

**SZEMERÉDI–TROTTER-TYPE THEOREMS
IN DIMENSION 3**

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ABSTRACT. We estimate the number of incidences in a configuration of m lines and n points in dimension 3. The main term is $mn^{1/3}$ if we work over the real or complex numbers but $mn^{2/5}$ over finite fields.

Let \mathcal{L} be a set of lines and \mathcal{P} a set of points in some affine or projective space. The papers [SzT83, SW04, ES11, EKS11, ST12] and [GK10, GK15, Gut14] point out the importance of bounding the number of special points of $\mathcal{L} \cup \mathcal{P}$.

Definition 1. Let \mathcal{L} be a set of lines in some ambient space. There are at least 3 sensible way to count the *number of intersection points* of \mathcal{L} . The smallest of these is to count each intersection point with multiplicity 1. Our formulas give naturally a larger number

$$I(\mathcal{L}) := \sum_p (r(p) - 1), \quad (1.1)$$

where $r(p)$ denotes the number of lines passing through p and the summation is over all points where at least 2 lines meet. The largest is to count all pairs of lines that intersect, this corresponds to $\sum_p \binom{r(p)}{2}$.

In addition, let \mathcal{P} be a set of points. An *incidence* is a pair $(p \in \ell)$ where $\ell \in \mathcal{L}$ and $p \in \mathcal{P}$. The total number of incidences is denoted by $I(\mathcal{L}, \mathcal{P})$. Thus

$$I(\mathcal{L}, \mathcal{P}) = \sum_{p \in \mathcal{P}} r(p). \quad (1.2)$$

The Szemerédi–Trotter theorem [SzT83] says that for m lines and n points in \mathbb{R}^2 the number of incidences satisfies

$$I(\mathcal{L}, \mathcal{P}) \leq 2.5m^{2/3}n^{2/3} + m + n, \quad (1.3)$$

where the constants are due to [Szé97, PRTT06]. This implies the same bound in any \mathbb{R}^d since projecting to \mathbb{R}^2 can only increase the number of incidences. Thus it is interesting to look for other bounds for point/line configurations in \mathbb{R}^d or \mathbb{C}^d that do not hold for planar ones. Any finite configuration of lines and points can be projected to \mathbb{R}^3 (resp. to \mathbb{C}^3) without changing the number of incidences, hence it is enough to study lines and points in 3-space.

The bounds we obtain are not symmetric in m, n . Note, however, that while lines and points have symmetric roles in \mathbb{R}^2 , they do not have symmetric behavior in \mathbb{R}^3 .

Theorem 2. *Let \mathcal{L} be a set of m distinct lines and \mathcal{P} a set of n distinct points in \mathbb{C}^3 . Let c be a constant such that no plane contains more than $c\sqrt{m}$ of the lines. Then the number of incidences satisfies*

$$I(\mathcal{L}, \mathcal{P}) \leq (3.66 + 0.91c^2)mn^{1/3} + 6.76n. \quad (2.1)$$

Example 3. Choose \mathcal{P} to be the integral points in the cube $[0, r-1]^3$ and \mathcal{L} to be the lines parallel to one of the coordinate axes meeting \mathcal{P} . Then $m = 3r^2$, $n = r^3$, $I(\mathcal{L}, \mathcal{P}) = 3r^3$ and each plane contains at most $2r$ lines.

We can do slightly better by tilting the above configuration. This is obtained by first taking the image of \mathcal{L} in \mathbb{R}^7 under the map $(x, y, z) \mapsto (xyz, xy, yz, zx, x, y, z)$ and then projecting generically to \mathbb{R}^3 . The incidences are unchanged but now any plane contains at most 2 of the lines. We can thus take c arbitrary small and (2.1) gives that $3r^3 = I(\mathcal{L}, \mathcal{P}) \leq 17.74r^3$. Hence, for this series of examples, (2.1) is a factor of < 6 away from an optimal bound.

In this example $mn^{1/3}$ and n are of the same size and the proof seems to work naturally in this case. Other cases are discussed in Paragraph 7.

A key step of the proof of Theorem 2 does not work over finite fields and we have the following estimate in general.

Theorem 4. *Let \mathcal{L} be a set of m distinct lines and \mathcal{P} a set of n distinct points in K^3 for an arbitrary field K . Let c be a constant such that no plane contains more than $c\sqrt{m}$ of the lines. Then the number of incidences satisfies*

$$I(\mathcal{L}, \mathcal{P}) \leq 2.45mn^{2/5} + 2.45n^{6/5} + 0.91c^2mn^{1/3} + 6.74n. \quad (4.1)$$

Example 35 gives a line/point configuration over \mathbb{F}_{q^2} where $m = (q+1)(q^3+1)$, $n = (q^2+1)(q^3+1)$ and $I(\mathcal{L}, \mathcal{P}) = (q+1)(q^2+1)(q^3+1)$. For this (4.1) gives an upper bound $2.45q^4(q^5)^{2/5} + 2.45(q^5)^{6/5} + \text{lower terms}$. Therefore

$$q^6 \leq I(\mathcal{L}, \mathcal{P}) \leq 4.9q^6 + (\text{lower terms}).$$

Hence (4.1) is a factor of < 5 away from an optimal bound.

Theorems 2–4 give the following form of Bourgain’s conjecture, proved in [GK10] over \mathbb{C} . As pointed out in [EH13], the exponent $5/4$ is optimal over finite fields; see also Example 35.

Corollary 5. *Let \mathcal{L} be a set of m distinct lines and \mathcal{P} a set of n distinct points in K^3 . Assume that every line contains at least \sqrt{m} points and no plane contains more than \sqrt{m} of the lines.*

- (1) *If K has characteristic 0 then $n \geq \frac{1}{50} \cdot m^{3/2}$.*
- (2) *If K has positive characteristic and $m \geq 10^4$ then $n \geq \frac{1}{20} \cdot m^{5/4}$.*

We get somewhat worse bounds for $I(\mathcal{L})$. Note that $I(\mathcal{L})$ is the largest when any 2 lines meet in distinct points; then we get $\binom{m}{2}$ intersection points. If this happens then all the lines are contained in a plane. One gets a similar quadratic growth if all lines are contained in a quadric surface. To avoid these cases, one should assume that no plane or quadric contains too many of the lines. The following is a strengthening of [GK15, Thm.2.10], which in turn was conjectured by [ES11].

Theorem 6. *Let \mathcal{L} be a set of m distinct lines in \mathbb{C}^3 . Let c be a constant such that no plane (resp. no quadric) contains more than $c\sqrt{m}$ (resp. more than $2c\sqrt{m}$) of the lines. Then the number of intersection points—with multiplicity as in (1.1)—is*

$$I(\mathcal{L}) \leq \left(29.1 + \frac{c}{2}\right) \cdot m^{3/2}.$$

Note that [GK15, Thm.2.10] gives a similar bound where each intersection point is counted with multiplicity 1. On the other hand, [GK15, Thm.2.11] gives a bound, under some restrictions, where each intersection point is counted with multiplicity

$r(p)^2$; see Remark 15 for details. For the applications in [GK15] the relevant value of c is ≤ 3.5 .

7 (Comparison with previous results). The idea of using algebraic surfaces to attack such problems is due to [GK10]. Our main observation is that the arithmetic genus of a line configuration provides a very efficient way to bound the number of incidences.

It was observed by Ellenberg and Hablicsek as well as by Guth and Katz (both unpublished) that, at least over \mathbb{R} , estimates similar to Theorem 2 could be deduced from the results of [GK15], though the resulting constants were never computed. (In this area, some proofs lead to quite large constants. For example, the coefficient 2.5 in (1.3) first appeared as $\leq 10^{60}$, and the complex version, due to [Tót03], still has a coefficient $\leq 10^{60}$.)

