EXPLICIT PROOF OF CONSTRAINTS AND EFFECTIVE THEORIES PRESENTED IN TABLE I

In this appendix, we give a general derivation the constraints and effective theories shown in Table I. On a C_m invariant line in a C_n invariant system (m being a factor of n), each band corresponds to a 1D representation of cyclic group C_m (spinless), or double cyclic group C_m^D (spinful). In either case, an eigenvalue of C_m is in the form:

$$\alpha_p = \exp(i2\pi p/m + iF\pi/m), \tag{S1}$$

where $p = 0, 1, ...m-1$. Now suppose the C_m eigenvalues of the conduction and valence bands are

$$\begin{aligned} u_c &= \alpha_p, \\ u_v &= \alpha_q. \end{aligned} \tag{S2}$$

We have if $p = q$, then the two bands will *not* cross on this C_m-invariant line because of the presence of a symmetry allowed a constant off-diagonal term $\delta|\psi_u\rangle\langle\psi_v| + h.c.$ which can always open a gap. If $p \neq q$, then the matrix representation of C_m, $\mathcal{C}_m$ is given by

$$\mathcal{C}_m = \exp(i\pi\frac{F+p+q}{m})\exp(i\pi\frac{p-q}{m}\sigma_z). \tag{S3}$$

The transform of $H_{eff}(\mathbf{q})$ under C_m is given by

$$\begin{aligned} \mathcal{C}_m H_{eff}(\mathbf{q})\mathcal{C}_m^{-1} &= g(\mathbf{q})\sigma_z + f(\mathbf{q})\exp(i\pi\frac{p-q}{m}\sigma_z)\sigma_+\exp(-i\pi\frac{p-q}{m}\sigma_z) + h.c. \\ &= g(\mathbf{q})\sigma_z + f(\mathbf{q})e^{-i2\pi(p-q)/m}\sigma_+ + h.c. \end{aligned} \tag{S4}$$

In the basis of $q_\pm$, the $R_m\mathbf{q}$ is given by

$$R_m(q_+, q_-) = (q_+e^{i2\pi/m}, q_-e^{-i2\pi/m}). \tag{S5}$$

Substituting Eq.(S4,S5) into Eq.(1) in main text, we obtain

$$\begin{aligned} e^{-i2\pi(p-q)/m}f(q_+, q_-) &= f(q_+e^{i2\pi/m}, q_-e^{-i2\pi/m}), \\ g(q_+, q_-) &= g(q_+e^{i2\pi/m}, q_-e^{-i2\pi/m}). \end{aligned} \tag{S6}$$

This is the general constraint on f, g by C_m symmetry for a given pair of (u_c, u_v). From Eq.(S6) we learn that the forms of f and g only depend on $p-q$, or u_c/u_v. These results comprise the central column of Table I.

To generate the last column from the general constraints Eq.(S6), we start from an expansion of

$$f(q_+, q_-) = \sum_{n_1 n_2} A_{n_1 n_2} q_+^{n_1} q_-^{n_2}, \tag{S7}$$

where $A_{n_1n_2}$ is an arbitrary complex coefficient. Eq.(S6) gives $A_{n_1n_2} = 0$ if $n_2 - n_1 \neq p - q \bmod m$. Then we pick up (n_1, n_2) with smallest $n_1 + n_2$ and nonzero $A_{n_1n_2}$ to obtain the last column of Table I.

Physically, one can easily understand Table I as the consequence of the 'semi-conservation' of total angular momentum J_z. Here 'semi-conservation' means that J_z is only conserved up to a multiple of m. This is because in a lattice the continuous rotation symmetry downgrades to discrete rotation symmetry of order m. If $u_c/u_v = e^{i2\pi(p-q)/m}$, if means the conduction and valence bands differ by $\delta J_z = p - q$. Therefore, the off-diagonal term of the effective Hamiltonian, $|\psi_c\rangle\langle\psi_v|$, must couple a k_-^{p-q} or k_+^{m-p+q} to conserve the total angular momentum up to a multiple of m.