schrödinger equation. We shall return to these connections later in this introduction.

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[1]Here and in the sequel, we allow all absolute constants C to depend on the dimension n and on exponents such as σ.

1.3. **Quantitative limiting absorption.** In this paper we study the problem of extending the limiting absorption principle (and its applications) in a quantitative manner to other Schrödinger operators H. More specifically, we shall limit our attention to operators of the form $H = -\Delta_M + V$, where M is an asymptotically conic manifold and V is a short-range potential; we now make these concepts more precise. Henceforth $n \geq 3$ and $\sigma, \sigma_0 > 0$ are fixed, and we allow all constants to depend on n, σ, and σ_0.

Definition 1.4 (Asymptotically conic manifold). An *asymptotically conic manifold* is a smooth connected n-dimensional Riemannian manifold (M, g) which, outside of a compact set $K_0 \subset M$, can be parameterized as $M \backslash K_0 = [R_0, \infty) \times \partial M = \{(r, \omega) : r > R_0, \omega \in \partial M\}$ for some $R_0 > 1$, where $(\partial M, h)$ is a smooth compact $n-1$-dimensional Riemannian manifold, and the metric g takes the scattering form[2]

$$(4) \qquad g = dr^2 + r^2 h[r]_{ab}(\omega) d\omega^a d\omega^b$$

where for each $r > R_0$, $h[r]$ is a metric of the form

$$(5) \qquad h[r] := (h_{ab}(\omega) + r^{-2\sigma_0} e_{ab}(r, \omega))$$

where $\sigma_0 > 0$ and the error e_{ab} obeys the first derivative estimates

$$(6) \qquad |e_{ab}(r, y)| + r|\partial_r e_{ab}(r, y)| + |\nabla_y e_{ab}(r, y)|_h \leq C$$

for all $(r, y) \in M \backslash K_0$. We also make the qualitative assumption that the function $e_{ab}(1/s, y)$ extends smoothly to a function on $[0, 1/R_0] \times \partial M$. We define a weight $\langle x \rangle$ on M by setting $\langle x \rangle := r$ on $M \backslash K_0$, and extending the weight smoothly to K_0 so that it stays between R_0 and $R_0/2$. We define the space $L^2(M)$ by using the measure $dg(x) := \sqrt{g}\, dx$ induced by the Riemannian metric, and let $\Delta_M := \nabla^\alpha \nabla_\alpha$ be the usual Laplace-Beltrami operator, where ∇_α denotes covariant differentiation with respect to the Levi-Civita connection, raised and lowered in the usual manner. We define the weighted Sobolev spaces $H^{s,m}(M)$ for $s = 0, 1, 2$ and $m \in \mathbf{R}$ by the formula

$$(7) \qquad \|f\|_{H^{s,m}(M)} := \sum_{k=0}^{s} \|\langle x \rangle^m |\nabla^k f|_g\|_{L^2(M)}.$$

Important convention. Henceforth, all constants C are allowed to depend on the manifold M and the quantities R_0, C, σ_0 appearing above, as well as on the dimension n.

Remark 1.5. The class of asymptotically conic manifolds includes the class of asymptotically flat manifolds (as studied for instance in [30], [40], [12], [69], [98]) as a sub-class, in which ∂M is the unit sphere S^{n-1} with the standard metric. However in general we do not assume any topological flatness of M, nor do we assume that ∂M is topologically a sphere. In particular it is certainly possible for M to contain trapped geodesics. The condition (6) seems to indicate that we only require C^1 control on our metric. However, this is because of our decision to use normal co-ordinates (4). A metric in a more general long-range form, e.g. $g = g_0 + r^{-2\sigma_0}\tilde{e}$, where $g_0 = dr^2 + r^2 h_{ab}$ and $\tilde{e}$ is a smooth function obeying symbol-type estimates of order 0, can be placed in the normal form (4) (see [52, Section 10.5]) but as

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[2]We use the indices a, b to parameterize the $n - 1$-dimensional space ∂M, and the indices i, j to parameterize the n-dimensional space M.

is well known, the use of normal forms costs one degree of regularity, so that one would need C^2 type bounds on the metric g in the original co-ordinates in order to get the C^1-type control in (4).

We define a *short-range potential* $V : M \to \mathbf{R}$ on M to be any real-valued function V such that $\langle x \rangle^{1+2\sigma_0} V(x) \in L^\infty(M)$ for some $\sigma_0 > 0$, and then define the *Schrödinger operator* $H := -\Delta_M + V$.

Remark 1.6. The spectral theory of the operator H in the short-range case is well understood; indeed, it is known, see e.g. [49], that H is essentially self-adjoint, so that the resolvents $R(\lambda \pm i\varepsilon) := (H - (\lambda \pm i\varepsilon))^{-1}$ are well-defined for $\varepsilon \neq 0$ and are bounded operators on $L^2(M)$ (with bounds depending on ε, of course). We note that the essential self-adjointness of H, without symbol behavior assumptions on the potential V and the metric coefficients of g, follows from a lower bound

$$(8) \qquad \int_M Hu\,\overline{u} \geq -C \int_M |u|^2$$

and uniqueness of solutions of the wave equation

$$u_{tt} + Hu = 0, \qquad u(0) = u_0, \quad \partial_t u(0) = u_1,$$

see [6], [26], [85] and references therein. The lower bound (8) easily follows from the L^∞ bound on the potential V and positivity of the Laplace-Beltrami operator $-\Delta_M$. The uniqueness statement is a consequence of the energy identity

$$\int_M \left(|\partial_t u(t)|^2 + |\nabla u(t)|_g^2 + V|u(t)|^2 \right) dg = \int_M \left(|\partial_t u(0)|^2 + |\nabla u(0)|_g^2 + V|u(0)|^2 \right) dg.$$

Furthermore, the spectrum $\sigma(H)$ consists of the right half-line $[0, +\infty)$ (where it is purely absolutely continuous, except possibly at 0 where one may have an eigenvalue), together with a finite number of eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq 0$ on the negative half-line $(-\infty, 0]$; the finite number of bound states is implied by the Cwikel-Lieb-Rozenblum inequality for manifolds, see e.g. [67], [68], since $V \in L^{n/2}$. In particular one can define spectral multipliers $m(H)$ of H for any bounded (or polynomially growing) m in the usual manner.

Our first main result establishes a limiting absorption principle with explicit control on the constants in this general setting:

Theorem 1.7 (Quantitative limiting absorption principle)**.** *Let $\lambda > 0$, $n \geq 3$, $\sigma, \sigma_0 > 0$, and $\pm$ be a sign. Let M be an asymptotically conic manifold, let $V : M \to \mathbf{R}$ be a short-range potential obeying the pointwise bounds*

$$(9) \qquad |V(x)| \leq A \left(\langle x \rangle^{-2-2\sigma_0} + \lambda^{1/2} \langle x \rangle^{-1-2\sigma_0} \right)$$

for all $x \in M$, and let $H := -\Delta_M + V$ with resolvents $R(\lambda \pm i\varepsilon) := (H - (\lambda \pm i\varepsilon))^{-1}$. Let $f \in H^{0,1/2+\sigma}(\mathbf{R}^n)$.

- *(Limiting absorption near infinity) There exists a compact set $K = K(M, A)$ such that for $s = 0, 1, 2$*

$$(10) \quad \|R(\lambda \pm i\varepsilon)f\|_{H^{s,-1/2-\sigma}(M\backslash K)} \leq \lambda^{s/2} C(M, A)(\lambda^{-C(M,A)} + 1)\|f\|_{H^{0,1/2+\sigma}(M)}.$$

- *(Global limiting absorption) We have the bound*

(11)
$$\|R(\lambda \pm i\varepsilon)f\|_{H^{2,-1/2-\sigma}(M)} \leq C(M,A)\left(\lambda^{-C(M,A)} + e^{C(M,A)\sqrt{\lambda}}\right)\|f\|_{H^{0,1/2+\sigma}(M)}.$$

The constant $C(M,A)$ can be explicitly computed in terms of M, A, n, σ, σ_0.

Remark 1.8. A key feature of this estimate is that it is *quantitative* (or *effective*), in the sense that the bounds in (10), (11) depend only on the underlying manifold M and on the bounds enjoyed by the potential V. In particular, no spectral assumptions on H (e.g. involving eigenfunctions or resonances at zero) are assumed. Later on we shall also discuss *qualitative* (or *ineffective*) limiting absorption principles, in which the bounds are obtained indirectly (via compactness arguments or Fredholm theory) and are allowed to depend on the potential V (and in particular on the spectral behavior of V). In order to obtain effective bounds in Theorem 1.7, we shall avoid the use of compactness methods, and instead rely on "elementary" methods such as integration by parts and ODE analysis.

Remark 1.9. The condition (9) on the potential demands $1/\langle x \rangle^{2+}$ type decay at low energies, but only $1/\langle x \rangle^{1+}$ decay at high energies. It appears to be essentially optimal in the class of pointwise bounds. It is almost scale invariant under the transformation $x \to \mu x$, $\lambda \to \mu^{-1/2}\lambda$. The limiting absorption principle in the Euclidean case $H = -\Delta_{\mathbf{R}^n} + V$ was established for $\lambda \geq c > 0$ by Agmon [1] essentially under the assumption that $|V(x)| \leq A\langle x \rangle^{-1-2\sigma_0}$. The global-in-time local smoothing estimate and Strichartz estimates for the Schrödinger group $e^{it(-\Delta_{\mathbf{R}^3}+V)}$ were proved in [5] and [45] respectively under the assumption that $|V(x)| \leq A\langle x \rangle^{-2-2\sigma_0}$. These results are qualitative as their proof relies on Fredholm theory. In [19] (see also the followup papers [90], [48]), these results were extended to the class of inverse square potentials $V(x) = A/|x|^2$; due to the explicit nature of the eigenfunctions in this case, Fredholm theory can be avoided, and the results here are quantitative.

It would be natural to replace pointwise conditions of (9) corresponding integral assumptions on V, i.e., on the scale-invariant $L^{n/2}$-norm of V. In the flat case there is a number of recent qualitative results, see [44], [58].

The second term in (9) suggests that one could weaken the decay hypotheses on V in exchange for more regularity. It is well known however that the limiting absorption principle can fail for potentials decaying like $1/|x|$ or slower. As the standard example with the Wigner-Von-Neumann potential shows, such potentials can create embedded eigenvalues in the continuous spectrum, which would clearly destroy the limiting absorption principle in this case.

Remark 1.10. The bound (10) shows that (at least for high energy λ) geometry of the "black box" compact region K essentially does not affect the behavior of the resolvent restricted to the complement of K. Such bounds for sufficiently large λ with the constant $C(M,A,\lambda) \leq C(M,A)\lambda^{-1/2}$ have been proven to hold in [22], see also [15]. It is also interesting to note that in this context, local in time Strichartz estimates, restricted to the complement of a compact set, for solutions of a time-dependent Schrödinger equation have been established in [10].

Remark 1.11. The high-energy case $\lambda \gg 1$ of the above theorem is essentially contained in earlier work of Burq [13] and Cardoso-Vodev [22], [24]. Thus the main novelty here is the ability to treat medium energies $\lambda \sim 1$ and low energies $\lambda \ll 1$, as well as the universality of the technique developed to treat all ranges of energies.

The above theorem only reprises one component of Proposition 1.2, namely the estimates (1), (2). Using those estimates, however, one can fairly easily obtain convergence properties of the resolvent as $\varepsilon \to 0$.

Proposition 1.12 (Limiting values of resolvent). *Let the notation and assumptions be as in Theorem 1.7. Let $0 < \sigma < \min(1, \sigma_0)$ and $f \in H^{0,1/2+\sigma}(M)$. Let $\lambda \pm i\varepsilon = z^2$ with $z = a \pm b$ such that $b > 0$, Then the functions $u_{\pm\varepsilon} := R(\lambda \pm i\varepsilon)f$ obey the Sommerfeld radiation condition*

$$
(12) \qquad
\begin{aligned}
&\|(\partial_r \mp iz)u_{\pm\varepsilon}\|_{H^{0,-1/2+\sigma}(M\setminus K_0)} + \|\nabla_\omega u_{\pm\varepsilon}\|_{H^{0,-3/2+\sigma}(M\setminus K_0)} \\
&\leq C(M, K_0, A)(\lambda^{-C(M,A)} + e^{C(M,A)\sqrt{\lambda}})\|f\|_{H^{0,1/2+\sigma}(M)}
\end{aligned}
$$

where $|\nabla_\omega u|_g^2 = r^2(|\nabla u|_g^2 - |\partial_r u|^2)$. Note that the constants are uniform in the choice of ε. Furthermore, for a fixed sign $\pm$ and fixed $\lambda > 0$, the functions $u_{\pm\varepsilon}$ converge in $H^{0,-1/2-\sigma}(M)$ to a limit $u_\pm$, which is the unique solution in $H^{2,-1/2-\sigma}(M)$ to the Helmholtz equation $(H - \lambda)u = f$ such that $(\partial_r \mp i\lambda^{1/2})u_\pm$ lies in $H^{0,-1/2+\sigma'}(M\setminus K_0)$ for at least one $0 < \sigma' < \sigma$.

We prove Theorem 1.7 and Proposition 1.12 in Section 3.

1.13. **Refinements at high energy.** At high energies $\lambda \gg 1$, Theorem 1.7 suffers an exponential loss in the constants. We now recall the standard quasimode example that shows that this loss is necessary for certain manifolds M:

Proposition 1.14 (Quasimode construction). *Let $C_0 > 0$ and $\sigma > 0$ be arbitrary. There exists a smooth Riemannian manifold M, which is equal to Euclidean space $\mathbf{R}^n$ outside of a compact set, and a sequence $\lambda_l \to +\infty$ and $\varepsilon_l \to 0^+$ of real numbers, as well as functions $f_l \in H^{0,1/2+\sigma}(M)$ such that if $H := -\Delta_M$ (i.e. $V \equiv 0$) then*

$$
\|R(\lambda_l \pm i\varepsilon_l)f_l\|_{H^{0,-1/2-\sigma}(M)} > e^{C_0\sqrt{\lambda_l}}\|f_l\|_{H^{0,1/2+\sigma}(M)}
$$

for all l.

We give this standard counterexample (based on the trapped geodesics in a sphere) in Section 13. It shows that we cannot hope to eliminate the exponential factor $e^{C(M,A)\sqrt{\lambda}}$ from Theorem 1.7, at least for manifolds which contain trapped geodesics.

However, as is well known, the limiting absorption properties improve substantially at high energies if one assumes a non-trapping condition on the manifold M. We give a (standard) result in this direction, with somewhat stronger assumptions on the manifold and potential than is strictly necessary:

Theorem 1.15 (Quantitative high-energy non-trapping limiting absorption principle)**.** *Let the notation and assumptions be as in Theorem 1.7. Suppose also that V obeys the bound*[3]

$$|V(x)| \leq A\langle x\rangle^{-1-2\sigma_0}$$

for some $A > 0$ and all $x \in M$, and that the metric perturbation error function e_{ab} obeys the symbol estimates

(13) $$|(r\partial_r)^\alpha \nabla_y^\beta \mathbf{e}_{ab}(r,y)| \leq C_{\alpha\beta}$$

for all multi-indices $\alpha, \beta \geq 0$. Suppose also that the manifold M is non-trapping in the sense that every unit speed geodesic $t \mapsto \gamma(t)$ in M escapes to the infinite boundary ∂M as $t \to \pm\infty$. Then, for λ sufficiently large depending on M and A, we have

(14) $$\|R(\lambda \pm i\varepsilon)f\|_{H^{s,-1/2-\sigma}(M)} \leq C(M,A)\lambda^{(s-1)/2}\|f\|_{H^{0,1/2+\sigma}(M)}$$

for all $0 \leq s \leq 2$.

We prove this Theorem in Section 7 using the positive commutator method and the scattering pseudodifferential calculus.

Remark 1.16. Such results for non-trapping metrics have a long history starting with the work [100], [66] for compact perturbations of the free Hamiltonian $H_0 = -\Delta_{\mathbf{R}^n}$. In the general case a related result is contained in [102], see also [13] for a qualitative version, and [82]. For the closely related task of establishing local smoothing estimates, results analogous to those above were established in [30].

Remark 1.17. In between the two extremes of non-trapping manifolds (for which one has the optimal $\lambda^{-1/2}$ decay in the limiting absorption principle at high energies) and the highly trapping counterexamples (for which one has exponential growth in $\sqrt{\lambda}$) there is a complicated range of intermediate behaviour, for instance if there is just one hyperbolic trapped orbit. We will not attempt to survey the substantial literature on this topic, but see [55], [56], [76], [16], [38], [27], [28], [29] for some work in this area.

Remark 1.18. If we replace the assumption on the potential by the weaker bound $|V(x)| \leq A\lambda^{1/2}\langle x\rangle^{-1-2\sigma_0}$ then one would have to modify the non-trapping condition as the potential term V has the same order as the kinetic term $-\Delta_M$ and one would have to consider the Hamilton flow associated to the symbol of $-\Delta_M + V$ rather than the geodesic flow. We omit the (standard) formalization of this fact.

Note that by combining this Theorem with Theorem 1.7 one can weaken the hypothesis that λ is sufficiently large, instead requiring that $\lambda > \lambda_0$ for some fixed $\lambda_0 > 0$ (but then the constant $C(M,A)$ in (14) may depend on λ_0).

Remark 1.19. The proof of Theorem 1.15 is microlocal. The result therefore can be strengthened to include the situation where there is trapping. In this case we consider the conical set

$$\mathcal{M}_{\text{nontrap}} = \{(x,\xi):\ \gamma(t)\ \gamma(0) = x,\ \dot\gamma(0) = \xi \text{ escapes to } \partial M,\ t \to \pm\infty\}$$

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[3]Alternatively we may assume that $|V(x)| + \langle x\rangle|\nabla V(x)|_g \leq A\langle x\rangle^{-2\sigma_0}$, which allows V to be a long range potential.

invariant under the geodesic flow in the phase space and a corresponding ΨDO P_{nontrap} with support in $\mathcal{M}_{\text{nontrap}}$. Then we can replace (14) with

$$\|R(\lambda \pm i\varepsilon)P_{\text{nontrap}}f\|_{H^{s,-1/2-\sigma}(M)} \leq C(M,A)\lambda^{s-1/2}\|f\|_{H^{0,1/2+\sigma}(M)}$$

1.20. **Refinements at low energy.** At the low energy limit $\lambda \to 0$, the constant in (11) (or (10)) blows up like a large negative power of λ. For the purposes of *effective* bounds, which are uniform for all potentials V obeying the bound (9), such blowup is necessary, as the following result shows:

Proposition 1.21 (Bessel equation matching construction). *Let $C_0 > 0$ and $\sigma, \sigma_0 > 0$ be arbitrary. Then there exists a sequence $V_m : \mathbf{R}^n \to \mathbf{R}$ of spherically symmetric potentials obeying the uniform bounds*

$$\sup_m \|\langle x \rangle^{1+\sigma_0} \nabla_x^\alpha V_m\|_{L^\infty(\mathbf{R}^n)} < \infty \text{ for all } \alpha \geq 0,$$

sequences $\lambda_m \to 0^+$ and $\varepsilon_m \to 0^+$ of real numbers with $0 < \varepsilon_m < \lambda_m$, as well as functions $f_m \in H^{0,1/2+\sigma}(\mathbf{R}^n)$ such that if $H_m := -\Delta + V_m$ then H_m has no resonance or eigenfunction at zero, and

$$\|R(\lambda_m \pm i\varepsilon_m)f_m\|_{H^{0,-1/2-\sigma}(M)} > \lambda_m^{-C_0}\|f_m\|_{H^{0,1/2+\sigma}(M)}$$

for all m.

We give this standard example in Section 14. This proposition shows that the large negative power in (11) is necessary. One can also show that a negative power is necessary in (10) by combining this example with the Carleman estimate (Proposition 2.6); we omit the details.

Despite this example, one can eliminate (or at least reduce) the low energy blowup in certain cases. Firstly, in the flat case $M \equiv \mathbf{R}^n$ the the work of Jensen-Kato [60] and Jensen [59] provides small energy resolvent expansions

$$(15) \qquad (H - \lambda)^{-1} = \sum_{j=-2}^{\ell} \lambda^{j/2} B_j + o(\lambda^{\ell/2})$$

for Hamiltonians $H = -\Delta_{\mathbf{R}^n} + V$ in odd[4] dimensions n, under appropriate conditions on V, dependent in particular on the index ℓ, even in the presence of zero eigenvalues and resonances. The expansion (15) shows that for a *fixed* potential V the limiting absorption constant for the resolvent $(H - \lambda)^{-1}$ blows up at worst as $C(M,V)\lambda^{-1}$ as $\lambda \to 0$. Note however that (15) is qualitative in a sense that the bounds on the norms B_j, as operators between weighted Sobolev spaces, depend on the potential V and not just on the bound A for that potential; thus there is no inconsistency between (15) and Proposition 1.21. Note however subtle distinction between effective bounds (which hold uniformly for all V obeying (9)) and ineffective bounds (which are not uniform in V).

It is quite likely that these expansions could be extended to asymptotically conic manifolds. We will not do so in full generality here, but just give the standard

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[4]In even dimensions resolvent expansions involves additional terms containing $\log \lambda$.

qualitative limiting absorption estimate in the case that there are no eigenfunctions or resonances at zero:

Definition 1.22 (Resonances at zero). Let $\lambda = \kappa^2$ for some $\kappa \geq 0$. We say that H has an outgoing (resp. incoming) *resonance at λ* if the equation $Hu = \lambda u$ has a solution $u \in H^{0,-3/2+\sigma}(\mathbf{R}^n)$ for some $\sigma > 0$ with the property that $u \notin \bigcup_{\sigma>0} H^{0,-1/2+\sigma}(\mathbf{R}^n)$ and $\partial_r u - i\kappa u \in H^{0,-1/2+\sigma'}$ (resp. $\partial_r u + i\kappa u \in H^{0,-1/2+\sigma'}$) for some $\sigma' > 0$. Note that when $\lambda = 0$, there is no distinction between incoming and outgoing resonances, and we shall talk simply about resonances at zero.

Proposition 1.23 (Qualitative low-energy limiting absorption principle). *Let the notation and assumptions be as in Theorem 1.7. Suppose also that V is strongly short-range in the sense that $\langle x \rangle^{2+2\sigma_0} V \in L^\infty(M)$. Assume also that the operator $H := -\Delta_M + V$ contains no eigenfunction or resonance at zero. Then for $\lambda, \varepsilon \neq 0$ sufficiently small depending on M and V, we have*

$$(16) \qquad \|R(\lambda \pm i\varepsilon)f\|_{H^{2,-1/2-\sigma}(M)} \leq C(M,V)|\lambda|^{-1/2}\|f\|_{H^{0,1/2+\sigma}(M)}.$$

Furthermore, if $\sigma > 1/2$, then we can strengthen this further, to

$$(17) \qquad \|R(\lambda \pm i\varepsilon)f\|_{H^{2,-1/2-\sigma}(M)} \leq C(M,V)\|f\|_{H^{0,1/2+\sigma}(M)}.$$

The proof of this result is essentially an application of the Fredholm alternative[5] and the theory for the free case $V = 0$; we give this proof at the end of this section. The estimates here should be compared with (11); the constants do not blow up as fast as $\lambda \to 0$, but on the other hand the bounds depend directly on V and not just on A.

Remark 1.24. As is well known, in asymptotically Euclidean case in high dimensions $n \geq 5$ no resonances at zero can occur; see [59]. Moreover, zero eigenfunctions and resonances are non-generic relative to a family of perturbations $H_\kappa = -\Delta_M + \kappa V$; see [60].

Remark 1.25. The above proposition is only stated for sufficiently small λ. But by combining this result with Theorem 1.7 we see that it in fact holds for all bounded λ. If we also assume non-trapping, then by Theorem 1.15 we can extend (16) and (17) to arbitrary energies $\lambda > 0$.

Remark 1.26. The subject of qualitative limiting absorption principles has a long and rich history. Most of the results have been focused on a high or a fixed non-zero energy regime in various geometric settings, such as the asymptotically Euclidean, conic, hyperbolic and obstacle cases; see for instance [66], [72], [57], [54], [53] for a representative set of results. The low energy results have been developed for example in [46], [47], [9]. For the operator $-\Delta_{\mathbf{R}^n} + V$ resolvent behavior have analyzed in particular in [1], [60], [59], [44]. Recently there have been a lot of interest in establishing sub-exponential, in fact logarithmic, bounds on the resolvent at high energies for geometries with sufficiently "small" or filamentary trapped sets, see [18], [76], [103], [39].

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[5]Strictly speaking, we do not apply the Fredholm alternative directly, as we will need to uniformly invert a *family* $1 + R_0(\lambda \pm i\varepsilon)V$ of compact perturbations of the identity, rather than a single such perturbation, but instead will be using the *proof* of the Fredholm alternative in our arguments.

In some cases we can make the absence of eigenfunctions or resonances at zero quite quantitative. For instance, when the potential is mostly positive we have the following result.

Proposition 1.27 (Quantitative low-energy limiting absorption principle for mostly positive potentials)**.** *Let the notation and assumptions be as in Theorem 1.7. Suppose also that V obeys the bounds*

$$|V(x)| \leq A\langle x \rangle^{-2-2\sigma_0}$$

for some $A > 0$, and suppose also that the negative part $V_- := \max(-V, 0)$ of V obeys the smallness condition

$$\int_M |V_-(x)|^{n/2} \leq \beta(M)$$

for some sufficiently small $\beta > 0$ depending only on M. Then H has no eigenfunctions or resonances at any energy $\lambda \in \mathbf{R}$ (and in particular at $\lambda = 0$). Furthermore, if $\lambda, \varepsilon \neq 0$ are sufficiently small depending on M and A, then

(18) $$\|R(\lambda \pm i\varepsilon)f\|_{H^{2,-1/2-\sigma}(M)} \leq C(M, A)|\lambda|^{-1/2}\|f\|_{H^{0,1/2+\sigma}(M)}.$$

Finally, for all sufficiently small $|\lambda|, \varepsilon$ and $\sigma < \min(\sigma_0, 1)$ we have the following uniform bound

(19) $$\|R(\lambda \pm i\varepsilon)f\|_{H^{2,-3/2+\sigma}(M)} \leq C(\mathcal{I}(M), A)\|f\|_{H^{0,1/2+\sigma}(M)},$$

where the constant $C(\mathcal{I}(M), A)$ depends on the manifold (M, g) only through the isoperimetric constant $\mathcal{I}(M)$.

We prove this result in Section 12.

Remark 1.28. Observe that the Euclidean limiting absorption principle, Proposition 1.2, now follows from Theorem 1.7, Proposition 1.12, Theorem 1.15 (since the Euclidean space $\mathbf{R}^n$ is non-trapping), and Proposition 1.27 (with $V \equiv 0$).

Note that Proposition 1.27 already applies in the free case $V = 0$. From this and Fredholm theory we can now prove Proposition 1.23:

Proof of Proposition 1.23. We begin with the proof of (17). We write $H = H_0 + V$ where $H_0 = -\Delta_M$ is the free Laplacian, and denote the resolvents for H_0 by $R_0(\lambda \pm i\varepsilon)$. We have the resolvent identity

$$R(\lambda \pm i\varepsilon) = (1 + R_0(\lambda \pm i\varepsilon)V)^{-1}R_0(\lambda \pm i\varepsilon).$$

From (19) (and duality and interpolation), the resolvent R_0 already obeys (17) for any $\sigma > 1/2$, and so it suffices to show that the operators $1 + R_0(\lambda \pm i\varepsilon)V$ are uniformly invertible on $H^{2,-1/2-\sigma}(M)$ for σ sufficiently close to (but larger than) $1/2$. From the Rellich embedding theorem and the strongly short-range nature of V, we know that V maps $H^{2,-1/2-\sigma}(M)$ compactly to $H^{0,1/2+\sigma}(M)$, and so by (17) for the free operator, the operators $R_0(\lambda \pm i\varepsilon)V$ are uniformly compact on $H^{2,-1/2-\sigma}(M)$ for λ sufficiently small.

Suppose that uniform invertibility fails, then we can find a sequence $\lambda_n \pm i\varepsilon_n$ going to zero, functions u_n of unit magnitude in $H^{2,-1/2-\sigma}(M)$, and f_n going to zero strongly in $H^{2,-1/2-\sigma}(M)$ such that

$$(1 + R_0(\lambda_n \pm i\varepsilon_n)V)u_n = f_n.$$

If we then write $\tilde{u}_n := R_0(\lambda \pm i\varepsilon)Vu_n = f_n - u_n$ and $\tilde{f}_n := R_0(\lambda \pm i\varepsilon)Vf_n$, then u_n has norm $1 + o(1)$ in $H^{2,-1/2-\sigma}(M)$ and lies in a fixed compact subset of that space, $\tilde{f}_n$ goes to zero strongly in $H^{2,-1/2-\sigma}(M)$, and

$$(1 + R_0(\lambda_n \pm i\varepsilon_n)V)\tilde{u}_n = \tilde{f}_n.$$

Applying $H_0 - (\lambda_n \pm i\varepsilon_n)$ to both sides, we conclude that

$$H\tilde{u}_n = (\lambda_n \pm i\varepsilon_n)\tilde{u}_n + (H_0 - (\lambda_n \pm i\varepsilon_n))\tilde{f}_n.$$

By compactness, we may pass to a subsequence such that $\tilde{u}_n$ converges strongly in $H^{2,-1/2-\sigma}(M)$ to a limit $\tilde{u}$ of norm 1. Taking distributional limits, we conclude that $H\tilde{u} = 0$ in the sense of distributions. But this contradicts the hypothesis that H has no resonance or eigenvalue at zero.

The proof of (16) is similar, but one takes σ close to zero instead of to $1/2$, and uses $H^{0,3/2-\sigma}(M)$ instead of $H^{0,1/2+\sigma}(M)$ (and relies on (18) and the adjoint of (19)) to mediate between $R_0(\lambda \pm i\varepsilon)$ and V; we omit the details. $\square$

1.29. **Applications.** We now give some applications of the above limiting absorption estimates. We begin with some (well-known) spectral consequences.

Proposition 1.30 (Absence of embedded eigenvalues or resonances). *Let the notation and assumptions be as in Theorem 1.7, and let $\lambda > 0$. Then there are no non-zero eigenfunctions or resonances (incoming or outgoing) at λ.*

Proposition 1.31 (Absence of embedded singular continuous spectrum). *The spectrum of H on $(0, +\infty)$ is purely absolutely continuous.*

We prove these results in Section 15.1.

In Section 15.5 we give a version of the celebrated RAGE theorem (cf. [83], [2], [42]).

Proposition 1.32 (RAGE theorem). *Let the notation and assumptions be as in Theorem 1.7. Let $f \in L^2(M)$ be orthogonal to all eigenfunctions of H. Then for any compact set K, we have*

$$(20) \qquad \lim_{t \to \pm\infty} \|e^{itH}f\|_{L^2(K)} = 0.$$

Similarly, we have

$$\lim_{t \to \pm\infty} \|u(t)\|_{H^1(K)} + \|\partial_t u(t)\|_{L^2(K)} = 0$$

for a solution of the wave equation $\partial_t^2 u + Hu = 0$ with initial data $(u(0), \partial_t u(0)) \in H^1(M) \times L^2(M)$, orthogonal to the eigenfunctions of H.

Next, we use the limiting absorption principle and the RAGE theorem, together with a closely related result that gives Hölder continuity bounds on the resolvent, to derive the *limiting amplitude principle* for the wave equation.

Proposition 1.33 (Limiting amplitude principle)**.** *Let the notation and assumptions be as in Theorem 1.7, let $f \in L^2(M)$ be compactly supported, and let $u_0 \in H^1(M)$, $u_1 \in L^2(M)$, and $\mu > 0$. Assume that f, u_0, u_1 are all orthogonal to all the eigenfunctions of H. Let $u : \mathbf{R} \times M \to \mathbf{C}$ be the solution to the inhomogeneous wave equation*

$$(21) \qquad \partial_t^2 u - \Delta_M u = e^{i\mu t} f, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1.$$

Then for any compact set $K \subset M$ we have

$$\|u(t) - e^{i\mu t} v\|_{H^1(K)} \to 0 \ as \ t \to +\infty$$

where v is the outgoing solution of the Helmholtz problem

$$(H - \mu^2)v = f,$$

i.e. v is such that $(\partial_r - i|\mu|)v \in H^{0,-1/2+\sigma'}(\mathbf{R}^n)$.

We prove this result in Section 15.6.

Now we give a global-in-time local smoothing estimate for the Schrödinger equation.

Proposition 1.34 (Global-in-time local smoothing estimate for H)**.** *Let the notation and assumptions be as in Theorem 1.7. Suppose that M is a non-trapping manifold, that V obeys the bounds*

$$|V(x)| \leq A\langle x \rangle^{-2-2\sigma_0},$$

and that $H := -\Delta_M + V$ has no eigenfunction or resonance at zero. Let $u : \mathbf{R} \times \mathbf{R}^n \to \mathbf{C}$ be a solution to the forced Schrödinger equation

$$iu_t - Hu = F.$$

and let P_H denote the projection on the continuous spectrum of H. Then we have the estimate
(22)
$$\int_{\mathbf{R}} \|H^{1/2} P_H u(t)\|_{H^{0,-1/2-\sigma}(M)}^2 \, dt \leq C(M,V,A) \left(\|u(0)\|_{H^{1/2}(M)}^2 + \int_{\mathbf{R}} \|F(t)\|_{H^{0,1/2+\sigma}(M)}^2 \, dt \right).$$

If furthermore $\sigma > 1/2$, then we have the variant estimate

$$\int_{\mathbf{R}} \left(\|H^{1/4} P_H u(t)\|_{H^{0,-1/2-\sigma}(M)}^2 + \|P_H u(t)\|_{H^{0,-1/2-\sigma}(M)}^2 \right) \, dt \leq C(M,V,A) \times$$

$$\left(\|u(0)\|_{L^2(M)}^2 + \int_{\mathbf{R}} \|F(t)\|_{H^{0,1/2+\sigma}(M)}^2 \, dt \right).$$

If we assume that V obeys the hypotheses of Proposition 1.27 then we may eliminate the projection P_H and replace the constants $C(M,V,A)$ here by $C(M,A)$.

Finally, if we assume symbol estimates on both the metric coefficients e_{ab} and the potential V then we can replace the last estimate with the family of bounds

$$\int_{\mathbf{R}} \|P_H u(t)\|^2_{H^{s+1/2,-1/2-\sigma}(M)} \, dt \le C(s, M, V, A) \times$$

$$\times \left(\|u(0)\|^2_{H^s(M)} + \int_{\mathbf{R}} \|F(t)\|^2_{H^{s-1/2,1/2+\sigma}(M)} \, dt \right)$$

for all $s \ge 0$.

We prove this proposition in Section 15.3.

Remark 1.35. The nontrapping assumption of Theorem 1.34 can be removed if one replaces the projection P_H by the projection P_H^Λ on the continuous spectrum with energies $\lambda < \Lambda < \infty$.

Remark 1.36. The close connection between limiting absorption principles and local smoothing (they are essentially Fourier transforms of each other, with t being the dual variable to λ) was first observed by Kato [63]. See [87] for some further discussion. In Euclidean space, the global-in-time local smoothing estimates were first established in [88], [101], [31]. In order to obtain estimates which are global in time, it is necessary (by the uncertainty principle) to establish limiting absorption principles at very low energies; high-energy analysis alone is only sufficient to establish local-in-time local smoothing estimates.

Remark 1.37. It is very likely that the above global-in-time local smoothing estimate will imply global-in-time Strichartz estimates, by adapting the arguments in [89], [94], [10], [11], [69]. This would allow one to create slightly more quantitative formulations of some of the Strichartz estimates in [11] and [69], however these improvements seem to be rather minor and so we will not detail them here.

The limiting absorption principle for the Hamiltonian H also leads to the integrated local energy decay for the wave equation.

Proposition 1.38 (Integrated local energy decay)**.** *Suppose that M is a non-trapping manifold, that V obeys the bounds*

$$|V(x)| \le A\langle x \rangle^{-2-2\sigma_0},$$

and that $H := -\Delta_M + V$ has no eigenfunction or resonance at zero. Let $u : \mathbf{R}^{n+1} \to \mathbf{C}$ be a solution of the wave equation

$$(23) \qquad\qquad \partial_t^2 u + Hu = F.$$

and let P_H denote the projection on the continuous spectrum of H. Then we have the estimate

$$\int_{\mathbf{R}} \left(\|\partial_t P_H u(t)\|^2_{H^{0,-1/2-\sigma}(M)} + \|\nabla P_H u(t)\|^2_{H^{0,-1/2-\sigma}(M)} \, dt \right) \le C(M, V, A) \times$$

$$(24)$$

$$\times \left(\|\nabla u(0)\|^2_{L^2(M)} + \|u_t(0)\|^2_{L^2(M)} + \int_{\mathbf{R}} \|F(t)\|^2_{H^{0,1/2+\sigma}(M)} \, dt \right).$$

For $\sigma < \min(1, \sigma_0)$ the retarded solution

$$u_{\mathrm{ret}}(t) := \int_{t' < t} \frac{\sin((t - t')\sqrt{H})}{\sqrt{H}} F(t') \, dt'$$

of the inhomogeneous problem (23) *obeys additional bounds*

(25)
$$\int_{\mathbf{R}} \left(\|(\partial_t - \partial_r) P_H u_{\text{ret}}(t)\|^2_{H^{0,-1/2+\sigma}(M)} + \|r^{-1}\nabla_\omega P_H u_{\text{ret}}(t)\|^2_{H^{0,-1/2+\sigma}(M)} \ dt \right)$$
$$\leq C(M, V, A) \int_{\mathbf{R}} \|F(t)\|^2_{H^{0,1/2+\sigma}(M)} \ dt.$$

A similar estimate holds for the advanced solution

$$u_{\text{adv}}(t) := - \int_{t'>t} \frac{\sin((t-t')\sqrt{H})}{\sqrt{H}} F(t') \ dt'$$

with $(\partial_t + \partial_r)$ *in place of* $(\partial_t - \partial_r)$.