To compare the various bounds, write $n = m^t$. For Theorem 2 the interesting cases are when each line contains at least 2 points and through each point there are at least 2 lines. Thus $t \in [\frac{1}{2}, 2]$ and Theorem 6 shows that in fact $t \in [\frac{1}{2}, \frac{3}{2}]$ is the important range. The easy planar bound $I(\mathcal{L}, \mathcal{P}) \leq m^{1/2}n$ (see Paragraph 8) is better than (2.1) for $t \in [\frac{1}{2}, \frac{3}{4}]$. Thus Theorem 2 gives new results when $t \in [\frac{3}{4}, \frac{3}{2}]$ and is sharp at $t = \frac{3}{2}$.

For real lines, [GK15] yields bounds that are smaller for $t < \frac{3}{2}$. I do not know whether these bounds also hold over \mathbb{C} , but it is not clear what should replace the ham sandwich theorems used in [GK15] to get a proof over \mathbb{C} or over finite fields.

By contrast, the proof of Theorem 6 given in [GK15] also works for complex lines, hence over any field of characteristic 0. Thus the new part of Theorem 6 is the explicit constant. The same applies to Corollary 5.1.

In all these results, the different behavior in positive characteristic is restricted to small values of p . This was observed in [EH13] where a positive characteristic version of Corollary 5 is also proved. We show in (39–40) that Theorem 6 holds over a field of characteristic p provided $p > \sqrt{m}$ and Theorem 2 holds provided $p > \sqrt[3]{6n}$; answering a question posed by Dvir in a conversation.

[ST12] proves higher dimensional analogs of the Szemerédi–Trotter theorem where lines are replaced by larger linear spaces.

The methods of this paper have been applied in [Hab14] to estimate the number of joints in higher dimensional line arrangements.

8 (Planar case). For planar configurations we clearly have

$$\sum \binom{r(p)}{2} = \binom{m}{2}, \quad (8.1)$$

where we sum over all intersection points, thus $\sum (r(p) - 1)^2 < m^2$. Using the Cauchy–Schwartz inequality as in (30.2), this implies that $I(\mathcal{L}, \mathcal{P}) \leq mn^{1/2}$. This is quite sharp since m general lines in a plane have $n = \binom{m}{2}$ intersection points and for these $I(\mathcal{L}, \mathcal{P}) \sim (1/\sqrt{2})mn^{1/2}$. Working with the dual configuration gives that $I(\mathcal{L}, \mathcal{P}) \leq m^{1/2}n$ and the two together imply that

$$I(\mathcal{L}, \mathcal{P}) \leq m^{3/4}n^{3/4}. \quad (8.2)$$

This is weaker than (1.3). Note, however, that (8.2) holds over any field and it is sharp over finite fields. If \mathcal{L} is the set of all lines and \mathcal{P} is the set of all points over \mathbb{F}_q then

$$q(q^2 + q + 1) = I(\mathcal{L}, \mathcal{P}) \leq m^{3/4}n^{3/4} = (q^2 + q + 1)^{3/4}(q^2 + q + 1)^{3/4} \quad (8.3)$$

shows that in (8.2) the exponents $3/4$ and the constant factor 1 are all optimal.

9 (Outline of the proofs). For all the theorems there are 4 steps, the first two follow [GK10, GK15].

(9.1) By an easy dimension count, all the lines and points lie on a low degree algebraic surface S . In general S is reducible; it is not hard to deal with the components that contain infinitely many lines. We recall the needed results in Section 7.

(9.2) In the remaining cases, old results of Monge, Salmon and Cayley are used to find another surface of low degree T that contains all the lines. Thus the union of all lines $C := \cup\{\ell : \ell \in \mathcal{L}\}$ is contained in the complete intersection curve $B := S \cap T$. Since the references are not easily accessible, we outline the proofs in Section 8.

(9.3) Sometimes we find a lower degree surface T . Some of the proofs work without this step but it improves the bounds substantially.

(9.4) Although B is a singular algebraic curve, the expected formula bounds its arithmetic genus, hence also the arithmetic genus of C . The key fact is that while a plane curve of degree d has genus $\asymp d^2$, a typical complete intersection curve of degree d in \mathbb{P}^3 has genus $\asymp d^{3/2}$. Finally the set of intersection points equals the set of singular points of C which in turn is controlled by the arithmetic genus of C .

These steps appear in the cleanest form in the proof of Theorem 6; we treat it in Section 2. The proofs of Theorems 2 and 4, presented in Sections 3–4, are slightly more involved.

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1. LOW DEGREE SURFACES

The following elementary lemmas show that any collection of lines or points is contained in a relatively low degree algebraic surface. Under some extra conditions there are even 2 such surfaces. We work in projective 3-space \mathbb{P}^3 over an arbitrary field. For a set of lines \mathcal{L} let $[\mathcal{L}] \subset \mathbb{P}^3$ denote their union. We view $[\mathcal{L}]$ as a (reducible) algebraic curve in \mathbb{P}^3 .

Lemma 10. *Let \mathcal{L} be m distinct lines in \mathbb{P}^3 .*

- (1) *There is a surface S of degree $d \leq \sqrt{6m} - 2$ that contains $[\mathcal{L}]$.*
- (2) *Let $U \subset \mathbb{P}^3$ be an irreducible surface of degree $g \leq \sqrt{6m}$. Then there is a surface T of degree e that contains $[\mathcal{L}]$, does not contain U and $ge \leq 6m$.*

Proof. Degree d homogeneous polynomials in 4 variables form a vector space of dimension $\binom{d+3}{3}$. For a surface of degree d it is $d+1$ linear conditions to contain a line. Thus if $\binom{d+3}{3} > m(d+1)$ then such a surface S exists, giving (1).

For $e \geq g$ the equations of surfaces that contain U form a vector space of dimension $\binom{e-g+3}{3}$. Thus if

$$\binom{e+3}{3} - \binom{e-g+3}{3} > m(e+1)$$

then we find a surface T of degree e that contains all the lines in \mathcal{L} but does not contain U . By expanding we see that

$$\binom{e+3}{3} - \binom{e-g+3}{3} > \frac{1}{6}g(e+1)(e+5),$$

so we are done if $g(e+5) \geq 6m$ since a vector space can not be a union of ≤ 2 lower dimensional vector subspaces. Finally note that if $e = \lfloor \frac{6m}{g} \rfloor \geq \frac{6m}{g} - 1$ then $g(e+5) \geq g(\frac{6m}{g} + 4) = 6m + 4g$. \square

Lemma 11. *Let \mathcal{P} be n distinct points in \mathbb{P}^3 .*

- (1) *There is a surface S of degree $d \leq \sqrt[3]{6n}$ that contains \mathcal{P} .*
- (2) *Let $U \subset \mathbb{P}^3$ be an irreducible surface of degree $g \leq \sqrt[3]{6n}$. Then there is a surface T of degree e that contains \mathcal{P} , does not contain U and $ge^2 \leq 6n$.*

Proof. We argue as in Lemma 10. For a surface of degree d it is 1 linear condition to contain a point. Thus if $\binom{d+3}{3} > n$ then such a surface S exists, giving (1).