Furthermore, even if M *does not satisfy a non-trapping condition there exists a compact set* $K \subset M$ *such that*

$$\int_{\mathbf{R}} \left(\|\partial_t P_H u(t)\|^2_{H^{0,-1/2-\sigma}(M\setminus K)} + \|\nabla P_H u(t)\|^2_{H^{0,-1/2-\sigma}(M\setminus K)} \ dt \right) \leq C(M, V, A)$$
$$\times \left(\|\nabla u(0)\|^2_{H^1(M)} + \|u_t(0)\|^2_{H^1(M)} + \int_{\mathbf{R}} \|F(t)\|^2_{H^{1,1/2+\sigma}(M)} \ dt \right).$$

If we assume that V *obeys the hypotheses of Proposition 1.27 then we may eliminate the projection* P_H *and replace the constants* $C(M, V, A)$ *by* $C(M, A)$.

We prove this in Section 15.3.

Remark 1.39. The statement of integrated local energy decay for $H_0 = -\Delta_{\mathbf{R}^n}$ goes back to Morawetz [74]. The proof of such estimates for solutions of the wave equation on black hole spacetimes, with geometries which are quite different from the ones considered in this paper, have been instrumental in understanding their global behavior. See [7], [33], [69] for Schwarzschild, [34], [96], [3], [36] for slowly rotating Kerr and [37] for the general sub-extremal Kerr case.

Our final results are pointwise decay estimates for the solutions of the Schrödinger and wave equations, obtained by commuting these equations with a Morawetz-type operator, applyign energy estimates, and using an iteration argument to amplify the resulting decay.

Proposition 1.40 (Decay for Schrödinger). *Let* M *be a non-trapping manifold with metric* g *given in* (4) *with* $\sigma_0 > 1/2$. *We assume that* $h[r] := h + e$ *satisfies the following assumptions.*

$$|(r\partial_r)^k (\nabla_\omega^\alpha) h[r]| \leq C_{k\alpha}, \qquad k \leq 3, \ |\alpha| \leq 2.$$

Let $\psi : \mathbf{R} \times \mathbf{R}^n \to \mathbf{C}$ *be a solution to the Schrödinger equation*

$$i\partial_t \psi + \Delta_M \psi = 0.$$

Then for any $t \geq 0$ we have the dispersive estimates

$$\|\psi(t)\|_{L^\infty(M)} \leq C_\varepsilon (1+t)^{-\frac{3}{2}+\varepsilon} \|\psi(0)\|_{H^{2,1}(M)}, \qquad \forall \varepsilon > 0, \quad n = 3,$$

$$\|\psi(t)\|_{L^p(M)} \leq C(1+t)^{-2+\frac{4}{p}} \|\psi(0)\|_{H^{2,1}(M)}, \qquad \forall 2 \leq p < \infty, \quad n = 4,$$

$$\|\psi(t)\|_{L^{\frac{2n}{n-4}}(M)} \leq C(1+t)^{-2} \|\psi(0)\|_{H^{2,1}(M)}, \qquad n \geq 5.$$

Remark 1.41. It is well known that a solution of the free Schrödinger equation $i\partial_t \psi + \Delta_{\mathbf{R}^n} \psi = 0$ satisfies the dispersive estimate

$$\|\psi(t)\|_{L^\infty(\mathbf{R}^n)} \leq Ct^{-n/2} \|\psi_0\|_{L^1(\mathbf{R}^n)}.$$

As a consequence, we do not believe the rates of decay given by Proposition 1.40 to be sharp, especially in higher dimensions. Nevertheless, they appear to be novel in such general setting, and we give them here to illustrate an application of how the limiting absorption principle (or more precisely, the global-in-time local smoothing estimate) can be used to obtain dispersive inequalities.

Remark 1.42. Dispersive estimates for solutions of the Schrödinger equation with $H = -\Delta_{\mathbf{R}^n} + V$, projected on the continuous spectrum of H and assuming absence of zero eigenvalues and resonances have been proved in [78], [61], [45], [4]; see also the survey [84] and the references therein.

Proposition 1.43 (Decay for wave)**.** *Let M satisfy the assumptions of Proposition 1.40 and let $u : \mathbf{R}^{n+1} \to \mathbf{C}$ be a solution of the wave equation*

$$\partial_t^2 u - \Delta_M u = 0.$$

Then in dimension $n = 3$ we have

$$\|u(t)\|_{L^\infty(M)} \leq C_\varepsilon (1+t)^{-1+\varepsilon} \left(\|\nabla u(0)\|_{H^{1,1}(M)} + \|u_t(0)\|_{H^{1,2}(M)} \right),$$

for all $\varepsilon > 0$.

Remark 1.44. A modification of our arguments also gives the variant estimate

$$\|\psi(t)\|_{L^{\frac{2(n-1)}{n-3}}(M)} \leq C(1+t)^{-1} \left(\|\nabla u(0)\|_{H^{1,1}(M)} + \|u_t(0)\|_{H^{1,1}(M)} \right)$$

in higher dimensions $n > 3$.

Remark 1.45. Decay estimates for solutions of the wave equation in Minkowski space, $M = \mathbf{R}^n$, have, of course, a very long history, including the Huygens principle in odd dimensions and the uniform t-decay with the rate of $t^{-(n-1)/2}$. A quantitative approach to decay in Minkowski space has been developed by Klainerman, in what is known as the vector field method, [64]. Qualitative decay results in non-trapping geometries have been obtained in the pioneering works [65], [99], [75], [70]. For the problem with $-\Delta_{\mathbf{R}^n} + V$ see e.g. [43]. Quantitative decay results for solutions of the wave equation on black hole spacetimes have been obtained in [7], [33], [34], [35], [3], [36], [95], [37]. A general approach to the derivation of decay from the integrated local energy decay statements have been developed in [35], where the arguments can be in particular adapted to the asymptotically conic case, and in [95] for stationary asymptotically Euclidean spacetimes. See also [104] for applications to nonlinear problems. The results here follow an earlier approach of [33] and serve as an illustration.

We prove these propositions in Section 16 and Section 17 respectively.

2. Key estimates

Throughout this paper we use the notation and assumptions of Theorem 1.7. All constants are henceforth allowed to depend on M, n, σ_0, A, and σ.

Fix $\lambda, \varepsilon > 0$, and suppose that we have a solution to the *resolvent equation*

$$(26) \qquad\qquad (H - (\lambda \pm i\varepsilon))u = f$$

(or equivalently that $u = R(\lambda \pm i\varepsilon)f$) and the closely related *Helmholtz equation*

$$(27) \qquad\qquad (-\Delta_M - z^2)u = F, \qquad z^2 = \lambda \pm i\varepsilon$$

Of course, the former equation can be viewed as a special case of the latter with $F := f - Vu$. To avoid technicalities, we shall always make the qualitative assumptions $u, f, F \in L^2(M)$.

In this section we lay out the fundamental estimates for these equations which we shall repeatedly use in our analysis. The proof of these estimates will be deferred to later sections. We begin with a simple charge estimate.

Lemma 2.1 (Charge estimate). *Let $\lambda, \varepsilon, \sigma > 0$, let $\pm$ be a sign, let $f \in H^{0,1/2+\sigma}(M)$, and let $u = R(\lambda \pm i\varepsilon)f \in L^2(M)$. Then we have*

$$(28) \qquad\qquad \varepsilon \int_M |u|^2 \, dg \leq \|f\|_{H^{0,1/2+\sigma}(M)} \|u\|_{H^{0,-1/2-\sigma}(M)}.$$

Moreover, for any $\alpha \in \mathbf{R}$ we have

$$(29) \qquad \varepsilon^2 \int_{\langle x \rangle \geq 2R} \langle x \rangle^{2\alpha} |u|^2 \, dg \leq C(\alpha) \int_{\langle x \rangle \geq R} \left(\langle x \rangle^{2\alpha} |f|^2 + \langle x \rangle^{-2+2\alpha} |\nabla u|_g^2 \right) \, dg,$$

This easy estimate can be established by integration by parts, and is proven in Section 4.2. Roughly speaking, this estimate allows us to handle most of the terms in our analysis which contain a factor of ε, and which do not have derivatives on u.

Another integration by parts gives the following standard elliptic estimates, proven in Section 5:

Lemma 2.2 (Elliptic regularity). *Let $H = -\Delta_M + V$ where V is a short-range potential obeying (9). Let $m \in \mathbf{R}$, and let $u, f \in H^{0,m}(M)$ which satisfies the Helmholtz equation $(H - z^2)u = f$ in the distributional sense (at least) with $z^2 = \lambda \pm i\epsilon$ and $\lambda \geq 0$.*

- We have $u \in H^{2,m}(M)$ with the elliptic regularity estimate

$$(30) \qquad \|u\|_{H^{2,m}(M)} \leq C(m)(\|f\|_{H^{0,m}(M)} + (1+\lambda)\|u\|_{H^{0,m}(M)}).$$

- For any $R > 2R_0$, we have the localised elliptic regularity estimates

$$(31) \qquad \int_{R \leq \langle x \rangle \leq 2R} |\nabla u|_g^2 \, dg \leq C \int_{R/2 \leq \langle x \rangle \leq 3R} (\lambda + R^{-2})|u|^2 + \frac{|f|^2}{\lambda + R^{-2}} \, dg,$$

and

$$(32) \qquad \int_{R \leq \langle x \rangle \leq 2R} |\nabla u|_g^2 \, dg \leq C \int_{R/2 \leq \langle x \rangle \leq 3R} (\lambda + R^{-2})|u|^2 + R^2|f|^2 \, dg$$

and

$$(33)$$
$$\int_{\langle x \rangle \geq R} |\nabla^2 u|_g^2 \langle x \rangle^{2m} \, dg \leq C(m,R)[\|f\|_{H^{0,m}(M)}^2 + (1+\lambda) \int_{\langle x \rangle \geq R/2} |u|_g^2 \langle x \rangle^{2m} \, dg].$$

- We have the energy estimate

$$(34) \qquad \lambda^{\frac{1}{2}} \|u\|_{H^{0,-1/2-\sigma}} \leq C \left(\|\nabla u\|_{H^{0,-1/2-\sigma}} + \|u\|_{H^{0,-3/2-\sigma}} + \|f\|_{H^{0,1/2+\sigma}} \right).$$

- If $\varepsilon \leq C\lambda$ for some constant $C > 0$ then we have the charge-type estimate

$$(35)$$
$$\varepsilon \|\nabla u\|_{L^2(M)} \|u\|_{L^2(M)} \leq C\|f\|_{H^{0,1/2+\sigma}} \left(\|\nabla u\|_{H^{0,-1/2-\sigma}} + \|u\|_{H^{0,-3/2-\sigma}} + \|f\|_{H^{0,1/2+\sigma}} \right).$$

as well as the localised estimates

$$(36)$$
$$\varepsilon^2 \int_{\langle x \rangle \geq 2R} |\nabla u|_g^2 \int_{\langle x \rangle \geq 2R} |u|_g^2 \leq C \int_{\langle x \rangle \geq R} \langle x \rangle^{1+2\sigma} |f|^2$$
$$\times \int_{\langle x \rangle \geq R} \left(\langle x \rangle^{-1-2\sigma} |\nabla u|_g^2 + \langle x \rangle^{-3-2\sigma} |u|^2 + \langle x \rangle^{1+2\sigma} |f|^2 \right).$$

and

$$(37)$$
$$\lambda \int_{\langle x \rangle \geq 2R} \langle x \rangle^{-1-2\sigma} |u|^2 \leq C \int_{\langle x \rangle \geq R} \left(\langle x \rangle^{-1-2\sigma} |\nabla u|_g^2 + \langle x \rangle^{-3-2\sigma} |u|^2 + \langle x \rangle^{1+2\sigma} |f|^2 \right).$$

Very roughly speaking, these estimates allow us to exchange derivatives on u with factors of $\lambda^{1/2}$ or $1 + \lambda^{1/2}$ whenever necessary.

Next, by applying the positive commutator method[6] to a first-order differential operator (a variant of ∂_r) we obtain a useful estimate which allows us to control the portion of u in the far region $r \gg 1$ by the portion of u in the intermediate region $r \sim 1$.

Lemma 2.3 (Pohozaev-Morawetz type estimate). *Let the notation and assumptions be as in Theorem 1.7. In addition we require that $\varepsilon \leq C\lambda$ for some positive*

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[6]One can also view this method as another application of integration by parts.

constant C. If $r_0 \geq R_0$ is a sufficiently large number (depending only on M and A) then we have

$$\int_{\langle x \rangle \geq 2r_0} \langle x \rangle^{-3-2\sigma} |u|^2 \, dg \leq$$

$$C(r_0) \left(\int_{r_0 \leq \langle x \rangle \leq 2r_0} (1+\lambda)|u|^2 \, dg + \|f\|_{H^{0,1/2+\sigma}(M)}^2 \right)$$

and

$$\int_{\langle x \rangle \geq 2r_0} \left(\langle x \rangle^{-1-2\sigma} |\nabla u|_g^2 \, dg + \lambda \langle x \rangle^{-1-2\sigma} |u|^2 \right) \, dg$$

$$\leq C(r_0) \left(\int_{r_0 \leq \langle x \rangle \leq 2r_0} (1+\lambda)|u|^2 \, dg + \|f\|_{H^{0,1/2+\sigma}(M)}^2 \right).$$

We prove this result in section 6.3. The positive commutator method, when applied to a suitable pseudodifferential operator, will also give Theorem 1.15; see Section 7.

We also need the following variant of the above estimate, which we prove in Section 11.3, using a more refined analysis based on spherical energies rather than on the positive commutator method.

Lemma 2.4. *Let $u \in H^2(M)$ be a solution to the resolvent equation*

$$(H - (\lambda \pm i\varepsilon))u = f, \qquad \lambda \pm i\varepsilon = z^2 = (a \pm ib)^2$$

for some $\lambda, \varepsilon > 0$ with the Hamiltonian $H = -\Delta_M + V$ satisfying the assumptions of Theorem 1.7. Then for any $0 < \sigma < \min(1, \sigma_0)$ and a sufficiently large r_0,

(38)
$$\|u\|_{H^{0,-3/2+\sigma}(M_{2r_0})} \leq C(r_0) \left(\|f\|_{H^{0,1/2+\sigma}(M_{r_0})} + (1+\lambda^{1/2})\|u\|_{L^2(M_{r_0} \setminus M_{4r_0})} \right),$$

$$|z|^{1/2}\|u\|_{H^{0,-1/2-\sigma}(M_{2r_0})} + \|\nabla u\|_{H^{0,-1/2-\sigma}(M_{2r_0})} \leq C(r_0)\|f\|_{H^{0,1/2+\sigma}(M_{r_0})}$$

(39)
$$+ C(r_0)(1+\lambda^{1/2})\|u\|_{L^2(M_{r_0} \setminus M_{4r_0})}$$

with $M_{r_0} = \{x \in M : \langle x \rangle \geq r_0\}$. Furthermore,

$$\|r^{-1}\nabla_\omega u\|_{H^{0,-1/2+\sigma}(M_{2r_0})} + \|u_r \mp izu\|_{H^{0,-1/2+\sigma}(M_{2r_0})} \leq C(r_0)\|f\|_{H^{0,1/2+\sigma}(M_{r_0})}$$

$$+ C(r_0)(1+\lambda^{1/2})\|u\|_{L^2(M_{r_0} \setminus M_{4r_0})}.$$

Remark 2.5. The result of Lemma 2.4 should be compared with Lemma 2.3. First, the analog of (38) in Lemma 2.3 provides control of the $\lambda^{1/2}\|u\|_{H^{0,-3/2-\sigma}(M_{2r_0})}$ norm. This improvement will be important in our analysis of the low energy regime. Furthermore, in Lemma 2.3 validity of the estimate (39) is restricted to the region $\varepsilon \leq C\lambda$. Removal of this condition is crucial for establishing a uniform (in the domain ε, λ) limiting absorption principle. Finally, (40) is the key ingredient in establishing the Sommerfeld radiation condition of Proposition 1.12.

We will need to complement Lemma 2.3 by the following estimate, which controls the portion of u in the near region $r \ll 1$ by the intermediate region $r \sim 1$.

Proposition 2.6 (Unique continuation estimate)*. Let K be a compact subset of M, and let K' be a compact set contained in the interior of K. Suppose that the potential V obeys the bound $|V(x)| \leq A$ for $x \in K$. Then if $u \in H^2(M)$ solves the Helmholtz equation $(H - \lambda \mp i\varepsilon)u = f$ on K, then we have the estimate*

$$\int_K |u|^2 + |\nabla u|_g^2 \, dg \leq C(K, K', A) e^{C(K, K', A)\sqrt{\lambda}} \left(\int_K |f|^2 \, dg + \int_{K \setminus K'} |u|^2 + |\nabla u|_g^2 \, dg \right).$$

One immediate consequence of this proposition is that any solution of the equation $(H - \lambda)u = 0$ which vanishes near the boundary of K, also vanishes on the interior of K, which explains the terminology "unique continuation". Proposition 2.6 is essentially due to Burq [12], but in the interest of self-containedness we provide a proof in Section 8 using the Carleman inequality method (which, again, can be viewed as a type of integration by parts). Notice the exponential dependence on the energy here, which will eventually lead to the exponential factors in Theorem 1.7.

To conclude the proof of limiting absorption for general manifolds, we will need to analyse how solution u decays at infinity. To this end we introduce the spherical masses

(40) $$\mathcal{M}[r] := r^{n-1} \int_{\partial M} |u(r, \omega)|^2 \, dh[r](\omega)$$

for $r > R_0$, where the angular metric $h[r]$ was defined in (5).

Remark 2.7. In the Euclidean case, at least, we expect u to decay like $O(r^{-(n-1)/2})$, and so we expect $\mathcal{M}[r]$ to stay bounded.

Lemma 2.8 (Dichotomy)*. Let $C_1 \gg R_0$ be a large number, and then let $C_2 \gg C_1$ be an even larger number. Let $M_{C_1/2} := \{x \in M : r > C_1/2\}$. Let $u \in H^2(M_{C_1/2})$ be a solution to the resolvent equation (26) for some $\lambda, \varepsilon > 0$ and sign $\pm$, so that u is also a solution to the Helmholtz equation (27) with $F := f - Vu \pm i\varepsilon u$. Suppose also that we have the normalization*

(41) $$\|u\|_{H^{0,-1/2-\sigma}(M_{C_1/2})} = 1; \quad \|f\|_{H^{0,1/2+\sigma}(M_{C_1/2})} = \delta$$

for some $\delta > 0$. Assume that $\sigma < 1/2$, that $\delta \leq 1$ and that $\varepsilon \leq \lambda$. If C_1 is sufficiently large (but not depending on λ), and C_2 is sufficiently large depending on C_1 (but not on λ), then one of the following must be true:

- *(Boundedness) There exists a radius $C_1 \leq r_0 \leq C(C_1, C_2)$ such that*

(42) $$\mathcal{M}[r] \leq C(C_1, C_2)(\lambda^{-C(C_1, C_2)} + 1)\delta \text{ for all } r_0/2 \leq r \leq 4r_0.$$

- *(Exponential growth) For all $C_1 \leq r \leq 10C_1$, we have*

(43) $$\frac{d}{dr}\mathcal{M}[r] \leq -C_2(1 + \lambda^{1/2})\mathcal{M}[r].$$

This lemma is proven by an ODE analysis of the equations of motion obeyed by the mass $\mathcal{M}[r]$ and several other related quantities; it is the main technical innovation of this paper and is proven in Section 10, after some important preliminaries in Section 9. Observe that this is a "black box" result, in that no assumptions whatsoever

are made concerning u in the region $r \leq C_1/2$. The dichotomy between (42) and (43) may seem strange, but one way to motivate it is by considering the free case $M = \mathbf{R}^n$, $V = 0$, $f = 0$, $\varepsilon = 0$ with an ansatz $u(r\omega) = r^{-(n-1)/2}v(r)Y_l(\omega)$ for some spherical harmonic Y_l of order l, in which case (26) simplifies to the Bessel equation

$$v_{rr} + (\lambda - \frac{L(L-1)}{r^2})v = 0$$

where $L := l + \frac{n-1}{2}$; the quantity $\mathcal{M}[r]$ is essentially just $|v(r)|^2$ in this case. The solutions to this ODE can be described in terms of Bessel and Hankel functions. By analysing the asymptotics of such functions one can observe that all solutions are either bounded or grow exponentially as r decreases from infinity[7], which helps explain the above dichotomy.

3. Proof of main theorem

We now show how the estimates in the previous section can quickly be used to deduce the main limiting absorption principles in our paper, namely Theorem 1.7 and Proposition 1.12. (The remaining limiting absorption principles will require slight modifications of the above estimates, which we shall address in later sections.)

The strategy will be as follows. Proposition 10.1 shows us that the spherical mass $\mathcal{M}[r]$ exhibits either boundedness (42) or exponential growth (43). The exponential growth scenario (43) will turn out to be incompatible with the unique continuation estimate in Proposition 2.6, and so we in fact have boundedness for u in the intermediate region $r \sim 1$. We will then use Lemma 2.3, Proposition 2.6, and Lemma 2.2 to recover control of u in other regions of space (and with higher derivatives), thus leading to the full limiting absorption principle.

3.1. **Proof of Theorem 1.7.** We begin by proving Theorem 1.7. Let the notation and assumptions be as in the theorem. We allow all constants to depend on M, A. Write $u := R(\lambda \pm i\varepsilon)f$, thus

$$(H - (\lambda \pm i\varepsilon))u = f$$

and $u \in H^2(M)$ (by (28) and (30)).

We begin by proving (10). We wish to show that

(44) $\|u\|_{H^{s,-1/2-\sigma}(M \setminus K)} \leq C(\lambda^{-C} + \lambda^{s/2})\|f\|_{H^{0,1/2+\sigma}(M)}$

for $s = 0, 1, 2$ and for sufficiently large K. By interpolation it suffices to establish the cases $s = 0, 2$. From (33) we see that the $s = 2$ case follows from the $s = 0$ case, so it suffices to take $s = 0$.

Henceforth $s = 0$. We may now assume $\varepsilon < \lambda$, since the claim (44) follows from (28) otherwise.

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[7]Of course, exponential growth as r decreases from infinity is the same thing as exponential decay as $r \to \infty$. But in our application, it is best to think of $r = \infty$ as the "initial" value of r for equations such as the Bessel equation.

Let r_0 be a large constant, and take $M_{r_0/2} := \{x \in M : \langle x \rangle \geq r_0/2\}$. We then normalize

$$(45) \qquad \|u\|_{H^{0,-1/2-\sigma}(M_{r_0/2})} = 1; \quad \|f\|_{H^{0,1/2+\sigma}(M)} = \delta$$

We may assume that $\delta \leq 1$, since the claim (44) is immediate otherwise.

Let $r_0 \ll C_1 \ll C_2$ be large numbers (depending on M and A, but independent of λ) to be chosen later. If one then applies Lemma 2.8, we conclude that we are either in the boundedness scenario (42) or the exponential growth scenario (43).

Suppose we are not in the boundedness scenario (42), so that we are necessarily in the exponential growth scenario (43). Then we see in particular from Gronwall's inequality that

$$\int_{C_1 \leq \langle x \rangle \leq 2C_1} |u|^2 \, dg \geq e^{C_2(1+\sqrt{\lambda})} \int_{3C_1 \leq \langle x \rangle \leq 10C_1} |u|^2 \, dg.$$

Using the energy inequality (32) and the normalization (45) we have

$$\int_{4C_1 \leq \langle x \rangle \leq 8C_1} |\nabla u|^2 \, dg \leq C(C_1) \left(\int_{3C_1 \leq \langle x \rangle \leq 10C_1} (1+\lambda)|u|^2 \, dg + \delta^2 \right).$$

We may assume that the first-term on the right-hand side dominates the second, since we are not in the boundedness scenario (42). We then conclude that

$$(46) \qquad \int_{C_1 \leq \langle x \rangle \leq 2C_1} |u|^2 \, dg \geq e^{\frac{1}{2}C_2(1+\sqrt{\lambda})} \int_{4C_1 \leq \langle x \rangle \leq 8C_1} \left(|u|^2 + |\nabla u|_g^2 \right) \, dg.$$

The constant C_2 is sufficiently large compared to C_1. Then using Proposition 2.6 with $K = \{x : \langle x \rangle \leq 8C_1\}$ and $K' = \{x : \langle x \rangle \leq 4C_1\}$ we obtain

$$\int_{\langle x \rangle \leq 8C_1} \left(|u|^2 + |\nabla u|_g^2 \right) \, dg \leq C(C_1) e^{C(C_1)\sqrt{\lambda}} \int_{\langle x \rangle \leq 8C_1} |f|^2 \, dg$$

$$\leq C(C_1) e^{C(C_1)\sqrt{\lambda}} \delta^2.$$

Applying (46) again we conclude (again taking C_2 large compared to C_1)

$$\int_{4C_1 \leq \langle x \rangle \leq 8C_1} |u|^2 \, dg \leq e^{-\frac{1}{4}C_2(1+\sqrt{\lambda})} \delta^2.$$

Applying Lemma 2.3 we conclude

$$\int_{\langle x \rangle \leq 8C_1} \lambda \langle x \rangle^{-1-2\sigma} |u|^2 \, dg \leq C(C_1)(1+\lambda) e^{-\frac{1}{4}C_2(1+\sqrt{\lambda})} \delta^2$$

and the claim (44) follows from (45).

Now suppose instead that we are in the boundedness half of the dichotomy, thus there exists a radius $C_1 \leq r_0 \leq C(C_1, C_2)$ such that

$$\int_{r_0/2 \leq \langle x \rangle \leq 4r_0} |u|^2 \, dg \leq C(C_1, C_2)(\lambda^{-C(C_1,C_2)} + 1)\delta.$$

Using Lemma 2.3 as before, we conclude

$$\int_{\langle x \rangle \geq 4r_0} \lambda \langle x \rangle^{-1-2\sigma} |u|^2 \, dg \leq C(r_0, C_1, C_2)(1+\lambda)(\lambda^{-C(C_1,C_2)} + 1)\delta$$

and (44) follows.

Finally, we prove (11). In view of (30), it suffices to show that

$$\|u\|_{H^{0,-1/2-\sigma}(M)} \leq C(\lambda^{-C} + e^{C\sqrt{\lambda}})\|f\|_{H^{0,1/2+\sigma}(M)}.$$

But this follows from (10) and Proposition 2.6. This completes the proof of Theorem 1.7.

3.2. **Proof of Proposition 1.12.** We now turn to the proof of Proposition 1.12.

The Sommerfeld radiation condition (12) follows immediately from Theorem 1.7 and (40). Their combination gives the estimate

$$\|r^{-1}\nabla_\omega u\|_{H^{0,-1/2+\sigma}(M_{2r_0})} + \|u_r - i\sqrt{\lambda}u\|_{H^{0,-1/2+\sigma}(M_{2r_0})} \leq C(\lambda^{-C} + e^{C\sqrt{\lambda}})\|f\|_{H^{0,1/2+\sigma}(M)}.$$

and completes the proof of the first part of Proposition 1.12.

To show that for fixed $\lambda > 0$ the the functions $u_{\pm_\varepsilon} = R(\lambda \pm i\varepsilon)f$ converge in $H^{0,-1/2-\sigma}$ to a unique solution $u_\pm$ of the Helmholtz equation $(H - \lambda)u = f$ selected by the requirement that $u \in H^{2,-1/2-\sigma}(M)$ and $(\partial_r \mp i\lambda^{1/2})u \in H^{0,-1/2+\sigma'}(M \setminus K_0)$ for some $\sigma' > 0$ we consider the difference

$$w(x) = e^{i(z_1-z_2)s(x)}u_{+_{\varepsilon_2}}(x) - u_{+_{\varepsilon_1}}(x)$$

with $z_{1,2}^2 = \lambda + i\varepsilon_{1,2}$ and $\mathrm{Im}(z_{1,2}) \approx \varepsilon_{1,2}\lambda^{-1/2} > 0$ (we assume that $\lambda \gg \varepsilon_{1,2}$) and $\varepsilon_1 > \varepsilon_2$ (convergence for u_{-_ε} can be treated in a similar fashion.) The function $s(x)$ is assumed to be smooth and obeys the requirement that $s(x) = r$ for $x \in M \setminus K_0$. The function w verifies the equation

$$(H - \lambda - i\varepsilon_1)w = G$$

where G is given by the formula

$$G := i(z_2-z_1)e^{i(z_1-z_2)s(x)}\Big(2(\nabla_k s\nabla^k - iz_2\,|\nabla s|^2) + \Delta_M s + i(z_1+z_2)(|\nabla s|^2 - 1)\Big)u_{+_{\varepsilon_2}}.$$

For values of $\varepsilon_2 < \varepsilon_1 \ll \lambda$ we have that

$$|z_1 - z_2| \leq 4\mathrm{Im}(z_1 - z_2) \approx \lambda^{-1/2}(\varepsilon_1 - \varepsilon_2).$$

Therefore the first part of Proposition 1.12 and the choice of function $s(x)$, which in particular gives that $|\Delta_M s(x)| \leq C\langle x\rangle^{-1}$, imply that for any $0 < \sigma' < \sigma$ and any $\alpha > 0$ we have the estimates

$$\|e^{i(z_1-z_2)s(x)}(\nabla_k s\nabla^k - iz_2\,|\nabla s|^2)u_{+_{\varepsilon_2}}\|_{H^{0,1/2+\sigma'}(M)}$$
$$\leq C|z_1 - z_2|^{-1+\sigma-\sigma'}(\lambda^{-C} + e^{C\sqrt{\lambda}})\|f\|_{H^{0,1/2+\sigma}(M)},$$
$$\|e^{i(z_1-z_2)s(x)}(|\Delta_M s| + |z_1 + z_2|\,|\nabla s|^2 - 1)u_{+_{\varepsilon_2}}\|_{H^{0,1/2+\sigma'}(M)}$$
$$\leq C\|e^{i(z_1-z_2)s(x)}\langle x\rangle^{-1}u_{+_{\varepsilon_2}}\|_{H^{0,1/2+\sigma'}(M)}$$
$$\leq C|z_1 - z_2|^{-\sigma'-\alpha}(\lambda^{-C} + e^{C\sqrt{\lambda}})\|f\|_{H^{0,1/2+\sigma}(M)}.$$

Thus for any $0 < \sigma' < \sigma < 1 - \alpha$ we have

$$\|G\|_{H^{0,1/2+\sigma'}(M)} \leq (\varepsilon_1 - \varepsilon_2)^{\sigma-\sigma'}C(\lambda^{-C} + e^{C\sqrt{\lambda}})\|f\|_{H^{0,1/2+\sigma}(M)},$$

By Theorem 1.7 and the first part of Proposition 1.12, the solution of the problem $(H - \lambda - i\varepsilon_2)w = G$ obeys
$$(47)$$
$$\|(\partial_r - \lambda^{1/2})w\|_{H^{0,-1/2+\sigma'}(M \setminus K_0)} + \|w\|_{H^{2,-1/2-\sigma}(M)} \leq (\varepsilon_1 - \varepsilon_2)^{\sigma - \sigma'} C(\lambda^{-C} + e^{C\sqrt{\lambda}})\|f\|_{H^{0,1/2+\sigma}(M)}.$$

Since by Theorem 1.7,
$$\|(e^{i(z_1 - z_2)s(x)} - 1)u_{+\varepsilon_2}\|_{H^{0,-1/2-\sigma}(M)} \leq C\|\langle x \rangle^{-\sigma + \sigma'}(e^{i(z_1 - z_2)s(x)} - 1)u_{+\varepsilon_2}\|_{H^{0,-1/2-\sigma'}(M)}$$
$$\leq C(\varepsilon_1 - \varepsilon_2)^{\sigma - \sigma'}\|u_{+\varepsilon_2}\|_{H^{0,-1/2-si'}(M)}$$
$$\leq (\varepsilon_1 - \varepsilon_2)^{\sigma - \sigma'} C(\lambda^{-C} + e^{C\sqrt{\lambda}})\|f\|_{H^{0,1/2+\sigma}(M)}.$$

we obtain
$$\|u_{+\varepsilon_2} - u_{+\varepsilon_1}\|_{H^{0,-1/2-\sigma}(M)} \leq \|w\|_{H^{0,-1/2-\sigma}(M)} + \|(e^{i(z_1 - z_2)s(x)} - 1)u_{+\varepsilon_2}\|_{H^{0,-1/2-\sigma}(M)}$$
$$\leq (\varepsilon_1 - \varepsilon_2)^{\sigma - \sigma'} C(\lambda^{-C} + e^{C\sqrt{\lambda}})\|f\|_{H^{0,1/2+\sigma}(M)},$$

which implies that $u_{+\varepsilon}$ is a Cauchy sequence in $H^{2,-1/2-\sigma}(M)$ converging to the limit $u_+ \in H^{2,-1/2-\sigma}(M)$. Moreover, (47) also implies $(\partial_r - i\lambda^{1/2})u_+ \in H^{0,-1/2+\sigma'}(M \setminus K_0)$ for any $\sigma' < \sigma$.

To prove uniqueness of solutions of the Helmholtz equation
$$(H - (\lambda \pm i\varepsilon))u = f$$

with the property that $u \in H^{2,1/2-\sigma}(M)$ and $(\partial_r \mp i\lambda^{1/2})u \in H^{0,-1/2+\sigma'}(M \setminus K_0)$ for some $\sigma' > 0$ we simply repeat the argument above with the functions
$$v(x) = e^{\pm i(\sqrt{\lambda \pm i\varepsilon} - \sqrt{\lambda})s(x)}u(x) - u_{\pm\varepsilon}(x)$$

and conclude that $u_{\pm\varepsilon}$ converges to u.

This concludes the proof of Theorem 1.7 and Proposition 1.12, except for the proofs of the estimates in Section 2. These estimates will be the focus of the next few sections.

4. CONSERVATION LAWS

All of the results stated in Section 2 rely, fundamentally, on integration by parts arguments. It will be convenient to present these arguments in a unified framework, namely that of exploiting the conservation laws for the Helmholtz equation (27) using the *abc method* of Friedrichs. While this framework is in practice too general to use directly, it does provide a convenient way to display the interrelationships between the more specialized methods we study below. Thus we shall devote this section to the general abc method for the Helmholtz equation (27). As a quick application, we will be able to establish the charge estimate, Lemma 2.1.

4.1. **Densities and currents.** Let $u \in H^2(M)$ be a (complex-valued) solution to the Helmholtz equation (27) for some $\lambda > 0$, so that $F \in L^2(M)$. We introduce a number of (real) tensor fields on M, namely

- the *charge density* $\mathbf{q} := |u|^2$,
- the *energy density* $\mathbf{e} := |\nabla u|_g^2 := \mathrm{Re}(\overline{\nabla^k u} \nabla_k u)$,
- the *charge current* $\mathbf{j}^k := \mathrm{Im}(\overline{u}\, \nabla^k u)$,
- the *charge gradient* $\mathbf{v}^k := \mathrm{Re}(\overline{u}\, \nabla^k u)$,
- the *energy-momentum tensor* $\mathbf{Q}^{mk} := \mathrm{Re}(\overline{\nabla^m u}\nabla^k u) - \frac{1}{2}g^{mk}(\mathbf{e} - |z|^2 \mathbf{q})$.

Note that from the assumption $u \in H^2(M)$ that all of these tensor fields are absolutely integrable. A direct computation (recalling that the Levi-Civita covariant derivative ∇ respects the metric g and is torsion free) yields the *charge gradient identity*

$$(48) \qquad \nabla_k \mathbf{q} = 2\mathbf{v}_k,$$

the *charge conservation law*

$$(49) \qquad \nabla_k \mathbf{j}^k + \mathrm{Im}(z^2)\mathbf{q} = -\mathrm{Im}(\overline{u}F)$$

the identity

$$(50) \qquad \nabla_k \mathbf{v}^k = \frac{1}{2}\Delta_M \mathbf{q} = \mathbf{e} - \mathrm{Re}(z^2)\mathbf{q} - \mathrm{Re}(\overline{u}F)$$

and the *conservation of energy-momentum*

$$(51) \qquad \nabla_k \mathbf{Q}^{mk} + \mathrm{Re}\left((\overline{z}^2 - |z|^2)\nabla^m u\,\overline{u}\right) = -\mathrm{Re}(\overline{\nabla^m u}F).$$

The energy density $\mathbf{e}$ does not directly obey a useful equation of the above type (the expression $\nabla_\alpha \mathbf{q}$ contains expressions involving other double derivatives of u than the Laplacian $\Delta_M u$). However, this energy density is clearly related to the other tensor fields, for instance we have the Cauchy-Schwarz inequalities $|\mathbf{j}|_g, |\mathbf{v}|_g \leq \mathbf{q}^{1/2}\mathbf{e}^{1/2}$ and also that $\mathbf{Q} = O(\mathbf{e}) + O(\lambda \mathbf{q})$.

The conservation laws (49), (50) and (51) are directly related to the conservation law of the energy-momentum tensor

$$\mathbf{Q}^{\alpha\beta} := \mathrm{Re}(\overline{\nabla^\alpha u}\nabla^\beta u) - \frac{1}{2}\mathbf{g}^{\alpha\beta}(\overline{\nabla^\mu u}\,\nabla_\mu u)$$

associated with the wave equation $\Box_{\mathbf{g}} u = 0$ with the metric $\mathbf{g}_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + g_{ij}dx^i dx^j$. Conservation of energy-momentum is expressed in the form

$$\nabla_m \mathbf{Q}^{km} = 0.$$

The relationship between conservation laws for the Helmholtz and wave equations becomes apparent under the formal rule

$$\frac{d}{dt} = iz$$

4.2. **Proof of Lemma 2.1.** As a quick application of the charge conservation law (49) to the resolvent equation (26), we now prove Lemma 2.1.

Let the notation and assumptions be as in that lemma. Then $u \in L^2(M)$. From the resolvent equation (26) (and the boundedness of V) we deduce that $\Delta_M u \in L^2(M)$ also. From elliptic regularity we conclude the qualitative fact that $u \in H^2(M)$. In particular from Cauchy-Schwarz we see that the charge current $\mathbf{j}^\alpha$ is absolutely integrable. Also if we set $F := f - Vu$ then $F \in L^2(M)$ and so the charge source

term $\mathrm{Im}(\overline{u}F)$ is also absolutely integrable. We may thus integrate (49) (by inserting a large cutoff function χ_R, adapted to the ball of radius R, and integrating by parts) to obtain the identity

$$\pm i\varepsilon \int_M \chi_R |u|^2 \, dg + \int_M \chi_R \mathrm{Im}(\overline{u}F) \, dg = \int_M \nabla\chi_R \mathrm{Im}(\nabla u \overline{u}).$$

Expanding out the definition of F and taking into account that V is real, we thus obtain

$$(52) \qquad \pm i\varepsilon \int_M \chi_R |u|^2 \, dg = -\int_M \chi_R \mathrm{Im}(\overline{u}f) \, dg + \int_M \nabla\chi_R \mathrm{Im}(\nabla u \overline{u}).$$

The first claim now follows after letting $R \to \infty$, applying the Cauchy-Schwarz inequality and using the assumption on V.

Similarly, integrating by parts with the function η_R, where η_R is a cut-off function adapted to the shell of radius R, we obtain

$$\pm i\varepsilon \int_M \eta_R^2 |u|^2 \, dg = -\int_M \eta_R^2 \mathrm{Im}(\overline{u}f) \, dg + 2\int_M \eta_R \nabla\eta_R \mathrm{Im}(\nabla u \overline{u}).$$

Therefore,

$$\varepsilon^2 \int_M \eta_R^2 |u|^2 \, dg \le C \int_{R \le \langle x \rangle \le 4R} (|f|^2 + R^{-2}|\nabla u|_g^2) \, dg.$$

The estimate (29) now follows after multiplying the above inequality by $R^{2\alpha}$ and summing over all dyadic $R \ge R_0$. This concludes the proof of Lemma 2.1.

4.3. The abc method.

Now we consider how to exploit the other conservation laws (48), (50), (51). The *abc method* of Friedrichs is a general way to exploit these identities. It proceeds by introducing arbitrary tensor fields a^k, b, c^k, and applying Stokes' theorem to the (momentum) vector field

$$(53) \qquad\qquad P^k = a^k \mathbf{q} + b\mathbf{v}^k + c_m \mathbf{Q}^{km}$$

to obtain (for a compactly supported cutoff function χ)

$$(54) \qquad\qquad \int_M (\mathrm{div}P)\chi \, dg = -\int_M P^k \partial_k \chi \, dg,$$

and in particular (if P has sufficient decay)

$$(55) \qquad\qquad \int_M \mathrm{div}P \, dg = 0.$$

From (48), (50), (51) we observe that the divergence $\mathrm{div}P = \nabla_k P^k$ can be computed explicitly as

(56)

$$\mathrm{div}P = \nabla_k a^k \mathbf{q} + (2a^k + \nabla^k b)\mathbf{v}_k + (b - \frac{1}{2}\nabla_k c^k)(\mathbf{e} - |z|^2\mathbf{q}) + (\nabla_k c_m)\mathrm{Re}(\overline{\nabla^k u}\nabla^m u)$$

$$+ b(|z|^2 - \lambda)\mathbf{q} + c_k \mathrm{Re}\left((z^2 - |z|^2)\overline{\nabla^k u}\, u\right) - b\mathrm{Re}(\overline{u}F) - c_k \mathrm{Re}(\nabla^k u \overline{F}).$$

One then hopes to select a^k, b, c^k, χ so that many of the terms in (54) or (55) carries a useful sign, so that one obtains a non-trivial estimate on u.

In practice, the full generality of abc method is difficult to use, as there are too many possibilities for a^k, b, c^k, χ which are available, and too many ways in which one could hope to exploit positivity. Instead, one typically specializes to a sub-case of the Friedrichs method which has fewer parameters. For instance, if one sets $a^\alpha = c^\alpha = 0$ and $b = 1$ one obtains the *Lagrangean identity*[8]

$$(57) \qquad \int_M \mathbf{e}\chi = \int_M \lambda \mathbf{q}\chi - \int_M \mathbf{v}^k \nabla_k \chi + \int_M \mathrm{Re}(\overline{u}F)\chi.$$

A further integration by parts using (48) yields

$$(58) \qquad \int_M \mathbf{e}\chi = \int_M \lambda \mathbf{q}\chi + \frac{1}{2}\int_M \mathbf{q}\Delta_M\chi + \int_M \mathrm{Re}(\overline{u}F)\chi$$

(or alternatively one can apply (55) with $a^k = -\frac{1}{2}\nabla^k\chi$, $b = \chi$, $c^\alpha = 0$). We exploit this identity in Section 5.

4.4. Pohozaev-Morawetz type identities.
Another way to reduce the parameters is to introduce the ansatz

$$c^k := \nabla^k W; \quad b = \frac{1}{2}\Delta_M W$$

for some scalar real-valued function W. Then the formula (56) simplifies (because the $(\mathbf{e} - |z|^2\mathbf{q})$ term disappears) to yield

$$\mathrm{div}P = \nabla_k a^k \mathbf{q} + (2a^k + \frac{1}{4}\nabla^k\Delta_M W)\mathbf{v}_k + \mathrm{Hess}_{mk}(W)\mathrm{Re}(\overline{\nabla^m u}\nabla^k u)$$

$$(59) \qquad\qquad + \frac{1}{2}\Delta W(|z|^2 - \lambda)\mathbf{q} + \nabla_k W\mathrm{Re}\left((z^2 - |z|^2)\overline{\nabla^k u}\,u\right)$$

$$\qquad\qquad - \frac{1}{2}(\Delta_M W)\mathrm{Re}(\overline{u}F) - (\nabla_k W)\mathrm{Re}(\nabla^k u\overline{F})$$

where $\mathrm{Hess}_{mk}(W) := \nabla^2_{mk}W$ is the Hessian of W. If one then sets[9] $a^k := -\frac{1}{4}\nabla^k\Delta_M W$ to cancel the $\mathbf{v}_k$ term, and introducing the modified inhomogeneous term

$$G := F + \lambda - |z|^2$$

one obtains the *Pohozaev-Morawetz identity*[10]

(60)
$$\int_M \mathrm{Hess}_{mk}(W)(\nabla^m u)(\nabla^k u) - \frac{1}{4}(\Delta^2_M W)\mathbf{q} + \frac{1}{2}(\Delta_M W)\mathrm{Re}(\overline{u}G) - (\nabla^k W)(\mathrm{Re}(\overline{\nabla^k u}G))\chi \; dg$$

$$= \int_M (-\frac{1}{4}(\nabla^k\Delta_M W)\mathbf{q} + \frac{1}{2}(\Delta_M W)\mathbf{v}^k + (\nabla_m W)\mathbf{Q}^{mk})\nabla_k\chi \; dg.$$

---

[8]The energy-momentum tensor $\mathbf{Q}^{mk}$ is associated with the Lagrangean $\mathcal{L} = \int_M(|\nabla u|^2_g - |z|^2|u|^2)$.

[9]To continue our analogy with the wave equation, the above choice of a^k, b, c^k is somewhat reminiscent of the construction of a modified momentum, associated with a conformal Killing vectorfield, for solutions of the wave equation.

[10]Particular cases of this identity have been used by Pohozaev to prove non-existence of solutions of certain nonlinear elliptic equations, [77], and by Morawetz in the study of long time behavior of solutions of a wave equation, [74].

This identity is particularly[11] useful when W is geodesically convex (i.e. the Hessian $\mathrm{Hess}(W)$ is positive definite) and the quantity $\Delta_M^2 W$ is non-positive, as this implies that the first two terms on the left-hand side of (60) become non-negative. We will also re-interpret this identity in terms of the positive commutator method, and give some consequences of this identity, in Section 6.

To account for the the difference between λ and $|z|^2$ we can modify the momentum P as follows

$$P^k = -\frac{1}{4}(\nabla^k \Delta_M W)\mathbf{q} + \frac{1}{2}(\Delta_M W)\mathbf{v}^k + (\nabla_m W)\mathbf{Q}^{mk} + \mathrm{Im}z\,\mathbf{v}^k - \mathrm{Re}z\,\mathbf{j}^k.$$

Then

$$\mathrm{div}P = \mathrm{Hess}_{mk}(W)\mathrm{Re}(\overline{\nabla^m u}\nabla^k u) - \frac{1}{4}\Delta_M^2 W\mathbf{q} + \frac{1}{2}\Delta_M W(|z|^2 - \lambda)\mathbf{q}$$
$$+ \mathrm{Im}z(\mathbf{e} - \lambda\mathbf{q}) + \mathrm{Re}z\mathrm{Im}(z^2)\mathbf{q} + 2\mathrm{Im}z\,\nabla_k W\mathrm{Re}\left(iz\,\overline{\nabla^k u}\,u\right)$$
$$- \frac{1}{2}(\Delta_M W)\mathrm{Re}(\overline{u}F) - (\nabla_k W)\mathrm{Re}(\nabla^k u\overline{F}) - \mathrm{Im}(zu\overline{F})$$

After some calculations we obtain

$$\mathrm{div}P = \mathrm{Hess}_{mk}(W)\mathrm{Re}(\overline{\nabla^m u}\nabla^k u) - \frac{1}{4}\Delta_M^2 W\mathbf{q} + \frac{1}{2}\Delta_M W(|z|^2 - \lambda)\mathbf{q}$$
$$+ \mathrm{Im}z\left(\mathbf{e} - |\nabla^k W\nabla_k u|^2 + |\nabla_k W\nabla^k u + izu|^2\right)$$
$$- \frac{1}{2}(\Delta W)\mathrm{Re}(\overline{u}F) - (\nabla_k W)\mathrm{Re}(\nabla^k u\overline{F}) - \mathrm{Im}(zu\overline{F}),$$

which after integration over M becomes

(61)
$$\int_M \left(\mathrm{Hess}_{mk}(W)\mathrm{Re}(\overline{\nabla^m u}\nabla^k u) - \frac{1}{4}\Delta_M^2 W\mathbf{q} + \frac{1}{2}\Delta_M W(|z|^2 - \lambda)\mathbf{q} + \mathrm{Im}z\left(\mathbf{e} - |\nabla^k W\nabla_k u|^2\right)\right.$$
$$\left.+ \mathrm{Im}z|\nabla_k W\nabla^k u + izu|^2 - \frac{1}{2}(\Delta W)\mathrm{Re}(\overline{u}F) - (\nabla_k W)\mathrm{Re}(\nabla^k u\overline{F}) - \mathrm{Im}(zu\overline{F})\right)\chi$$
$$= -\int_M \left(-\frac{1}{4}(\nabla^k \Delta_M W)\mathbf{q} + \frac{1}{2}(\Delta_M W)\mathbf{v}^k + (\nabla_m W)\mathbf{Q}^{mk} + \mathrm{Im}z\,\mathbf{v}^k - \mathrm{Re}z\,\mathbf{j}^k\right)\nabla_k\chi$$

This identity is useful if the Hessian of W is positive definite, $\Delta_M^2 W \leq 0$, $\Delta_M W \geq 0$ and $|\nabla W| \leq 1$. For instance in Euclidean space $\mathbf{R}^n$ with the standard metric, the choice of $W = |x| - (1 + |x|)^{1-\delta}$ satisfies all the above requirements and effectively leads to the proof of the limiting absorption principle. We will use a version of (61) in Section 6 and later in the derivation of the Sommerfeld radiation condition.

4.5. **Carleman type identities.** Returning to (59), the choice $a^k := \frac{1}{4}\nabla^k \Delta_M W$ is not the only possible choice for a^k. If we set $W := e^{2w}$ in the above ansatz, and then set

$$a^k := \frac{1}{2}(\nabla^k w)\Delta_M e^{2w} - \frac{1}{2}(\nabla^k \Delta_M w)e^{2w},$$

---

[11]We should note that this identity is most applicable when $\lambda = |z|^2$, i.e. $\varepsilon = 0$, in which case it leads to interesting statements even for nonlinear $F = F(u)$.

then after some computation using (59) we obtain[12]

$$\text{div}P = ((\Delta_M w)^2 + 4|\nabla w|_g^2 \Delta_M w + 4\text{Hess}_{mk}(w)\nabla^m w \nabla^k w + 4|\nabla w|_g^4 - \frac{1}{2}\Delta_M^2 w)e^{2w}\mathbf{q}$$

$$+ (4(\nabla_k w)\Delta_M w + 4\text{Hess}_{mk}(w)\nabla^m w + 8|\nabla w|_g^2 \nabla_k w)e^{2w}\mathbf{v}^k$$

$$+ (2\text{Hess}_{mk}(w) + 4\nabla_m w \nabla_k w)e^{2w}\nabla_m u \nabla_k u$$

$$- \frac{1}{2}(\Delta e^{2w})\text{Re}(\overline{u}G) - (\nabla_k e^{2w})\text{Re}(\overline{\nabla^k u}G)$$

which after completion of the square becomes

$$\text{div}P = e^{2w}|2\nabla_k w \nabla^k u + (2|\nabla w|_g^2 + \Delta_M w)u|^2 + 2\text{Hess}_{mk}(w)\nabla^m(e^w u)\nabla^k(e^w u)$$

$$+ 2\text{Hess}_{mk}(w)(\nabla^m w)(\nabla^k w)e^{2w}|u|^2 - \frac{1}{2}(\Delta_M^2 w)e^{2w}|u|^2$$

$$- \frac{1}{2}(\Delta e^{2w})\text{Re}(\overline{u}G) - (\nabla_k e^{2w})\text{Re}(\overline{\nabla^k u}G).$$

This identity may seem like an algebraic miracle, but we can interpret it as another positive commutator estimate in the more general setting of Carleman inequalities, see Section 8. If u is compactly supported, then one can integrate this identity on M (as in (55)) to obtain the *Carleman identity*[13]

$$\int_M |2\nabla_k w \nabla^k u + (2|\nabla w|_g^2 + \Delta_M w)u|^2 e^{2w} \, dg$$

$$+ 2\int_M \text{Hess}_{mk}(w)\nabla^m(e^w u)\nabla^k(e^w u) \, dg$$

$$+ 2\int_M \text{Hess}_{mk}(w)(\nabla^m w)(\nabla^k w)|u|^2 e^{2w} \, dg = \int_M \frac{1}{2}(\Delta_M^2 w)|u|^2 e^{2w} \, dg$$

$$+ \int_M \frac{1}{2}(\Delta_M e^{2w})\text{Re}(\overline{u}G) + (\nabla_k e^{2w})\text{Re}(\overline{\nabla^k u}G) \, dg.$$

If we write $U := (2\nabla_k w \nabla^k u + (2|\nabla w|_g^2 + \Delta_M w)u)e^w$ and note that

$$\frac{1}{2}\Delta_M e^{2w}u + (\nabla_k e^{2w})\nabla^k u = e^w U$$

we can rewrite the previous expression as

$$\|U\|_{L^2(M)}^2 + 2\int_M \text{Hess}_{mk}(w)\nabla^m(e^w u)\nabla^k(e^w u) \, dg$$

$$+ 2\int_M \text{Hess}_{mk}(w)(\partial^m w)(\nabla^k w)|u|^2 e^{2w} \, dg$$

$$= \int_M \frac{1}{2}(\Delta_M^2 w)|u|^2 e^{2w} \, dg + \langle e^w F, U \rangle_{L^2(M)}$$

---

[12]Note that no curvature terms appear here, because the Levi-Civita connection is torsion free, and at no stage do we need to commute the Laplacian Δ_M with a derivative.

[13]It may seem remarkable at first that the energy λ makes no appearance in this identity. But note that if one increments G by λu then the resulting changes in the two terms involving F cancel each other out, thanks to (48) and integration by parts. In practice, the requirement that u be compactly supported means that we will have to apply a cutoff function to truncate u, and this will cause λ to enter the identity in a non-trivial manner.

and hence by the Cauchy-Schwarz inequality $\langle e^w F, U \rangle_{L^2(M)} \le \frac{1}{4} \|e^w F\|_{L^2(M)} + \|U\|_{L^2(M)}$, we conclude

(62)
$$\int_M \mathrm{Hess}_{mk}(w) \nabla^m(e^w u) \nabla^k(e^w u) \, dg$$
$$+ \int_M \mathrm{Hess}_{mk}(w)(\nabla^m w)(\nabla^k w)|u|^2 e^{2w} \, dg \le \int_M \frac{1}{4}(\Delta_M^2 w)|u|^2 e^{2w} \, dg + \frac{1}{8}\|e^w F\|^2_{L^2(M)}.$$

This identity is useful when w is geodesically convex (so that both terms on the left-hand side is positive) and large (so that one can absorb some of the right-hand side terms into the left-hand side). We shall exploit it in Section 8.

5. Elliptic regularity

In this section we prove the various claims in Lemma 2.2. One of the key tools here will be the energy identities (57), (58).

5.1. **Proof of** (30). Let the notation and assumptions be as in Lemma 2.2. We begin with the proof of the elliptic regularity estimate (30). By absorbing $z^2 u$ into the f term we may assume that $\lambda = 0$; similarly, by absorbing $V u$ into the f term we may assume that $V \equiv 0$. From classical elliptic regularity we know that u is locally in H^2. Our task is thus to show that

$$\int_M \langle x \rangle^{2m}(|\nabla u|_g^2 + |\nabla^2 u|_g^2) \, dg \le C_m \int_M \langle x \rangle^{2m}(|\Delta_M u|^2 + |u|^2) \, dg.$$

Let us first verify this claim for the first derivatives ∇u. Let $R \gg 1$ be a large number, and let φ_R to the region $\langle x \rangle \le R$ which equals one for $\langle x \rangle \le R/2$. From (58) we have

$$\int_M |\nabla u|_g^2 \langle x \rangle^{2m} \varphi_R = \frac{1}{2} \int_M |u|^2 \Delta_M(\langle x \rangle^{2m} \varphi_R) + \int_M \mathrm{Re}(\overline{u} \Delta_M u) \langle x \rangle^{2m} \varphi_R;$$

applying Cauchy-Schwarz (and the very crude estimate $\Delta_M(\langle x \rangle^{2m} \varphi_R) = O_m(\langle x \rangle^{2m})$) and then letting $R \to \infty$ we obtain the estimate for first derivatives. Thus it will suffice to show that

$$\int_M \langle x \rangle^{2m} |\nabla^2 u|_g^2 \, dg \le C_m \int_M \langle x \rangle^{2m} |\Delta_M u|^2 \, dg + C_m \|u\|^2_{H^{1,m}(M)}.$$

The standard Bochner integration by parts argument to give this estimate would require boundedness of the Riemann curvature tensor. Let us present a slight variant of that argument that only requires boundedness on first derivatives of the metric. To illustrate the method let us first work locally, using a compactly supported bump function χ. We claim the estimate

$$\||\nabla^2 u|\chi\|_{L^2(M)} \le C_m(\|\Delta_M u\|_{H^{0,m}(M)} + \|u\|_{H^{1,m}(M)}).$$

To see this, we integrate by parts, computing

$$
\begin{aligned}
\||\nabla^2 u|\chi\|^2_{L^2(M)} &= \langle \nabla_k \nabla^\ell u, \chi^2 \nabla_\ell \nabla^k u \rangle_{L^2(M)} \\
&= -\langle \nabla^\ell u, \chi^2 \nabla_k \nabla_\ell \nabla^k u \rangle_{L^2(M)} - 2\langle \nabla^\ell u, \chi(\nabla_k \chi) \nabla_\ell \nabla^k u \rangle_{L^2(M)} \\
&= -\langle \nabla^\ell u, \chi^2 \nabla_\ell \Delta_M u \rangle_{L^2(M)} - \langle \nabla^\ell u, \chi^2 [\nabla_k, \nabla_\ell] \nabla^k u \rangle_{L^2(M)} \\
&\quad + O\left(\||\nabla\chi|_g |\nabla u|_g\|_{L^2(M)} \||\nabla^2 u|_g \chi\|_{L^2(M)} \right) \\
&= O_m(\|\Delta_M u\|^2_{H^{0,m}(M)}) + O(\|u\|_{H^{1,m}(M)} \|\Delta_M u\|_{H^{0,m}(M)}) \\
&\quad + O(\||\nabla^2 u|_g \chi\|_{L^2(M)} \|u\|_{H^{1,m}(M)}) - \langle \nabla^\ell u, \chi^2 [\nabla_k, \nabla_\ell] \nabla^k u \rangle_{L^2(M)}.
\end{aligned}
$$

Thus to verify the claim, it will suffice to show that

$$
|\langle \nabla_\ell u, \chi^2 [\nabla_k, \nabla_\ell] \nabla_k u \rangle_{L^2(M)}| \leq C_m(\|u\|^2_{H^{1,m}(M)} + \||\nabla^2 u|\chi\|_{L^2(M)} \|u\|_{H^{1,m}(M)}).
$$

If we exploited the boundedness of the Riemann curvature tensor at this point (recalling that g was assumed to be smooth), then we would be done (and we would not need the second term on the right-hand side). However, one can instead work in local co-ordinates, writing[14]

$$
([\nabla_k, \nabla_\ell] X)^\gamma = (\partial_k (\Gamma^s_{\ell p} X^p) + \Gamma^s_{kp} (\nabla_\ell X)^p) - (\partial_\ell (\Gamma^s_{kp} X^p) + \Gamma^s_{\ell p} (\nabla_k X)^p)
$$

for any vector field X, where Γ are the Christoffel symbols. Applying this identity and integrating by parts as necessary to prevent any derivatives falling on the Christoffel symbols, we obtain the above claim, where we only needed the boundedness of the Christoffel symbols Γ (which are essentially one derivative of the metric g, in contrast to the Riemann curvature which is two derivatives).

One can perform a similar argument with χ replaced by $\langle x \rangle^m \varphi_R (1 - \chi)$, where $1 - \chi$ localizes to the asymptotic region $r \gg R_0$; the point being that the conditions (6) ensures the appropriate boundedness of the Christoffel symbols in this region. Indeed, there is some additional decay and vanishing properties of these symbols arising from the normal form co-ordinate structure, which we shall simply discard; similarly we shall discard any gains of $\langle x \rangle$ when a derivative hits the $\langle x \rangle^{2m}$ factor. We omit the standard details. This concludes the proof of (30).

5.2. **Proof of** (33). We now localise (30) to prove (33). Let χ be a smooth cutoff to the region $\{\langle x \rangle \geq R/2\}$ which equals 1 when $\langle x \rangle \geq R$. Applying (30) to the function $\chi^3 u$ we have

$$
\|\chi^3 u\|_{H^{2,m}(M)} \leq C(m,\chi)(\|\chi^3 f\|_{H^{0,m}(M)} + \|\chi^2 |\nabla u|\|_{H^{0,m}(M)} + (1+\lambda)\|\chi^3 u\|_{H^{0,m}(M)})
$$

which (by the product rule) implies

$$
\|\chi^3 |\nabla^2 u|\|_{H^{0,m}(M)} \leq C(m,\chi)(\|\chi^3 f\|_{H^{0,m}(M)} + \|\chi^2 |\nabla u|\|_{H^{0,m}(M)} + (1+\lambda)\|\chi^3 u\|_{H^{0,m}(M)}).
$$

By using the interpolating inequality

$$
\|\chi^2 |\nabla u|\|^2_{H^{0,m}(M)} \leq C(m,\chi)\|\chi^3 |\nabla^2 u|\|_{H^{0,m}(M)} \|\chi u\|_{H^{0,m}(M)}
$$

(which is easily proven by integration by parts and Cauchy-Schwarz) we conclude that

$$
\|\chi^3 |\nabla^2 u|\|_{H^{0,m}(M)} \leq C(\chi)(\|f\|_{H^{0,m}(M)} + (1+\lambda)\|\chi u\|_{H^{0,m}(M)}).
$$

---

[14]Schematically, what we are doing is observing that any integral expression of the form $\nabla u \nabla^2 g \nabla u$ can be rewritten using integration by parts as $\nabla u \nabla g \nabla^2 u$ plus lower order terms.

The claim (33) follows.

5.3. Proof of (31), (32), (34).

5.3. **Proof of** (31), (32), (34). If we apply the identity (58) with χ equal to a smooth cutoff function adapted to the annulus $\frac{1}{2}R \leq \langle x \rangle \leq 3R$ which equals one when $R \leq \langle x \rangle \leq 2R$, we obtain the estimate

$$\int_{R \leq \langle x \rangle \leq 2R} |\nabla u|_g^2 \, dg \leq C \int_{R/2 \leq \langle x \rangle \leq 3R} (\lambda + R^{-2})|u|^2 + |u|^2|V| + |u||f| \, dg,$$

and hence by Cauchy-Schwarz (and (9)) we establish (31) and (32). Similarly, applying (58) with $\chi = \langle x \rangle^{-1-2\sigma}$ we obtain (34).

5.4. **Proof of** (35), (36), (37). Interpolating (34) $\|u\|_{H^{0,-3/2-\sigma}}$ we have

$$\lambda^{\frac{1}{4}}\|u\|_{H^{0,-1-\sigma}} \leq C(A)\left(\|\nabla u\|_{H^{0,-1/2-\sigma}} + \|u\|_{H^{0,-3/2-\sigma}} + \|f\|_{H^{0,1/2+\sigma}}\right).$$

Combining (34) with the charge estimate (28) we have that

$$\varepsilon\lambda^{\frac{1}{2}}\|u\|_{L^2(M)}^2 \leq C(A)\|f\|_{H^{0,1/2-\sigma}}\left(\|\nabla u\|_{H^{0,-1/2-\sigma}} + \|u\|_{H^{0,-3/2-\sigma}} + \|f\|_{H^{0,1/2+\sigma}}\right).$$

At this point we require the condition $\varepsilon \leq C\lambda$ for some constant $C > 0$. Then

$$\varepsilon^{\frac{3}{2}}\|u\|_{L^2(M)}^2 \leq C(A)\|f\|_{H^{0,1/2+\sigma}}\left(\|\nabla u\|_{H^{0,-1/2-\sigma}} + \|u\|_{H^{0,-3/2-\sigma}} + \|f\|_{H^{0,1/2+\sigma}}\right).$$

The identity (58) (with $\chi = 1$) implies that

$$\|\nabla u\|_{L^2(M)} \leq \lambda^{\frac{1}{2}}\|u\|_{L^2(M)} + C(A)\left(\lambda^{-\frac{1}{4}}\|f\|_{H^{0,1/2+\sigma}} + \lambda^{\frac{1}{4}}\|u\|_{H^{0,-1/2-\sigma}} + \|u\|_{H^{0,-1-\sigma}}\right).$$

Combining the above four inequalities and using that $\varepsilon \leq C\lambda$ we conclude (35).

By inserting smooth cutoffs localised to the region $\{x : \langle x \rangle \gtrsim R\}$ to the above argument we also obtain (36) and (37); we omit the standard details.

The proof of Lemma 2.2 is now complete.

6. THE POSITIVE COMMUTATOR METHOD

We continue our study of the Helmholtz equation (27), and recall the well-known *positive commutator method* to analyze this equation. We let S be an arbitrary pseudo-differential operator, and consider the expression $\langle i[-\Delta_M, S]u, u\rangle_{L^2(M)}$, where $i[S, B] := i(SB - BS)$ is the usual Lie commutator and $\langle u, v \rangle := \text{Re} \int_M \overline{u}v \, dg$ is the real inner product. Then from the self-adjointness of $-\Delta_M - \lambda$ and (27) we have the *commutator identity*[15]

$$\begin{aligned}
\langle i[-\Delta_M, S]u, u\rangle_{L^2(M)} &= \langle i[-\Delta_M - \lambda, S]u, u\rangle_{L^2(M)} \\
&= \langle iSu, (-\Delta_M - \lambda)u\rangle_{L^2(M)} - \langle i(-\Delta_M - \lambda)u, S^*u\rangle_{L^2(M)} \\
&= \langle iSu, F \pm i\varepsilon u\rangle_{L^2(M)} - \langle F \pm i\varepsilon u, (iS)^*u\rangle_{L^2(M)}
\end{aligned}$$

---

[15]In applications, u will be in $H^2(M)$, F will be in $L^2(M)$, and S will be first order, and so there is no difficulty justifying the manipulations below.

and in particular by Cauchy-Schwarz

(63)
$$\langle i[-\Delta_M, S]u, u\rangle_{L^2(M)} \leq \left(\|Su\|_{H^{0,-1/2-\sigma}(M)} + \|S^*u\|_{H^{0,-1/2-\sigma}(M)}\right)\|F\|_{H^{0,1/2+\sigma}(M)}$$
$$+ 2\varepsilon\|Su\|_{L^2(M)}\|u\|_{L^2(M)}.$$

This identity is useful when S is chosen so that $i[-\Delta_M, S]$ is a positive operator (at least to top order), which is why the use of this identity is known as the positive commutator method.

There are a number of ways to generate an operator S whose commutator $i[-\Delta_M, S]$ is positive. One is to make A itself equal to a commutator $S := i[-\Delta_M, W]$, where W is a real-valued scalar weight function, interpreted as a pointwise multiplication operator $u \mapsto Wu$. Then S is the self-adjoint first-order differential operator

$$Su := -2i(\nabla^k W)\nabla_k u - i(\Delta_M W)u$$

and $i[-\Delta_M, S]$ can be computed to be the second-order operator[16]

$$i[-\Delta_M, S]u = -4\nabla_k(\text{Hess}^{km}(W)\nabla_m u) - (\Delta_M^2 W)u$$

The positive commutator estimate (63) then becomes

(64)
$$\int_M 4\text{Hess}^{km}(W)\text{Re}(\overline{\nabla_k u}\nabla_m u) - (\Delta_M^2 W)|u|^2 \, dg$$
$$\leq 2\|2(\nabla^k W)\nabla_k u + (\Delta_M W)u\|_{H^{0,-1/2-\sigma}(M)}\|F\|_{H^{0,1/2+\sigma}(M)}$$
$$+ 2\varepsilon \|2(\nabla^k W)\nabla_k u + (\Delta_M W)u\|_{L^2(M)}\|u\|_{L^2(M)},$$

which should be compared with the $\chi = 1$ case of (60). Indeed (60) can be interpreted as the positive commutator identity applied to the operator $S := \chi i[-\Delta_M, W]$; we omit the details of this calculation.

6.1. **A cheap proof of limiting absorption in the free case.** Using this inequality we can easily prove the limiting absorption principle[17] (or at least the main estimates (1), (2)) for the operator $H_0 = -\Delta$ in Euclidean space. We choose the function

$$W(x) = |x| - (1 + |x|)^{1-2\sigma}.$$

Direct calculation shows that for dimensions $n \geq 3$

$$|x|\,|\Delta W| + |\nabla W| \leq C, \quad -\Delta^2 W \geq c(1+|x|)^{-3-2\sigma}, \quad \text{Hess}_{km}(W) \geq (1+|x|)^{-1-2\sigma}\delta_{km}.$$

We then obtain from (64)

$$\int_{\mathbf{R}^n} \left(\langle x\rangle^{-1-2\sigma}|\nabla u|^2 + \langle x\rangle^{-3-2\sigma}|u|^2\right) \leq \left(\|\nabla u\|_{H^{0,-1/2-\sigma}} + \||x|^{-1}u\|_{H^{0,-1/2-\sigma}}\right)\|f\|_{H^{0,1/2+\sigma}}$$
$$+ 2\varepsilon \left(\|\nabla u\|_{L^2} + \||x|^{-1}u\|_{L^2}\right)\|u\|_{L^2},$$

---

[16]This can either be seen by expanding everything out using the Leibnitz rule, and then exploiting the symmetry properties of the Riemann curvature tensor, or alternatively noting that both sides of this identity are second-order self-adjoint operators with real coefficients and agree both at top order and at the constant term, and thus must be identical. See e.g. [51].

[17]This "cheap" proof of the limiting absorption principle applies to the region where $\varepsilon \leq C\lambda$ for some positive constant C. Analysis of the remaining region requires the use of an additional conservation law (61). This issue will be addressed in Section 11 in a more general context.

Now combining Hardy type inequalities

$$\||x|^{-1}u\|_{L^2} \le C\|\nabla u\|_{L^2}, \qquad \||x|^{-1}u\|_{H^{0,-1/2-\sigma}} \le C\left(\|\nabla u\|_{H^{0,-1/2-\sigma}} + \|u\|_{H^{0,-3/2-\sigma}}\right)$$

with (35) and (34) we obtain

$$\|\nabla u\|_{H^{0,-1/2-\sigma}} + \lambda^{\frac{1}{2}}\|u\|_{H^{0,-1/2-\sigma}} + \|u\|_{H^{0,-3/2-\sigma}} \le C\|f\|_{H^{0,1/2+\sigma}}$$

with second derivative estimates following from the elliptic regularity estimate (30).

Remark 6.2. A proof of the Sommerfeld radiation condition and the estimate (3) will require the use of the identity (61) in place of (60). We will return to this point in a more general context when we establish Lemma 11.1.

6.3. **Proof of Lemma 2.3.** Now we can prove the Pohozaev-Morawetz type estimate in Lemma 2.3. Let the notation and assumptions be as in that Lemma.

For this argument, the function W is chosen to be

$$W = \chi(\langle x\rangle x)(|x| - |x|^{1-2\sigma}),$$

where $\chi(r)$ is a smooth cut-off function vanishing for $r \le r_0$ and equal to 1 for $r \ge 2r_0$ for some $r_0 \ge R_0$ to be chosen later. On the support of W the metric g has the form

$$g = dr^2 + r^2(h_{ab}(\omega) + r^{-2\sigma_0}e_{ab}(r,\omega))d\omega^a d\omega^b$$

The second fundamental form Θ_{ab} of the hypersurfaces $r =$const for a metric in this form is equal to the expression

$$\Theta_{ab} = rh_{ab} + \frac{1}{2}\partial_r\left[r^{2-2\sigma_0}e_{ab}(r,\omega)\right] = rh_{ab} + O(r^{1-2\sigma_0})$$

and its mean curvature Θ (i.e. the trace of Θ_{ab}) has the asymptotic

$$\theta = r^{-2}(h(\omega) + r^{-2\sigma_0}e(r,\omega))^{ab}\Theta_{ab} = \frac{n-1}{r} + O(r^{-1-2\sigma_0}).$$

The Laplacian Δ_M can be written in this coordinate system as

(65) $$\Delta_M = \partial_r^2 + \Theta\,\partial_r + \Delta_{\partial M_r},$$

where $\Delta_{\partial M_r}$ is the Laplace-Beltrami operator of the hypersurface $r =$const. In particular, Δ_M applied to the function $W(r)$ can be computed as

$$\Delta_M W = W_{rr} + \Theta\,W_r = W_{rr} + \frac{n-1}{r}\,W_r + O(r^{-1-2\sigma_0})W_r.$$

Furthermore,

$$\Delta_M^2 W = (\partial_r^2 + \frac{n-1}{r}\partial_r)^2 W + \Delta_M\left(O(r^{-1-2\sigma_0})W_r\right).$$

Finally the Hessian of W,

$$\text{Hess}_{rr}(W) = W_{rr}, \quad \text{Hess}_{ar}(W) = 0, \qquad \text{Hess}_{ab}(W) = \Theta_{ab}W_r.$$

From this we easily conclude that

$$r|\Delta_M W| + |\nabla W|_g \le C,$$
$$-\Delta_M^2 \ge C\chi r^{-3-2\sigma} + \Delta_M\left(O(r^{-1-2\sigma_0})W_r\right) + \zeta O(1),$$
$$\text{Hess}_{mk}(W) \ge Cr^{-1-2\sigma} + \zeta O(1).$$

Here ζ is a smooth cut-off function with support in the region $r_0 \leq \langle x \rangle \leq 2r_0$ introduced to account for the derivatives falling on χ in the expression for W. Let M_r denote the set $\{x \in M : \langle x \rangle \geq r\}$. Then

$$c \int_{M_{2r_0}} \left(\langle x \rangle^{-1-2\sigma} |\nabla u|_g^2 + \langle x \rangle^{-3-2\sigma} |u|^2 \right) \leq \int_M \Delta_M \left(O(r^{-1-2\sigma_0}) W_r \right) |u|^2$$

$$+ \int_{M_{r_0} \backslash M_{2r_0}} \left(|\nabla u|_g^2 + |u|^2 \right) + \left(\|\nabla u\|_{H^{0,-1/2-\sigma}(M_{r_0})} + \|u\|_{H^{0,-3/2-\sigma}(M_{r_0})} \right) \|f\|_{H^{0,1/2+\sigma}(M_{r_0})}$$

$$+ \varepsilon \left(\|\nabla u\|_{L^2(M_{r_0})} + \||x|^{-1}u\|_{L^2(M_{r_0})} \right) \|u\|_{L^2(M_{r_0})},$$

Arguing as in the proof of the limiting absorption principle in Euclidean space above we can, with the help of (36) and (37), reduce the inequality further,

$$c \int_{M_{2r_0}} \left(\langle x \rangle^{-1-2\sigma} |\nabla u|_g^2 + \langle x \rangle^{-1-2\sigma} \lambda |u|^2 + \langle x \rangle^{-3-2\sigma} |u|^2 \right) \leq \int_M \Delta_M \left(O(r^{-1-2\sigma_0}) W_r \right) |u|^2$$

$$+ \int_{M_{r_0/2} \backslash M_{2r_0}} \left(|\nabla u|_g^2 + |u|^2 \right) + \|f\|_{H^{0,1/2+\sigma}(M_{r_0/2})}^2.$$

It remains to deal with the term $\int_M \Delta_M \left(O(r^{-1-2\sigma_0}) W_r \right) |u|^2$. We avoid applying the Laplacian Δ_M to the term $O(r^{-1-2\sigma_0}) W_r$, as this would require higher differentiability of the metric g than required in Theorem 1.7. Instead we integrate by parts to obtain

$$\int_M \Delta_M \left(O(r^{-1-2\sigma_0}) W_r \right) |u|^2 = -2 \int_M \left(O(r^{-1-2\sigma_0}) W_r \right) \left(\mathrm{Re}(\Delta_M u \overline{u}) + |\nabla u|_g^2 \right)$$

$$= -2 \int_M \left(O(r^{-1-2\sigma_0}) W_r \right) \left(\lambda |u|^2 + \mathrm{Re}(F\overline{u}) + |\nabla u|_g^2 \right)$$

It then follows that

$$c \int_{M_{2r_0}} \left(\langle x \rangle^{-1-2\sigma} |\nabla u|_g^2 + \langle x \rangle^{-1-2\sigma} \lambda |u|^2 + \langle x \rangle^{-3-2\sigma} |u|^2 \right)$$

$$\leq \int_{M_{r_0/2} \backslash M_{2r_0}} \left(|\nabla u|_g^2 + |u|^2 \right) + \|f\|_{H^{0,1/2+\sigma}(M_{r_0/2})}^2.$$

provided that $\sigma < \sigma_0$ and r_0 is sufficiently large. The desired conclusion now is a consequence of (32). This concludes the proof of Lemma 2.3.