In order to prove (2) we need to find e such that

$$\binom{e+3}{3} - \binom{e-g+3}{3} > n.$$

As before, the left hand side is $\frac{1}{6}g(e+1)(e+5) > \frac{1}{6}g(e+1)^2$. Thus we can choose $e := \lfloor \sqrt{6n/g} \rfloor$. \square

Remark 12. Over infinite fields, both lemmas can be extended to the case when we want to avoid any finite collection of surfaces U_i whose degrees are between g and $\sqrt{6m}$ in Lemma 10 (resp. between g and $\sqrt[3]{6n}$ in Lemma 11). We just need to take a general linear combination of the equations obtained for the individual U_i .

The conclusions of the second part of Lemmas 10–11 get weaker as g gets smaller. I believe that Lemma 11 can not be improved, but a quite different method works for Lemma 10.

Let $S \subset \mathbb{P}^3$ be a surface of degree d . In 1849 Salmon wrote down an equation of degree $11d - 24$ that cuts out on S the locus of points where there is a triple tangent line; see [Sal1865, pp.277–291] for a detailed treatment based on [Cle1861]. This locus clearly contains the union of all lines contained in S . Cayley noted that every point has a triple tangent line iff S is ruled. The latter assertion is already in the fourth edition of Monge’s book [Mon1809, §XXI], see especially p.225. (I could not find the 1801 first edition *Feuilles d’analyse appliquée à la géométrie*; it is much shorter than the 1809 fourth edition.)

Theorem 13 (Monge–Salmon–Cayley). *Let $S \subset \mathbb{C}\mathbb{P}^3$ be a surface of degree d without ruled irreducible components. Then there is a surface T of degree $11d - 24$ such that S and T do not have common irreducible components and every line on S is contained in $S \cap T$.*

For the reader’s convenience, I give a—partly analytic—proof of this in Section 8. Strictly speaking, I only show that $\deg T \leq 11d - 18$. In the applications I use only that $\deg T \leq 11d$, so this is not a problem. For an algebraic approach see [Vol03], where the emphasis is on understanding what happens in positive characteristic.

Once we have two surfaces S, T , we use the following bound on the number of intersections. This is the observation that makes the estimates in the Theorems readily computable.

Let C be a reduced curve. For a point $p \in C$, let $r(p)$ denote the *multiplicity* of C at p . For line configurations, this equals the number of lines passing through p . Since we use only the line configuration case, we do not discuss the extra complications that appear in general.

Proposition 14. *Let $S, T \subset \mathbb{P}^3$ be two surfaces of degrees a and b that have no common irreducible components. Set $C = S \cap T$ (with reduced structure). Then*

- (1) C has at most ab irreducible components.
- (2) $\sum_{p \in C} (r(p) - 1) \leq \frac{1}{2}ab(a + b - 2)$.
- (3) $\sum_{p \in C} (r(p) - 1)^{3/2} \leq \frac{1}{\sqrt{2}}ab(a + b - 2)$.
- (4) $\sum_{p \in C}^{(sm)} r(p)(r(p) - 1) \leq ab(a + b - 2)$ where the sum is over those points where either S or T is smooth.

Outline of proof. We repeatedly use the *theorem of Bézout* which says that if $H_1, \dots, H_n \subset \mathbb{P}^n$ are hypersurfaces of degrees d_1, \dots, d_n then their intersection $H_1 \cap \dots \cap H_n$ either contains an algebraic curve or it consist of at most $d_1 \cdots d_n$ points; cf. [Sha74, Sec.IV.2.1].

Using this for S, T and a general hyperplane we see that C has degree $\leq ab$, thus $\leq ab$ irreducible components; if equality holds then all irreducible components are lines, proving (14.1).

The proof of (14.2–4) has 2 main steps.

Note that $\frac{1}{2}ab(a + b - 4) + 1$ is the genus of a smooth complete intersection curve of two surfaces of degrees a and b . This is a well known formula; see for example [Sha74, Sec.VI.1.4] (especially Exercise 9 on p.68 of volume 2) or [Har77, Exrc.I.7.2]. The key claim is that even very singular complete intersection curves have arithmetic genus $\leq \frac{1}{2}ab(a + b - 4) + 1$; see Section 6 for details.

Note that the arithmetic genus frequently jumps up for singular curves in families. (Historically, schemes and flatness were introduced to understand similar phenomena.) For instance, all rational curves of degree d in \mathbb{P}^3 form a single family. General members are smooth, thus with genus 0. At the other extreme we get plane rational curves of degree d , these have arithmetic genus $\binom{d-1}{2}$.

The second step is to use the arithmetic genus of a curve to control its singularities and convert this information into the estimates (14.2–4). This is done in Section 5.

Remark 15. [GK15] suggests (see especially Proposition 2.2 and the Appendix) that, at least over \mathbb{R} , for line configurations the optimal bound is of the form

$$\sum_p^* (r(p) - 1)^2 \leq (\text{constant}) \cdot ab(a + b - 2) \log(a + b), \quad (15.1)$$

where summation is over the points satisfying $1 \leq r(p) \leq a + b$. The appearance of \log on the right hand side is surprising from the point of view of algebraic geometry.

I do not know if (15.1) holds over \mathbb{C} or not, but over finite fields the exponent $3/2$ is optimal as shown by Example 41. The exponent $3/2$ is also optimal for complete intersection curves in general, even when the singularities locally look like unions of lines.

As an example, pick general homogeneous polynomials f, g, h of degree n . For general $\alpha, \beta, \gamma \in \mathbb{R}$ set

$$S := (f^m + g^m + h^m = 0) \quad \text{and} \quad T := (\alpha f^m + \beta g^m + \gamma h^m = 0).$$

Then $C := S \cap T$ has n^3 singular points (where $f = g = h = 0$) and, at each of these points C has m^2 smooth branches. Thus

$$\sum_{p \in C} (r(p) - 1)^{3/2} = n^3(m^2 - 1)^{3/2} \leq (nm)(nm)(2nm - 2)$$

indeed holds but the exponent $3/2$ can not be replaced with anything bigger.

We can even arrange all the singular points to be real.

Remark 16. The maximum possible number of lines on a degree d non-ruled surface is not known. The Fermat-type surface

$$F_d := (x_0^d + x_1^d + x_2^d + x_3^d = 0)$$

contains $3d^2$ lines. There are a few examples with more lines, for instance there are degree 20 surfaces with $4 \cdot 20^2$ lines. See [BS07, RS12, RS13] and the references there for further examples. Over finite fields one can have many more lines, see Example 35.

2. COUNTING INTERSECTIONS

17. In order to prove Theorem 6, let S be a surface of smallest possible degree that contains our m lines $[\mathcal{L}]$. By Lemma 10 we know that $d := \deg S \leq \sqrt{6m}$.

Fix an ordering of the irreducible components $S_i \subset S$ and let $\mathcal{L}_i \subset \mathcal{L}$ denote those lines that are contained in S_i but are not contained in $S_1 \cup \dots \cup S_{i-1}$. We can write $I(\mathcal{L})$ in the form

$$I(\mathcal{L}) = \sum_{i=1}^r I(\mathcal{L}_i) + \sum_{i=2}^r \#(|\mathcal{L}_i| \cap |\mathcal{L}_1 \cup \dots \cup \mathcal{L}_{i-1}|). \quad (17.1)$$

The second sum counts intersections of lines that lie on different irreducible components; these are easy to bound, see (18).

For the first sum we need to work with one irreducible component at a time. We treat 3 cases separately: planes and quadrics in (19), ruled surfaces of degree ≥ 3 in (20) and non-ruled surfaces in (23).

18 (External intersections). Let $\ell \in \mathcal{L}$ be a line and $S'(\ell) \subset S$ the union of those irreducible components of S that do not contain ℓ . Any intersection point of ℓ with a line that is in $S'(\ell)$ is also contained in $\ell \cap S'(\ell)$. By Bézout, this is a set of at most $\deg S'(\ell) \leq \deg S = d$ elements. This gives at most md such intersections. We can do better if we order the S_i such that $\#\mathcal{L}_i / \deg S_i$ is a non-increasing function of i . (Note that the set \mathcal{L}_i depends on the ordering of the surfaces, but we can choose at each step the surface that maximizes the quotient.) With such a choice there are at most $\frac{1}{2}md$ external intersections.

We are thus left to work with the surfaces S_i separately and estimate the number of intersections of the lines in \mathcal{L}_i .

19 (Planes and quadrics). Let $\{P_i : i \in I\}$ be the planes and $\{Q_j : j \in J\}$ the quadrics in S . Set $m_i = \#\mathcal{L}_i$ for $i \in I$ and $n_j = \#\mathcal{L}_j$ for $j \in J$.

A line on P_i intersects all the other lines on P_i thus $I(\mathcal{L}_i) \leq \frac{1}{2}m_i^2$. By assumption $m_i \leq c\sqrt{m}$ thus $I(\mathcal{L}_i) \leq \frac{1}{2}m_i c\sqrt{m}$.

On a singular quadric, any two lines meet at the vertex, thus $I(\mathcal{L}_j) = n_j - 1$. On a smooth quadric, there are 2 families of lines. Correspondingly write $n_j = n'_j + n''_j$. Then

$$I(\mathcal{L}_i) \leq n'_j n''_j \leq \frac{1}{4}(n'_j + n''_j)^2 = \frac{1}{4}n_j^2 \leq \frac{1}{4}n_j 2c\sqrt{m}.$$

Thus the number of internal intersections on planes and quadrics is at most

$$\sum_{i \in I} \frac{1}{2}m_i c\sqrt{m} + \sum_{j \in J} \frac{1}{2}n_j c\sqrt{m} = \frac{1}{2}c\sqrt{m} \left(\sum_{i \in I} m_i + \sum_{j \in J} n_j \right) \leq \frac{1}{2}cm^{3/2}.$$

20 (Ruled surfaces). Basic results on ruled surfaces are discussed in Section 7.

Let $\{R_i : i \in I\}$ be the irreducible ruled surface in S and set $d_i = \deg R_i$.

By (55.4) there are at most 2 lines, called *special lines*, that intersect infinitely many other lines. For each irreducible ruled surface these contribute at most $2m_i$. By (55.5), every non-special line intersects at most $d_i := \deg R_i$ other non-special lines, hence these contribute at most $\frac{1}{2}m_i d_i$.

Thus all together we get at most $\frac{1}{2}md + 2m$ intersections.

Finally we give two different bounds for the non-ruled irreducible components. First, combining Theorem 13 with Proposition 14.2 we get the following.

Corollary 21. *Let $S \subset \mathbb{C}\mathbb{P}^3$ be a surface of degree d without ruled irreducible components and \mathcal{L} the set of lines on S . Then*

- (1) \mathcal{L} contains at most $d(11d - 24)$ lines and
- (2) $I(\mathcal{L}) \leq \frac{1}{2}d(11d - 24)(12d - 28) + d(11d - 24) \leq 66d^3$. □

The above bound does not involve m , so it is best when the degree of the surface S is small compared to the number of lines. When $\deg S$ is close to the bound $\sqrt{6m}$ given in Lemma 10.1, we get a better estimate using Lemma 10.2.

Proposition 22. *Let \mathcal{L} be a set of m distinct lines in \mathbb{P}^3 and $S \subset \mathbb{P}^3$ a minimal degree surface containing $[\mathcal{L}]$. Assume that S is irreducible and has degree d . Then $I(\mathcal{L}) \leq 3m(d + \frac{6m}{d})$.*

Proof. By Lemma 10, there is another surface T of degree $\leq \frac{6m}{d}$ that contains $[\mathcal{L}]$. Applying Proposition 14.2 to S, T we get our bound. □

Corollary 23. *Let \mathcal{L} be a set of m distinct lines in \mathbb{P}^3 and $S \subset \mathbb{P}^3$ a minimal degree surface containing $[\mathcal{L}]$. Assume that S is irreducible and non-ruled. Then $I(\mathcal{L}) \leq 26.6 \cdot m^{3/2}$.*

Proof. Set $d := \deg S$ and write it as $d = \alpha\sqrt{m}$. Note that $\alpha \leq \sqrt{6}$ by Lemma 10. Both Corollary 21 and Proposition 22 give bounds, thus

$$I(\mathcal{L}) \leq \min\{66\alpha^3, 3(\alpha + \frac{6}{\alpha})\} \cdot m^{3/2}.$$

The minimum reaches its maximum when the two quantities are equal. This happens at $\alpha_0 = \sqrt{6/11} \approx 0.738$ and $66\alpha_0^3 < 26.6$. □

24 (Adding up). Starting with m distinct lines \mathcal{L} , let S be the smallest degree surface that contains $[\mathcal{L}]$. Note that each irreducible component $S_i \subset S$ has minimal degree among those surfaces that contain every line of \mathcal{L}_i (as in Paragraph 18). We have 4 sources of intersection points.

External intersections (18) contribute $\leq \frac{1}{2}md$, planes and quadrics (19) contribute $\leq \frac{1}{2}cm^{3/2}$ and the other ruled surfaces (20) contribute $\leq \frac{1}{2}md + 2m$.

Let $\{S_i : i \in I\}$ be the non-ruled irreducible components and m_i denote the number of lines in \mathcal{L}_i . By Corollary 23 these lines have at most $26.6m_i^{3/2}$ intersections with each other. Thus the non-ruled irreducible components contribute at most

$$\sum_{i \in I} 26.6m_i^{3/2} \leq 26.6m^{3/2}.$$

So the total number of intersection points is at most

$$\frac{1}{2}md + \frac{1}{2}cm^{3/2} + \frac{1}{2}md + 2m + 26.6m^{3/2}$$

Since $d \leq \sqrt{6m} - 2$ by Lemma 10, this is at most

$$(\sqrt{6} + 26.6 + \frac{1}{2}c)m^{3/2} < (29.1 + \frac{c}{2})m^{3/2}.$$

This completes the proof of Theorem 6. \square

3. COUNTING INCIDENCES OVER \mathbb{C}

25. Let \mathcal{L} be a set of m distinct lines and \mathcal{P} a set of n distinct points in \mathbb{P}^3 . Instead of $I(\mathcal{L}, \mathcal{P})$ it is more convenient to work with the smaller quantity

$$I^\circ(\mathcal{L}, \mathcal{P}) := \sum_{p \in [\mathcal{L}] \cap \mathcal{P}} (r(p) - 1) \quad (25.1)$$

which is better suited to induction thanks to the subadditivity property:

$$I^\circ(\mathcal{L} \cup \{\ell\}, \mathcal{P}) \leq I^\circ(\mathcal{L}, \mathcal{P}) + \#([\mathcal{L}] \cap \ell) \quad \text{provided } \ell \notin \mathcal{L}. \quad (25.2)$$

The two variants are related by the formula $I(\mathcal{L}, \mathcal{P}) = I^\circ(\mathcal{L}, \mathcal{P}) + \#([\mathcal{L}] \cap \mathcal{P})$.

As a preliminary step toward proving Theorem 2 we reduce to the case when every line meets \mathcal{P} in many points.