Remark 6.4. From the Hessian bound

$$\mathrm{Hess}_{ab}(W) = \Theta_{ab} W_r = (r h_{ab} + O(r^{1-2\sigma_0})) W_r$$

we observe that a more precise estimate is available on the angular part $|\nabla_\omega u|_g^2 = r^2(|\nabla u|_g^2 - |\partial_r u|^2)$ of the the gradient of u:

$$c \int_{M_{2r_0}} \langle x \rangle^{-3} |\nabla_\omega u|_g^2 \leq \int_{M_{r_0/2} \backslash M_{2r_0}} \left(|\nabla u|_g^2 + |u|^2 \right) + \|f\|_{H^{0,1/2+\sigma}(M_{r_0/2})}^2.$$

We will revisit this analog of the angular Morawetz estimate for solutions of the time-dependent wave and Schrödinger equations in Lemma 11.1, where we will prove an even stronger result.

Remark 6.5. As Proposition 1.14 shows, it is not possible to remove the error term from Lemma 2.3 in general. However, under the non-trapping assumption it is possible. For examples of such results see [100], [66], [102], [13], [82].

7. High energy limiting absorption for non-trapping metrics

We are now ready to prove Theorem 1.15. This case resembles the local-in-time theory of Craig-Kappeler-Strauss [30] and Doi [40], and indeed our main tool here will be the positive commutator method applied to a certain pseudo-differential operator, exploiting the non-trapping hypothesis to ensure that the symbol of the pseudo-differential operator increases along geodesic flow. As we shall now be working in the high frequency setting, we will not need to take as much care with lower order terms as in previous sections. It will be convenient to use the *scattering pseudo-differential calculus*, which is an extension of the standard pseudo-differential calculus which keeps track of the decay of the symbol at infinity. We briefly summarize the relevant features of this calculus here, referring the reader to [30] for more complete details. (This material will not be used outside of this section.)

For any $m, l \in \mathbf{R}$, we define a *symbol* $s : T^*M \to \mathbf{C}$ *of order* (m, l) to be any smooth function obeying the bounds

$$|\nabla_x^\alpha \nabla_\xi^\beta s(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|} \langle x \rangle^{-l-|\alpha|};$$

the function $s(x, \xi) = \langle x \rangle^{-l} \langle \xi \rangle^m$ is a typical example of such a symbol. Note that we assume that each derivative in x gains a power of $\langle x \rangle$, in contrast to the standard symbol calculus in which no such gain is assumed. We let $S^{m,l}(\overline{M})$ denote the space of such symbols. Given any such symbol $s \in S^{m,l}(\overline{M})$, we can define an associated pseudo-differential operator $S = \text{Op}(s)$ by the usual Kohn-Nirenberg quantization formula

$$\text{Op}(a)u(x) := (2\pi)^{-n} \int e^{i\langle x-y, \xi \rangle} s(x, \xi) u(y) \, dy \, d\xi.$$

We sometimes denote s by $\sigma(S)$ and refer to it as the *symbol* of S. Heuristically speaking, we have $S = \sigma(s)(x, \frac{1}{i}\nabla_x)$. We refer to the class of pseudo-differential operators of order (m, l) as $\Psi_{\text{sc}}^{m,l}$. Also, if $h : \mathbf{R} \to \mathbf{C}$ is any spectral symbol of order $m/2$, the corresponding spectral multiplier $h(H)$ is a pseudo-differential operator of order $(m, 0)$. In particular, $(1 + H)^{m/2}$ has order $(m, 0)$, and the Littlewood-Paley type operators P_{lo}, P_{med}, P_{hi} have order $(0, 0)$. We caution however that the Schrödinger propagators e^{-itH} are not pseudo-differential operators.

The composition of an operator $S = \text{Op}(s)$ of order (m, l) with an operator of $B = \text{Op}(b)$ order (m', l') is an operator SB of order $(m + m', l + l')$, whose symbol $\sigma(SB)$ is equal to $\sigma(S)\sigma(B)$ plus an error of order $(m + m' - 1, l + l' + 1)$; note the additional gain of 1 in the decay index l, which is not present in the classical calculus. Similarly, the commutator $i[S, B]$ will be an operator of order $(m + m' - 1, l + l' - 1)$ with symbol $\sigma(i[S, B])$ equal to the Poisson bracket

$$\{\sigma(S), \sigma(B)\} := \nabla_x \sigma(S) \cdot \nabla_\xi \sigma(B) - \nabla_\xi \sigma(S) \cdot \nabla_x \sigma(B),$$

plus an error of order $(m + m' - 2, l + l' + 2)$. We shall write the above facts schematically as

$$\sigma(AB) = \sigma(A)\sigma(B) + S^{m+m'-1, l+l'+1};$$
$$\sigma(i[A, B]) = \{\sigma(A), \sigma(B)\} + S^{m+m'-2, l+l'+2}$$

or equivalently as

$$\mathrm{Op}(s)\,\mathrm{Op}(b) = \mathrm{Op}(sb) + \Psi_{\mathrm{sc}}^{m+m'-1,l+l'+1},$$

$$i[\mathrm{Op}(s),\mathrm{Op}(b)] = \mathrm{Op}(\{s,b\}) + \Psi_{\mathrm{sc}}^{m+m'-2,l+l'+2}.$$

In particular, since H has order $(2,0)$ and has principal symbol $\frac{1}{2}|\xi|^2_{g(x)}$ plus lower order terms of order $(1,1)$ and $(0,2)$, we see that if $a \in S^{m,l}$, then we have

$$i[H,\mathrm{Op}(s)] = \mathrm{Op}(Xs) + \Psi_{\mathrm{sc}}^{m,l+2},$$

where Xa denotes the derivative of s along geodesic flow in the cotangent bundle T^*M.

Associated with the scattering calculus are the weighted Sobolev spaces $H^{m,\ell}(M)$ defined (for instance) by

$$\|u\|_{H^{m,\ell}(M)} := \|\langle x\rangle^\ell (1+H)^{m/2} u\|_{L^2(M)}$$

(many other equivalent expressions for this norm exist, of course); when $\ell = 0$ this corresponds to the usual Sobolev space $H^m(M)$ and for $m = 0,1,2$ the space $H^{m,\ell}(M)$ coincides with the weighted Sobolev spaces previously defined in (7). It is easy to verify that a scattering pseudo-differential operator of order (m,l) maps $H^{m',l'}(M)$ to $H^{m'-m,l'+l}(M)$ for any m', l'.

In [30] (see also [40]) it was shown that the non-trapping hypothesis on M allows one to construct a real-valued symbol $s \in S^{1,0}$ (depending on φ) which was non-decreasing along geodesic flow, $Xs \geq 0$, and in fact obeyed the more quantitative estimate

$$Xs(x,\xi) = \varphi(x)|\xi|^2_g + |b|^2$$

for some symbol b of order $(1, 1/2 + \sigma)$. The function $\phi(x)$ belongs to the class $S^{0,1+2\sigma}$ and can be chosen to obey a lower bound

$$\phi(x) \geq \langle x\rangle^{-1-2\sigma}.$$

In Euclidean space, an example of such a symbol a is $C_\varphi \frac{x}{\langle x\rangle - \alpha\langle x\rangle^{1-2\sigma}} \cdot \xi$ for some sufficiently large constant C_φ and a small constant α. Quantizing this, we obtain

$$i[H,S] = \nabla^j \varphi(x)\nabla_j + B^*B + \Psi_{\mathrm{sc}}^{1,2+2\sigma},$$

where $S := \mathrm{Op}(s)$ is a symbol of order $(1,0)$, and $B := \mathrm{Op}(b)$ is a symbol of order $(1, 1/2 - \sigma)$. Applying (63), discarding the positive term B^*B and using that $\Psi_{\mathrm{sc}}^{1,2+2\sigma}$ maps $H^{1,-1/2-\sigma}$ to $H^{0,-3/2-\sigma}$ we obtain

$$\int_M \varphi|\nabla u|^2_g\, dg \leq \left(\|Su\|_{H^{0,-1/2-\sigma}(M)} + \|S^*u\|_{H^{0,-1/2-\sigma}(M)}\right)\|F\|_{H^{0,1/2+\sigma}(M)}$$
$$+ \varepsilon\|Su\|_{L^2(M)}\|u\|_{L^2(M)} + \|\nabla u\|_{H^{0,-1/2-\sigma}(M)}\|u\|_{H^{0,-3/2-\sigma}(M)}$$

Since S is of order $(1,0)$, it maps $H^{1,-1/2-\sigma} \to H^{0,-1/2-\sigma}$ and $H^1(M) \to L^2(M)$. Recalling that $F = f + Vu$ with V satisfying the bound $|V(x)| \leq \Gamma\langle x\rangle^{-1-2\sigma_0}$ and $\phi(x) \geq \langle x\rangle^{-1-2\sigma}$, we have

$$\|\nabla u\|^2_{H^{0,-1/2-\sigma}(M)} \leq C(A)\left(\|f\|^2_{H^{0,1/2+\sigma}(M)} + \|u\|^2_{H^{0,-1/2-\sigma}(M)} + \varepsilon\|u\|_{H^1(M)}\|u\|_{L^2(M)}\right).$$

The term $\varepsilon\|\nabla u\|_{L^2(M)}\|u\|_{L^2(M)}$ can be controlled with the help of (35), while the term $\varepsilon\|u\|_{L^2(M)}^2$ can be bounded from the charge estimate (28). As a consequence,

$$\|\nabla u\|_{H^{0,-1/2-\sigma}(M)} \leq C(A)\left(\|f\|_{H^{0,1/2+\sigma}(M)} + \|u\|_{H^{0,-1/2-\sigma}(M)}\right).$$

To prove the desired result it would be sufficient to show that

$$\|u\|_{H^{0,-1/2-\sigma}(M)} \leq c\left(\|\nabla u\|_{H^{0,-1/2-\sigma}(M)} + \|u\|_{H^{0,-3/2-\sigma}(M)}\right) + \|f\|_{H^{0,1/2+\sigma}(M)}$$

with a sufficiently small constant c. This follows immediately from (34) with $c = C(\Gamma)\lambda^{-1/2}$ provided that λ is sufficiently large. We now have that

$$\|\nabla u\|_{H^{0,-1/2-\sigma}(M)} + \lambda^{\frac{1}{2}}\|u\|_{H^{0,-1/2-\sigma}(M)} \leq C(\Gamma)\|f\|_{H^{0,1/2+\sigma}(M)}.$$

The remaining estimate for the second derivative follows from the elliptic regularity estimate (30).

The proof of Theorem 1.15 under an alternative assumption that the potential V satisfies $|V(x)| + \langle x\rangle|\nabla V(x)|_g \leq A\langle x\rangle^{-2\sigma_0}$ follows simply requires running the positive commutator argument with $H = -\Delta_M + V$ in place of $-\Delta_M$. We omit the details.

8. CARLEMAN INEQUALITIES

In this section we prove the unique continuation estimate in Proposition 2.6. Let the notation and assumptions be as in that proposition.

The standard way to prove such unique continuation estimates is via *Carleman inequalities*, which are inequalities of the form

$$(66) \qquad \int_K (t^3|u|^2 + t|\nabla u|^2)e^{2tw}\,dg \leq C(K,w)\int_K |f|^2 e^{2tw}\,dg$$

for some suitable smooth weight function $w : K \to \mathbf{R}$ and some large real parameter t (typically $t \geq C(K,w)(1+\sqrt{\lambda})$). However, as observed in Burq [12], such inequalities are not available when K has a non-trivial topology, due to the fact that w can be forced to contain stationary points. However, it is still possible to obtain a *two-weight* Carleman inequality of the form

(67)
$$\int_K (t^3|u|^2 + t|\nabla u|^2)(e^{2tw_1}+e^{2tw_2})\,dg \leq C(K,w_1,w_2)\int_K (|(-\Delta_M-\lambda)u|^2)(e^{2tw_1}+e^{2tw_2})\,dg,$$

the point being that we can choose w_1 and w_2 to have stationary points at different locations (and furthermore that $w_2 > w_1$ at the stationary points of w_1, and $w_1 > w_2$ at the stationary points of w_2). Such an inequality will allow us to obtain the above proposition.

We now turn to the details. We begin by reviewing the standard approach to Carleman inequalities in the literature; for a more detailed survey see [93]. Let u be a solution to the Helmholtz equation (27) which is supported on the compact set K. We rewrite the equation in the form

$$(-\Delta_M - \lambda)u = F := f \pm i\varepsilon u - Vu$$

If we multiply this equation by a weight e^w for some smooth real-valued w, we obtain

$$(e^w(-\Delta_M - \lambda)e^{-w})e^w u = e^w F$$

We now split $e^w \Delta_M e^{-w}$ into real and imaginary parts

$$e^w(-\Delta_M - \lambda)e^{-w} := \mathrm{Re}(e^w(-\Delta_M - \lambda)e^{-w}) + i\mathrm{Im}(e^w(-\Delta_M - \lambda)e^{-w}),$$

where $\mathrm{Re}(A) := \frac{A+A^*}{2}$ and $\mathrm{Im}(A) := \frac{A-A^*}{2i}$. Using the general identity

$$\|Av\|_{L^2(M)}^2 = \|\mathrm{Re}(A)v\|_{L^2(M)}^2 + \|\mathrm{Im}(A)v\|_{L^2(M)}^2 + \langle i[\mathrm{Re}(A), \mathrm{Im}(A)]v, v\rangle_{L^2(M)}$$

we thus obtain

$$\|\mathrm{Re}(e^w(-\Delta_M - \lambda)e^{-w})e^w u\|_{L^2(M)}^2 + \|\mathrm{Im}(e^w(-\Delta_M - \lambda)e^{-w})e^w u\|_{L^2(M)}^2$$

(68) $$+ \langle i[\mathrm{Re}(e^w(-\Delta_M - \lambda)e^{-w}), \mathrm{Im}(e^w(-\Delta_M - \lambda)e^{-w})]e^w u, e^w u\rangle_{L^2(M)}$$

$$= \|e^w f\|_{L^2(M)}^2;$$

this identity is closely related to (62). In particular we have the inequality

$$\langle i[\mathrm{Re}(e^w(-\Delta_M - \lambda)e^{-w}), \mathrm{Im}(e^w(-\Delta_M - \lambda)e^{-w})]e^w u, e^w u\rangle_{L^2(M)} \le \|e^w f\|_{L^2(M)}^2.$$

Since w was arbitrary, we may replace w by tw for some arbitrary real parameter t (which we shall think of as being large and positive) to obtain

$$\langle i[\mathrm{Re}(e^{tw}(-\Delta_M - \lambda)e^{-tw}), \mathrm{Im}(e^{tw}(-\Delta_M - \lambda)e^{-tw})]e^{tw} u, e^{tw} u\rangle_{L^2(M)} \le \|e^{tw} f\|_{L^2(M)}^2.$$

The strategy is then to select the weight w so that the commutator $i[\mathrm{Re}(e^{tw}(-\Delta_M - \lambda)e^{-tw}), \mathrm{Im}(e^{tw}(-\Delta_M - \lambda)e^{-tw})]$ is positive definite[18], at least to top order in t, to obtain a useful inequality such as (66).

It is certainly possible to adapt the above scheme to prove a two-weight inequality such as (67), and this was essentially what was done in [12]. Let us however pursue a slightly different (though closely related) approach, based on the inequality (62) as a substitute for (68), to emphasize the fact that certain applications of the Carleman method can be viewed as special cases of the general abc method[19]. Substituting w by tw as before, the identity (62) becomes

(69)

$$t \int_K \mathrm{Hess}_{\alpha\beta}(w) \partial^\alpha(e^{tw} u) \partial^\beta(e^{tw} u) \, dg$$

$$+ t^3 \int_K \mathrm{Hess}_{\alpha\beta}(w)(\partial^\alpha w)(\partial^\beta w)|u|^2 e^{2tw} \, dg = t \int_K \frac{1}{4}(\Delta_M^2 w)|u|^2 e^{2tw} \, dg + \frac{1}{8}\|e^{tw} F\|_{L^2(M)}^2.$$

It is easiest to apply this inequality when w is strictly geodesically convex (i.e. $\mathrm{Hess}(w) > 0$) and non-stationary (i.e. $\nabla w \ne 0$) on the support K of u, as the left-hand side is then positive, and the first term on the right-hand side can be

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[18]Actually, one only needs this commutator to be positive definite on the region of phase space where the operators $\mathrm{Re}(e^{tw}(-\Delta_M - \lambda)e^{-tw})$ and $\mathrm{Im}(e^{tw}(-\Delta_M - \lambda)e^{-tw})$ vanish, thanks to the other two terms in the identity (68).

[19]One advantage of doing so is that the abc method does not require more than one degree of regularity on the metric; in particular, the Riemann curvature tensor does not appear here, whereas this tensor will appear when computing the above commutator.

absorbed into the second term on the left-hand side if t is large enough. One would then obtain a *Carleman inequality* of the form

$$t\|\nabla(e^{tw}u)\|^2_{L^2(M)} + t^3\|e^{tw}u\|^2_{L^2(M)} \leq C(w,K)\|e^{tw}F\|^2_{L^2(M)}$$

for sufficiently large t, where the key point is that the constant $C(w,K)$ is independent of t. For instance, in the Euclidean setting $M = \mathbf{R}^n$ one could take $w = \langle x \rangle = (1 + |x|^2)^{1/2}$ to obtain an estimate of this form.

For more general manifolds M, however, it is not possible to find a weight function w which is both geodesically convex and non-stationary, for two reasons. Firstly, if K contains a closed geodesic[20], then it is clearly impossible for w to be strictly geodesically convex on this geodesic. Secondly, if K contains a non-trivial topology (e.g. K contains a handle), then from Morse theory we see that w must contain at least one stationary point. As mentioned earlier, the latter difficulty will be overcome by considering a pair of weight functions w_1, w_2 rather than a single weight w. To overcome the former difficulty, we can replace the weight functions w_1, w_2 by $h(w_1), h(w_2)$ for "sufficiently convex" functions $h : \mathbf{R} \to \mathbf{R}$, however this only gives us convexity of the Hessian in the gradient directions $\nabla w_1, \nabla w_2$ respectively (note though that this is the most important direction that we need convexity in, as one can already see from the second term on the left-hand side of (69)). To deal with the possible failure of convexity in the non-gradient directions, we will use energy identity (58). More precisely, we have

Lemma 8.1 (Preliminary Carleman inequality). *Let $w : K \to \mathbf{R}$ be a smooth function and let $B \subset K$ be a open subset of K. Suppose that w is non-stationary in the set $K \backslash B$, so that the unit vector field $n^\alpha := (\nabla^\alpha w)/|\nabla^\alpha w|_g$ is well defined. Suppose that w is convex in the direction of n^α, and more precisely*

$$\mathrm{Hess}_{nn}(w) := \mathrm{Hess}_{\alpha\beta}(w)n^\alpha n^\beta > 0 \ on \ K\backslash B$$

for some $c > 0$. Suppose also that for any unit vector field X^α we have the estimate

$$\mathrm{Hess}_{\alpha\beta}(w)X^\alpha X^\beta > -\frac{1}{10}\mathrm{Hess}_{nn}(w)|X|^2_g \ on \ K\backslash B.$$

Then for any solution $u \in H^2(K)$ to the Helmholtz equation (27) which is supported in K, and any $t \geq C(K,B,w)(1 + \sqrt{\lambda})$ we have

$$\int_K (t^3|u|^2 + t|\nabla u|^2)e^{2tw} \, dg \leq C(K,B,w)\left(\int_K |F|^2 e^{2tw} \, dg + \int_B (t^3|u|^2 + t|\nabla u|^2)e^{2tw} \, dg\right).$$

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[20]It is natural to conjecture that this implication can be reversed, i.e. that if K was geodesically non-trapping and topologically trivial then there exists a weight function w which is geodesically convex and non-stationary. This would allow one to replace the pseudo-differential calculus considerations in the previous section by a more elementary integration by parts argument. Unfortunately, there exist manifolds which are geodesically non-trapping, but for which no such weight function w exists; this can be seen by a minor modification of the example in [50, Section 6]. The point is that geodesic convexity is equivalent to the assertion that the function $\xi_\alpha \nabla^\alpha w$ is increasing with respect to geodesic flow on the cotangent bundle $T^*K = \{(x, \xi_\alpha) : \xi_\alpha \in T^*_x M\}$, but this function $\xi_\alpha \nabla^\alpha w$ is necessarily linear in ξ, which places additional constraints on solvability beyond the mere non-existence of trapped geodesics. If one uses pseudo-differential operators instead then one does not have this geometrically unnatural linearity constraint.

Proof. Suppose that we are in the region $K\backslash B$. Then by hypothesis, we have

$$\text{Hess}_{\alpha\beta}(w)(\partial^\alpha w)(\partial^\beta w)|u|^2 e^{2tw} = \text{Hess}_{nn}(w)|\nabla w|_g^2 |u|^2 e^{2tw} \geq 0$$

and

$$\text{Hess}_{\alpha\beta}(w)\partial^\alpha(e^{tw}u)\partial^\beta(e^{tw}u) \geq -\frac{1}{10}\text{Hess}_{nn}(w)|\nabla(e^{tw}u)|_g^2$$

$$\geq -\frac{1}{5}\text{Hess}_{nn}(w)(t^2|\nabla w|_g^2 e^{2tw}|u|^2 + |\nabla u|_g^2 e^{2tw})$$

and hence

$$t\text{Hess}_{\alpha\beta}(w)\partial^\alpha(e^{tw}u)\partial^\beta(e^{tw}u) + t^3\text{Hess}_{\alpha\beta}(w)(\partial^\alpha w)(\partial^\beta w)|u|^2 e^{2tw}$$

$$\geq \text{Hess}_{nn}(w)(\frac{4}{5}t^3|\nabla w|_g^2 e^{2tw}|u|^2 - \frac{1}{8}t|\nabla u|_g^2 e^{2tw}).$$

Integrating this on $K\backslash B$ and using (69), we see that

$$\frac{4}{5}t^3\int_{K\backslash B}\text{Hess}_{nn}(w)|\nabla w|_g^2 e^{2tw}|u|^2\ dg \leq Ct\int_K(\Delta_M^2 w)|u|^2 e^{2tw}\ dg + C\|e^{tw}F\|_{L^2(M)}^2$$

$$+\frac{1}{5}t\int_K\text{Hess}_{nn}(w)\varphi(w)|\nabla u|_g^2 e^{2tw}\ dg$$

$$+C(K,B,w)\int_B(t^3|u|^2 + t|\nabla u|^2)e^{2tw}\ dg,$$

where φ is a cutoff function that equals 1 on $K\backslash B$ and vanishes near the stationary points of w. But if we apply the energy identity (58) with $\chi = \text{Hess}_{nn}(w)\varphi(w)e^{2tw}$, and using the Cauchy-Schwarz inequality $\text{Re}(\overline{u}F) \leq t|u|^2 + t^{-1}|F|^2$, we obtain

$$\frac{1}{5}t\int_K\text{Hess}_{nn}(w)\varphi(w)|\nabla u|_g^2 e^{2tw}\ dg \leq \frac{1}{5}\lambda t\int_K\text{Hess}_{nn}(w)\varphi(w)|u|^2 e^{2tw}\ dg$$

$$+\frac{2}{5}t^3\int_K\text{Hess}_{nn}(w)\varphi(w)|\nabla w|_g^2|u|^2 e^{2tw}\ dg$$

$$+C(K,B,w)t^2\int_K|u|^2 e^{2tw}\ dg$$

$$+\|e^{tw}F\|_{L^2(M)}^2.$$

The contribution of the first two terms on $K\backslash B$ can be bounded by

$$\frac{3}{5}t^3\int_{K\backslash B}\text{Hess}_{nn}(w)|\nabla w|_g^2 e^{2tw}|u|^2\ dg$$

term since t^2 is assumed to be large compared with λ. Thus we have

$$\frac{1}{5}t^3\int_{K\backslash B}\text{Hess}_{nn}(w)|\nabla w|_g^2 e^{2tw}|u|^2\ dg \leq C\|e^{tw}F\|_{L^2(M)}^2$$

$$+C(K,B,w)t^2\int_K|u|^2 e^{2tw}\ dg$$

$$+C(K,B,w)\int_B(t^3|u|^2 + t|\nabla u|^2)e^{2tw}\ dg.$$

(absorbing the $\Delta_M^2 w$ term into the $C(K,B,w)t^2\int_K|u|^2 e^{2tw}\ dg$ error). But since $\text{Hess}_{nn}(w)|\nabla w|_g^2$ is non-zero on the compact set $K\backslash B$, and t is assumed large compared with $C(K,B,w)$, we can absorb the $C(K,B,w)t^2\int_K|u|^2 e^{2tw}\ dg$ error into

the right-hand side (plus the error on B) and conclude that

$$t^3 \int_K e^{2tw}|u|^2 \, dg \le C(K,B,w)\|e^{tw}F\|_{L^2(M)}^2 + C(K,B,w) \int_B (t^3|u|^2 + t|\nabla u|^2)e^{2tw} \, dg.$$

Applying (58) again (with $\chi = e^{2tw}$, and using the assumption that t is large compared with $\sqrt{\lambda}$) we obtain the result. $\qquad\square$

The above lemma has the drawback that there is an error term involving u on the right hand side. However, it is localized to a smaller set B than K. We can exploit this localization by using two weights instead of one, whose critical points are at different locations. We begin with a Morse theory lemma (first observed by Burq [12]):

Lemma 8.2 (Construction of Morse function pair). *There exist smooth functions $a_1 : K \to \mathbf{R}$ and $a_2 : K \to \mathbf{R}$ which each have finitely many critical points on K, which are all in the interior of K. Furthermore, whenever x is a critical point of a_1 we have $a_2(x) > a_1(x)$, and whenever x is a critical point of a_2 we have $a_1(x) > a_2(x)$. (In particular, the critical points of a_1 are at distinct locations from the critical points of a_2).*

Proof. By enlarging K if necessary we may assume that the boundary ∂K of K is a sphere $\partial K = \{(r,y) : r = R\}$ for some $R \gg 1$; in particular, K is now connected (since M is connected by hypothesis). We construct a_1 to be the solution to the Dirichlet problem

$$\Delta_M a_1 = 1 \text{ on } K; \quad a_1 = 0 \text{ on } \partial K$$

which can be constructed for instance by a standard variational procedure (or the spectral theory of the Laplacian on a compact manifold with Dirichlet boundary conditions). By standard elliptic theory, this function is smooth on K, and all critical points lie in the interior of K and are non-degenerate (since $\text{Hess}(a_1)$ is clearly non-vanishing), so in particular the number of critical points is finite. Furthermore none of the critical points can be local maxima, since $\Delta_M a_1$ is positive. In particular if we enumerate the critical points of a_1 as $x_1, \ldots, x_m$, and let $\varepsilon > 0$ be a radius so small that the closed balls $\overline{B(x_1,\varepsilon)}, \ldots, \overline{B(x_m,\varepsilon)}$ are disjoint from each other and from the boundary ∂K, then we can find $x_1', \ldots, x_m'$ such that $d_M(x_i, x_i') < \varepsilon$ and $a_1(x_i') > a_1(x_i)$.

We now let $\phi : K \to K$ be a diffeomorphism which is equal to the identity outside of the balls $\overline{B(x_1,\varepsilon)}, \ldots, \overline{B(x_m,\varepsilon)}$, and which swaps x_i and x_i' for each i. If we then set $a_2 := a_1 \circ \phi$ then it is clear that a_2 has critical points precisely at $x_1', \ldots, x_m'$, and the claims follow. $\qquad\square$

We can now combine these two lemmas to obtain a two-weight Carleman inequality which avoids the B errors.

Corollary 8.3 (Two weight Carleman inequality). *There exist smooth functions $w_1 : K \to \mathbf{R}$ and $w_2 : K \to \mathbf{R}$ with the property that for any solution $u \in H^2(K)$ to*

the Helmholtz equation (27) *which is supported in* K, *and any* $t \geq C(K, w_1, w_2)(1 + \sqrt{\lambda})$ *we have*

$$\int_K (t^3 |u|^2 + t|\nabla u|_g^2)(e^{2tw_1} + e^{2tw_2}) \, dg \leq C(K, w_1, w_2) \int_K |F|^2 (e^{2tw_1} + e^{2tw_2}) \, dg.$$

Proof. Let a_1, a_2 be constructed by the previous lemma, and let $x_1, \ldots, x_m$ be the critical points of a_1 and $x'_1, \ldots, x'_{m'}$ be the critical points of a_2. If $\varepsilon = \varepsilon(K, a_1, a_2) > 0$ is a sufficiently small radius, then we have $a_2 > a_1$ on the set

$$B_1 := B(x_1, \varepsilon) \cup \ldots \cup B(x_m, \varepsilon) \subset K$$

and $a_1 > a_2$ on the set

$$B_2 := B(x'_1, \varepsilon) \cup \ldots \cup B(x'_{m'}, \varepsilon) \subset K$$

for some $c = c(K, a_1, a_2) > 0$.

We now let $A = A(K, a_1, a_2, \varepsilon) \gg 1$ be a large parameter to be chosen later, and set $w_1 := \exp(Aa_1)$ and $w_2 := \exp(Aa_2)$. Observe that for $j = 1, 2$, w_j has no critical points outside of B_j. Furthermore, on $K \backslash B_j$ we compute

$$n_j^\alpha := \nabla^\alpha w_j / |\nabla w_j|_g = \nabla^\alpha a / |\nabla a|_g$$

and

$$\mathrm{Hess}_{\alpha\beta}(w_j) = (A^2 \nabla_\alpha a_j \nabla_\beta a_j + A\mathrm{Hess}_{\alpha\beta}(a)) \exp(Aa_j)$$

so in particular

$$\mathrm{Hess}_{n_j n_j}(w_j) = (A^2 |\nabla a_j|_g^2 + A\mathrm{Hess}_{n_j n_j}(a_j)) \exp(Aa_j)$$

and

$$\mathrm{Hess}_{XX}(w_j) = (A^2 |X^\alpha \nabla_\alpha a_j|^2 + A\mathrm{Hess}_{XX}(a_j)) \exp(Aa_j).$$

Thus if A is large enough, the hypotheses of Lemma 8.1 are obeyed, and we have

$$\int_K (t^3 |u|^2 + t|\nabla u|^2) e^{2tw_j} \, dg \leq C(K, B_1, B_2, w_1, w_2) \times$$

$$\times \left(\int_K |F|^2 e^{2tw_j} \, dg + \int_{B_j} (t^3 |u|^2 + t|\nabla u|^2) e^{2tw_j} \, dg \right)$$

for $j = 1, 2$. Adding the two inequalities together we obtain

$$\int_K (t^3 |u|^2 + t|\nabla u|^2)(e^{2tw_1} + e^{2tw_2}) \, dg \leq C(K, B_1, B_2, w_1, w_2) \left(\int_K |F|^2 (e^{2tw_1} + e^{2tw_2}) \, dg \right.$$

$$+ \int_{B_1} (t^3 |u|^2 + t|\nabla u|^2) e^{2tw_1} \, dg$$

$$\left. + \int_{B_2} (t^3 |u|^2 + t|\nabla u|^2) e^{2tw_2} \, dg \right).$$

But observe that $w_2 \geq w_1 + c$ on B_1 and $w_1 \geq w_2 + c$ on B_2 for some $c = c(K, w_1, w_2, B_1, B_2) > 0$, and thus for $j = 1, 2$ we have $e^{2tw_j} \leq e^{-2ct}(e^{2tw_1} + e^{2tw_2})$ on B_j. Thus if t is large enough, we can absorb the last two terms on the right-hand side into the left-hand side, and the claim follows. $\qquad\square$

Remark 8.4. Morally speaking, the above two-weight Carleman inequality can be viewed heuristically as a single-weight Carleman inequality on the product manifold $M \times M$, with u replaced by the tensor product $u \otimes u$ and w replaced by the tensor sum $w_1 \oplus w_2$ (and λ replaced by 2λ). The point is that the critical points (x_i, x_j') of $w_1 \oplus w_2$ lie off the diagonal, and the contribution of the weights at those points can be dominated by the contributions of the weights at the diagonal points (x_i, x_i) and (x_j', x_j'), for instance by the Cauchy-Schwarz inequality. Intriguingly, this perspective shares many similarities with the philosophy underlying the interaction Morawetz inequalities, used for instance in [52], where the positive commutator method of the previous section was also extended to a product setting.

We can now quickly prove Proposition 2.6.

Proof of Proposition 2.6. Let χ be a smooth cutoff function which equals 1 on K' and is supported on K. Then χu is supported on K and obeys the equation

$$(-\Delta_M - \lambda)(\chi u) = -2(\nabla_\alpha \chi)\nabla^\alpha u - (\Delta_M \chi)u + \chi(f \pm i\varepsilon u - Vu) + \chi Vu.$$

In particular, $(-\Delta_M - \lambda)(\chi u)$ is less than $\chi(|f| + O(A|u|))$ on K', and obeys the bound

$$(-\Delta_M - \lambda)(\chi u) \leq C(\chi)(A|u| + |\nabla u|_g + |f|)$$

on $K \backslash K'$. Applying Corollary 8.3 to χu we thus obtain

$$
\begin{aligned}
\int_K (t^3|u|^2 + t|\nabla u|_g^2)(e^{2tw_1} + e^{2tw_2})\, dg \leq & C(K, w_1, w_2)C(\chi)\int_K |f|^2(e^{2tw_1} + e^{2tw_2})\, dg \\
& + AC(\chi)\int_K |u|^2(e^{2tw_1} + e^{2tw_2})\, dg \\
& + C(\chi)\int_{K\backslash K'}(|u|^2 + |\nabla u|_g^2)(e^{2tw_1} + e^{2tw_2})\, dg
\end{aligned}
$$

for $t \geq C(K, w_1, w_2)(1 + \sqrt{\lambda})$, and the claim follows by choosing t to be a large multiple of $C(A, K, \chi, w_1, w_2)(1+\sqrt{\lambda})$ (to absorb the second term on the right-hand side) and noting that w_1 and w_2 are smooth and hence bounded above and below on the compact set K. $\qquad\square$

Remark 8.5. Proposition 2.6 gives good control on solutions to the Helmholtz equation on a compact set K. To obtain a limiting absorption principle, we will have to combine this proposition with more "global" estimates, such as the Morawetz estimates of the previous section, or the Bessel ODE analysis in Section 9.

9. Conservation laws and differential inequalities of Bessel type

The results of Lemma 2.3 and Proposition 2.6 provide us with the following very useful dichotomy. On one hand Lemma 2.3 shows that if the limiting absorption principle can be proved for the restriction, to a certain dyadic region $r_0/2 \leq \langle x \rangle \leq 2r_0$, of a solution u of the Helmholtz equation, then it also holds on the set $\langle x \rangle \geq 2r_0$. Control of the region $r_0/2 \leq \langle x \rangle \leq 2r_0$ together with the unique continuation principle of Proposition 2.6 also imply that the limiting absorption principle can

be extended to the set $\langle x \rangle \leq r_0/2$. On the other hand Proposition 2.6 allows us an alternative scenario. To prove the limiting absorption principle on a compact set K, which can be thought of as the region $\langle x \rangle \leq r_0/2$, rather than proving unconditional control on the solution in the region $r_0/2 \leq \langle x \rangle \leq 2r_0$ it would be sufficient to show instead that the solution varies super-exponentially through that region, i.e., its rate of change is given by $e^{-C(1+\sqrt{\lambda})}$ with a sufficiently large constant C.

To show that either of these scenarios must be realized we need to perform analysis near the asymptotic end of the manifold M; this is the purpose of Lemma 2.8, which we shall prove in this section and in the next.

We begin with an informal discussion. For simplicity consider the case when M is asymptotically Euclidean manifold. Then we can heuristically approximate the Helmholtz equation by its Euclidean version

$$\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}} - V + \lambda \right) u = -F.$$

Assuming that V is radial, $V = V(r)$, and applying the ansatz

$$u(r, \omega) = r^{-(n-1)/2} v(r) Y_l(\omega), \quad F(r, \omega) = -r^{-(n+1)/2} G(r) Y_l(\omega)$$

where Y_l is a spherical harmonic of order l on S^{n-1}, normalized to have $L^2(S^{n-1})$ norm equal to 1, the Helmholtz equation becomes the *Bessel ordinary differential equation*

$$v_{rr} - \frac{L(L-1)}{r^2} v - Vv + \lambda v = G,$$

where $L := l + \frac{n-1}{2}$. As we shall see in Section 14 this equation will play an important role in a counterexample construction of Proposition 1.21. Despite providing good insight into behavior of solutions of the Helmholtz equation near infinity, the use of the Bessel equation approximation has several drawbacks. At first glance, it seems that this equation only emerges when the solution u has a specific structure, namely that it decouples as the product of a radial function and a spherical harmonic. Of course, one could orthogonally decompose an arbitrary function u into such products and work on each harmonic separately (as is done in a number of places in the literature, e.g. [12]), but this becomes difficult if the metric and potential only decay slowly at infinity (although such analysis well suited for compact perturbations of the standard Euclidean metric). Also, such an approach often requires detailed analysis of the asymptotics of Bessel or Hankel functions. Here, we present a more "energy-based" method to simulate differential equations (or differential inequalities) Bessel type for solutions to the Helmholtz equation (27), without requiring an explicit decomposition into spherical harmonics, and without requiring any knowledge of Bessel or Hankel functions (although such functions are in some sense lurking in the background in what follows).

In this section we shall work purely in the asymptotic region $r > R_0$; thus the analysis here may be viewed as a "black box" analysis, requiring no knowledge of the manifold, solution, or potential in the interior region $r \leq R_0$. Eventually we will combine this black box analysis with the Carleman analysis in the near region $r = O(1)$ from previous sections to obtain the full limiting absorption principle.