26 (Lines with few points). Assume that under the assumptions of Theorems 2 or 4 we want to prove a bound of the form

$$I(\mathcal{L}, \mathcal{P}) \leq mA(n) + (c^2m)B(n) + C(n) \quad (26.1)$$

for some functions $A(n), B(n), C(n)$. Let $\ell \in \mathcal{L}$ be a line that meets \mathcal{P} in $\leq A(n)$ points. Remove ℓ from \mathcal{L} . Note that we may need to increase c to $c(\frac{m}{m-1})^{1/2}$. Thus the left hand side of (26.1) decreases by $\leq A(n)$ and the right hand side by

$$m(A(n) + c^2B(n)) - (m-1)(A(n) + c^2\frac{m}{m-1}B(n)) = A(n).$$

Hence it is sufficient to prove (26.1) for line/point configurations where every line meets \mathcal{P} in $> A(n)$ points.

This step makes the proof less direct. In Section 2 we just wrote down the estimates and got a final result. Here we need to know in advance the final result we aim at and use the corresponding value of $A(n)$.

27 (Decomposing S and \mathcal{P}). Let S be a surface of smallest possible degree that contains our set of n distinct points \mathcal{P} . By Lemma 11 we know that $d := \deg S \leq \sqrt[3]{6n}$.

We would like to ensure that S contains all the lines in \mathcal{L} . If a line ℓ is not contained in S then, by Bézout, it meets S in at most $d \leq \sqrt[3]{6n}$ points. Thus if ℓ passes through more than $\sqrt[3]{6n}$ points of \mathcal{P} then $\ell \subset S$. This suggests that we use (26) with $A(n) = 1.82n^{1/3} > \sqrt[3]{6n}^{1/3}$. Thus we may assume that each line in \mathcal{L} contains $\geq 1.82n^{1/3}$ points of \mathcal{P} hence $[\mathcal{L}]$ is contained in S .

We will also need to divide the points among the irreducible components of S . Let $S_i \subset S$ be an irreducible component of degree d_i . Let $\mathcal{P}_i^* \subset \mathcal{P}$ denote the subset

of points that are on S_i but not on any other irreducible component of S . There is at most 1 component, call it S_0 , for which $|\mathcal{P}_0^*| > \frac{1}{2}n$. Let $\mathcal{P}_0 \subset \mathcal{P}$ denote the subset of points that are on S_0 ; for $i \neq 0$ set $\mathcal{P}_i = \mathcal{P}_i^*$. The \mathcal{P}_i are disjoint subsets of \mathcal{P} , thus $\sum n_i \leq n$ where $n_i := |\mathcal{P}_i|$. Since S has minimal degree, we know that each S_i has minimal degree among those surfaces that contain \mathcal{P}_i , hence $d_i \leq \sqrt[3]{6n_i}$.

Next we use S to estimate $I^\circ(\mathcal{L}, \mathcal{P})$. As before we try to find another surface T that contains \mathcal{L} but does not contain S or at least some of the irreducible components of S .

28 (Contributions from singular points of S). We start with lines contained in $\text{Sing } S$ and their intersection points. If S is defined by an equation $(f(x_0, \dots, x_3) = 0)$ then we can take T to be defined by a general linear combination

$$\sum_i a_i \frac{\partial f}{\partial x_i} = 0.$$

Thus $\deg T = d - 1$ and, using (14.2), we get a contribution to $I^\circ(\mathcal{L}, \mathcal{P})$ that is $\leq \frac{1}{2}d(d-1)(2d-3) \leq d^3 \leq 6n$. This is the contribution from lines that are contained in $\text{Sing } S$.

We can do better using (14.3) which says that

$$\sum (r(p) - 1)^{3/2} \leq 6\sqrt{2} \cdot n. \quad (28.1)$$

Since we have at most n summands on the left, the convexity of $x^{3/2}$ implies that

$$\sum (r(p) - 1) \leq (6\sqrt{2})^{2/3} \cdot n < 4.2n. \quad (28.2)$$

Now we add to this lines ℓ_i not contained in $\text{Sing } S$ one at a time. Each line intersects $\text{Sing } S$ in at most $d - 1$ points. Repeatedly using (25.2) we get a contribution of $\leq m(d - 1)$.

These two account for all the contributions in (25.1) coming from those points of \mathcal{P} that are singular on S .

29 (Contributions from smooth points of S ; ruled case). A smooth point is contained in a unique irreducible component of S , thus we can treat the irreducible components separately. We start with the ruled components.

29.1 *Planes*. By assumption, each plane contributes $\leq \frac{1}{2}c^2m$. Since there are $\leq d$ planes, all together they contribute $\leq \frac{1}{2}c^2md$.

29.2 *Other ruled surfaces*. On a smooth quadric, there are 2 lines through each point. On other ruled surfaces there is usually only 1 line through a smooth point, except on the special lines (54) when there can be 2 by (55). Thus we get a total contribution $\leq n$.

As in Section 2, we again use 2 methods to control non-ruled irreducible components.

30 (Contributions from smooth points of S ; non-ruled case I). Let $S_i \subset S$ be a non-ruled irreducible component of degree d_i . As we noted in (27), $d_i \leq \sqrt[3]{6n_i}$.

By Theorem 13 there is another surface T_i of degree $\leq 11d_i$ that contains every line lying on S_i . Using (14.4) we get that

$$\sum_i^{(sm)} (r(p) - 1)^2 \leq 11 \cdot 12 \cdot d_i^3, \quad (30.1)$$

where summation is over all smooth points of S_i that are in \mathcal{P}_i . Since $d_i^3 \leq 6n_i$ and $\sum n_i \leq n$, adding these up gives that

$$\sum_S^{(sm)} (r(p) - 1)^2 \leq 11 \cdot 12 \cdot 6 \cdot n.$$

where summation is over all smooth points of the non-ruled irreducible components of S that are in \mathcal{P} . Since we have at most n summands on the left, by Cauchy–Schwartz

$$\sum_S^{(sm)} (r(p) - 1) \leq \sqrt{11 \cdot 12 \cdot 6} \cdot n < 28.2 \cdot n. \quad (30.2)$$

31 (First estimate). Adding these together we get that

$$\begin{aligned} I(\mathcal{L}, \mathcal{P}) &\leq n + I^\circ(\mathcal{L}, \mathcal{P}) \\ &\leq n + 4.2n + m(d-1) + \frac{1}{2}c^2md + n + 28.2n \\ &\leq \left(1 + \frac{1}{2}c^2\right)md + 34.6n \\ &\leq \sqrt[3]{6}\left(1 + \frac{1}{2}c^2\right)mn^{1/3} + 34.6n. \end{aligned} \quad (31.1)$$

This is different from the bound claimed in Theorem 2. The coefficient of $mn^{1/3}$ is smaller but the coefficient of n is bigger. For some applications this may be a better variant but (31.1) gives a worse constant for Corollary 5.

We need to look at the non-ruled components again.

32 (Contributions from smooth points of S ; non-ruled case II). Here we are aiming to get an estimate as in (26.1) with $A(n) = 3.66n^{1/3}$ which is chosen to be an upper bound for $\sqrt{6\sqrt[3]{11}n^{1/3}}$. Thus we may assume that each line contains $\geq 3.66n^{1/3}$ points of \mathcal{P} .

Write $d_i = \alpha_i \cdot n_i^{1/3}$ and note that $\alpha_i \leq \sqrt[3]{6}$. We improve the previous estimate if $\alpha_i \geq 1/\sqrt[3]{11}$. By Lemma 11 there is a surface T_i of degree $\leq \sqrt{6n_i/d_i} = \sqrt{6/\alpha_i} \cdot n_i^{1/3}$ that contains \mathcal{P}_i but not S_i .

If $i = 0$ then every line in \mathcal{L} that is contained in S_0 meets \mathcal{P} , and hence also $\mathcal{P}_0 = \mathcal{P} \cap S_0$, in at least $3.66n^{1/3}$ points. Thus these lines are also contained in T_0 .

If $i > 0$ then let $T^{(i)}$ be the surface obtained from S by replacing S_i with T_i . Note that $T^{(i)}$ contains \mathcal{P} and its degree is

$$\leq \sqrt{6/\alpha_i} n_i^{1/3} + d - d_i \leq (\sqrt{6/\alpha_i} - \alpha_i) n_i^{1/3} + \sqrt[3]{6} n^{1/3}.$$

Since $n_i < \frac{1}{2}n$, this is less than $3.66n^{1/3}$. Thus $T^{(i)}$ contains \mathcal{L} and hence T_i contains every line in \mathcal{L} that is not contained in any other S_j .

Since $\alpha_i \leq \sqrt[3]{6}$, this gives a bound

$$\begin{aligned} \sum_i^{(sm)} (r(p) - 1)^2 &\leq \alpha_i n_i^{1/3} \sqrt{6/\alpha_i} n_i^{1/3} (\alpha_i n_i^{1/3} + \sqrt{6/\alpha_i} n_i^{1/3}) \\ &= (6 + \sqrt{6}\alpha_i^{3/2}) n_i \leq 12n_i. \end{aligned} \quad (32.1)$$

If $\alpha_i \leq 1/\sqrt[3]{11}$ then $d_i \leq (1/\sqrt[3]{11})n_i^{1/3}$ and so (30.1) and (32.1) together show that

$$\sum_i^{(sm)} (r(p) - 1)^2 \leq 12n_i \quad (32.2)$$

holds for every non-ruled surface. Adding up all cases gives that

$$\sum_S^{(sm)} (r(p) - 1)^2 \leq 12n. \quad (32.3)$$

As before, by Cauchy–Schwartz this implies that $\sum_S^{(sm)} (r(p) - 1) \leq \sqrt{12}n$.

33 (Final estimate II). Adding these together we get that

$$\begin{aligned} I(\mathcal{L}, \mathcal{P}) &\leq n + I^\circ(\mathcal{L}, \mathcal{P}) \\ &\leq n + 2 \cdot 3^{2/3}n + m(d-1) + \frac{1}{2}c^2md + n + 12^{1/2}n \\ &\leq \left(1 + \frac{1}{2}c^2\right)md + 9.9n \\ &\leq \sqrt[3]{6}\left(1 + \frac{1}{2}c^2\right)mn^{1/3} + 9.9n. \end{aligned} \quad (33.1)$$

There is one place where it is easy to improve the estimate. Assume that there are xn points used in (28.1), yn points used in (29.2) and zn points used in (32.3). Then $x + y + z \leq 1$ and the total contribution coming from these points is at most

$$(2 \cdot 3^{2/3} x^{1/3} + y + 2 \cdot 3^{1/2} z^{1/2}) \cdot n \quad \text{where } x + y + z \leq 1.$$

A straightforward computation using Lagrange multipliers shows that this is always $\leq 5.76n$. Thus we get that

$$I(\mathcal{L}, \mathcal{P}) \leq \sqrt[3]{6}(1 + \frac{1}{2}c^2)mn^{1/3} + 6.76n. \quad (33.2)$$

Note however, that we assumed that each line contains $\geq 3.66n^{1/3}$ points of \mathcal{P} . By the first reduction step (26) this requires us to have $A(n) \geq 3.66n^{1/3}$ in (26.1), thus we can not use the smaller coefficient $\sqrt[3]{6} \leq 1.82$ in general. Thus we can only conclude that

$$I(\mathcal{L}, \mathcal{P}) \leq \max\{3.66, 1.82 + 0.91c^2\}mn^{1/3} + 6.76n. \quad (33.3)$$

This is stronger than Theorem 2. \square

34 (Proof of Corollary 5). We start with the characteristic 0 case. Note that $50 > 3.66^3$, thus if $n < \frac{1}{50}m^{3/2}$ then $\sqrt{m} > 3.66n^{1/3}$. This means that we do not need to go through the first reduction step (26), hence the stronger conclusion (33.2) applies.

Choose x such that $n = \frac{1}{x^3}m^{3/2}$. Since we assume that $c = 1$, (33.2) becomes

$$I(\mathcal{L}, \mathcal{P}) \leq (2.73\frac{1}{x} + 6.76\frac{1}{x^3})m^{3/2}.$$

We compute that if $x^3 \geq 50$ then $2.73\frac{1}{x} + 6.76\frac{1}{x^3} < 1$ hence $I(\mathcal{L}, \mathcal{P}) < m^{3/2}$. On the other hand, by assumption each line contributes at least $m^{1/2}$, hence $I(\mathcal{L}, \mathcal{P}) \geq m^{3/2}$. This is a contradiction if $n < \frac{1}{50} \cdot m^{3/2}$.

The positive characteristic case follows from Theorem 4 similarly. For large m the proof gives a coefficient $\geq \frac{1}{13}$; the smaller value $\frac{1}{20}$ and the $m \geq 10^4$ assumption are there to account for the contribution of the two lower degree terms in (4.1). \square

4. COUNTING INCIDENCES OVER \mathbb{F}_q

In this section we work with arbitrary fields, but the main interest is understanding what happens over finite fields.

While Salmon's argument applies over any field, Monge's proof only works in characteristic 0. As a replacement, (53) shows that a surface of degree d without ruled irreducible components contains at most d^4 lines. The following example shows that this is quite sharp.

Example 35. Let q be a p -power and consider the surface

$$S_{q+1} := (x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1} = 0) \subset \mathbb{P}^3$$

over the field \mathbb{F}_q . Linear spaces on such *Hermitian hypersurfaces* have been studied in detail [Seg65, BC66]. These examples have also long been recognized as extremal for the Gauss map. The failure of Monge's theorem has been noted in [KP91] for surfaces and in [Wal56] for curves. See [Kle86, CRS08] for surveys of the Gauss map. Other extremal properties are discussed in [HK13]. Kleiman observed that the affine Heisenberg surface in [MT04, §8] is, after taking its closure in \mathbb{P}^3 , a Hermitian surface, so isomorphic, under an \mathbb{F}_q -linear transformation, to the surface above.

The configuration of lines on S_{q+1} is quite interesting.

- (1) S_{q+1} contains $(q+1)(q^3+1)$ lines, all defined over \mathbb{F}_{q^2} .
- (2) S_{q+1} contains $(q^2+1)(q^3+1)$ points in \mathbb{F}_{q^2} .
- (3) $\text{PSU}_4(q)$ acts transitively on the lines and on the \mathbb{F}_{q^2} -points.
- (4) There are $q+1$ lines through every \mathbb{F}_{q^2} -point.

All of these are easy to do by hand as in see [Seg65, BC66] or can be obtained from the general description of finite unitary groups; see for instance [Car72].

More generally consider any equation of the form

$$\sum_{0 \leq i, j \leq n} c_{ij} x_i^q x_j = 0. \quad (35.5)$$

If we substitute $x_i = a_i t + b_i s$ then we get

$$\sum_{ij} c_{ij} (a_i t + b_i s)^q (a_j t + b_j s) = \sum_{ij} c_{ij} (a_i^q t^q + b_i^q s^q) (a_j t + b_j s) = 0,$$

which involves only the monomials $t^{q+1}, t^q s, t s^q, s^{q+1}$. Thus, arguing as in (53), we expect many more lines than usual. It was proved by [Has36] that if the hyper-surface given by (35.5) is smooth then it is isomorphic to the Hermitian example, though the coordinate change is usually defined only over a field extension of \mathbb{F}_q .