We write the metric in the form

$$g = dr^2 + r^2 h_{jk}[r](\omega) d\omega^j d\omega^k$$

where

$$h_{jk}[r](\omega) := h_{jk}(\omega) + r^{-2\sigma_0} e_{jk}(r, \omega)$$

and work on the hypersurfaces $S_r := \{(r, \omega) : \omega \in \partial M\}$, which are naturally endowed with the metric $h[r]$ and the corresponding measure $dh[r] := \sqrt{h[r]} d\omega$; note that this differs from the induced measure $dg|_{S_r}$ by a factor of r^{n-1}. In particular we have the co-area formula

$$(70) \qquad \int_{r > R_0} f \, dg = \int_{R_0}^{\infty} \int_{S_r} f(r, \omega) \, dh[r](\omega) r^{n-1} dr.$$

We now rewrite the resolvent equation (26) in polar co-ordinate form as

$$\frac{\partial^2}{\partial r} u + \frac{n-1}{r} \frac{\partial}{\partial r} u + \frac{\frac{\partial}{\partial r} dh[r]}{dh[r]} \frac{\partial}{\partial r} u + \frac{1}{r^2} \Delta_{h[r]} u + \lambda u = V u \mp i\varepsilon u - f.$$

Note that

$$\theta_{jk}(r, \omega) = \frac{1}{2} \partial_r \theta_{jk}(r, \omega)$$

is the second fundamental form of the surface S_r relative to the renormalized metric

$$\overline{g} = dr^2 + h_{jk} d\omega^j \, d\omega^k.$$

with mean curvature

$$(71) \qquad \theta = \frac{\frac{\partial}{\partial r} dh[r]}{dh[r]}$$

It follows from (6) that

$$(72) \qquad \theta = O(r^{-1-2\sigma_0}), \quad |\theta|_{h[r]} = O(r^{-1-2\sigma_0}).$$

We expect solutions u of the Helmholtz equation to decay like $r^{-(n-1)/2}$ as $r \to \infty$. Thus we can renormalize u by defining

$$(73) \qquad v := r^{(n-1)/2} u$$

(cf. (124)) and observe that v obeys a Bessel-like equation

$$(74) \quad v_{rr} + \frac{1}{r^2}\Big(\Delta_{h[r]} - \frac{(n-1)(n-3)}{4}\Big) v + \lambda v = -\theta v_r + (V + \frac{n-1}{r}\theta) v \mp i\varepsilon v + r^{(n-1)/2} f$$

with the operator $-\Delta_{h[r]}$ playing the role of the parameter $l(l + n - 2)$.

We now define the "spherical energies"

$$\text{Mass} \quad \mathcal{M}[r] := \int_{S_r} |v|^2 \, dh[r]$$

$$\text{Radial energy} \quad \mathcal{R}[r] := \int_{S_r} |v_r|^2 \, dh[r]$$

$$\text{Angular energy} \quad \mathcal{A}[r] := \int_{S_r} \frac{1}{r^2}\Big(|\nabla_\omega v|^2_{h[r]} + \frac{(n-1)(n-3)}{4}|v|^2\Big) \, dh[r]$$

$$\text{Mass flux} \quad \mathcal{F}[r] := \int_{S_r} \text{Re}(\overline{v} v_r) \, dh[r],$$

where $\nabla_\omega v$ is the angular gradient. Note that the quantity $\mathcal{M}[r]$ was already introduced in Lemma 2.8. We also need the "forcing term"

$$(75) \qquad \mathbf{G}[r] := r^{(n-1)/2} \int_{S_r} (|v| + \frac{|\nabla v|_g}{r^{-1} + \lambda^{1/2}})(|f| + \varepsilon|u|) \, dh[r],$$

and record the following three "equations of motion" for the spherical energies, which are closely related to the conservation laws (48), (50), (59).

Lemma 9.1 (Equations of motion). *We have positivity properties*

$$(76) \qquad \mathcal{M}[r], \mathcal{R}[r], \mathcal{A}[r] \geq 0$$

the Cauchy-Schwarz estimate

$$(77) \qquad |\mathcal{F}[r]| \leq \mathcal{M}[r]^{1/2}\mathcal{R}[r]^{1/2}$$

and the equations of motion

$$(78) \qquad \frac{d}{dr}\mathcal{M} = 2\mathcal{F} + O(r^{-1-2\sigma_0})\mathcal{M}$$

$$(79) \qquad \frac{d}{dr}\mathcal{F} = \mathcal{R} + \mathcal{A} - \lambda\mathcal{M} + O(r^{-2-2\sigma_0})\mathcal{M} + O(r^{-1-\sigma_0})\lambda^{1/2}\mathcal{M} + O(\mathbf{G})$$

$$\frac{d}{dr}(\mathcal{R} + \lambda\mathcal{M} - \mathcal{A}) = \frac{2}{r}\mathcal{A} + O(r^{-1-2\sigma_0})(\mathcal{A} + \mathcal{R} + \lambda\mathcal{M})$$

$$(80) \qquad\qquad + O(r^{-3-2\sigma_0})\mathcal{M} + O(r^{-1} + \lambda^{1/2})\mathbf{G}.$$

Here the implicit constants are allowed to depend on M and A.

Proof. While in dimensions $n \geq 3$ positivity of the spherical energies and the Cauchy-Schwarz estimate are obvious, equations (78)-(80) follow from the identities:

$$\frac{d}{dr}|v|^2 = 2\mathrm{Re}(v_r\overline{v}),$$

$$\frac{d}{dr}(v_r\overline{v}) = |v_r|^2 - \frac{1}{r^2}\left(\Delta_{h[r]} - \frac{(n-1)(n-3)}{4}\right)v\overline{v} - \lambda|v|^2 - \theta v_r\overline{v}$$

$$\qquad + \left((V + \frac{n-1}{r}\theta)v \mp i\varepsilon v + r^{(n-1)/2}f\right)\overline{v},$$

$$\frac{d}{dr}(|v_r|^2 + \lambda|v|^2 - \frac{1}{r^2}|\nabla_\omega v|_h^2 - \frac{(n-1)(n-3)}{4r^2}|v|^2) = \frac{2}{r}\left(\frac{1}{r^2}|\nabla_\omega v|_h^2 + \frac{(n-1)(n-3)}{4r^2}|v|^2\right)$$

$$\qquad + \frac{2}{r^2}\mathrm{div}_\omega\mathrm{Re}(\nabla_\omega v\,\overline{v_r}) - 2\theta|v_r|^2 - \frac{1}{r^2}\theta_{jk}\nabla_\omega^j v\overline{\nabla_\omega^k v}$$

$$\qquad + 2\mathrm{Re}\left((V + \frac{n-1}{r}\theta)v \mp i\varepsilon v + r^{(n-1)/2}f\right)\overline{v_r},$$

identity (71) and the assumptions (72) and $|V(x)| \leq A(r^{-2-2\sigma_0} + \lambda^{1/2}r^{-1-2\sigma_0})$. $\quad\square$

Remark 9.2. It is helpful to keep in mind the model case (see (124)), where v solves the Bessel differential equation

$$v_{rr} - \frac{L(L-1)}{r^2}v + \lambda v = 0,$$

in which case

$$\mathcal{M}[r] = |v(r)|^2; \quad \mathcal{R}[r] = |v_r(r)|^2; \quad \mathcal{A}[r] = \frac{L(L-1)}{r^2}|v(r)|^2;$$

$$\mathcal{F}(r) = \mathrm{Re}(\overline{v(r)}v_r(r)); \quad \mathbf{G}[r] = 0.$$

The reader may wish to verify the above equations of motion (with all error terms set to zero) in this special case. It may also be useful to keep in mind the dimensional analysis

$$r \sim \mathrm{length}^1; \quad \mathcal{M} \sim \mathrm{length}^0; \quad \mathcal{F} \sim \mathrm{length}^{-1}; \quad \mathcal{R}, \mathcal{A}, \lambda, \mathbf{G} \sim \mathrm{length}^{-2},$$

noting that the above equations then become dimensionally consistent up to errors involving σ_0.

Remark 9.3. Note that while we have four energies, we only have three equations of motion; we do not control the evolution of $\mathcal{R}$ and $\mathcal{A}$ separately, but only have an equation for a certain combination $\mathcal{R} - \mathcal{A}$ of these two. However, we can obtain a lower bound on $\mathcal{R}$ from (77). This system of three equations and one inequality is thus still underdetermined, but we will still be able to extract enough control out of this system to establish all the estimates we need.

Remark 9.4. Of course, the three equations of motion can also be interpreted in terms of the Friedrichs abc method, where the cutoff χ is now the surface measure on a sphere S_r. We omit the details.

Now we obtain some preliminary estimates on the above energies, in the setting of Lemma 2.8.

Proposition 9.5 (Preliminary estimates). *Let the notation and assumptions be as in Lemma 2.8. Then we have the integral estimate*

$$\tag{81} \int_{R_0}^{\infty} \mathbf{G}[r] \, dr = O(\delta).$$

and the boundary condition

$$\tag{82} \lim_{r \to \infty} \frac{1}{r} \int_r^{2r} \mathcal{M}[s] + \mathcal{R}[s] + \mathcal{A}[s] \, ds = 0.$$

Proof. The boundary condition (82) follows from the normalization $\|u\|_{H^{0,-1/2-\sigma}(M)} = 1$ and (31). To prove (81), we first see from (75), (124) that

$$\mathbf{G}[r] \leq Cr^{n-1} \int_{S_r} (|u| + \frac{|\nabla u|_g}{r^{-1} + \lambda^{1/2}})(|f| + \varepsilon|u|) \, dh[r]$$

and so by dyadic decomposition it would suffice to show that

$$\tag{83} \sum_{R \geq R_0} \int_{R \leq \langle x \rangle \leq 2R} \left(|u| + \frac{|\nabla u|_g}{R^{-1} + \lambda^{1/2}} \right) (|f| + \varepsilon|u|) \, dg = O(\delta).$$

From (41) we have the bounds

$$\int_{R \leq \langle x \rangle \leq 2R} |u|^2 \, dg \leq c_R R^{1+2\sigma}$$

and

$$\int_{R \leq \langle x \rangle \leq 2R} |f|^2 \, dg \leq \delta^2 c_R R^{-1-2\sigma}$$

where $c_R > 0$ are numbers such that $\sum_R c_R = O(1)$. From the charge estimate (Lemma 2.1) we also have

$$\varepsilon \int_M |u|^2 \, dg \leq \|f\|_{H^{0,1/2+\sigma}(M)} \|u\|_{H^{0,-1/2-\sigma}(M)} \leq \delta$$

and thus

$$\int_{R \leq \langle x \rangle \leq 2R} |u|^2 \, dg \leq c_R \delta/\varepsilon$$

(after adjusting c_R if necessary). Finally, from (31) we have

$$\int_{R \leq \langle x \rangle \leq 2R} \left(|u| + \frac{|\nabla u|_g}{R^{-1} + \lambda^{1/2}} \right)^2 dg \leq C \int_{R/2 \leq \langle x \rangle \leq 4R} \left(|u|^2 + \frac{f^2}{(\lambda + R^{-2})^2} \right) dg$$

and so after adjusting c_R a bit more we obtain

$$\int_{R \leq \langle x \rangle \leq 2R} (|u| + \frac{|\nabla u|_g}{R^{-1} + \lambda^{1/2}})^2 \, dg \leq C c_R \left(\min(R^{1+2\sigma}, \delta/\varepsilon) + \frac{\delta^2 R^{-1-2\sigma}}{(\lambda + R^{-2})^2} \right).$$

On the other hand, we have

$$\int_{R \leq \langle x \rangle \leq 2R} (|f| + \varepsilon|u|)^2 \, dg \leq C c_R (\delta^2 R^{-1-2\sigma} + \min(\varepsilon\delta, k\varepsilon^2 R^{1+2\delta})).$$

Applying Cauchy-Schwarz, we will obtain (83) if we can show that

$$\left(\min(R^{1+2\sigma}, \delta/\varepsilon) + \frac{\delta^2 R^{-1-2\sigma}}{(\lambda + R^{-2})^2} \right) (\delta^2 R^{-1-2\sigma} + \min(\varepsilon\delta, k\varepsilon^2 R^{1+2\delta})) \leq C\delta^2$$

for all $R \geq R_0$. This is clear for the terms involving $\min(R^{1+2\sigma}, \delta/\varepsilon)$. For the term

$$\frac{\delta^2 R^{-1-2\sigma}}{(\lambda + R^{-2})^2} \delta^2 R^{-1-2\sigma}$$

we bound $\lambda + R^{-2}$ from below by R^{-2} to obtain a bound of $\delta^4 R^{2-4\sigma}$, which is acceptable since $\sigma < 1/2$ and $\delta \leq C$. Finally, for the term

$$\frac{\delta^2 R^{-1-2\sigma}}{(\lambda + R^{-2})^2} \min(\varepsilon\delta, k\varepsilon^2 R^{1+2\delta})$$

we bound $\lambda + R^{-2}$ from below by λ, and use the second term in the minimum, to obtain a bound of $\varepsilon^2 \delta^2/\lambda^2$, which is acceptable since $\varepsilon < \lambda$. $\square$

10. An ODE lemma

In view of Lemma 9.1 and Proposition 9.5, we see that Lemma 2.8 will follow immediately from the following ODE lemma.

Lemma 10.1 (ODE Lemma). *Let $C_1 \gg R_0$ be a large number, and then let $C_2 \gg C_1$ be an even larger number. For all $r \geq R_0$, let $\mathcal{M}, \mathcal{R}, \mathcal{A}, \mathcal{F}, \mathbf{G}$ be real-valued functions obeying the differential inequalities in Lemma 9.1 as well as the properties (81), (82). Then if C_1 is sufficiently large (but not depending on λ), and C_2 is sufficiently large depending on C_1 (but not on λ), then one of the following must be true:*

- *(Boundedness) There exists a radius $C_1 \le r_0 \le C(C_1, C_2)$ such that (42) holds.*
- *(Exponential growth) For all $C_1 \le r \le 10C_1$, we have (43).*

Of course, the magnitude of C_1, C_2, $C(C_1, C_2)$ will depend on the implicit constants in Lemma 9.1 and (81) (which in practice will depend on M and A).

The proof of this Lemma is lengthy and will occupy the remainder of this section.

10.2. **Heuristics.** We first describe in informal terms why one would expect there to be a dichotomy of the type asserted in Lemma 10.1. We first observe that (80) is an approximate monotonicity formula for the quantity $\mathcal{R} + \lambda\mathcal{M} - \mathcal{A}$. Since this quantity is zero at infinity by (82), we expect it to be negative (up to errors of size of $O(\delta)$, thanks to (81)) at other values of r. Thus we have a lower bound on $\mathcal{A}$, heuristically of the form

$$(84) \qquad\qquad \mathcal{A} \ge \mathcal{R} + \lambda\mathcal{M} - O(\delta).$$

This converts (79) into a monotonicity formula as well, roughly of the form

$$(85) \qquad\qquad \frac{d}{dr}\mathcal{F} \ge 2\mathcal{R} - O(\delta).$$

Now the dichotomy in Lemma 10.1 rests on whether the forcing terms such as $\mathcal{R}$ or $\frac{2}{r}\mathcal{A}$ in the monotonicity formulae are large enough to dominate the error terms such as $O(\delta)$. If this domination never occurs (or only occurs when r is relatively small), then one ends up in the "boundedness" scenario (42). On the other hand, if at least one of the forcing terms becomes large, one expects that this will eventually force the other forcing term to be large as well (as r decreases towards C_1), causing a positive feedback loop which will eventually lead to the "exponential growth" scenario (43). For instance, if $\mathcal{R}$ gets large (compared to $\mathcal{M}$ and δ), this should force $\mathcal{M}$ to be similarly large thanks to (85); from (78) one then expects $\mathcal{M}$ to grow exponentially (but slightly less fast than $\mathcal{F}$) as r decreases; from (77) one then expects $\mathcal{R}$ to stay large, thus creating a self-sustaining feedback loop. Similarly, if $\mathcal{A}$ gets large, then from (80) we expect the quantity $\mathcal{R} + \lambda\mathcal{M} - \mathcal{A}$ to get large and negative, which adds an additional positive term to the right-hand side of (85), which as mentioned earlier should cause $\mathcal{F}$, $\mathcal{M}$, and $\mathcal{R}$ to grow; using (84), this should eventually sustain the growth of $\mathcal{A}$, thus creating another self-sustaining feedback loop. If these loops start far enough away from the origin (e.g. at $r > C(C_1, C_2)$) then one might hope to expect the growth to become exponential with growth rate C_2 by the time r reaches C_1, which is the "exponential growth" half of the dichotomy.

Remark 10.3. An oversimplified model of this dichotomy can be seen by considering plane wave solutions to the Helmholtz equation $\Delta u = \lambda u$ in the (flat) cylinder $\mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}^+ := \{(\theta, r) : \theta \in \mathbf{R}/2\pi\mathbf{Z}, r \in \mathbf{R}^+\}$, which should be thought of as a caricature of polar co-ordinates; we assume some boundedness on u at $r = +\infty$ (e.g. $u(r) = O(\delta)$ for sufficiently large r) but not when r is small. One has "bounded" solutions of the form $u(\theta, r) = Ce^{ia\theta}e^{ibr}$ where a is an integer such that $|a|^2 \le \lambda$ and $a^2 + b^2 = \lambda$ and $C = O(\delta)$. Then there are "exponential growth" solutions of the form $u(x, y) = Ce^{ia\theta}e^{-br}$ where $|a|^2 > \lambda$ and $a^2 - b^2 = \lambda$ and C is arbitrarily large. Thus one expects the solution to stay bounded if the "angular energy" a^2

stays smaller than λ, and to grow exponentially otherwise. This dichotomy roughly corresponds in our setting to the case when $\mathcal{A}$ stays controlled by $\mathcal{R} + \lambda\mathcal{M}$ (which will basically ensure the bounded scenario) or is larger than this quantity (which will ensure the exponential growth scenario). See [21] for some rigorous formulations of these heuristics, where the cylinder has now been replaced by a stadium.

10.4. **Step 1: a Pohozaev bound.** We now begin the rigorous proof of Lemma 10.1. We begin by giving a rigorous version of (84), based primarily on (80) and the positivity of $\mathcal{A}$.

Lemma 10.5 (Pohozaev bound). *Let $\mathcal{P}$ denote the "Pohozaev flux"*

$$\mathcal{P}[r] := \lambda\mathcal{M}[r] + \mathcal{R}[r] - \mathcal{A}[r].$$

Then for all $r \geq C_1$, we have

$$\mathcal{P}[r] \leq O((r^{-1} + \lambda^{1/2}))\delta + O(r^{-2-2\sigma})\mathcal{M}[r].$$

Proof. We rewrite (80) in terms of the Pohozaev flux as

$$\frac{d}{dr}\mathcal{P} = \frac{2}{r}\mathcal{A} + O(r^{-1-2\sigma_0})(2\mathcal{A} + \mathcal{P}) + O(r^{-3-2\sigma_0})\mathcal{M} + O((r^{-1} + \lambda^{1/2}))\mathbf{G},$$

To eliminate the $O(r^{-3-2\sigma_0})\mathcal{M}$ error[21] we shall consider the modified Pohozaev flux

$$\mathcal{P}^* := \mathcal{P} - C_0 r^{-2-2\sigma_0}\mathcal{M}$$

for constant $C_0 \gg 1$, and observe using (78) that

$$\begin{aligned}
\frac{d}{dr}\mathcal{P}^* = {} & \frac{2}{r}\mathcal{A} + O(r^{-1-2\sigma_0})(2\mathcal{A} + \mathcal{P}^* + C_0 r^{-2-2\sigma_0}\mathcal{M}) \\
& + C_0 r^{-2-2\sigma_0}(\mathcal{F} + O(r^{-1-2\sigma_0})\mathcal{M}) \\
& + (2 + 2\sigma)C_0 r^{-3-2\sigma_0}\mathcal{M} + O(r^{-3-2\sigma_0})\mathcal{M} + O(r^{-1} + \lambda^{1/2})\mathbf{G}.
\end{aligned}$$

We can use (77) to bound

$$|\mathcal{F}| \leq r^{-1-\sigma_0}\mathcal{M} + r^{1+\sigma_0}\mathcal{R} = r^{-1-\sigma_0}(1 + C_0)\mathcal{M} + 2r^{1+\sigma_0}\mathcal{A} + r^{1+\sigma_0}\mathcal{P}^*.$$

If C_0 is suitably large (and $C_1 \leq r$ is also suitably large) then the net $\mathcal{M}$ term on the right-hand side is positive, as is the net $\mathcal{A}$ term. Thus we have

$$\frac{d}{dr}\mathcal{P}^* \geq O(C_0 r^{-1-\sigma_0})\mathcal{P}^* + O(r^{-1} + \lambda^{1/2})\mathbf{G}.$$

On the other hand, from (82) we know that $\mathcal{P}^*[r] \to 0$ as $r \to \infty$. The claim then follows from (81) and Gronwall's inequality. $\square$

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[21]In dimensions $n \geq 4$ we can use the $\mathcal{A}$ term to control the $\mathcal{M}$ error since $(n-1)(n-3)/4 > 0$ in that case; this leads to some minor simplifications in the proof of Lemma 10.1.

10.6. **Step 2: Dimensionless formulation.** In order to analyze our system further it is convenient to make a number of changes of variable to a more scale-invariant or "dimensionless" formulation. At present we have three equations of motion (78), (79), (80) and one inequality (77) (as well as the positivity properties (76)) for four unknowns. We shall now use the inequality (77) to replace two of the unknowns $\mathcal{R}$, $\mathcal{A}$ by a single unknown $\mathcal{P}$, at the cost of replacing the equalities in (79), (80) by inequalities. Indeed, we can rewrite (79) using (77) as

$$\frac{d}{dr}\mathcal{F} = 2\mathcal{R} - \mathcal{P} + O(r^{-2-2\sigma_0})\mathcal{M} + O(\lambda^{1/2}r^{-1-2\sigma_0})\mathcal{M} + O(\mathbf{G})$$

$$\geq 2\frac{\mathcal{F}^2}{\mathcal{M}} - \mathcal{P} + O(r^{-2-2\sigma_0})\mathcal{M} + O(\lambda^{1/2}r^{-1-2\sigma_0})\mathcal{M} + O(\mathbf{G}).$$

As for (80), we rewrite it (for C_1 sufficiently large) as

$$\frac{d}{dr}\mathcal{P} = \frac{2}{r}\mathcal{A} + O(r^{-1-2\sigma_0})(2\mathcal{A} + \mathcal{P}) + O(r^{-3-2\sigma_0})\mathcal{M} + O(r^{-1} + \lambda^{1/2})\mathbf{G}$$

$$= \left(\frac{2}{r} + O(r^{-1-2\sigma_0})\right)\mathcal{A} + O(r^{-1-2\sigma_0})\mathcal{P} + O(r^{-3-2\sigma_0})\mathcal{P} + O(r^{-1} + \lambda^{1/2})\mathbf{G}$$

and then write

$$\mathcal{A} = \mathcal{R} + \lambda\mathcal{M} - \mathcal{P} \geq \frac{\mathcal{F}^2}{\mathcal{M}} + \lambda\mathcal{M} - \mathcal{P}.$$

We thus have the new equations of motion

$$\frac{d}{dr}\mathcal{M} = 2\mathcal{F} + O(r^{-1-2\sigma_0})\mathcal{M}$$

$$\frac{d}{dr}\mathcal{F} \geq 2\frac{\mathcal{F}^2}{\mathcal{M}} - \mathcal{P} + O(r^{-2-2\sigma_0})\mathcal{M} + O(\lambda^{1/2}r^{-1-2\sigma_0})\mathcal{M} + O(\mathbf{G})$$

$$\frac{d}{dr}\mathcal{P} \geq \left(\frac{2}{r} + O(r^{-1-2\sigma_0})\right)(\frac{\mathcal{F}^2}{\mathcal{M}} + \lambda\mathcal{M} - \mathcal{P}) + O(r^{-1-2\sigma_0})\mathcal{P}$$

$$+ O(r^{-3-2\sigma_0})\mathcal{M} + O(r^{-1} + \lambda^{1/2})\mathbf{G}.$$

To analyze these equations, we now adjust the quantities $\mathcal{F}$, $\mathcal{P}$ slightly to handle the forcing terms involving $\mathbf{G}$. Define

$$\mathcal{F}^*[r] := \mathcal{F}[r] - C\int_r^\infty |\mathbf{G}[s]|\, ds = \mathcal{F}[r] + O(\delta)$$

and

$$\mathcal{P}^*[r] := \mathcal{P}[r] - C\int_r^\infty (s^{-1} + \lambda^{1/2})|\mathbf{G}[s]|\, ds = \mathcal{P}[r] + O(r^{-1} + \lambda^{1/2})\delta,$$

where we have used (81). If the constant C appearing above is large enough, then we can dominate the $\mathbf{G}[r]$ forcing terms on the right-hand sides of the above equations to obtain

$$\frac{d}{dr}\mathcal{M} = 2\mathcal{F} + O(r^{-1-2\sigma_0})\mathcal{M}$$

$$\frac{d}{dr}\mathcal{F}^* \geq 2\frac{\mathcal{F}^2}{\mathcal{M}} - \mathcal{P} + O(r^{-2-2\sigma_0})\mathcal{M} + O(\lambda^{1/2}r^{-1-2\sigma_0})\mathcal{M}$$

$$\frac{d}{dr}\mathcal{P}^* \geq \left(\frac{2}{r} + O(r^{-1-2\sigma_0})\right)(\frac{\mathcal{F}^2}{\mathcal{M}} + \lambda\mathcal{M} - \mathcal{P}) + O(r^{-1-2\sigma_0})\mathcal{P} + O(r^{-3-2\sigma_0})\mathcal{M}.$$

Writing $\mathcal{F} = \mathcal{F}^* + O(\delta)$ and so $\mathcal{F}^2 \geq (\mathcal{F}^*)^2 + O(\delta)\mathcal{F}^*$, and similarly writing $\mathcal{P} = \mathcal{P}^* + O(r^{-1} + \lambda^{1/2})\delta$, we thus obtain

(86)
$$\frac{d}{dr}\mathcal{M} = 2\mathcal{F}^* + O(r^{-1-2\sigma_0})\mathcal{M} + O(\delta)$$
(87)
$$\frac{d}{dr}\mathcal{F}^* \geq 2\frac{(\mathcal{F}^*)^2 + O(\delta)\mathcal{F}^*}{\mathcal{M}} - \mathcal{P}^* + O(1 + \lambda^{1/2}r)r^{-2-2\sigma_0}\mathcal{M} + O(r^{-1} + \lambda^{1/2})\delta$$
(88)
$$\frac{d}{dr}\mathcal{P}^* \geq \left(\frac{2}{r} + O(r^{-1-2\sigma_0})\right)\left(\frac{(\mathcal{F}^*)^2 + O(\delta)\mathcal{F}^*}{\mathcal{M}} + \lambda\mathcal{M} - \mathcal{P}^*\right) + O(r^{-1-2\sigma_0})\mathcal{P}^*$$
(89) $$+ O(r^{-3-2\sigma_0})\mathcal{M} + O(r^{-2} + \lambda^{1/2}r^{-1})\delta.$$

To analyze this system of equations, it is convenient to work in the "dimensionless" co-ordinates[22]

$$\mu(r) := r\frac{\delta}{\mathcal{M}[r]}$$
$$\alpha(r) := -r\frac{\mathcal{F}^*[r]}{\mathcal{M}[r]}$$
$$\beta(r) := -r^2\frac{\mathcal{P}^*[r]}{\mathcal{M}[r]}$$

and to introduce the "dimensionless" derivative $D := -r\frac{d}{dr}$. We then have

$$D\mu = -\mu + r^2\frac{\delta}{\mathcal{M}^2}\frac{d}{dr}\mathcal{M}$$
$$= -\mu - 2r^2\frac{\delta}{\mathcal{M}^2}\mathcal{F} + O(r^{1-2\sigma_0})\frac{\delta}{\mathcal{M}} + O(r^2\frac{\delta^2}{\mathcal{M}^2})$$
$$= -\mu - 2\alpha\mu + O(r^{-2\sigma_0})\mu + O(\mu^2)$$

and

$$D\alpha = -\alpha + \frac{r^2}{\mathcal{M}^2}(\mathcal{M}\frac{d}{dr}\mathcal{F}^* - \mathcal{F}^*\frac{d}{dr}\mathcal{M})$$
$$\geq -\alpha + \frac{r^2}{\mathcal{M}^2}\left(2(\mathcal{F}^*)^2 + O(\delta)\mathcal{F}^* - \mathcal{M}\mathcal{P}^* + O(1 + r\lambda^{1/2})r^{-2-2\sigma_0}\mathcal{M}^2\right.$$
$$\left. + O(r^{-1} + \lambda^{1/2})\delta\mathcal{M} - 2(\mathcal{F}^*)^2 + O(r^{-1-2\sigma_0})\mathcal{M}\mathcal{F}^* + O(\delta)\mathcal{F}^*\right)$$
$$= \beta - \alpha + O(\mu)\alpha + O(1 + r\lambda^{1/2})r^{-2\sigma_0} + O(1 + \lambda^{1/2}r)\mu + O(r^{-2\sigma_0})\alpha$$

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[22]Admittedly, these co-ordinates have a singularity when $\mathcal{M}[r] = 0$, but this will not be relevant for us as we shall only perform the remainder of the analysis in the case when μ is small (and hence $\mathcal{M}$ is large).

and

$$D\beta = -2\beta + \frac{r^3}{\mathcal{M}^2}(\mathcal{M}\frac{d}{dr}\mathcal{P}^* - \mathcal{P}^*\frac{d}{dr}\mathcal{M})$$

$$\geq -2\beta + \frac{r^3}{\mathcal{M}^2}\left(\frac{2}{r} + O(r^{-1-2\sigma_0})\right)((\mathcal{F}^*)^2 + O(\delta)\mathcal{F}^* + \lambda\mathcal{M}^2 - \mathcal{M}\mathcal{P}^*)$$

$$+O(r^{-1-2\sigma_0})\mathcal{M}\mathcal{P}^* + O(r^{-3-2\sigma_0})\mathcal{M}^2 + O(r^{-2} + \lambda^{1/2}r^{-1})\delta\mathcal{M}$$

$$-2\mathcal{P}^*\mathcal{F}^* + O(r^{-1-2\sigma_0})\mathcal{M}\mathcal{P}^* + O(\delta)\mathcal{P}^*)$$

$$= -2\beta + (2 + O(r^{-2\sigma_0}))(\alpha^2 + O(\mu)\alpha + \lambda r^2 + \beta) + O(r^{-2\sigma_0})\beta + O(r^{-2\sigma_0})$$

$$+ O(1 + \lambda^{1/2}r)\mu - 2\alpha\beta + O(r^{-2\sigma_0})\beta + O(\mu)\beta$$

$$\geq 2\alpha(\alpha - \beta) + \lambda r^2 + O(\mu(1 + \lambda^{1/2}r + |\alpha| + |\beta|)) + O(r^{-2\sigma_0}(1 + |\beta| + \alpha^2)).$$

Meanwhile, from (10.5) we have

$$\mathcal{P}^* \leq O(r^{-1} + \lambda^{1/2})\delta + O(r^{-2-2\sigma})\mathcal{M}$$

and hence

$$\beta \geq -O(1 + \lambda^{1/2}r)\mu - O(r^{-2\sigma_0}).$$

To summarize, the functions $\alpha(r), \beta(r), \mu(r)$ obey the differential inequalities

(90)
$$D\alpha \geq \beta - \alpha - O((\mu + r^{-2\sigma_0})(1 + \lambda^{1/2}r + |\alpha|))$$

(91)
$$D\beta \geq 2\alpha(\alpha - \beta) + \lambda r^2 - O(\mu(1 + \lambda^{1/2}r + |\alpha| + |\beta|)) - O(r^{-2\sigma_0}(1 + |\beta| + \alpha^2))$$

(92)
$$D\mu = -\mu - 2\alpha\mu + O(r^{-2\sigma_0}\mu) + O(\mu^2)$$

(93)
$$\beta \geq -O(1 + \lambda^{1/2}r)\mu - O(r^{-2\sigma_0}).$$

10.7. **More heuristics.** Recall that we are trying to establish a dichotomy between boundedness and exponential growth. In our new co-ordinates, boundedness roughly corresponds to μ being bounded away from zero for small values of r, while exponential growth corresponds to α being large (as can be seen either from (92) or (78)). So, heuristically speaking, we have to rule out the scenario in which μ is small and α is also small. If μ is very small, however, then we expect to be able to ignore most of the error terms in (90), (91), and thus we reduce (heuristically) to the model equations

(94) $$D\alpha \geq \beta - \alpha; \quad D\beta \geq 2\alpha(\alpha - \beta) + \lambda r^2; \quad \beta \geq 0.$$

The intuition here is that as r moves backwards from ∞ to C_1, the first inequality will lift up α if α is below β, while the second inequality will lift up β if β is below α (note that this forces both α and β to be non-negative, by the third equation); indeed, when λr^2 is large there some additional lift applied to β (and thus indirectly to α, by the first equation). These equations then suggest that if α and β are both large at some radius r, then they will stay large for all smaller radii also; this is

what will lead to the exponential growth scenario. On the other hand, if α and β stay small for all time then we can hope to obtain the boundedness scenario.

10.8. **Step 3: A condition for igniting exponential growth.** We now make the above heuristics rigorous.

Lemma 10.9 (Exponential growth is self-sustaining). *Suppose that $r \geq C_1$ is such that*

$$\mu(r) \leq 1/C_1 \text{ and } \alpha(r) \geq C_2^2.$$

Then for all $C_1 \leq s \leq r$ we have

(95) $$\mu(s) \leq 1/C_1 \text{ and } \alpha(s) \geq 2C_2.$$

Proof. From (92) we have the somewhat crude estimate

(96) $$D\mu(s) \leq -\frac{1}{2}\mu(s) \text{ whenever } \alpha(s) \geq 0 \text{ and } \mu(s) \leq 1/C_1.$$

From (96) and the continuity method it will thus suffice to show that

$$\alpha(s) > 2C_2 \text{ for all } C_1 < s \leq r.$$

Suppose this claim is false, then there exists $C_1 < s_* < r$ such that

(97) $$\alpha(s_*) = 2C_2$$

and

(98) $$\alpha(s) \geq 2C_2 \text{ for all } s_* \leq s \leq r.$$

From (96) and Gronwall's inequality (and the continuity method) we conclude in particular that

(99) $$\mu(s) \leq (s/r)^{1/2}\mu(r) \leq (s/r)^{1/2}/C_1 \leq 1/C_1 \text{ for all } s_* \leq s \leq r.$$

It is convenient to introduce the quantity

$$\kappa(s) := \alpha^2(s) + \beta(s).$$

From (90), (91) we have

$$\begin{aligned}
D\kappa(s) &= 2\alpha(s)D\alpha(s) + D\beta(s) \\
&\geq -O((\mu(s) + s^{-2\sigma_0})(\alpha(s) + \lambda^{1/2}s\alpha(s) + \alpha^2(s))) \\
&\quad + \lambda s^2 - O(\mu(s)(1 + \lambda^{1/2}s + \alpha(s) + |\beta(s)|)) - O(s^{-2\sigma_0}(1 + |\beta(s)| + \alpha^2(s))) \\
&\geq \lambda s^2 + O((\mu(s) + s^{-2\sigma_0})(1 + \alpha(s) + \alpha^2(s) + \lambda^{1/2}s + \lambda^{1/2}s\alpha(s) + |\beta(s)|)) \\
&\geq \lambda s^2 - O((\mu(s) + s^{-2\sigma_0})(1 + \alpha^2(s) + \lambda s^2 + |\beta(s)|)) \\
&\geq (1 - O(\mu(s) + s^{-2\sigma_0}))\lambda s^2 - O((\mu(s) + s^{-2\sigma_0})(1 + \kappa(s) + 2\min(-\beta(s), 0))).
\end{aligned}$$

But from (93) we have

$$\min(-\beta(s), 0) \leq C\mu(1 + \lambda s^2) + Cs^{-2\sigma_0}$$

and thus

$$D\kappa(s) \geq (1 - O((\mu + s^{-2\sigma_0})(1 + \mu)))\lambda s^2 - O((\mu(s) + s^{-2\sigma_0})(1 + \kappa(s) + \mu(s))).$$

Using (99), we conclude

$$(100) \quad D\kappa(s) \geq \frac{1}{2}\lambda s^2 - O((C_1^{-1}(s/r)^{1/2} + s^{-2\sigma_0})(1 + \kappa(s))) \text{ for all } s_* \leq s \leq r.$$

On the other hand, from (93) we have the crude estimate

$$\beta(r) \geq -O(1 + \lambda^{1/2}r)$$

and hence

$$\kappa(r) \geq C^{-1}C_2^4 - C\lambda^{1/2}r.$$

From this, (100), and Gronwall's inequality we see that

$$(101) \qquad \kappa(s) \geq C^{-1}(\lambda r(r-s) + C_2^4 - C\lambda^{1/2}r) \text{ for all } s_* \leq s \leq r.$$

On the other hand, from (90) and (98), and writing $\beta = \kappa - \alpha^2$, we have the rather crude estimate

$$(102) \qquad D\alpha(s) \geq \kappa(s) - O(C_1^{-1}(1 + \lambda^{1/2}s)) - O(\alpha^2(s)) \text{ for all } s_* \leq s \leq r.$$

From (97) we have $\alpha(s_*) < 3C_2 < \alpha(r)$. Thus we can find $s_* < r_* < s$ such that $\alpha(r_*) = 3C_2$ and $2C_2 < \alpha(s) < 3C_2$ for all $s_* < s < r_*$. Then by (101), (102) we have

$$D\alpha(s) \geq C^{-1}C_2^2 + C^{-1}\lambda r(r-s) - C\lambda^{1/2}r$$

Observe that the expression on the right is positive unless $r \geq C_2\lambda^{-1/2}$ and $s = r + O(C_1\lambda^{-1/2})$, in which case it is bounded below by $-C\lambda^{1/2}r$. From this and the fundamental theorem of calculus we see that

$$-C_2 = \alpha(r_*) - \alpha(s_*) \geq \int_{r_*}^{s_*} D\alpha(s) \frac{ds}{s} \geq -C_1,$$

a contradiction. The claim follows. $\qquad\square$

Corollary 10.10. *There exists $C_3 = C(C_1, C_2) \gg C_2$ such that if there exists $r \geq C_3$ for which $\mu(r) \leq 1/C_1$ and $\alpha(r) \geq C_2^2$ are both true, then we are in the exponential growth scenario (43).*

Proof. Suppose first that we are in the low energy case $\lambda \leq 1$. Then from the above Lemma we have $\alpha(r) \geq 2C_2$ for all $C_1 \leq r \leq 10C_1$. From (92) we then have

$$D\mu(s) \leq -2C_2\mu(s) \text{ for all } C_1 \leq s \leq 10C_1$$

which by definition of $\mu(s)$, yields the mass growth estimate (43).