The arguments in Section 3 are independent of the characteristic, save (30) where we started considering non-ruled irreducible components. We show below how to modify the estimates in (30–33) to work over any field.

36 (Contributions from smooth points of S ; non-ruled case I). Let $S_i \subset S$ be a non-ruled irreducible component of degree d_i and $\mathcal{P}_i \subset \mathcal{P}$ as in (27).

By Proposition 53 and Lemma 10 there is another surface T_i of degree $\sqrt{6d_i^3}$ that contains every line lying on S_i . Using (14.4) we get that

$$\sum_i^{(sm)} (r(p) - 1)^2 \leq 6d_i^7 + \sqrt{6}d_i^5, \quad (36.1)$$

where summation is over all smooth points of S_i that are in \mathcal{P}_i .

Assume for now that $d_i \leq n_i^{1/5}$. Then $6d_i^7 + \sqrt{6}d_i^5 \leq 6n_i^{7/5} + \sqrt{6}n_i$. Since we have at most n_i summands on the left, by Cauchy–Schwartz

$$\sum_S^{(sm)} (r(p) - 1) \leq \sqrt{6}n_i^{6/5} + \frac{1}{2}n_i. \quad (36.2)$$

37 (Contributions from smooth points of S ; non-ruled case II). Here we deal with the other possibility $d_i \geq n_i^{1/5}$ using (26) with $A(n) = \sqrt{6}n^{2/5}$.

By Lemma 11 there is a surface T_i of degree $\leq \sqrt{6n_i/d_i} \leq \sqrt{6}n_i^{2/5}$ that contains \mathcal{P}_i . If $i = 0$ then T_0 contains every line of \mathcal{L} that lies only on S_0 .

If $i > 0$ then $n_i \leq \frac{1}{2}n$ and, as in (32), we get a surface $T^{(i)}$ of degree

$$\leq \sqrt{6}n_i^{2/5} + d - d_i < \sqrt{6}n_i^{2/5} + \sqrt[3]{6}n^{1/3} < \sqrt{6}n^{2/5} \quad (37.2)$$

that contains \mathcal{P} . Therefore again T_i contains every line of \mathcal{L} that lies only on S_i . The rest of (32) works as before and we get that

$$\sum_i^{(sm)} (r(p) - 1) \leq \sqrt{12}n_i. \quad (37.3)$$

For $n_i \geq 2$ the right hand side of (36.2) is bigger than $\sqrt{12}n_i$, thus we can always use (36.2).

38 (Final estimate). Adding these together we get that

$$\begin{aligned} I(\mathcal{L}, \mathcal{P}) &\leq n + I^\circ(\mathcal{L}, \mathcal{P}) \\ &\leq n + m(d-1) + \frac{1}{3}c^2md + \\ &\quad 12^{2/3}n + n + \sqrt{6}n^{6/5} + \frac{1}{2}n. \end{aligned} \quad (38.1)$$

Note, however, that we have used (26) with $A(n) = \sqrt{6}n^{2/5}$, thus the leading term $\sqrt[3]{6}mn^{1/3}$ needs to be increased to $\sqrt{6}mn^{2/5}$, resulting in the final estimate

$$I(\mathcal{L}, \mathcal{P}) \leq 2.45mn^{2/5} + 2.45n^{6/5} + 0.91c^2mn^{1/3} + 6.74n. \quad (38.2)$$

This completes the proof of (4). As in (33) we could improve the coefficient of n a little but I see no immediate application for it. \square

39 (Bourgain's conjecture over finite fields). We prove in Corollary 40 that Theorem 6 holds over a field of characteristic p provided $p > \sqrt{m}$. This implies that Theorem 6 holds for all line configurations in $\mathbb{F}_p\mathbb{P}^3$ where p is a prime. For $p < \sqrt{m}$ the methods seem to yield only a weaker variant with exponent $7/4$.

Similarly, Theorem 2 holds in characteristic p provided $p > \sqrt[3]{6n}$. If we work over \mathbb{F}_q then $I(\mathcal{L}, \mathcal{P}) \leq q^3 + q^2 + q + 1$, hence the estimate (2.1) is obvious if $q + 1 \leq \sqrt[3]{6.76n}$. Thus Theorem 2 holds over \mathbb{F}_p . (Note that [EH13] gives counter examples over \mathbb{F}_{p^2} , building on [MT04].)

The key to these is that Monge's theorem holds in characteristic $p > 0$ if the degree is less than the characteristic. [Vol03, Thm.1] proves this for smooth surfaces but essentially the same argument works in general.

Corollary 40. *Let \mathcal{L} be a set of m distinct lines in $\mathbb{F}_q\mathbb{P}^3$ where $q = p^a$. Let c be a constant such that no plane (resp. no quadric) contains more than $c\sqrt{m}$ (resp. more than $2c\sqrt{m}$) of the lines.*

Assume that either $m < \frac{11}{6}p^2$ or $q = p$. Then the number of points where at least two of the lines in \mathcal{L} meet is $\leq (29.1 + \frac{c}{2}) \cdot m^{3/2}$.

Proof. In the proof of Theorem 6 we used Theorem 13 only during the proof of Corollary 23 where we applied it to a surface of degree $\leq \alpha_0\sqrt{m}$ with $\alpha_0 = \sqrt{6/11}$. If $m < \frac{11}{6}p^2$ then $\alpha_0\sqrt{m} < p$ hence, as noted above, Theorem 13 still applies.

If $q = p$ then we are done if $m < \frac{11}{6}p^2 = \frac{11}{6}q^2$. If $m \geq \frac{11}{6}q^2$ then we are done trivially since $\mathbb{F}_q\mathbb{P}^3$ has $q^3 + q^2 + q + 1$ points, hence there are at most $2m^{3/2}$ possible intersection points. \square

Example 41. Let $L_1, L_2 \subset \mathbb{P}^3$ be a pair of skew lines. For every point $p \in \mathbb{P}^3 \setminus (L_1 \cup L_2)$ there is a unique line ℓ_p passing through p that intersects both L_1, L_2 .

The picture becomes especially simple when we work over a field K and L_1, L_2 is a conjugate pair defined over a quadratic extension K'/K . Thus we get that $\mathbb{P}^3(K)$ is a disjoint union of lines naturally parametrized by the K' -points of L_1 . If $P \subset \mathbb{P}^3$ is a K -plane then $P \cap (L_1 \cup L_2)$ consists of 2 points; the line connecting them is the only line in our family that is contained in P .

For $K = \mathbb{F}_q$ we get a family of $q^2 + 1$ disjoint lines $\{\ell_i\}$ that cover $\mathbb{F}_q\mathbb{P}^3$.

A different pair of skew lines L'_1, L'_2 gives a different covering family of lines $\{\ell'_i\}$. If L_1, L_2, L'_1, L'_2 do not lie on a quadric surface then they have ≤ 2 common transversals. (These are sometimes K -lines, sometimes conjugate pairs.)

Thus if we have r different pairs of skew lines in general position then their union gives a family of m lines where

$$r(q^2 + 1) \geq m \geq r(q^2 + 1) - 2\binom{r}{2}.$$

The number of points where r lines meet is at least

$$q^3 + q^2 + q + 1 - 2(q + 1)\binom{r}{2}.$$

Thus, for $r \ll \sqrt{m}$ we have

$$m \text{ lines and } \asymp \frac{m^{3/2}}{r^{3/2}} \text{ } r\text{-fold intersections.}$$

Furthermore, any plane contains at most r of the lines.