Now suppose that we are in the high energy case $\lambda > 1$. In this case we observe from (102), (101) (with r now being replaced by C_3) that

$$D\alpha(s) \geq C^{-1}(\lambda C_3(C_3 - s) + C_2^4) - CC_3\lambda^{1/2} - O(\alpha^2(s)) \text{ for all } C_1 \leq s \leq C_3$$

and in particular

$$D\alpha(s) \geq C^{-1}\lambda C_3^2 \text{ for all } C_1 \leq s \leq C_3/2 \text{ such that } \alpha(s) \leq C_2^2\lambda^{1/2}.$$

We also have $\alpha(s) \geq 2C_2$ in this region. We thus conclude (if C_3 is large) that

$$\alpha(s) \geq C_2^2\lambda^{1/2} \text{ for all } C_1 \leq s \leq 10C_1$$

and then by arguing as before we obtain (43). $\qquad\square$

10.11. **Step 4: The case of no exponential growth.** In light of the above corollary, we may assume without loss of generality that for any $r \geq C_3$, at least one of

$$(103) \qquad \mu(r) \geq 1/C_1 \text{ or } \alpha(r) \leq C_2^2$$

is true.

We can now remove the second half of of the dichotomy (103) at large distances.

Lemma 10.12. *Suppose that (103) holds. Let $C_4 = C(C_1, C_2, C_3) \gg C_3$ be a sufficiently large constant depending on C_1, C_2 and C_3. Then we have $\mu(r) > 1/C_4^2$ for all $r \geq C_4(1 + \lambda^{-1/2})$.*

Proof. By (82) we have $\mu(r) \to \infty$ as $r \to \infty$. Thus if the claim is false, then we can find $r_1 \geq r_0 \geq C_4(1 + \lambda^{-1/2})$ such that

$$(104) \qquad \mu(r_0) = C_4^{-2}$$

$$(105) \qquad \mu(r_1) = C_4^{-1}$$

$$(106) \qquad C_4^{-2} \leq \mu(r) \leq C_4^{-1} \text{ for all } r_0 \leq r \leq r_1.$$

In particular from (103) we have

$$(107) \qquad \alpha(r) \leq C_2^2 \text{ for all } r_0 \leq r \leq r_1$$

From (105) we have

$$D\mu(r_1) \leq 0$$

which by (92), (105) forces

$$(108) \qquad \alpha(r_1) \geq -1$$

(for instance). Also, from (92), (106), (107) we have

$$D\mu(r) \geq -2C_2^2 \mu(r)$$

which by Gronwall's inequality and (104), (105) forces a certain largeness bound in the interval $[r_0, r_1]$:

$$\int_{r_0}^{r_1} \frac{dr}{r} \geq C^{-1} C_2^{-2} \log C_4,$$

which implies that $r_1 \geq r_0 \, C_4^{C^{-1}C_2^{-2}}$. In particular, if C_4 is large enough then we have

$$(109) \qquad r_1 - r_0 \geq C_4(1 + \lambda^{-1/2}),$$

(for instance). Now we control β. If $r_0 \leq r \leq r_1$, then $r \geq C_4 \lambda^{-1/2}$ and $\mu(r) \leq C_4^{-1}$, which by (91) implies the crude bound

$$D\beta \geq C^{-1}\alpha^2 - O((1 + |\alpha|)(1 + |\beta|)) + C^{-1}\lambda r^2;$$

using the crude bound $O((1 + |\alpha|)(1 + |\beta|)) = O(1 + C^{-1}|\alpha|^2 + C|\beta|^2)$ we conclude

$$D\beta \geq C^{-1}\lambda r^2 \text{ whenever } |\beta(r)| \leq C^{-1} r \lambda^{1/2}.$$

On the other hand, from (93) we have

$$\beta(r) \geq -C\lambda^{1/2} r \text{ whenever } r_0 \leq r \leq r_1.$$

Combining these two equations we see that

$$\beta(r) \geq C^{-1}\lambda^{1/2}r \text{ whenever } r_0 \leq r \leq r_1 - C\lambda^{-1/2}.$$

On the other hand, from (90) and the estimates $r \geq C_4\lambda^{-1/2}$, $\mu \leq C_4^{-1}$ we have

$$D\alpha \geq \beta - O(\alpha) - O(C_4^{-2\sigma_0}\lambda^{1/2}r),$$

and thus we have

$$D\alpha(r) \geq C^{-1}r\lambda^{1/2} - O(\alpha(r)) \text{ whenever } r_0 \leq r \leq r_1 - C\lambda^{-1/2}$$

and

$$D\alpha(r) \geq -O(C\lambda^{1/2}r) - O(\alpha(r)) \text{ whenever } r_1 - C\lambda^{-1/2} \leq r \leq r_1.$$

Moreover, since $r \geq C_4\lambda^{-1/2}$ and $\alpha(r) \leq C_2^2$, we can replace the first estimate by

$$D\alpha(r) \geq C^{-1}r\lambda^{1/2} \text{ whenever } r_0 \leq r \leq r_1 - C\lambda^{-1/2}$$

From these estimates and the initial condition (108), applying Gronwall's inequality, we see that

$$\alpha(r) \geq -C \text{ whenever } r_1 - C\lambda^{-1/2} \leq r \leq r_1$$

and then by a further application of Gronwall we see that

$$\alpha(r) \geq C^{-1}(r_1 - r)\lambda^{1/2} - O(1) \text{ whenever } r_0 \leq r \leq r_1 - C\lambda^{-1/2}.$$

Since $r_1 - r_0 \geq C_4\lambda^{-1/2}$ this contradicts (109) and (107), and the claim follows. $\square$

In the high-energy case $\lambda \geq 1$ this lemma immediately gives the boundedness half (42) of the dichotomy, by choosing $r := 4C_4$ (for instance) and using the definition of μ. In the low-energy case one observes from (103) and (92) that

$$D\mu(r) \geq -2C_2^2\mu(r)$$

whenever $C_3 \leq r \leq C_4(1 + \lambda^{-1/2})$ and $\mu(r) \leq 1/C_1$. From this, Lemma 10.12, and Gronwall's inequality we see that

(110)
$$\mu(r) \geq C(C_1, C_2, C_3, C_4)^{-1}\lambda^{C(C_1,C_2)}$$

for all $C_3 \leq r \leq C_4(1 + \lambda^{1/2})$, and by definition of μ we are again in the boundedness half (42) of the dichotomy. The claim follows. This completes the proof of Lemma 10.1, and thus Lemma 2.8. ∎

11. ADDITIONAL EQUATIONS OF MOTION

As we have already seen in Section 3.1, the combined results of Lemma 2.1, Lemma 2.2, Lemma 2.3, Proposition 2.6, and Lemma 2.8 are already sufficient to prove Theorem 1.7. But to prove the Sommerfeld radiation conditions of Proposition 1.12, it turns out that we must study further spherical energies in addition to the energies $\mathcal{M}, \mathcal{R}, \mathcal{A}, \mathcal{F}$ introduced in Section 9. We take a quick detour to define these energies, study additional equations of motions and derive their consequences, in particular establishing Lemma 2.4. The results of this section will also be useful in the proof of Proposition 1.27.

Using the notations of Section 9, we define the following additional energies:

$$\text{Outgoing null energy} \qquad \mathcal{N}[r] := \int_{S_r} |v_r \mp izv|^2 \, dh[r],$$

$$\text{Complex flux} \qquad \mathcal{Z}[r] := \int_{S_r} v_r \overline{v} \, dh[r],$$

where $z = a \pm ib$ is the complex number such that $z^2 = \lambda \pm i\varepsilon$ and $\text{Im}(z) = \pm b > 0$.

Lemma 11.1 (Additional equation of motion). *Let $u \in H^2(M)$ be a solution to the resolvent equation*

$$(H - (\lambda \pm i\varepsilon))u = f$$

for some $\lambda, \varepsilon > 0$ with the Hamiltonian $H = -\Delta_M + V$, satisfying assumptions of Proposition 1.12. Then (with the notation of the previous two sections)

$$\frac{d}{dr}(\mathcal{N} - \mathcal{A}) = \frac{2}{r}\mathcal{A} + 2b\,(\mathcal{N} + \mathcal{A}) + O(r^{-2-2\sigma_0})b\mathcal{M}$$

(111)
$$+ O\Big(r^{-1-2\sigma_0}(\mathcal{R} + \mathcal{A} + |z|^2\mathcal{M})\Big) + O(F[r]^{1/2}\mathcal{N}^{1/2})$$

$$+ O\big(r^{-2-2\sigma_0} + \lambda^{1/2}r^{-1-2\sigma_0}\big)\mathcal{M}^{1/2}\mathcal{N}^{1/2}$$

where $F[r] := r^{n-1}\int_{\partial M} |f|^2 dh[r]$.

Remark 11.2. A similar equation of motion appeared in [57] for the proof of a qualitative limiting absorption principle for a Schödinger operator with magnetic potential in $\mathbf{R}^n$. This equation can be interpreted as a special case of the identity (61).

Proof. For simplicity assume that $(H - \lambda - i\varepsilon)u = f$. Note that $\mathcal{F} = \text{Re}\,\mathcal{Z}$. We have as in Lemma 9.1

$$\frac{d}{dr}\mathcal{M} = 2\text{Re}\mathcal{Z} + O(r^{-1-2\sigma_0}\mathcal{M}),$$

$$\frac{d}{dr}\mathcal{Z} = \mathcal{R} + \mathcal{A} - z^2\mathcal{M} + O(r^{-2-2\sigma_0})\mathcal{M} + r^{\frac{n-1}{2}}\int_{S_r} (f + Vv)\overline{v}dh[r]$$

Note that

$$|v_r - izv|^2 = |v_r|^2 + |z|^2|v|^2 + 2\text{Im}(zv\overline{v_r})$$
$$= |v_r|^2 + z^2|v|^2 + (|z|^2 - z^2)|v|^2 + 2\text{Im}(zv\overline{v_r}).$$

Thus,

$$\mathcal{N} = \mathcal{R} + |z|^2\mathcal{M} + 2\text{Im}(z\overline{\mathcal{Z}}).$$

Since

$$\frac{d}{dr}\Big(\mathcal{R} - \mathcal{A}\Big) = \frac{2}{r}\mathcal{A} - 2\text{Re}(z^2\overline{\mathcal{Z}}) + O(r^{-1-2\sigma_0})(\mathcal{R} + \mathcal{A})$$

(112)
$$+ 2\int_{S_r} \text{Re}\big((f + Vv)\overline{v_r}\big)dh[r]$$

we derive the following equation for $\mathcal{N} - \mathcal{A}$:

$$\frac{d}{dr}\left(\mathcal{N} - \mathcal{A}\right) = \frac{2}{r}\mathcal{A} + 2\mathrm{Re}\left((|z|^2 - z^2)\overline{\mathcal{N}}\right) + 2\mathrm{Im}(z)(\mathcal{R} + \mathcal{A})$$
$$+ 2|z|^2\,\mathrm{Im}(z)\mathcal{M} + O(r^{-1-2\sigma_0})(\mathcal{R} + \mathcal{A} + |z|^2\mathcal{M}) + O(F[r]^{1/2}\mathcal{N}^{1/2})$$
$$+ O(r^{-2-2\sigma_0})\mathrm{Im}(z)\mathcal{M} + O(r^{-2-2\sigma_0} + \lambda^{1/2}r^{-1-2\sigma_0})\mathcal{M}^{1/2}\mathcal{N}^{1/2}$$
$$= \frac{2}{r}\mathcal{A} + 2\mathrm{Re}\left((|z|^2 - z^2)\overline{\mathcal{Z}}\right) - 4\mathrm{Im}(z)\,\mathrm{Im}(z\overline{\mathcal{Z}})$$
$$+ 2\mathrm{Im}(z)\,(\mathcal{N} + \mathcal{A}) + O(r^{-1-2\sigma_0})(\mathcal{R} + \mathcal{A} + |z|^2\mathcal{M}) + O(r^{-2-2\sigma_0})\mathrm{Im}(z)\mathcal{M}$$
$$+ O(F[r]^{1/2}\mathcal{N}^{1/2}) + O(r^{-2-2\sigma_0} + \lambda^{1/2}r^{-1-2\sigma_0})\mathcal{M}^{1/2}\mathcal{N}^{1/2}$$

Since

$$2\mathrm{Re}\left((|z|^2 - z^2)\overline{\mathcal{Z}}\right) = 4\mathrm{Im}(z)\,\mathrm{Re}(-iz\overline{\mathcal{Z}}) = 4\mathrm{Im}(z)\,\mathrm{Im}(z\overline{\mathcal{Z}}),$$

we have

$$\frac{d}{dr}(\mathcal{N} - \mathcal{A}) = \frac{2}{r}\mathcal{A} + 2b\,(\mathcal{N} + \mathcal{A}) + O(r^{-1-2\sigma_0})(\mathcal{R} + \mathcal{A} + |z|^2\mathcal{M}) + O(r^{-2-2\sigma_0})b\mathcal{M}$$
$$+ O(F[r]^{1/2}\mathcal{N}^{1/2}) + O(r^{-2-2\sigma_0} + \lambda^{1/2}r^{-1-2\sigma_0})\mathcal{M}^{1/2}\mathcal{N}^{1/2}.$$

$\square$

11.3. **Proof of Lemma 2.4.** We are now ready to establish Lemma 2.4. Let the notation and assumptions be as in that lemma.

Multiplying equation (111) for $(\mathcal{N} - \mathcal{A})$ by $r^{2\sigma}$ we obtain

$$\frac{d}{dr}r^{2\sigma}(\mathcal{N} - \mathcal{A}) = \frac{1}{r^{1-2\sigma}}\left((2 - 2\sigma)\mathcal{A} + \mathcal{N}\right) + 2b\,r^{2\sigma}\,(\mathcal{N} + \mathcal{A})$$
$$+ O(r^{-1-2\sigma_0+2\sigma})(\mathcal{R} + \mathcal{A} + |z|^2\mathcal{M}) + r^{2\sigma}O(F[r]^{1/2}\mathcal{N}^{1/2})$$
$$+ O(r^{-2-2\sigma_0+2\sigma})b\mathcal{M} + O\left(r^{2\sigma}(r^{-2-2\sigma_0} + \lambda^{1/2}r^{-1-2\sigma_0})\right)\mathcal{M}^{1/2}\mathcal{N}^{1/2}.$$

The presence of the $2b\,r^{2\sigma}\,(\mathcal{N} + \mathcal{A})$ term will allow us to ignore the boundary term at $r = \infty$ arising after integration in r. Therefore, for some universal positive constant c.

$$c\int_{2r_0}^{\infty}(r^{-1+2\sigma} + b\,r^{2\sigma})\,(\mathcal{N} + \mathcal{A})\,dr \leq \int_{r_0}^{\infty}r^{-1-2\sigma_0+2\sigma}\left(\mathcal{R} + |z|^2\mathcal{M}\right)dr$$

(113)
$$+ r_0^{-1+2\sigma}\int_{r_0}^{4r_0}\mathcal{A}\,dr + \int_{r_0}^{\infty}r^{-3-2\sigma_0+2\sigma}\mathcal{M}\,dr + \int_{r_0}^{\infty}r^{1+2\sigma}F[r]\,dr.$$

Note that in view of Lemma 2.3, the bound (113) already gives the estimate (40) in the region $\varepsilon \leq C\lambda$.

We now claim that u verifies the Poincaré type inequality
(114)
$$\|u\|_{H^{0,-3/2+\sigma}(M_{2r_0})}^2 + b\|u\|_{H^{0,-1+\sigma}(M_{2r_0})}^2 \leq C(r_0)\left(\int_{2r_0}^{\infty}\mathcal{N}[r] + \|u\|_{L^2(M_{2r_0}\setminus M_{4r_0})}^2\right).$$

Note that (114) together with (113) and Lemma 2.3 imply all the statements of Lemma 2.4 for $\varepsilon \leq C\lambda$.

To establish (114), observe that

$$\mathcal{N}[r] = \int_{S_r} |v_r - izv|^2 dh[r] = \int_{S_r} e^{-2br} |\partial_r(e^{-izr}v)|^2 dh[r]$$

Our assumptions on the metric g imply that the area form $dh[r]$, which corresponds to the metric $(h_{ab}(\omega) + r^{-2\sigma_0}e_{ab}(r,\omega))d\omega^a d\omega^b$, is equivalent to the area form dh of the metric $h_{ab}(\omega)d\omega^a d\omega^b$. Integrating by parts the expression

$$
\begin{aligned}
\int_{2r_0}^\infty dr \, r^{-2+2\sigma}((1-\sigma)r^{-1} + b) &\int_{S_r} |v|^2 dh \\
&= \int_{2r_0}^\infty dr \, r^{-1+2\sigma} e^{-2br} \int_{S_r} \mathrm{Re}\left(\partial_r(e^{-izr}v)(\overline{e^{-izr}v})\right) dh \\
&\quad + \frac{1}{2}(2r_0)^{-2+2\sigma}((1-\sigma)(2r_0)^{-1} + b) \int_{S_{2r_0}} |v|^2 dh \\
&\leq 2(1-\sigma)^{-1} \int_{2r_0}^\infty dr \, r^{-1+2\sigma} e^{-2br} \int_{S_r} |\partial_r(e^{-izr}v)|^2 dh \\
&\quad + \frac{1}{2}(1-\sigma) \int_{2r_0}^\infty dr \, r^{-3+2\sigma} \int_{S_r} |v|^2 dh \\
&\quad + \frac{1}{2}(2r_0)^{-2+2\sigma}((1-\sigma)(2r_0)^{-1} + b) \int_{S_{2r_0}} |v|^2 dh
\end{aligned}
$$

and averaging over r_0 we immediately obtain (114).

It remains to consider the case $\varepsilon \geq C\lambda$. Under this assumption we have $|z| \leq 2\varepsilon$ (for instance). In view of (114) and a trivial inequality

$$\mathcal{N} \geq \frac{1}{2}\mathcal{R} - 7|z|^2\mathcal{M}$$

the bound (113) can be reduced to the estimate

$$
\begin{aligned}
\|\nabla u\|_{H^{0,-1/2+\sigma}(M_{2r_0})} + \|u\|_{H^{0,-3/2+\sigma}(M_{2r_0})} &\leq C|z|\|u\|_{H^{0,-1/2-\sigma_0+\sigma}(M_{2r_0})} \\
&+ C(r_0)\left(\|u\|_{L^2(M_{r_0}\backslash M_{4r_0})} + \|\nabla u\|_{L^2(M_{r_0}\backslash M_{4r_0})} + \|f\|_{H^{0,1/2+\sigma}(M_{r_0})}\right).
\end{aligned}
$$

According to (29)

$$\varepsilon\|u\|_{H^{0,-1/2-\sigma_0+\sigma}(M_{2r_0})} \leq C\|f\|_{H^{0,-1/2-\sigma_0+\sigma}(M_{r_0})} + C\|\nabla u\|_{H^{0,-3/2-\sigma_0+\sigma}(M_{r_0})},$$

which implies that

$$
\begin{aligned}
\|\nabla u\|_{H^{0,-1/2+\sigma}(M_{2r_0})} + \|u\|_{H^{0,-3/2+\sigma}(M_{2r_0})} &\leq C|z|\|u\|_{H^{0,-1/2-\sigma_0+\sigma}(M_{2r_0})} \\
&+ C(r_0)\left(\|u\|_{L^2(M_{r_0}\backslash M_{4r_0})} + \|\nabla u\|_{L^2(M_{r_0}\backslash M_{4r_0})} + \|f\|_{H^{0,1/2+\sigma}(M_{r_0})}\right).
\end{aligned}
$$

provided that r_0 is sufficiently large. The bounds (38), (39) follow immediately, while (40) can be recovered from (113). The proof of Lemma 2.4 is now complete.

12. The low energy regime

We now prove Proposition 1.27. We assume that the potential V obeys the bounds

$$|V(x)| \leq A\langle x \rangle^{-2-2\sigma_0}, \qquad \int_M |V_-(x)|^{\frac{n}{2}} \leq \beta$$

with a small constant $\beta = \beta(M)$ and show that for all sufficiently small $\varepsilon, |\lambda| < \lambda_0(M)$ and $\sigma < \min(1, \sigma_0)$,

(115) $\qquad \|R(\lambda \pm i\varepsilon)f\|_{H^{2,-1/2-\sigma}(M)} \leq C(M, A)\lambda^{-1/2}\|f\|_{H^{0,1/2+\sigma}(M)}$,

(116) $\qquad \|R(\lambda \pm i\varepsilon)f\|_{H^{2,-3/2+\sigma}(M)} \leq C(M, A)\|f\|_{H^{0,1/2+\sigma}(M)}$.

We start by recalling the *Friedrichs inequality*:

(117) $\qquad\qquad \|u\|_{L^2(M)} \leq C_s|\mathrm{supp}u|^{\frac{1}{n}}\|\nabla u\|_{L^2(M)}$,

which holds for any smooth function of compact support on M. The Friedrichs inequality is a direct consequence (by Hölder's inequality) of the Sobolev inequality (with the same constant C_s):

(118) $\qquad\qquad \|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C_s\|\nabla u\|_{L^2(M)}$,

which holds for all smooth functions of compact support on M with the constant $C_s = C_s(M)$ is related to the the isoperimetric constant $\mathcal{I}(M)$ for M:

$$\mathcal{I}(M) = \inf_N \frac{A(\partial N)^n}{|N|^{n-1}}, \qquad C_s^{-1} = \mathcal{I}(M)^{\frac{2}{n}}\left(\frac{n-2}{2(n-1)}\right)^2.$$

The infimum above ranges over all open submanifolds N with compact closure and smooth boundary ∂N, and $A(\partial N)$ denotes the area of ∂N. A proof of this Sobolev inequality can be found for instance in [25].

Let $u := R(\lambda \pm i\varepsilon)f$, thus

$$(-\Delta_M + V(x) - \lambda \mp i\varepsilon)u = f.$$

Let $v = \mathrm{Re}\, u$ then

(119) $\qquad\qquad (-\Delta_M + V(x))v = F = \mathrm{Re}(f + (\lambda \pm i\varepsilon)u)$.

We normalize

(120) $\qquad\qquad \|u\|_{H^{0,-3/2+\sigma}(M)} = 1, \qquad \|f\|_{H^{0,1/2+\sigma}(M)} = \delta$

with a small constant δ. We will then show that a sufficiently small λ_0 and all $|\lambda|, \varepsilon < \lambda_0$

(121) $\quad \|u\|_{H^{0,-3/2+\sigma}(M)} \leq \frac{1}{2} + C(M, A)\delta, \qquad \|u\|_{H^{0,-1/2-\sigma}(M)} \leq C(M, A)\lambda^{-1/2}\delta$.

By Lemma 2.4 it will suffice to show that for a sufficiently large[23] compact set K

$$\|\nabla u\|_{L^2(K)} + \|u\|_{L^2(K)} \leq (2C_K)^{-1} + C(M, A)\delta.$$

with the constant C_K from (38). Let $\mu > 0$ be a small constant to be determined later and set

$$\mu_\ell = \mu + \frac{1}{2}(1 + \gamma)^{-\ell}\mu$$

---

[23]In what follows the set K will be fixed and the small constants λ_0, δ will be allowed to depend on K.

Define a sequence of increasing nested sets

$$K_\ell := B(2^\ell R_0) \cap \{x : \ v(x) \geq \mu_\ell\}$$

Let χ_ℓ be smooth cut-off functions on M adapted to $B(2^\ell R_0)$, i.e., $\chi_\ell(x) = 1$ for $x \in B(2^\ell R_0)$ and $\chi_\ell(x) = 0$ for $x \in B^c(2^{\ell+1} R_0)$.

Multiplying (119) by[24] $\chi_\ell^2(v - \mu_\ell)_+$ and integrating by parts we obtain

$$\int_M \chi_\ell^2\big(|\nabla(v - \mu_\ell)_+|^2 + V(x)u(v - \mu_\ell)_+\big)dg = -2\int_M \chi_\ell \nabla\chi_\ell \cdot \nabla(v - \mu_\ell)_+(v - \mu_\ell)_+ \ dg$$

$$+ \int_M \chi_\ell^2 F(v - \mu_\ell)_+ \ dg$$

By Cauchy-Schwarz and a simple rearrangement,

$$\int_M \chi_\ell^2\big(|\nabla(v - \mu_\ell)_+|^2 + V_+(x)v(v - \mu_\ell)_+\big)dg \leq 4\int_M |\nabla\chi_\ell|^2(v - \mu_\ell)_+^2 \ dg$$

$$+ 2\int_M \chi_\ell^2|F|(v - \mu_\ell)_+ \ dg$$

$$+ 2\mu_\ell \int_M \chi_\ell^2 V_-(x)(v - \mu_\ell)_+ \ dg + 2\int_M \chi_\ell^2 V_-(x)(v - \mu_\ell)_+^2 dg$$

Using the smallness assumption on the negative part of the potential V we have

$$\int_M \chi_\ell^2 V_-(x)(v - \mu_\ell)_+^2 dg \leq \beta\Big(\int_M (\chi_\ell(v - \mu_\ell)_+^2)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}$$

$$\leq \beta C\Big(\int_M \big(\chi_\ell^2|\nabla(v - \mu_\ell)_+|^2 + |\nabla\chi_\ell|^2(v - \mu_\ell)_+^2\big) \ dg\Big),$$

where the last line follows from the Sobolev inequality (118). Also, choosing a sufficiently small (universal) constant a,

$$\mu_\ell \int_M \chi_\ell^2 V_-(x)(v - \mu_\ell)_+ dg \leq \mu_\ell\beta|K_{\ell+1}|^{\frac{n-2}{2n}} \Big(\int_M (\chi_\ell(v - \mu_\ell)_+^2)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{2n}}$$

$$\lesssim a^{-1}C\mu_\ell^2\beta^2|K_{\ell+1}|^{\frac{n-2}{n}} + a\Big(\int_M \big(\chi_\ell^2|\nabla(v - \mu_\ell)_+|^2 + |\nabla\chi_\ell|^2(v - \mu_\ell)_+^2\big) \ dg\Big).$$

Similarly, by Cauchy-Schwarz,

$$\int_M \chi_\ell^2|F|(v - \mu_\ell)_+ dg \leq a^{-1}|K_{l+1}|^{\frac{2}{n}}\int_M \chi_\ell^2|F|^2 dg + a|K_{l+1}|^{-\frac{2}{n}}\int_M \chi_\ell^2(v - \mu_\ell)_+^2 dg$$

Thus, assuming that β, a are sufficiently small,

(122)

$$c\int_M \chi_\ell^2|\nabla(v - \mu_\ell)_+|^2 dg \leq \int_M |\nabla\chi_\ell|^2(v - \mu_\ell)_+^2 \ dg + a|K_{l+1}|^{-\frac{2}{n}}\int_M \chi_\ell^2(v - \mu_\ell)_+^2 dg$$

$$+ a^{-1}|K_{l+1}|^{\frac{2}{n}}\int_M \chi_\ell^2|F|^2 dg + a^{-1}\mu_\ell^2\beta^2|K_{\ell+1}|^{\frac{n-2}{n}}.$$

for some universal constant $c > 0$, which depends only on the Sobolev constant C_s.

---

[24]Here of course we use the notation $x_+ := \max(x, 0)$.

By Friedrichs' inequality (117), we have

$$\int_M \chi_\ell^2 (v - \mu_\ell)_+^2 \lesssim |K_{\ell+1}|^{\frac{2}{n}} \left(\int_M \chi_\ell^2 |\nabla (v - \mu_\ell)_+|^2 + \int_M |\nabla \chi_\ell|^2 (v - \mu_\ell)_+^2 \right).$$

Therefore, substituting the bound for $\int_M \chi_\ell^2 |\nabla (v - \mu_\ell)_+|^2 dg$ from (122) and using smallness of a, we obtain

$$c \int_M \chi_\ell^2 (v - \mu_\ell)_+^2 dg \leq |K_{l+1}|^{\frac{2}{n}} \int_M |\nabla \chi_\ell|^2 (v - \mu_\ell)_+^2 \, dg$$

$$+ |K_{l+1}|^{\frac{4}{n}} \int_M \chi_\ell^2 |F|^2 dg + \mu_\ell^2 \beta^2 |K_{\ell+1}|.$$

We can further simplify this by using the bound $|K_\ell| \leq C R_0^n 2^{n\ell}$.

$$c(R_0) \int_M \chi_\ell^2 (v - \mu_\ell)_+^2 dg \leq |K_{l+1}|^{\frac{2}{n}} \int_M |\nabla \chi_\ell|^2 (v - \mu_\ell)_+^2 \, dg$$

$$+ 2^{4\ell} \int_M \chi_\ell^2 |F|^2 dg + \mu_\ell^2 \beta^2 2^{n\ell}.$$

It is important however to keep the factor $|K_{l+1}|^{\frac{2}{n}}$ in front of the first term!

By the Chebyshev inequality we have

$$|K_{\ell+1}| \leq \frac{1}{(\mu_{\ell+1} - \mu_{\ell+2})^2} \int_M \chi_{\ell+2}^2 (v - \mu_{\ell+2})_+^2.$$

Observe that

$$|\nabla \chi_\ell| \leq 2^{-\ell} R_0^{-1} \chi_{\ell+2}, \qquad (\mu_{\ell+1} - \mu_{\ell+2}) = \frac{1}{2} \mu \gamma (1+\gamma)^{-\ell-2}, \qquad (v - \mu_{\ell+2})_+ \geq (v - \mu_\ell)_+.$$

We then obtain

$$c(R_0) \int_M \chi_\ell^2 (v - \mu_\ell)_+^2 dg \leq C(\mu, \gamma)(1+\gamma)^{\frac{4\ell}{n}} 2^{-2\ell} \left(\int_M \chi_{\ell+2}^2 (v - \mu_{\ell+2})_+^2 dg \right)^{1+\frac{2}{n}}$$

$$+ 2^{4\ell} \int_M \chi_\ell^2 |F|^2 dg + \mu^2 \beta^2 2^{n\ell}.$$

The constant $C(\mu, \gamma)$ is essentially $(\mu\gamma)^{-4/n}$. In particular, $C(\mu, \gamma)$ becomes large as $\mu, \gamma \to 0$.

We now define

$$A_\ell = 2^{-(3-2\sigma)\ell} \int_M \chi_\ell^2 (v - \mu_\ell)_+^2 dg$$

so that, with the normalization (120)

$$A_\ell \leq c(R_0) \|u\|_{H^{0,-3/2+\sigma}(M)}^2 = c(R_0) \leq 1.$$

Then

$$A_\ell \leq C(\mu, \gamma, R_0) \left((1+\gamma)^{\frac{4\ell}{n}} 2^{(-2 + \frac{2}{n}(3-2\sigma))\ell} A_{\ell+2}^{1+\frac{2}{n}} + 2^{(1+2\sigma)\ell} \int_M \chi_\ell^2 |F|^2 dg + \mu^2 \beta^2 2^{(n-3+2\sigma)\ell} \right).$$

Observe that for $n \geq 3$

$$-2 + \frac{2}{n}(3 - 2\sigma) \leq -\frac{2\sigma}{3n},$$

which allows us to conclude that with an appropriate choice of a sufficiently small $\gamma = \gamma(\sigma)$ (explicitly, we need $6\ln(1+\gamma) < \sigma\ln 2$) there exists a positive constant $\omega > 0$ such that

$$A_\ell \leq C(\mu,\gamma,R_0)\left(2^{-2\omega\ell}A_{\ell+2}^{1+\frac{2}{n}} + 2^{(1+2\sigma)\ell}\int_M \chi_\ell^2 |F|^2 dg + \mu^2\beta^2 2^{(n-3+2\sigma)\ell}.\right)$$

We now recall our normalizations (120). In particular,

$$\int_M \chi_\ell^2 |F|^2 dg \leq \int_M \chi_\ell^2(|f|^2 + (\lambda^2+\varepsilon^2)|u|^2)dg \leq C\delta^2 + 2^{(3-2\sigma)\ell}C(R_0)(\lambda^2+\varepsilon^2)$$

and $A_\ell \leq 1$. Thus
(123)
$$A_\ell \leq C(\mu,\gamma,R_0)\left(2^{-2\omega\ell}A_{\ell+2} + 2^{(1+2\sigma)\ell}\left(\delta^2 + 2^{(3-2\sigma)\ell}(\lambda^2+\varepsilon^2)\right) + \mu^2\beta^2 2^{(n-3+2\sigma)\ell}\right).$$

Iterating (123) we obtain that for any $k \ll L$

$$A_k \leq C(\mu,\gamma,R_0,k)^L(2^{-\omega L^2} + \delta^2 + \lambda^2 + \varepsilon^2 + \beta^2).$$

The constants γ and R_0 have been fixed, independently of λ,β,δ. Constant μ will be chosen to depend only on the compact set K. Therefore, using the smallness of λ,δ and β we can find a large integer $L_0 = L_0(\mu,\gamma,R_0)$ such that for all $k \leq k_0 \ll L_0$

$$(\varepsilon^2 + \lambda^2 + \beta^2) \leq C(\mu,\gamma,R_0,k)^{L_0}2^{-100nL_0}, \qquad C(\mu,\gamma,R_0,k)^{L_0}2^{-\omega L_0^2} \leq 2^{-100nL_0}.$$

Thus, for all $k \leq k_0$

$$A_k \leq 2^{-100nL_0} + C(L_0)\delta^2.$$

We can assume that the set K is contained in the ball $B(2^{k_0}R)$. Therefore,

$$\int_K (v - \mu_{k_0})_+^2 \leq 2^{(3-2\sigma)k_0}(2^{-100nL_0} + C(L_0)\delta^2).$$

The above inequality implies that

$$\int_K v_+^2 \leq 4\mu^2|K| + 2^{(3-2\sigma)k_0}(2^{-100nL_0} + C(L_0)\delta^2) \leq (8C_K)^{-2} + C\delta^2,$$

provided that μ is chosen to be sufficiently small relative to the size of K. Repeating the above arguments for the function $-v = -\mathrm{Re}u$ and similarly for $\mathrm{Im}u$ we conclude that

$$\int_K |u|^2 \leq (2C_K)^{-2} + C\delta^2.$$

as desired. The remaining estimate for ∇u is straightforward and follows immediately by integrating the equation $(-\Delta_M + V(x) - \lambda \mp i\varepsilon)u = f$ against $\bar{u}$ over the set K. We omit the details.

13. THE QUASIMODE COUNTEREXAMPLE

We now present the (standard) counterexample in Proposition 1.14 which shows that the losses of $\exp(C\sqrt{\lambda})$ which arise for instance in the unique continuation estimates in Section 8 are in fact sharp. For more sophisticated versions of this type of construction in the more general context of an elliptic trapped geodesic, see [23].

We begin by reviewing some basic facts about spherical harmonics. Let $(S^n, g_n) \subset \mathbf{R}^{n+1}$ be the unit sphere with the standard metric, which we parameterize in Euler polar co-ordinates as

$$S^n := \{((\sin\theta)\omega, \cos\theta) : 0 \leq \theta \leq \pi; \omega \in S^{n-1}\}$$

Observe that the Laplace-Beltrami operators on S^n and on S^{n-1} are related by the formula

$$\Delta_{S^n} = \frac{\partial^2}{\partial\theta^2} + \frac{(n-1)\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{S^{n-1}},$$

where $\Delta_{S^{n-1}}$ is of course applied to the ω variable. Thus if $l \geq 0$ is an integer, and $Y_l(\omega)$ is a spherical harmonic of order l on S^{n-1}, i.e.

$$\Delta_{S^{n-1}}Y_l(\omega) = -l(l+n-2)Y_l(\omega),$$

then the *sectorial harmonic*

$$U_l(\theta, \omega) := \sin^l(\theta)\cos(\theta)Y_l(\omega)$$

is a spherical harmonic on S^n, in fact we have

$$(-\Delta_{S^n} - \lambda)U_l = 0, \text{ where } \lambda := (l+1)(l+n).$$

We shall think of being l as being much larger than n, so $\lambda \sim l^2$. Because of the $\sin^l(\theta)$ factor, U_l will be highly concentrated near the equator $\theta = \pi/2$ (which is a stable trapped geodesic of the sphere S^n). Indeed, if we normalize Y_l to have L^2 norm equal to 1 on S^{n-1}, then a simple computation shows that

$$\int_{S^n:|\theta-\pi/2|<\pi/4} |U_l|^2 \sim l^{-1} \sim \lambda^{-1/2},$$

while

$$\int_{S^n:|\theta-\pi/2|>\pi/4} |U_l|^2 + |\nabla U_l|^2 = O(e^{-cl}) = O(e^{-c\sqrt{\lambda}})$$

for some constant $c = c_n > 0$ depending only on the dimension.

We now transfer this phenomenon to the setting in Proposition 1.14. Let $M \subset \mathbf{R}^{n+1}$ be any smooth n-dimensional manifold isometrically embedded in $\mathbf{R}^{n+1}$, which is equal to the plane $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$ outside of a compact set, and contains the portion of the sphere

$$S^n_{\geq\pi/8} := \{((\sin\theta)\omega, \cos\theta) : \theta \geq \pi/8; \omega \in S^{n-1}\} \subset S^n \subset \mathbf{R}^{n+1}.$$

In other words, M is formed by gluing a large portion of the sphere (which contains the equator $\theta = \pi/2$) to the Euclidean space $\mathbf{R}^n$. Note that M thus inherits the equator of S^n as a stable trapped geodesic. Now let l be a large integer parameter, set $\lambda = \lambda_l := (l+1)(l+n)$, and consider the "quasimode" $u : M \to \mathbf{C}$ defined by $u := U_l\chi$, where χ is a smooth cutoff supported on the set $S^n_{\geq\pi/8}$ which equals one on $S^n_{\geq\pi/4}$. Then from the previous computations we see that

$$\|u\|_{H^{0,-1/2-\sigma}(M)} \sim \lambda^{-1/2}$$

and

$$\|(-\Delta_M - \lambda)u\|_{H^{0,1/2+\sigma}(M)} = O(e^{-c\sqrt{\lambda}}),$$

and thus for $\varepsilon = \varepsilon_l$ sufficiently small

$$\|(-\Delta_M - (\lambda \pm i\varepsilon))u\|_{H^{0,1/2+\sigma}(M)} = O(e^{-c\sqrt{\lambda}}).$$

Proposition 1.14 follows (with a small value of C_0) by setting $f_l := (-\Delta_M - (\lambda \pm i\varepsilon))u$. To obtain a larger value of C_0, we simply scale this example, replacing the unit sphere S^n by a sphere of larger radius; we omit the standard details. ∎

14. The Bessel matching counterexample

We now present the proof of Proposition 1.21. We begin by considering the Helmholtz equation $(H - \lambda)u = f$ in polar co-ordinates (r, ω) on $\mathbf{R}^n$, thus

$$(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}} - V + \lambda)u = -F,$$

where we ignore the singularity at $r = 0$ for now. If we now assume V to be radial, $V = V(r)$, and apply the ansatz

(124) $u(r, \omega) = r^{-(n-1)/2}v(r)Y_l(\omega), \quad F(r, \omega) = -r^{-(n+1)/2}G(r)Y_l(\omega)$

where $l \geq 0$ is a large even integer parameter and Y_l is a spherical harmonic of order l on S^{n-1}, normalized to have $L^2(S^{n-1})$ norm equal to 1, then the Helmholtz equation becomes the *Bessel ordinary differential equation*

(125) $$v_{rr} - \frac{L(L-1)}{r^2}v - Vv + \lambda v = G,$$

where $L := l + \frac{n-1}{2}$. Also, we observe that

$$\|u\|_{H^{0,-1/2-\sigma}(M)} \sim \int_0^\infty r^{-2\sigma}|v(r)|^2 \, dr$$

and

$$\|F\|_{H^{0,1/2+\sigma}(M)} \sim \int_0^\infty r^{2\sigma}|G(r)|^2 \, dr.$$

Let us temporarily ignore the contributions of the λ, V, and G factors, and consider the ODE

(126) $$v_{rr} - \frac{L(L-1)}{r^2}v = 0.$$

This equation has two linearly independent solutions, r^L and r^{-L+1}. The former function decays quickly as $r \to 0$, and the latter decays quickly as $r \to \infty$. To exploit this, let us define v by *fiat* to be a smooth function on $(0, \infty)$ which equals r^L when $r < 1/2$, equals r^{-L+1} when $r > 1$, and is smooth and positive in between (of course, the smoothness bounds on v will depend on L). Then the function $v_{rr} - \frac{L(L-1)}{r^2}v$ is smooth and supported on the annulus $\{1/2 \leq r \leq 1\}$. Note also that u is then equal to $r^l Y_l(\omega)$ near the origin, which is a smooth function (indeed, it is a harmonic polynomial of degree l).

Now let us define the spherically symmetric potential $V = V_l$ by

$$V := \frac{v_{rr} - \frac{L(L-1)}{r^2}v}{v}.$$

By construction, V is smooth and supported on the annulus $\{1/2 \leq r \leq 1\}$ (though the bounds on this potential will get worse as l increases), and we have

$$v_{rr} - \frac{L(L-1)}{r^2}v - Vv = 0.$$

Reversing the above steps, this means that the corresponding function u is an eigenfunction of the Schrödinger operator $H := -\Delta + V$, with eigenvalue zero, thus $\Delta u = Vu$. Also, u decays at infinity like $|u(r,\omega)| = O(r^{-(n-1)/2-L+1}) = O(r^{-l-n})$, and in particular will be bounded in the space $H^{0,1/2+\sigma}(M)$ for l large enough.

Next, we let $m \geq 1$ be an arbitrary integer and set $u_m := u\chi_m$, where χ_m is a smooth cutoff to the region $\{r \leq 2m\}$ which equals one when $\{r \leq m\}$. We also set $\lambda = \lambda_m := m^{-l/10}$, and set $V_m := V + \lambda\chi_m$. Then we see that

$$(-\Delta - V_m - \lambda)u_m$$

vanishes outside of the annulus $m \leq r \leq 2m$, and has magnitude $O_l(m^{-l+O(1)})$ on this annulus, thus

$$\|(-\Delta + V_m - \lambda_m)u_m\|_{H^{0,1/2+\sigma}(M)} \leq O_l(m^{-l+O(1)}).$$

In particular for ε_m sufficiently small

$$\|(-\Delta + V_m - (\lambda_m \pm i\varepsilon_m))u_m\|_{H^{0,1/2+\sigma}(M)} \leq O_l(m^{-l+O(1)}).$$

Also, by construction of u_m and V_m we see that (if l is sufficiently large)

$$\|u_m\|_{H^{0,-1/2-\sigma}(M)} \geq c_l > 0.$$

and

$$\sup_m \|\langle x \rangle^{1+\sigma_0} \nabla_x^\alpha V_m\|_{L^\infty(\mathbf{R}^n)} < \infty \text{ for all } \alpha \geq 0,$$

and the claim follows (by taking l large enough depending on C_0). To complete the construction we need to show that the hamiltonian $H_m = -\Delta + V_m$ does not contain a small negative eigenvalue or an eigenfunction or resonance at zero. (Of course, the potentials V_m are very close to a fixed potential V which *does* have an eigenfunction at zero, which is indeed the cause of the bad behavior of low frequency limiting absorption for the perturbed potentials.) By the Cwikel-Rozenblum bound, the Hamiltonian $H = -\Delta + V$ has only finitely many negative eigenvalues labeled $\lambda_1 \leq \ldots \leq \lambda_k < c < 0$. Their number and location is independent of m. The operator norm of the perturbation $H_m - H$ is bounded by $m^{-l/10}$, which implies, by perturbation theory, that for sufficiently large values of m the hamiltonian H_m has at least k negative eigenvalues $\lambda'_1 \leq \ldots \leq \lambda'_k < c + m^{-l/10} < 0$. Moreover, denoting the linear span the associated eigenfunctions of H_m by P_m, we have

$$\int_{\mathbf{R}^n} H\phi\,\bar\phi \geq 0$$

for any $\phi \in H^2(\mathbf{R}^n)$ in the orthogonal complement of P_m. Recall that the potential V is supported in the annulus $1/2 \leq r \leq 1$. We can also assume without loss of generality that $|V(x)| \leq 2$. As a consequence, for any $\phi \in H^2(\mathbf{R}^n)$ in the orthogonal

complement of P_m,

$$\int_{\mathbf{R}^n} H_m \phi \, \bar{\phi} = \frac{\lambda}{4} \int_{\mathbf{R}^n} H_m \phi \, \bar{\phi} + (1 - \frac{\lambda}{4}) \int_{\mathbf{R}^n} (H + \lambda \chi_m) \phi \, \bar{\phi}$$

$$\geq \frac{\lambda}{4} \int_{\mathbf{R}^n} \left(|\nabla \phi|^2 + V |\phi|^2 + \lambda \chi_m |\phi|^2 \right) + (1 - \frac{\lambda}{4}) \int_{\mathbf{R}^n} \lambda \chi_m |\phi|^2$$

$$(127) \qquad \geq \int_{\mathbf{R}^n} \left(\frac{\lambda}{4} |\nabla \phi|^2 + \frac{\lambda}{2} \chi_m |\phi|^2 \right) \geq C^{-1} \lambda \int_{\mathbf{R}^n} \frac{|\phi(x)|^2}{|x|^2}.$$

This estimate already implies that H_m has precisely k negative eigenvalues $\lambda_1' \leq \ldots \leq \lambda_k' < c + m^{-l/10} < 0$ and that 0 is not an eigenvalue. On the other hand if $\phi \in \cap_{\alpha>0} H^{2,-\frac{1}{2}-\alpha}(\mathbf{R}^n)$ is a zero resonance it is formally (the corresponding eigenfunctions are exponentially decaying) orthogonal to the subspace P_m. Moreover, $\nabla \phi \in \cap_{\alpha>0} H^{1,\frac{1}{2}-\alpha}(\mathbf{R}^n)$, $\Delta \phi \in \cap_{\alpha>0} H^{0,\frac{3}{2}-\alpha}(\mathbf{R}^n)$, which implies that (127) holds and thus ϕ vanishes identically. ∎

Remark 14.1. It is possible to sharpen this example a little bit by making the potentials V_m compactly supported in the annulus $1/2 \leq r \leq 1$. This is achieved by replacing the approximating ODE (126) by the minor variant $v_{rr} - \frac{L(L-1)}{r^2} v + (\lambda_m \pm i\varepsilon_m) v = 0$, and replacing r^L and r^{-L-1} by Hankel functions (which have similar decay behavior at zero and at $r \sim m$ respectively). We omit the details.

15. Applications

15.1. Spectral applications.
We now prove Proposition 1.30 and Proposition 1.31.

Proof. (Proof of Proposition 1.30) Suppose first that we have an outgoing resonance u for some $\lambda = \kappa^2$ for some $\kappa > 0$, thus $u \in H^{0,-1/2-\sigma}(M)$, $Hu = \lambda u$, and $(\partial_r u - i\kappa u) \in H^{0,-1/2+\sigma'}(M \backslash K_0)$ for some $\sigma' > 0$ and all $\sigma > 0$. By shrinking σ, σ' if necessary we can take $0 < \sigma < \sigma' < 1/2$. By elliptic regularity, $u \in H^{2,-1/2-\sigma}(M)$.

Write $\kappa := \sqrt{\lambda}$, let $0 < \delta \ll \min(1, \kappa)$, and let $z := \kappa + i\delta$. Consider the function

$$u_\delta(x) := e^{-\delta \langle x \rangle} u$$

where $\langle r \rangle := (1 + r^2)^{1/2}$.

Since $Hu = \lambda u$, one has

$$(H - z^2) u = -2i\delta \kappa u + \delta^2 u,$$

while from (65) one has

$$\Delta_M e^{-\delta \langle x \rangle} = -\delta^2 e^{-\delta \langle x \rangle} + O(\frac{1}{\langle x \rangle} \delta e^{-\delta \langle x \rangle})$$

in the exterior of K_0. A straightforward application of the product rule then gives the equation

$$(H - z^2) u_\delta = \delta e^{-\delta \langle x \rangle} f_\delta$$

where f_δ takes the form

$$f_\delta = 2(\partial_r u - i\kappa u) + O\left(\frac{1}{\langle x \rangle}|u|\right) + O\left(\frac{1}{\langle x \rangle}|\partial_r u|\right)$$

in the exterior region $M \backslash K_0$, and takes the form

$$f_\delta = O(u) + O(|\nabla u|)$$

in the compact region K_0; here all implied constants are allowed to depend on M and κ. Applying the limiting absorption principle (Theorem 1.7) we conclude that

$$\|u_\delta\|_{H^{0,-1/2-\sigma}(M)} \leq C(\lambda)\delta(\|e^{-\delta\langle x\rangle}(\partial_r u - i\kappa u)\|_{H^{0,1/2+\sigma}(M\backslash K_0)}$$
$$+ \|e^{-\delta\langle x\rangle}u\|_{H^{1,-1/2+\sigma}(M)}).$$

Now observe that

$$\|e^{-\delta\langle x\rangle}u\|_{H^{1,-1/2+\sigma}(M)} \leq C\delta^{-2\sigma}\|u\|_{H^{1,-1/2-\sigma}(M)}$$

and

$$\|e^{-\delta\langle x\rangle}(\partial_r u - i\kappa u)\|_{H^{0,1/2+\sigma}(M\backslash K_0)} \leq C\delta^{\sigma'-\sigma-1}\|\partial_r u - i\kappa u\|_{H^{0,-1/2+\sigma'}(M\backslash K_0)}.$$

We thus have

$$\|u_\delta\|_{H^{0,-1/2-\sigma}(\mathbf{R}^n)} \leq C(\lambda)(\delta^{1-2\sigma}\|u\|_{H^{1,-1/2-\sigma}(M)} + \delta^{\sigma'-\sigma}\|\partial_r u - i\kappa u\|_{H^{0,-1/2+\sigma'}(M\backslash K_0)}).$$

Taking limits as $\delta \to 0$ and using monotone convergence we conclude that u is identically zero, which is absurd.

The same argument rules out incoming resonances and eigenfunctions at any positive energy $\lambda > 0$. $\qquad\square$

Remark 15.2. If the Hamiltonian H obeyed the limiting absorption principle (17) (note in particular that the constant here does not blow up as $\lambda \to 0$), a modification of above argument would also rule out an eigenvalue or resonance at zero; we omit the details[25]. This provides a converse to Proposition 1.23.

Proof of Proposition 1.31. The absence of singular continuous spectrum in an interval (a, b) is guaranteed by the condition that

$$\sup_{0 < \epsilon < 1} \int_a^b |\text{Im}\langle R(\lambda + i\epsilon)\phi, \phi\rangle_{L^2(M)}|^2 \, d\lambda < \infty$$

for any function $\phi \in C_0^\infty(M)$, see [80, Theorem XIII.20]. (Indeed, the spectral measure associated to ϕ will have an L^2 density with respect to Lebesgue measure.) The result now easily follows from the resolvent estimates established in Proposition 1.7. $\qquad\square$

---

[25]In fact, by taking z to be a number such as $2\delta + i\delta$, it suffices to have the limiting absorption principle (17) in a sector such as $\{z = \kappa + i\varepsilon : 0 < \varepsilon < \kappa\}$.

15.3. **Local smoothing estimates and integrated local energy decay.** We now prove Propositions 1.34 and 1.38. As the arguments are standard (dating back to [63]; see also [16], [27], [28]), we shall be somewhat brief in our discussion. In particular, we will work formally (assuming that all exchanges of integrals, etc. are justified); one can make these formal computations rigorous by standard approximation arguments which we omit here.

We begin with the proof of (22) from Proposition 1.34. Henceforth we fix M, V, A and allow all constants to depend on these quantities.

From Duhamel's formula we have

$$H^{1/2}P_H u(t) = H^{1/2}P_H e^{itH} u(0) - i \int_0^t H^{1/2}P_H e^{i(t-t')H} \nabla^* F(t') \, dt'.$$

If we can establish the retarded estimate

$$(128) \quad \| \int_{t'<t} H^{1/2}P_H e^{i(t-t')H} F(t')\|_{L_t^2 H^{0,-1/2-\sigma}(\mathbf{R}\times M)} \le C\|F\|_{L_t^2 H^{0,1/2+\sigma}(\mathbf{R}\times M)}$$

then by time reversal we also have the corresponding advanced estimate in which the constraint $t' < t$ in (128) is replaced by $t' > t$. Summing, we obtain

$$\| \int_{\mathbf{R}} H^{1/2}P_H e^{i(t-t')H} F(t')\|_{L_t^2 H^{0,-1/2-\sigma}(\mathbf{R}\times M)} \le C\|F\|_{L_t^2 H^{0,1/2+\sigma}(\mathbf{R}\times M)};$$

taking the inner product of the expression inside the left-hand norm with F and rearranging, we obtain the inhomogeneous estimate

$$\| \int_{\mathbf{R}} H^{1/4}P_H e^{-it'H} F(t') \, dt'\|_{L^2(M)} \le C\|F\|_{L_t^2 H^{0,1/2+\sigma}(\mathbf{R}\times M)}.$$

which by duality gives

$$\|\nabla H^{1/2}P_H e^{itH} u_0\|_{L_t^2 H^{0,-1/2-\sigma}(\mathbf{R}\times M)} \le C\|H^{1/4}P_H u_0\|_{L^2(M)}$$

for any test function u_0. From all these estimates we see that to prove (22) it suffices to establish the retarded estimate (128) and the standard elliptic estimate

$$\|H^{1/4}P_H u_0\|_{L^2(M)} \le C\|u_0\|_{H^{1/2}(M)}.$$

To prove the latter estimate, we observe from the TT^* method that it is equivalent to $H^{1/2}P_H$ mapping $H^{1/2}(M)$ to $H^{-1/2}(M)$, which by interpolation and duality follows from $H^{1/2}P_H$ mapping $H^1(M)$ to $L^2(M)$, which by another application of the TT^* method (and the boundedness of P_H on $L^2(M)$) follows from the Dirichlet form $\langle Hu, v \rangle$ being bounded in $H^1(M)$.

Now we turn to (128). By a limiting argument, it suffices to prove the damped retarded estimate
(129)
$$\liminf_{\varepsilon \to 0^+} \| \int_{t'<t} H^{1/2}P_H e^{i(t-t')(H+i\varepsilon)} F(t')\|_{L_t^2 H^{0,-1/2-\sigma}(\mathbf{R}\times M)} \le C\|F\|_{L_t^2 H^{0,1/2+\sigma}(\mathbf{R}\times M)}.$$

Following Kato [63], we perform a Fourier transform in the time variable

$$F(t) = \int_{\mathbf{R}} e^{i\lambda t} \hat{F}(\lambda) \, d\lambda.$$

The expression inside the left-hand norm of (129) can then be expressed as[26]

$$\int_{\mathbf{R}} e^{i\lambda t} H^{1/2} P_H R(\lambda - i\varepsilon)\hat{F}(\lambda)\,d\lambda.$$

Applying Plancherel's theorem (and Fatou's lemma) to both sides, we see that it suffices to establish the estimates

$$\limsup_{\varepsilon \to 0^+} \|H^{1/2} P_H R(\lambda - i\varepsilon)f\|_{H^{0,-1/2-\sigma}(M)} \leq C\|f\|_{H^{0,1/2+\sigma}(M)}$$

for all $\lambda \in \mathbf{R}$ and all test functions f. If λ is negative and bounded away from the origin, then $H^{1/2} P_H R(\lambda - i\varepsilon)$ is bounded on $L^2(M)$ (thanks to the spectral theorem), so we may assume that λ is either positive, or close to the origin.

Using the elliptic estimate

$$\|H^{1/2}u\|_{H^{0,-1/2-\sigma}(M)} \leq C\|u\|_{H^{1,-1/2-\sigma}(M)}$$

it suffices to show that

$$\|P_H R(\lambda - i\varepsilon)f\|_{H^{1,-1/2-\sigma}(M)} \leq C\|f\|_{H^{0,1/2+\sigma}(M)}$$

whenever $\lambda \in \mathbf{R}$ and ε is sufficiently small.

For λ sufficiently close to the origin, this follows from Proposition 1.23. For λ positive and bounded away from the origin, the claim follows instead from Theorem 1.15. This completes the proof of Proposition 1.34.

Remark 15.4. It is clear from the above argument that the global-in-time local smoothing estimates are in fact *equivalent* to the limiting absorption principle (with the appropriate decay in λ in the constants).

Proof of Proposition 1.38. As before, we allow all constants to depend on C, M, A.

We rewrite the wave equation in the form

$$\Phi_t - \mathcal{A}\Phi = \tilde{F},$$

where $\Phi := \begin{pmatrix} u \\ u_t \end{pmatrix}$, $\tilde{F} := \begin{pmatrix} 0 \\ F \end{pmatrix}$, and $\mathcal{A}$ is the matrix operator

$$\mathcal{A} := \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}$$

We begin with the proof of (24). As in the proof of Proposition 1.34, it suffices to consider the solution of the retarded inhomogeneous problem

$$\Phi(t) = \int_{s<t} e^{(t-s)A}\tilde{F}(s)\,ds.$$

Applying Plancherel as before, we see that bounds on the retarded solution are equivalent to bounds on the resolvent $(i\mathcal{A} - \mu - i0^-)^{-1}$. More precisely, if $\Psi = (\psi_1, \psi_2)$ is a solution of the equation

$$(i\mathcal{A} - \mu + i\varepsilon)\Psi = (0, g)$$

---

[26]One can also shift the contour, taking advantage of the projection away from any negative eigenvalues to work with resolvents in a neighbourhood of the positive real axis, if desired.

then we need to show that uniformly in $\varepsilon > 0$ and $\mu \in \mathbf{R}$ that

$$\|\nabla\psi_1\|_{H^{0,-1/2-\sigma}(\mathbf{R}^n)} + \|\psi_2\|_{H^{0,-1/2-\sigma}(\mathbf{R}^n)} \leq C\|g\|_{H^{0,1/2+\sigma}(\mathbf{R}^n)}.$$

Since

$$(130) \qquad (i\mathcal{A} - \mu + i\varepsilon)^{-1} = R((\mu - i\varepsilon)^2) \begin{pmatrix} \mu - i\varepsilon & i \\ -iH & \mu - i\varepsilon \end{pmatrix}$$

we have

$$\psi_1 = iR((\mu - i\varepsilon)^2)g; \quad \psi_2 = (\mu - i\varepsilon)R((\mu - i\varepsilon)^2)g$$

and the desired estimate (24) follows immediately from Theorem 1.7.

The proof of (25) follows from a similar argument using Lemma 2.4; we omit the details. $\qquad\square$

15.5. **The RAGE theorem.** We now prove Proposition 1.32. We establish the claim just for the Schrödinger equation; the claim for the wave equation is analogous and is left to the reader.

By L^2 stability of the Schrödinger equation it suffices to establish (20) on a dense subset of $L^2(M)$. Thus we may assume that $u(0) \in H^2(M)$. Moreover, Proposition 1.30 together with the assumption that $u(0) = f$ is orthogonal to all eigenfunction of H implies continuity of the measure $d\mu_f(\lambda) = (dE_\lambda(H)f, f)$ defined from the spectral measure $dE_\lambda(H)$ associated to H. As a consequence, by a density argument, we can assume that $u(0)$ is spectrally supported on an interval $\lambda \in (a, b)$ with $0 < a < b < \infty$ of energies λ. For such functions we have both the limiting absorption principle and the local smoothing estimate

$$\int_0^\infty \|u(t)\|^2_{H^{2,-1/2-\sigma}(M)}\, dt < \infty$$

for some $\sigma > 0$.

Let $\varepsilon > 0$. Applying the monotone convergence theorem, we can find $T_0 > 0$ such that

$$\int_{T_0}^\infty \|u(t)\|^2_{H^{2,-1/2-\sigma}(M)}\, dt < \varepsilon.$$

Now for any $T > T_0 + 1$ and $g \in H^{0,1/2+\sigma}(M)$ we have

$$\int_{\mathbf{R}^n} \overline{u(T,x)}g(x) = \int_{T-1}^T \int_M \frac{d}{dt}\left(\overline{u(t,x)}(t - T + 1)g(x)\right)\, dxdt$$

$$= \int_{T-1}^T \int_{\mathbf{R}^n} \left(iH\overline{u(t,x)}(t - T + 1)g(x) + \overline{u(t,x)}g(x)\right)\, dxdt$$

$$\leq C\|g\|_{H^{0,1/2+\sigma}(M)} \left(\int_{T-1}^T \|u(t)\|^2_{H^{2,-1/2-\sigma}(M)}\, dt\right)^{1/2}$$

$$\leq C\varepsilon\|g_0\|_{H^{0,1/2+\sigma}(M)}.$$

This implies that $u(t) \to 0$ in $H^{0,-1/2-\sigma}(M)$ for some $\sigma > 1/2$, and the result follows.

15.6. **The limiting amplitude principle.** We now prove Proposition 1.33. We will follow an approach of Eidus [41] and deduce the limiting amplitude principle from the limiting absorption principle, combined with a Hölder continuity property of the resolvent. More precisely, we will use the following fact:

Proposition 15.7 (Hölder continuity). *Let the assumptions be as in Theorem 1.7. Let K be a compact subset of M, and let F be a compact subset of the right half-plane $\{z : \mathrm{Re}z > 0\}$ that avoids 0. Then there exists $C > 0$ and $\sigma > 0$ such that one has the Hölder continuity bound*

$$\|R(z)f - R(z')f\|_{H^1(K)} \leq C|z - z'|^\sigma \|f\|_{L^2(K)}$$

for all $z, z' \in F$ with $\mathrm{Im}z, \mathrm{Im}z' > 0$, and all $f \in L^2(K)$.

Let us assume this proposition for the moment and prove Proposition 1.33. For simplicity of notation we will assume that H has no eigenfunctions; the general case is analogous and proceeds by inserting a projection P_H to the absolutely continuous portion of the spectrum of H (which commutes with all spectral multipliers of H, or of related operators such as the matrix operator $\mathcal{A}$ introduced below) throughout the argument. By Proposition 1.32 and linearity, it suffices to handle the homogeneous case $u_0 = u_1 = 0$. As in the proof of Proposition 1.38, we introduce the vector $\Phi = \begin{pmatrix} u \\ u_t \end{pmatrix}$ and rewrite the wave equation in the form

$$\Phi_t - \mathcal{A}\Phi = e^{i\mu t}F,$$

where $F := \begin{pmatrix} 0 \\ f \end{pmatrix}$ and $\mathcal{A}$ is the matrix operator

$$\mathcal{A} := \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}.$$

It will suffice to show that

$$e^{-i\mu t}\Phi(t) \to \begin{pmatrix} v \\ i\mu v \end{pmatrix}$$

in $H_1(K) \times L^2(K)$ as $t \to +\infty$.

By the Duhamel formula, we have

$$e^{-i\mu t}\Phi(t) = \int_0^t e^{(t-s)(\mathcal{A}-i\mu)}dsF$$

$$= \lim_{\varepsilon \to 0^+} \int_0^t e^{(t-s)(\mathcal{A}-i\mu-\varepsilon)}dsF$$

$$= e^{t(\mathcal{A}-i\mu)}(\mathcal{A} - i\mu - 0^+)^{-1}F - (\mathcal{A} - i\mu - 0^+)^{-1}F,$$

writing $(\mathcal{A} - i\mu - 0^+)^{-1}$ as the weak limit of the $(\mathcal{A} - i\mu - \varepsilon)^{-1}$ as $\varepsilon \to 0^+$. From (130) we have

$$-(\mathcal{A} - i\mu - 0^-)^{-1}F = R((\mu - i0^+)^2)\begin{pmatrix} u \\ i\mu u \end{pmatrix}$$

and so from Proposition 1.12 (and elliptic regularity) one has

$$(\mathcal{A} - i\mu - 0^+)^{-1}F \to \begin{pmatrix} v \\ i\mu v \end{pmatrix}$$

in $H^1(K) \times L^2(K)$. Discarding the bounded phase $e^{-it\mu}$, it thus remains to establish that

$$e^{t\mathcal{A}}(\mathcal{A} - i\mu - 0^+)^{-1}F$$

converges strongly in $H^1(K) \times L^2(K)$ to zero as $t \to +\infty$.

Let C be a semicircular contour of the form

$$\{iy : |y - \mu| \leq r\} \cup \{\mu + re^{i\theta} : \pi/2 \leq \theta \leq 3\pi/2\}$$

for some radius $0 < r < \mu$ (e.g. one can select $r := \mu/2$). From the Cauchy integral formula and the spectral theorem (using the fact that the spectrum of $\mathcal{A}$ is the imaginary axis) we have

$$e^{t\mathcal{A}}(\mathcal{A} - i\mu - 0^+)^{-1}F = \lim_{\varepsilon \to 0^+} \lim_{\varepsilon' \to 0^+} \frac{1}{2\pi i} \int_C \frac{e^{tz}}{z - i\mu - \varepsilon}(\mathcal{A} - z - \varepsilon')^{-1}f \; dz$$
$$+ \; e^{t\mathcal{A}}1_{|\mathcal{A}-i\mu|\geq r}(\mathcal{A} - i\mu - 0^+)^{-1}F$$

strongly in $H^1(K) \times L^2(K)$. From the spectral theorem, $1_{|\mathcal{A}-i\mu|\geq r}(\mathcal{A}-i\mu-0^+)^{-1}F$ lies in $H^1(M) \times L^2(M)$, and so by the RAGE theorem, the second term on the right-hand side goes to zero strongly in $H^1(K) \times L^2(K)$ as $t \to \pm\infty$. It thus suffices to show that

$$\lim_{t \to +\infty} \lim_{\varepsilon \to 0^+} \lim_{\varepsilon' \to 0^+} \int_C \frac{e^{tz}}{z - i\mu - \varepsilon}(\mathcal{A} - z - \varepsilon')^{-1}f \; dz = 0$$

strongly in $H^1(K) \times L^2(K)$.

From the Cauchy integral formula we have

$$\int_C \frac{e^{tz}}{z - i\mu + \varepsilon} \; dz = 0$$

for ε small enough, and so it suffices to show that

(131) $$\lim_{t \to +\infty} \lim_{\varepsilon \to 0^+} \lim_{\varepsilon' \to 0^+} \int_C \frac{e^{tz}}{z - i\mu - \varepsilon}G_{\varepsilon'}(z) \; dz = 0$$

strongly in $H^1(K) \times L^2(K)$, where

$$G_{\varepsilon'}(z) := (\mathcal{A} - z - \varepsilon')^{-1}F - (\mathcal{A} - i\mu - \varepsilon')^{-1}F.$$

From (130) and the Hölder continuity property in Proposition 15.7, $G_{\varepsilon'}$ is Hölder continuous in $H^1(K) \times L^2(K)$, uniformly in ε'; in particular, as G'_ε vanishes at $i\mu$, we have the bound

(132) $$\|G_{\varepsilon'}(z)\|_{H^1(K) \times L^2(K)} = O(|z - i\mu|^\delta)$$

for some $\delta > 0$. This is already enough, when combined with Minkowski's integral inequality and the dominated convergence theorem, to control the semicircular portion of the contour C; it remains to demonstrate that

$$\lim_{t \to +\infty} \lim_{\varepsilon \to 0^+} \lim_{\varepsilon' \to 0^+} \int_{-r}^{r} \frac{e^{ity}}{iy - \varepsilon}G_{\varepsilon'}(i\mu + iy) \; dy = 0$$

strongly in $H^1(K) \times L^2(K)$. From (132) and Minkowski's integral inequality we have

$$\left\| \int_{-s}^{s} \frac{e^{ity}}{iy - \varepsilon}G_{\varepsilon'}(i\mu + iy) \; dy \right\|_{H^1(K) \times L^2(K)} = O(s^\delta)$$

for any $0 < s < r$, so it suffices to show that

$$\lim_{t \to +\infty} \lim_{\varepsilon \to 0^+} \lim_{\varepsilon' \to 0^+} \int_I \frac{e^{ity}}{iy - \varepsilon} G_{\varepsilon'}(i\mu + iy) \ dy = 0$$

for any compact interval I in $[-r, r]$ avoiding the origin. However, from (132) we see that for any unit vector w in $H^1(K) \times L^2(K)$, the scalar function

$$y \mapsto \left\langle \frac{1}{iy - \varepsilon} G_{\varepsilon'}(i\mu + iy), w \right\rangle_{H^1(K) \times L^2(K)}$$

is uniformly Hölder continuous in $\varepsilon, \varepsilon'$ on I, and in particular (by the Arzela-Ascoli theorem) is precompact in the uniform norm. From the Riemann-Lebesgue lemma we thus have

$$\lim_{t \to +\infty} \lim_{\varepsilon \to 0^+} \lim_{\varepsilon' \to 0^+} \left\langle \int_I \frac{e^{ity}}{iy - \varepsilon} G_{\varepsilon'}(i\mu + iy) \ dy, w \right\rangle_{H^1(K) \times L^2(K)} = 0$$

uniformly in w, and the claim follows by duality.

It remains to prove Proposition 15.7. We first observe from the limiting absorption principle that

$$\|R(z)f\|_{H^2(K)} \leq C\|f\|_{L^2(K)}$$

uniformly for all z in a compact set avoiding the origin. Thus, by interpolation, it suffices to obtain an $L^2(K) \to L^2(K)$ Hölder continuity bound, thus we need to show that

$$\|R(z)f - R(z')f\|_{L^2(K)} \leq C|z - z'|^\sigma \|f\|_{L^2(K)}$$

for all z, z' in F with positive imaginary part, and some sufficiently small $\sigma > 0$ (which may be different from the one in Proposition 15.7). Clearly, we may assume that $|z - z'|$ is smaller than any given absolute constant.

Let $\sigma > 0$ be a small exponent to be chosen later. By the triangle inequality, it suffices to prove this claim under the additional assumption that $\text{Im}(z') \geq 0.1|z - z'|$ (say). Write $\eta := \text{Im}(z')$, then $\eta > 0$ is small and $|z - z'| = O(\eta)$, with $|z|, |z'|$ comparable to 1, and with $\text{Re}(z), \text{Re}(z')$ positive and bounded away from the origin, and our task is to now show that

$$\|R(z)f - R(z')f\|_{L^2(K)} \leq C\eta^\sigma \|f\|_{L^2(K)}.$$

Using the resolvent identity

$$R(z) - R(z') = (z - z')R(z)R(z')$$

it thus suffices to show that

$$\|R(z)R(z')f\|_{L^2(K)} \leq C\eta^{-1+\sigma}\|f\|_{L^2(K)},$$

so by duality it suffices to show that

$$|\langle R(z')f, R(\overline{z})h \rangle_{L^2(M)}| \leq C\eta^{-1+\sigma}$$

whenever $f, h \in L^2(K)$ have unit norm.

Fix f, h. Recall that z' has imaginary part η. This effectively localises $R(z')f$ to the region $\{x : \langle x \rangle = O(1/\eta)\}$. Indeed, if we write $z' := -\omega^2$, where $\text{Im}\,\omega > 0$, then

$|\omega|$ is comparable to 1 and $\mathrm{Im}\,\omega$ is comparable to η (with constants depending on the compact region F). Using the elementary identity

$$\frac{1}{x^2 + \omega^2} = \frac{1}{\omega} \int_0^\infty e^{-\omega t} \cos tx \; dt$$

we see that

$$R(z')f = \frac{1}{\omega} \int_0^\infty e^{-\omega t} \cos(t\sqrt{-H})f \; dt.$$

The wave propagators $\cos(t\sqrt{-H})$ are contractions on $L^2(M)$, and $\cos(t\sqrt{-H})f$ is supported in the region where $\langle x \rangle \leq t + O(1)$. From the exponential decay in the $e^{-\omega t}$ factor, we thus see that

$$(133) \qquad \int_{|z| \geq \eta^{-1+\sigma}} \langle x \rangle^{100} |R(z')f(x)|^2 \; dg \leq C\eta^{100}$$

(say), for any fixed $\sigma > 0$ (allowing C to depend on σ). Thus, it will suffice to show that

$$(134) \qquad |\int_M \psi(R(z')f)\overline{R(\overline{z})h} \; dg| \leq C\eta^{-1+\sigma}$$

for a smooth cutoff ψ to the region $\langle x \rangle \leq 2\eta^{1-\sigma}$ that equals 1 when $\langle x \rangle \leq \eta^{1-\sigma}$, as the error term caused by $1 - \psi$ can be estimated by (133) and the limiting absorption principle (with plenty of powers of η to spare).

Write $u := R(z')f$ and $v := R(\overline{z}h)$. From the limiting absorption principle we have

$$(135) \qquad \|u\|_{H^{0,-1/2-\sigma}(M)}, \|v\|_{H^{0,-1/2-\sigma}(M)} \leq C.$$

If we apply these bounds and Cauchy-Schwarz to estimate (133) directly, we obtain a bound of $O(\eta^{-1-2\sigma})$, which barely fails to be adequate for our purposes. To obtain the additional powers of η needed to close the argument, we take advantage of the fact that z' and $\overline{z}$ lie on different sides of the real axis, and so $R(z')f$ and $R(\overline{z})h$ obey opposing radiation conditions. Indeed, if we write $u := R(z')f$ and $v := R(\overline{z}h)$, then from Proposition 1.12 we have

$$(136) \qquad \|(\partial_r - iz')u\|_{H^{0,-1/2+3\sigma}(M \setminus K_0)} \leq C$$

and

$$(137) \qquad \|(\partial_r + i\overline{z})v\|_{H^{0,-1/2+3\sigma}(M \setminus K_0)} \leq C$$

(say), if σ is small enough. To use these facts, denote the integral in (134) by I. From (135) and Cauchy-Schwarz, we can write

$$I = \int_M (\psi - \psi_0)u\overline{v} \; dg + O(1),$$

where ψ_0 is a cutoff to a fixed compact region that equals 1 on K_0. Applying (136) (and using (135) to estimate the error), we can then write

$$I = \frac{1}{iz'} \int_M (\psi - \psi_0)(\partial_r u)\overline{v} \; dg + O(\eta^{-1+\sigma}).$$

We integrate by parts to then obtain

$$I = -\frac{1}{iz'} \int_M (\psi - \psi_0) u \overline{(\partial_r v)} \, dg + O(\eta^{-1+\sigma})$$
$$+ O\left(\int_M \frac{1}{\langle x \rangle} \psi |u||v| \, dg\right).$$

Using (135) and Cauchy-Schwarz, the final error term is $O(\eta^{-1+\sigma})$ for σ small enough. For the main term, we use (137), estimating the error again using (135) and Cauchy-Schwarz, to obtain

$$I = -\frac{1}{iz'} \int_M (\psi - \psi_0) u \overline{(-i\overline{z}v)} \, dg + O(\eta^{-1+\sigma})$$

which simplifies (using (135) and Cauchy-Schwarz one last time to remove the ψ_0 cutoff) to

$$I = -\frac{\overline{z}}{z'} I + O(\eta^{-1+\sigma}).$$

As z, z' are both in the upper half-plane, have magnitude comparable to 1, and are within η of each other, we see that $-\frac{\overline{z}}{z'}$ is a bounded distance away from 1 for η small enough. We thus conclude that $I = O(\eta^{-1+\sigma})$ as required. This proves Proposition 15.7 and thus Proposition 1.33.

Remark 15.8. It should be clear from the above argument that one also has a similar limiting absorption principle for the Schrödinger evolution (replacing $H^1(K) \times L^2(K)$ by $L^2(K)$). We leave the details to the interested reader. It should also be clear from the argument that one can strengthen the convergence in $H^1(K) \times L^2(K)$ to convergence in $H^{1,-s}(M) \times H^{0,-s}(M)$ for some sufficiently large s, and dually one can also relax the hypothesis $f \in L^2(K)$ to $f \in H^{0,s}(M)$.

Remark 15.9. The hypothesis in Proposition 15.7 that F lie in the right half-plane and avoid zero was needed in order to invoke the limiting absorption principle. If one assumes that H has no eigenfunctions (or if f is assumed to be orthogonal to such eigenfunctions), then one can extend this proposition to the left half-plane as well (and indeed the claim follows easily from the spectral theorem in that caase), and if H has no eigenfunction or resonance at zero, then one no longer needs to avoid the origin (thanks to Proposition 1.23). It is likely that one can upgrade the Hölder continuity bound to a stronger bound, such as differentiability, under further regularity and decay hypotheses on the metric g and potential V, but we will not pursue this matter here.

Remark 15.10. We sketch here an alternate approach to the limiting amplitude principle that does uses wave equation energy estimates instead of Hölder continuity properties of the resolvent, but requires the σ_0 parameter to be large (in particular, potential needs to be strongly short-range), and also requires H to have no bound states. The main task, as noted above, is to establish decay of $e^{t\mathcal{A}}(\mathcal{A} - i\mu - 0^+)^{-1} F$ in $H^1(K) \times L^2(K)$ as $t \to +\infty$. If the data $\begin{pmatrix} v \\ i\mu v \end{pmatrix} = (\mathcal{A} - i\mu - 0^+)^{-1} F$ had finite total energy, then this would follow from the RAGE theorem; however, from the spectral theorem we see that we do not expect this data to have finite energy. However, one can use Lemma 2.4 (or more precisely, a variant of this lemma) to show (if σ_0 is large enough) that $\begin{pmatrix} v \\ i\mu v \end{pmatrix}$ has finite *incoming* energy, in the sense that $r^{-1}\nabla_\omega v$ and

$v_r + i\mu v$ lie in $L^2(M)$. It turns out that these types of estimates, together with wave equation energy estimates formed by contracting the stress-energy tensor against a well-chosen vector field (essentially outgoing vector field $\partial_t + \partial_r$), shows that the energy of $e^{t\mathcal{A}}(\mathcal{A} - i\mu - 0^+)^{-1}F$ on a forward light cone is bounded, which by energy estimates implies that $e^{t\mathcal{A}}(\mathcal{A} - i\mu - 0^+)^{-1}F$ stays bounded in $H^1(K) \times L^2(K)$. By truncating away a compactly supported component of $(\mathcal{A} - i\mu - 0^+)^{-1}F$ (whose contribution decays by the RAGE theorem), one can upgrade this boundedness to decay.

16. Decay estimates for the time-dependent Schrödinger equation

In this section we prove Proposition 1.40. We may assume that $t \geq 1$, since the case $0 \leq t \leq 1$ follows from Sobolev embedding and $H^2(M)$ energy estimates. Our initial analysis will be valid in all dimensions three or greater, but we will eventually specialize to the three-dimensional case for sake of concreteness.

Let ψ be a solution of the time-dependent Schrödinger equation

$$i\partial_t \psi + \Delta_M \psi = 0,$$
$$\psi|_{t=0} = \psi_0.$$

Consider the following second order self-adjoint operator

$$P := t^2 \Delta_M - it\chi(r\partial_r + \frac{n}{2} + \frac{r}{2}\theta) - \frac{1}{4}\chi^2 r^2$$

with a smooth cut-off function $\chi(r)$ supported in the region $\langle x \rangle \geq r_0$ and equal to 1 for $r \geq 2r_0$ for some sufficiently large r_0. As we shall see later, this operator can essentially be viewed as a conjugate of $t^2 \Delta_M$, via the heuristic[27]

(138) $$P \approx e^{i\chi r^2/4t} t^2 \Delta_M e^{-i\chi r^2/4t}.$$

We will establish the decay estimate by commuting the Schrödinger equation with P.

Recall that in the region $\langle x \rangle \geq r_0$ the metric g on M takes the form

$$g = dr^2 + r^2 h[r]_{ab} d\omega^a d\omega^b, \qquad h[r]_{ab} = h_{ab}(\omega) + r^{-2\sigma_0} e_{ab}(r, \omega)$$

and $\theta = \frac{1}{2} h[r]^{ab} \partial_r h[r]_{ab}$. By hypothesis, we are assuming that

$$|(r\partial_r)^k (\nabla_\omega^\alpha) h| \leq C_{k\alpha}, \qquad k \leq 3, \ |\alpha| \leq 2.$$

Let $\phi := P\psi$, then ϕ solves the forced Schrödinger equation

$$i\partial_t \phi + \Delta_M \phi = F$$

where

$$F := (i\partial_t P + [\Delta_M, P])\psi.$$

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[27]In the Euclidean case $H = \Delta_{\mathbf{R}^n}$, setting χ equal to 1, P is the Laplacian conjugated by the pseudoconformal transformation $u(t, x) \mapsto \frac{1}{t^{n/2}} e^{ir^2/4t} u(\frac{1}{t}, \frac{x}{t})$, which explains why we expect P to approximately commute with the Schrödinger operator $i\partial_t + \Delta_M$.

To obtain bounds for F, we compute

$$i\partial_t P = 2it\Delta_M + \chi(r\partial_r + \frac{n}{2} + \frac{r}{2}\theta)$$

and

$$[\Delta_M, P] = -it[\Delta_M, \chi(r\partial_r + \frac{n}{2} + \frac{r}{2}\theta)] - \frac{1}{4}[\Delta_M, \chi^2 r^2] = -it(\Delta_M\chi)(r\partial_r + \frac{n}{2} + \frac{r}{2}\theta)$$

$$- 2\chi' it\partial_r - 2\chi' itr\partial_r^2 - \chi' it\partial_r(r\theta) - 2\chi' it(r\partial_r + \frac{n}{2} + \frac{r}{2}\theta)\partial_r - \chi it[\Delta_M, r\partial_r + \frac{r}{2}\theta]$$

$$- \frac{1}{4}(\Delta_M\chi^2)r^2 - 2r\chi\chi' - \chi\chi'r^2\partial_r - \frac{1}{4}\chi^2[\Delta_M, r^2]$$

with the understanding that all expressions involving χ or its derivatives vanish on K_0.

In polar coordinates (outside of K_0), the Laplace-Beltrami operator has the following representation

$$\Delta_M = \partial_r^2 + \left(\frac{n-1}{r} + \theta\right)\partial_r + \frac{1}{r^2}\Delta_{h[r]}.$$

As a consequence, we may compute commutators:

$$[\Delta_M, r\partial_r] := 2\Delta_M - (r\partial_r\theta + \theta)\partial_r + \frac{1}{r}\partial_r(\Delta_{h[r]}),$$

$$[\Delta_M, r^2] := 4r\partial_r + 2 + 2(n-1) + 2\theta r,$$

$$[\Delta_M, r\theta] := 2(\theta + r\partial_r\theta)\partial_r + \frac{2}{r}\nabla_\omega\theta\nabla_\omega + \left(\frac{n-1}{r} + \theta\right)\theta + 2\partial_r\theta + r(\Delta_M\theta).$$

Putting all these estimates together, we obtain the pointwise bound

$$|F| \leq C \left((1+t)\zeta|\partial_r\psi| + t\zeta|\partial_r^2\psi|\right.$$
$$\left. + r^{-2-2\sigma_0}t\left(|\nabla_\omega^2\psi| + |\nabla_\omega\psi|\right) + (1+t)r^{-2-2\sigma_0}|\psi|\right),$$

where ζ is a smooth cut-off supported in the region $r_0 \leq \langle x \rangle \leq 2r_0$.

Fix $T > 0$ and let η_T be the characteristic function of the interval $[1, T]$. Then

$$(i\partial_t + \Delta_M)\eta_T\phi = \eta_T F + i(\phi(1)\delta(t-1) - \phi(T)\delta(t-T))$$

By the global in time local smoothing estimate for $\sigma > 1/2$,

$$\|\eta_T\phi\|_{L_t^2 H^{0,-1/2-\sigma}(M)} \leq C \left(\|\eta_T F\|_{L_t^2 H^{0,1/2+\sigma}(M)} + \|\phi(1)\|_{L^2(M)} + \|\phi(T)\|_{L^2(M)}\right),$$

which means that

$$\|\phi\|_{L_{[1,T]}^2 H^{0,-1/2-\sigma}(M)} \leq C \left(\|F\|_{L_{[1,T]}^2 H^{0,1/2+\sigma}(M)} + \|\phi(1)\|_{L^2(M)} + \|\phi(T)\|_{L^2(M)}\right).$$

We also have the standard L^2 estimate, which implies that for any $t \in [1, T]$,

$$\|\phi(t)\|_{L^2(M)} \leq C \left(\|F\|_{L_{[0,T]}^2 H^{0,1/2+\sigma}(M)} + \|\phi(1)\|_{L^2(M)}\right).$$

Adding the two estimates above we obtain the standard bound
(139)

$$\|\phi(t)\|_{L^2(M)} + \|\phi\|_{L_{[1,T]}^2 H^{0,-1/2-\sigma}(M)} \leq C \left(\|F\|_{L_{[1,T]}^2 H^{0,1/2+\sigma}(M)} + \|\phi(1)\|_{L^2(M)}\right).$$

To relate $\phi = P\psi$ back to ψ, we now work on developing the heuristic (138). We compute

$$\Delta_M \left(e^{-i\chi \frac{r^2}{4t}} \psi \right) = e^{-i\chi \frac{r^2}{4t}} \left(\Delta_M - \frac{i}{2t}(2\chi r + \chi' r^2)\partial_r \right) \psi + \psi \Delta_M \left(e^{-i\chi \frac{r^2}{4t}} \right).$$

We expand the final term $\Delta_M \left(e^{-i\chi \frac{r^2}{4t}} \right)$:

$$\Delta_M \left(e^{-i\chi \frac{r^2}{4t}} \right) = -\frac{i}{4t}\partial_r \left(e^{-i\chi \frac{r^2}{4t}}(2r\chi + \chi' r^2) \right) - \frac{i}{4t}(\frac{n-1}{r} + \theta)(2r\chi + \chi' r^2)e^{-i\chi \frac{r^2}{4t}}$$

$$= -\frac{i}{4t}e^{-i\chi \frac{r^2}{4t}} \left(-\frac{i}{4t}(2r\chi + \chi' r^2)^2 + 4\chi' r + 2\chi + \chi'' r^2 \right.$$

$$\left. + (\frac{n-1}{r} + \theta)(2r\chi + \chi' r^2) \right)$$

We thus obtain the pointwise estimate

$$\left| \phi - e^{i\chi \frac{r^2}{4t}} t^2 \Delta_M \left(e^{-i\chi \frac{r^2}{4t}} \psi \right) \right| \leq C(t\zeta|\partial_r \psi| + (1+t)\zeta|\psi|).$$

Inserting this bound into (139) and letting

$$u := e^{-i\chi \frac{r^2}{4t}} \psi$$

we obtain

$$\|t^2 \Delta_M u\|_{L^2(M)} + \|t^2 \Delta_M u\|_{L^2_{[1,T]} H^{0,-1/2-\sigma}(M)} \leq C \left(\|F\|_{L^2_{[1,T]} H^{0,1/2+\sigma}(M)} \right.$$

$$\left. + t \, \|\zeta|\partial_r \psi| + (1+t)\zeta|\psi| \, \|_{L^2_{[1,T]} H^{0,-1/2-\sigma}(M)} + \|\phi(1)\|_{L^2(M)} \right)$$

We now observe that

$$|\psi| = |u|, \quad |\nabla_\omega \psi| = |\nabla u|, \quad |\partial_r \psi| \leq |\partial_r u| + \frac{r}{t}|u|,$$

$$|\nabla_\omega^2 \psi| = |\nabla_\omega^2 u|, \quad |\partial_r^2 \psi| \leq |\partial_r^2 u| + \frac{r}{t}|\partial_r u| + \frac{r^2}{t^2}|u|^2.$$

Therefore,

$$|F| + t\,\zeta|\partial_r \psi| + t\zeta|\psi| \leq C\,t \, \left(\zeta|\partial_r^2 u| + r^{-2-2\sigma_0}|\nabla_\omega^2 u| \right.$$

$$\left. + \zeta|\partial_r u| + r^{-2-2\sigma_0}|\nabla_\omega u| + r^{-2-2\sigma_0}|u| \right)$$

We thus have

$$\|t^2 \Delta_M u\|_{L^2(M)} + \|t^2 \Delta_M u\|_{L^2_{[1,T]} H^{0,-1/2-\sigma}(M)}$$

$$\leq C \Big(\|t\partial_r^2 u\|_{L^2_{[1,T]} H^{0,-1/2-\sigma}(M)} + \|t\partial_r u\|_{L^2_{[1,T]} H^{0,-1/2-\sigma}(M)}$$

$$+ \|tr^{-1}\nabla_\omega u\|_{L^2_{[1,T]} H^{0,-1/2-2\sigma_0+\sigma}(M)} + \|tr^{-2}\nabla_\omega^2 u\|_{L^2_{[1,T]} H^{0,1/2-2\sigma_0+\sigma}(M)}$$

$$+ \|tu\|_{L^2_{[1,T]} H^{0,-3/2-2\sigma_0+\sigma}(M)} + \|\phi(1)\|_{L^2(M)} \Big).$$

We can couple this bound with the local smoothing estimate for the original solution ψ, which implies (as u has the same magnitude as ψ) that

$$\|u\|_{L^2(M)} + \|u\|_{L^2_{[1,T]} H^{0,-1/2-\sigma}(M)} \leq C\|u(1)\|_{L^2(M)}.$$

By combining these two estimates, we arrive at the estimate

(140)
$$\|t^2 \Delta_M u\|_{L^2(M)} + \|t^2 \Delta_M u\|_{L^2_{[1,T]} H^{0,-1/2-\sigma}(M)}$$
$$\leq C \Big(\|t r^{-2} \nabla^2_\omega u\|_{L^2_{[1,T]} H^{0,1/2-2\sigma_0+\sigma}(M)} + \|t u\|_{L^2_{[1,T]} H^{0,-3/2-2\sigma_0+\sigma}(M)}$$
$$+ \|u(1)\|_{L^2(M)} + \|\phi(1)\|_{L^2(M)} \Big)$$

The first two terms on the right hand side require special care. We introduce a radius parameter R and split

$$\|t r^{-2} \nabla^2_\omega u\|_{L^2_{[1,T]} H^{0,1/2-2\sigma_0+\sigma}(M)} \leq \|t r^{-2} \nabla^2_\omega u\|_{L^2_{[1,T]} H^{0,1/2-2\sigma_0+\sigma}(M_R)}$$
$$+ \|t r^{-2} \nabla^2_\omega u\|_{L^2_{[1,T]} H^{0,1/2-2\sigma_0+\sigma}(M_R^c)}$$
$$\leq C(R) \|t r^{-2} \nabla^2_\omega u\|_{L^2_{[1,T]} H^{0,-1/2-\sigma}(M)}$$
$$+ C R^{1/2-2\sigma_0+\sigma} \sup_{t \in [1,T]} \|t^2 r^{-2} \nabla^2_\omega u\|_{L^2(M)}$$

For $\sigma_0 > \sigma > 1/2$ the second term can be absorbed by the $\|t^2 \Delta_M u\|_{L^2(M)}$ in the left-hand side of (140), while the first term, via Bochner identities, can be partially absorbed by the second term[28] on the left hand side of (140) with the remaining residual term of the same form as $\|t u\|_{L^2_{[1,T]} H^{0,-3/2-2\sigma_0+\sigma}(M)}$.

Therefore, we have that

(141)
$$\|t^2 \Delta_M u\|_{L^\infty_{[1,T]} L^2(M)} + \|t^2 \Delta_M u\|_{L^2_{[1,T]} H^{0,-1/2-\sigma}(M)} \leq C \Big(\|t u\|_{L^2_{[1,T]} H^{0,-3/2-2\sigma_0+\sigma}(M)}$$
$$+ \|u(1)\|_{L^2(M)} + \|\phi(1)\|_{L^2(M)} \Big)$$

In dimension $n = 3$ we now proceed as follows. Using a local smoothing estimate for u we get the bound

$$\|t^2 \Delta_M u(t)\|_{L^2(M)} \leq C t (\|u(1)\|_{L^2(M)} + \|\phi(1)\|_{L^2(M)}),$$

which gives the preliminary bound

(142)
$$\|\Delta_M u(t)\|_{L^2(M)} \leq \frac{C}{t} (\|u(1)\|_{L^2(M)} + \|\phi(1)\|_{L^2(M)})$$

for all $t \geq 1$. By the Sobolev estimate,

$$\|u(t)\|_{L^\infty(M)} \leq C \|\Delta_M u(t)\|_{L^2(M)}^{\frac{3}{4}} \|u(t)\|_{L^2(M)}^{\frac{1}{4}} \leq \frac{C}{t^{\frac{3}{4}}} (\|u(1)\|_{L^2(M)} + \|\phi(1)\|_{L^2(M)})$$

---

[28]Note that Bochner identity and our assumptions on the metric g imply that

$$\|\nabla^2 u\|_{L^2(M)} + \|\langle x \rangle^{-1} |\nabla u|_g\|_{L^2(M)} \leq C \|\Delta_M u\|_{L^2}.$$

The dependence of constant C on the manifold M is implicit as this inequality proved with the help of compactness arguments. This bound allows us to prove Sobolev multiplicative inequalities of the form

$$\|u\|_{L^p(M)} \leq C \|\Delta_M u\|_{L^2(M)}^\alpha \|u\|_{L^2(M)}^{1-\alpha}, \quad 2\alpha = (\frac{n}{2} - \frac{n}{p}), \ 0 \leq \alpha \leq 1, \ (\alpha, p) \neq (1, \infty).$$

Therefore,

$$\|tu\|_{L^2_{[1,T]}H^{0,-3/2-2\sigma_0+\sigma}(M)} \le T^{\frac{1}{2}}\|tu\|_{L^\infty_{[1,T]}L^\infty(M)} \le T^{\frac{1}{4}}(\|u(1)\|_{L^2(M)} + \|\phi(1)\|_{L^2(M)}).$$

We can thus bootstrap the preliminary bound (142) to the improved bound

$$\|\Delta_M u(t)\|_{L^2(M)} \le \frac{C}{t^{\frac{5}{4}}}(\|u(1)\|_{L^2(M)} + \|\phi(1)\|_{L^2(M)})$$

for any $t \ge 1$.

We now iterate this process. If on the nth step the decay rate of $\|\Delta_M u\|_{L^2(M)}$ is $t^{-\alpha_n}$, then

$$\alpha_{n+1} = \frac{3}{4}\alpha_n + \frac{1}{2}$$

and thus $\alpha_n \to 2$. It then follows that

$$\|\psi(t)\|_{L^\infty(M)} \le C_\varepsilon t^{-\frac{3}{2}+\varepsilon}(\|\psi(1)\|_{L^2(M)} + \|\phi(1)\|_{L^2(M)})$$

for any $\epsilon > 0$ and $t \ge 1$.

In higher dimensions the norm $\|tu(t)\|_{H^{0,-3/2-2\sigma_0+\sigma}(M)}$ is L^2 integrable and can be dealt with directly in one step by interpolating between $\|\Delta_M u\|_{L^2(M)}$ and $\|u\|_{L^2(M)}$. We omit the details.

17. Decay estimates for the wave equation

In this section we prove[29] Proposition 1.43. For $t \le 2$ the claim is easily established from $H^2 \times H^1$ energy estimates and Sobolev embedding, so we shall limit ourselves to the case when $t > 2$.

Let u be a solution of the wave equation

$$\Box_M u := u_{tt} - \Delta_M u = 0,$$
$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1.$$

In Section 4 we defined the energy-momentum tensor $\mathbf{Q}_{mk}$ associated with the Helmholtz equation $(H - z^2)u = 0$. Here we use its spacetime counterpart[30]

$$\mathbf{Q}_{\alpha\beta} := \partial_\alpha u \partial_\beta u - \frac{1}{2}\mathbf{g}_{\alpha\beta}\,\mathbf{g}^{\mu\nu}\partial_\mu u\,\partial_\nu u.$$

As is well known, the energy-momentum tensor $\mathbf{Q}_{\alpha\beta}$ is divergence free:

$$D^\beta \mathbf{Q}_{\alpha\beta} = 0.$$

Let $K = K^\alpha \partial_\alpha$ be an arbitrary (smooth) vectorfield. We form the quantity

$$P_\alpha := \mathbf{Q}_{\alpha\beta}K^\beta$$

---

[29]The argument below is an optimal but has been retained for illustrative purposes. The following decay rates can be improved via an alternative approach as in [35].

[30]Greek indices $\alpha, \beta = 0, \ldots, n$ with index 0 corresponding to the t coordinate. Operations of raising and lowering of indices are done with respect to the space-time metric $\mathbf{g} = -dt^2 + g$. Finally, D will denote the Levi-Civita connection of metric $\mathbf{g}$.

with the property that

$$D^\alpha P_\alpha = \mathbf{Q}_{\alpha\beta}\pi^{\alpha\beta}, \quad \pi_{\alpha\beta} = \frac{1}{2}(D_\alpha K_\beta + D_\beta K_\alpha),$$

where $\pi_{\alpha\beta} := \frac{1}{2}\mathcal{L}_K g_{\alpha\beta}$ is the deformation tensor of K. We note two more identities:

$$\partial_\alpha(u^2) = 2\partial_\alpha u\, u,$$
$$D^\alpha(\partial_\alpha u\, u) = \partial^\alpha u\, \partial_\alpha u.$$

Then choosing additional smooth function a^α and b, we have that

$$D^\alpha\left(a_\alpha u^2 + b\,\partial_\alpha u\, u + \mathbf{Q}_{\alpha\beta}K^\beta\right) = D^\alpha a_\alpha u^2 + (2a_\alpha + \partial_\alpha b) + b\partial^\alpha u\,\partial_\alpha u + \mathbf{Q}_{\alpha\beta}\pi^{\alpha\beta}$$

or more explicitly

$$D^\alpha\left(a_\alpha u^2 + b\,\partial_\alpha u\, u + \mathbf{Q}_{\alpha\beta}K^\beta\right) = D^\alpha a_\alpha u^2 + (2a_\alpha + \partial_\alpha b)\partial^\alpha u\, u$$
$$+ \left(\pi^{\alpha\beta} + (b - \frac{1}{2}\mathrm{tr}\pi)\mathbf{g}^{\alpha\beta}\right)\partial_\alpha u\, \partial_\beta u,$$

where $\mathrm{tr}\pi = \mathbf{g}^{\mu\nu}\pi_{\mu\nu}$.

It is clear that one of the choices of a^α, b to simplify the right hand side is

$$a_\alpha = -\frac{1}{2}\partial_\alpha b$$

whence

$$D^\alpha\left(-\frac{1}{2}\partial_\alpha b\, u^2 + b\,\partial_\alpha u\, u + \mathbf{Q}_{\alpha\beta}K^\beta\right) = \Box_M b\, u^2 + \left(\pi^{\alpha\beta} + (b - \frac{1}{2}\mathrm{tr}\pi)\mathbf{g}^{\alpha\beta}\right)\partial_\alpha u\, \partial_\beta u$$

Integrating this expression in the space-time slab $[t, 0] \times M$ we obtain

(143)
$$\int_{M\times\{0\}}\left(-\frac{1}{2}\partial_t b\, u^2 + b\,\partial_t u\, u + \mathbf{Q}_{0\beta}K^\beta\right)dg - \int_{M\times\{t\}}\left(-\frac{1}{2}\partial_t b\, u^2 + b\,\partial_t u\, u + \mathbf{Q}_{0\beta}K^\beta\right)dg$$
$$= \int_0^t \int_M \left(\Box_M b\, u^2 + \left(\pi^{\alpha\beta} + (b - \frac{1}{2}\mathrm{tr}\pi)\mathbf{g}^{\alpha\beta}\right)\partial_\alpha u\, \partial_\beta u\right) dg\, dt$$

We now make the remaining choices of the vector field K and function b. For K we choose a modification of the Morawetz vectorfield $(t^2 + r^2)\partial_t + 2tr\partial_r$, known to play an important role in the study of the decay properties of solutions of the wave equation in Minkowski space and obstacle problems (see [74]). Define

$$K := (t^2 + \chi r^2)\partial_t + 2tr\chi\partial_r$$

with a cut-off function χ of the previous section supported in $\langle x\rangle \le r_0$ and equal to 1 for $\langle x\rangle \ge 2r_0$. The deformation tensor of K can be computed as follows:

$$\pi_{00} = -2t, \quad \pi_{rr} = 2t\chi + 2tr\chi',$$
$$\pi_{ab} = 2t\chi(h[r]_{ab} + r\theta_{ab}), \quad \pi_{0r} = -\chi' r^2,$$
$$\mathrm{tr}\pi = 2t + 2t\chi(n-1) + 2t\chi r\theta + 2tr\chi'$$

We then choose

$$b = (n-1)t + 2t\chi r\theta$$

so that

$$(144) \quad \left(\pi^{\alpha\beta} + (b - \frac{1}{2}\mathrm{tr}\pi)\mathbf{g}^{\alpha\beta} \right) \nabla_\alpha u \, \nabla_\beta u = 2tr^{-1}\chi\theta_{ab}\nabla^a_\omega u \, \nabla^b_\omega u + O(\zeta)(1+t)|\nabla u|^2,$$

where ζ is a smooth cut-off supported in $r_0 \leq \langle x \rangle \leq 2r_0$. Furthermore,

$$(145) \qquad\qquad \Box_M b = 2t\Delta_g(\chi r\theta) = t \, O(r^{-2-2\sigma_0})$$

We now analyze the expression $\left(-\frac{1}{2}\partial_t b \, u^2 + b \, \partial_t u \, u + \mathbf{Q}_{0\beta}K^\beta \right)$.

Lemma 17.1. *Let $n \geq 3$. Then for sufficiently large r_0 and all $t \gg r_0$*

$$(146)$$
$$\int_M \left(-\frac{1}{2}\partial_t b \, u^2 + b \, \partial_t u \, u + \mathbf{Q}_{0\beta}K^\beta \right) dg \geq C(r_0) \, t^2 \int_{M^c_{4r_0}} (|\partial_t u|^2 + |\nabla u|^2 + u^2) \, dg$$

$$+ C(r_0) \int_{M_{4r_0}} \left((t+r)^2|(\partial_t + \partial_r)u|^2 + (t-r)^2|(\partial_t - \partial_r)u|^2 \right) \, dg$$

$$+ C(r_0) \int_{M_{4r_0}} \left((t^2+r^2)|r^{-1}\nabla_\omega u|^2 + (1 + t^2 r^{-2})u^2 \right) \, dg$$

$$- C(r_0) \int_{M_{2r_0}} O(r^{-2\sigma_0})(|\partial_t u|^2 + |\nabla u|^2) \, dg.$$

Remark 17.2. Similar arguments can be found in [64] for the wave equation in Minkowski space and [33] for the wave equation on Schwarzschild background.

Proof. First observe that with our choices of b, K^α

$$Q := -\frac{1}{2}\partial_t b \, u^2 + b \, \partial_t u \, u + \mathbf{Q}_{0\beta}K^\beta = \frac{1}{2}(t^2 + \chi r^2)(|\partial_t u|^2 + |\nabla u|_g^2) + 2tr\chi\partial_t u \, \partial_r u$$

$$+ t(n - 1 + 2\chi r\theta)\partial_t u \, u - \frac{1}{2}(n - 1 + 2\chi r\theta)u^2.$$

On the complement of the set M_{2r_0} we can thus bound Q as follows.

$$(147) \qquad\qquad Q \geq \frac{t^2}{4}(|\partial_t u|^2 + |\nabla u|_g^2) - C(r_0)u^2, \qquad \text{on } M^c_{4r_0}$$

provided that $t \gg r_0$. Let $\tilde{\chi}$ be a cut-off function supported in M_{2r_0} and equal to 1 on M_{4r_0}. To prove the desired result it will be sufficient to show that

$$\int_M \tilde{\chi} \left(\frac{1}{2}(t^2 + r^2)(|\partial_t u|^2 + |\nabla u|_g^2) + 2tr\partial_t u \, \partial_r u + t(n - 1 + 2r\theta)\partial_t u \, u - \frac{1}{2}(n - 1 + 2r\theta)u^2 \right)$$

$$\geq C(r_0) \int_{M_{4r_0}} \left((t+r)^2|(\partial_t + \partial_r)u|^2 + (t-r)^2|(\partial_t - \partial_r)u|^2 \right) \, dg$$

$$+ C(r_0) \int_{M_{4r_0}} \left((t^2+r^2)|r^{-1}\nabla_\omega u|^2 + (1 + t^2 r^{-2})u^2 \right) \, dg$$

$$- C(r_0) \int_{M_{2r_0}} O(r^{-2\sigma_0})(|\partial_t u|^2 + |\nabla u|^2) \, dg$$

Note that the $t^2 u^2$ term on $M^c_{4r_0}$ can be obtained via a Poincaré inequality from the estimate above and (147).

Define

$$Su = (t\partial_t + r\partial_r)u, \qquad \underline{S} = (t\partial_r + r\partial_t u).$$

Then

$$\frac{1}{2}(t^2 + r^2)(|\partial_t u|^2 + |\nabla u|_g^2) + 2tr\partial_t u\, \partial_r u = \frac{1}{2}\left((Su)^2 + (\underline{S}u)^2\right) + \frac{1}{2}(t^2 + r^2)|r^{-1}\nabla_\omega u|_g^2.$$

Furthermore,

$$t\partial_t u\, u = Suu - \frac{1}{2}r\partial_r(u)^2, \quad t\partial_t u\, u = \frac{t}{r}\underline{S}uu - \frac{t^2}{2r}\partial_r(u^2).$$

As a consequence,

$$t(n-1)\int_M \tilde{\chi}\partial_t u\, u = (n-1)\int_M \tilde{\chi}Su\, u + \frac{n-1}{2}\int_M (\tilde{\chi}(n + r\theta) + r\tilde{\chi}')\, u^2,$$

$$t(n-1)\int_M \tilde{\chi}\partial_t u\, u = (n-1)\frac{t}{r}\int_M \tilde{\chi}\underline{S}u\, u + \frac{n-1}{2}t^2\int_M (\tilde{\chi}(n-2 + r\theta) + r\tilde{\chi}')\, \frac{u^2}{r^2}.$$

It is important to note that $\tilde{\chi}' \geq 0$ and $(n-2) \gg r\theta$ on the support of $\tilde{\chi}$ provided that r_0 is sufficiently large. It is not difficult to show that for $n \geq 3$ one can find constants A, B such that

$$A + B = n - 1, \quad A^2 + B^2 < (n - \frac{5}{2})(n - 3) + \frac{5}{4}(n-1)A.$$

(in dimension $n = 3$ the choice of $A = 3/2, B = 1/2$ is sufficient). As a consequence,

$$\int_M \tilde{\chi}\left(\frac{1}{2}(t^2 + r^2)(|\partial_t u|^2 + |\nabla u|_g^2) + 2tr\partial_t u\, \partial_r u + t(n - 1 + 2r\theta)\partial_t u\, u - \frac{1}{2}(n - 1 + 2r\theta)u^2\right)$$

$$\geq \frac{1}{2}\int_M \tilde{\chi}\left((Su + Au)^2 + (\underline{S}u + \frac{t}{r}Bu)^2 + (t^2 + r^2)|r^{-1}\nabla_\omega u|_g^2\right)\, dg$$

$$+ \frac{1}{2}\int_M \tilde{\chi}\left(\left((n-2)(n-3) + (n-1)A - A^2 - B^2 + \frac{n-1}{2r^2}t^2 B\right)u^2\right)\, dg$$

$$- \int_M \tilde{\chi}O(r^{-2\sigma_0})(u^2 + t|\partial_t u|\,|u|)\, dg.$$

Running the argument again we obtain that

$$\int_M \tilde{\chi}\left(\frac{1}{2}(t^2 + r^2)(|\partial_t u|^2 + |\nabla u|_g^2) + 2tr\partial_t u\, \partial_r u + t(n - 1 + 2r\theta)\partial_t u\, u - \frac{1}{2}(n - 1 + 2r\theta)u^2\right)$$

$$\geq c\int_M \tilde{\chi}\left((Su)^2 + (\underline{S}u)^2 + (t^2 + r^2)|r^{-1}\nabla_\omega u|_g^2 + (1 + \frac{t^2}{r^2})u^2\right)\, dg - \int_M \tilde{\chi}O(r^{-2\sigma_0})(\partial_t u)^2\, dg.$$

The result now follows immediately from the identity

$$(t + r)^2((\partial_r + \partial_r)u)^2 + (t - r)^2((\partial_t - \partial_r)u)^2 = \left((Su)^2 + (\underline{S}u)^2\right)$$

$$\square$$

We introduce notations for the energy associated with the vectorfield K and a conserved energy associated with the vectorfield $\partial/\partial t$:

$$(148) \quad E_K(t) = E_{K,u}(T) := t^2 \int_{M_{4r_0}^c} \left(|\partial_t u|^2 + |\nabla u|^2 + u^2 \right) dg$$

$$(149) \qquad\qquad + \int_{M_{4r_0}} \left((t+r)^2 |(\partial_t + \partial_r)u|^2 + (t-r)^2 |(\partial_t - \partial_r)u|^2 \right) dg$$

$$(150) \qquad\qquad + \int_{M_{4r_0}} \left((t^2 + r^2)|r^{-1}\nabla_\omega u|^2 + (1 + t^2 r^{-2})u^2 \right) dg$$

$$E(t) = E_u(T) := \int_M \left(|\partial_t u|^2 + |\nabla u|^2 \right) dg.$$

Combining Lemma 17.1 and (143), we derive the following

Proposition 17.3. *Let u be a solution of $\Box_g u = 0$ and let T be sufficiently large. Then there exists a sufficiently large $R \ll T$ and a smooth cut-off function ζ supported in the region $M_{r_0} \setminus M_R = \{x : r_0 \leq \langle x \rangle \leq R\}$ such that*

$$E_K(T) \leq C(r_0) \left(E_K(0) + \int_0^T (1+t) \int_M \tilde{\zeta} |\nabla u|^2 \, dg \, dt \right).$$

Proof. Lemma 17.1 together with (143), (144) and (145) imply that all $T \gg r_0 \gg 1$

$$E_K(T) \leq C(r_0)(E_K(0) + E(T))$$

$$+ C(r_0) \int_0^T \int_{M_{r_0}} \left(O(r^{-2\sigma_0}) t \left(|r^{-1}\nabla_\omega u|_g^2 + r^{-2} u^2 \right) + O(\zeta)(1+t)|\nabla u|^2 \right) dg \, dt.$$

Since the expression for the energy $E_K(t)$ contains both $t^2 |r^{-1}\nabla_\omega u|^2$ and $t^2 r^{-2} u^2$ we can find a sufficiently large R such that the above inequality can be simplified to

$$E_K(T) \leq C(r_0)(E_K(0) + E(0)) + C(r_0) \int_0^T (1+t) \int_M \tilde{\zeta} |\nabla u|^2 \, dg \, dt$$

with $\tilde{\zeta}$ supported in $M_{r_0} \setminus M_R$. Note that to get to the last inequality we also used a Poincaré inequality to convert the u^2 term into $|\nabla u|_g^2$, and conservation of the energy $E(t)$. To conclude the proof of the proposition it remains to observe that the energy $E_K(0)$ easily dominates $E(0)$. $\qquad\square$

Henceforth we specialize to the three-dimensional case $n = 3$.

The integrated local energy decay estimate of Proposition 1.38 gives the bound

$$\int_0^T (1+t) \int_M \tilde{\zeta} |\nabla u|^2 \, dg \, dt \leq (1+T) \int_0^T \int_M \tilde{\zeta} |\nabla u|^2 \, dg \, dt \leq C(1+T)E(0).$$

This immediately implies that

$$E_K(T) \leq C(1+T)E_K(0).$$

In particular, as can be seen from (150), inequality above leads to the $1/t$ decay of the local energy. Precisely, for any compact set $B \subset M$

$$(151) \qquad \int_B ((\partial_t u)^2 + |\nabla u|_g^2) \, dg \le \frac{C(B)}{t} E_K(0).$$

Applying in addition the same argument to the function $v = \partial_t u$ we then have

$$\int_B ((\partial_t^2 u)^2 + |\partial_t \nabla u|_g^2) \, dg \le \frac{C(B)}{t} E_{K, \partial_t u}(0).$$

From the Gagliardo-Nirenberg inequality we thus obtain uniform decay for u:

$$\|u(t)\|_{L_x^\infty} \le C_\varepsilon t^{-\frac{1}{2}+\varepsilon} \left(E_{K,u}(0) + E_{K, \partial_t u}(0) \right).$$

for any $\varepsilon > 0$.

This however is not an optimal result (in particular, as suggested by our experience with the wave equation in Minkowski space in dimension $n = 3$). To improve on it one observes that Proposition 17.3 can be iterated, [33]. Heuristically (for more details see [33]), the argument is as follows. First partition the time interval $[0, T]$ dyadically into subintervals of the form $[t, \Gamma t]$ with some $\Gamma > 1$ sufficiently close to 1. By finite speed of propagation, on the set $[t, \Gamma t] \times M_R^c$, solution u is completely determined by its values at time t on the set $M_{R+C(\Gamma-1)t}^c$ for some universal constant C dependent on the manifold M. In particular, Γ can be chosen in such a way that $R + C(\Gamma - 1)t < 1/2t$. Now, the estimate $E_K(t) \le C(1 + t) E_K(0)$ in fact gives more than just the local energy decay. One can easily show that (151) holds for any set B such that $B \subset M_{1/2t}^c$ with a constant $C(B)$ independent of B (in particular of t). In view of the above discussion the integrated local energy decay estimate for u on the interval $[t, \Gamma t]$ should take the form

$$\int_t^{\Gamma t} \int \tilde{\zeta}((\partial_t u)^2 + |\nabla u|_g^2) \, dg \, dt \le C \int_{\{t\} \times M_{R+C(\Gamma-1)t}^c} ((\partial_t u)^2 + |\nabla u|_g^2) \, dg \le \frac{C}{t} E_K(0).$$

Adding these estimates over all such time subintervals we obtain that

$$\int_0^T (1 + t) \int \tilde{\zeta}((\partial_t u)^2 + |\nabla u|_g^2) \, dg \, dt \le \log T E_K(0).$$

Using Proposition 17.3 now implies that

$$E_K(T) \le C(r_0) \log T E_K(0)$$

and

$$\int_B ((\partial_t u(t))^2 + |\nabla u(t)|_g^2) \, dg \le \frac{C(B) \log t}{t^2} E_K(0)$$

for all $t \le T$. Repeating the whole argument one more time eliminates the remaining $\log t$ term and shows that

$$E_K(T) \le C(r_0) E_K(0)$$

Finally, combining the bound for u with the similar bound for $v = \partial_t u$ and using Sobolev inequalities we arrive at the desired estimate

$$\|u(t)\|_{L_x^\infty} \le C_\varepsilon t^{-1+\varepsilon} \left(E_{K,u}(0) + E_{K, \partial_t u}(0) \right).$$

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