Given any set of rq^2 lines in $\mathbb{F}_q\mathbb{P}^3$, in average r of them pass through a point and r of them are contained in a plane. The interesting aspect of the example is that for both of these, the expected value is the maximum.

All of these examples either cover a positive proportion of $\mathbb{F}_q\mathbb{P}^3$ or can be derived by a linear transformation from a configuration defined over a subfield of \mathbb{F}_q . It would be interesting if these turned out to be the only cases that behave differently from characteristic 0.

5. GENUS AND SINGULAR POINTS OF CURVES

42 (Hilbert polynomials). See [AM69, Chap.11] or [Har77, Sec.I.7] for proofs of the following results.

Let k be a field, $R := k[x_0, \dots, x_n]$ and $I \subset R$ a homogeneous ideal. The quotient ring R/I is graded, that is, it is the direct sum of its homogeneous pieces $(R/I)_d$. Hilbert proved that there is a polynomial $H_{R/I}(t)$, called the *Hilbert polynomial* of R/I such that

$$\dim(R/I)_d = H_{R/I}(d) \quad \text{for } d \gg 1. \quad (42.1)$$

If $X \subset \mathbb{P}^n$ is a closed algebraic subvariety and $I(X)$ the ideal of homogeneous polynomials that vanish on X then $H_{R/I(X)}(t)$ is also called the *Hilbert polynomial* of X and denoted by $H_X(t)$.

The degree of $H_{R/I}(t)$ equals the dimension of the corresponding variety $V(I)$ and the leading coefficient of $H_X(t)$ equals $\deg X / (\dim X!)$. The constant coefficient is the (holomorphic) *Euler characteristic* of X , denoted by $\chi(X, \mathcal{O}_X)$.

If $\dim X = 1$ then, for historical reasons, one usually uses the *arithmetic genus* $p_a(X) := 1 - \chi(X, \mathcal{O}_X)$. (If X is a smooth curve over \mathbb{C} (=Riemann surface), the arithmetic genus equals the topological genus.)

Let $g \in k[x_0, \dots, x_n]$ be homogeneous of degree a and set $H := (g = 0)$. It is easy to see that if g is not a zero-divisor on X then

$$H_{X \cap H}(t) = H_X(t) - H_X(t - a). \quad (42.2)$$

Assume next that we have hypersurfaces $H_i := (g_i = 0) \subset \mathbb{P}^n$ of degree a_i such that $B := H_1 \cap \dots \cap H_{n-1}$ has dimension 1. (Such a B is called a *complete intersection curve*.) Starting with

$$H_{\mathbb{P}^n}(t) = \binom{t+n}{n}, \quad (42.3)$$

and using (42.2) one can compute the Hilbert polynomial of B :

$$H_B(t) = \prod_i a_i \cdot t - \frac{1}{2} (\sum_i a_i - n - 1) \cdot \prod_i a_i; \quad (42.4)$$

see [Har77, Exrc.II.8.4]. Thus the arithmetic genus of B is

$$p_a(B) := 1 + \frac{1}{2}(\sum_i a_i - n - 1) \cdot \prod_i a_i. \quad (42.5)$$

The formulas (42.4–5) compute the Hilbert polynomial and the arithmetic genus scheme-theoretically, that is, we work with the Hilbert polynomial of the quotient ring $k[x_0, \dots, x_n]/(g_1, \dots, g_{n-1})$ and this ring may contain nilpotents.

As a simple example, consider $B = (xy - zt = 0) \cap (x(x + y) - zt = 0)$, the intersection of two hyperboloids. Then $x \in k[x, y, z, t]/(xy - zt, x(x + y) - zt)$ is non-zero yet $x^2 \in (xy - zt, x(x + y) - zt)$. The geometric picture is that B consists of 2 lines $L_1 \cup L_2 = (x = z = 0) \cup (x = t = 0)$, but B “counts” both with multiplicity 2. The ideal corresponding to $L_1 \cup L_2$ is $I(L_1 \cup L_2) = (x, zt)$.

Thus the ideals $(xy - zt, x(x + y) - zt)$ and (x, zt) define the same algebraic set. Given an algebraic curve $B \subset \mathbb{P}^n$, it is usually not hard to find some of the equations satisfied by B and to write down an ideal J that defines B set-theoretically. However, we can not compute the arithmetic genus of B using J .

It is usually much harder to write down the ideal $I(B)$ of *all* equations satisfied by B .

We prove the following basic inequality in the next section.

Proposition 43. *For $i = 1, \dots, n - 1$ let $H_i \subset \mathbb{P}^n$ be a hypersurface of degree a_i such that the intersection $B := H_1 \cap \dots \cap H_{n-1}$ is 1-dimensional. Let $C \subset B$ be a reduced subcurve. Then*

$$p_a(C) \leq p_a(B) = 1 + \frac{1}{2}(\sum_i a_i - n - 1) \cdot \prod_i a_i.$$

44 (Arithmetic genus of a union of lines I). Let $C \subset \mathbb{P}^n$ be a union of m lines L_i . We compute its Hilbert polynomial in 2 ways. Let $I \subset k[x_0, \dots, x_n]$ be the ideal of all homogeneous polynomials that vanish on C . Then, for $d \gg 1$, $H_C(d) = md + 1 - p_a(C)$ is the dimension of the quotient

$$W_C(d) := \frac{(\text{degree } d \text{ homogeneous polynomials in } k[x_0, \dots, x_n])}{(\text{degree } d \text{ homogeneous polynomials that vanish on } C)}.$$

Let $W_{\mathbb{P}^n}(d)$ denote the vector space of degree d homogeneous polynomials on \mathbb{P}^n . Let $\bar{C} := \coprod L_i$ denote the disjoint union of the lines L_i and $\pi : \bar{C} \rightarrow C$ the natural map.

A degree d homogeneous polynomial in $k[x_0, \dots, x_n]$ restricts to a degree d homogeneous polynomial on each $L_i \cong \mathbb{P}^1$. This gives a restriction map

$$\text{rest}_d : W_{\mathbb{P}^n}(d) \hookrightarrow \sum_{i=1}^m W_{L_i}(d) \cong \sum_{i=1}^m W_{\mathbb{P}^1}(d) \cong k^{m(d+1)} \quad (44.2)$$

that induces an injection

$$W_C(d) \rightarrow \sum_{i=1}^m W_{L_i}(d). \quad (44.3)$$

The linear terms of the Hilbert polynomials of the two sides of (44.3) are equal, hence we conclude that

$$p_a(C) = \dim(\text{coker}(\text{rest}_d)) - m + 1 \quad \text{for } d \gg 1. \quad (44.4)$$

We aim to rewrite the cokernel of rest_d in terms of intersection points of the lines. We start with the special case when there is only 1 intersection point. The general formula will then be just a sum of such local terms.

45 (Local genus formula). Let $C_r^n \subset \mathbb{A}^n \subset \mathbb{P}^n$ be a union of r lines L_i through the origin.

A (parametrized, affine) line is given by $q : t \mapsto (a_1 t, \dots, a_n t)$ and the corresponding restriction map is $q^* : f(x_1, \dots, x_n) \mapsto f(a_1 t, \dots, a_n t)$. Given r different lines through the origin corresponds to r maps q_i^* . A homogeneous polynomial of degree d on \mathbb{P}^n can be identified with a polynomial of degree $\leq d$ on \mathbb{A}^n . Thus the cokernel of rest_d is identified with

$$\dim \text{coker} \left(k[x_1, \dots, x_n]_{(d)} \xrightarrow{\oplus q_i^*} \oplus_{i=1}^r k[t_i]_{(d)} \right) \quad (45.1)$$

where the subscript (d) denotes the subspace of polynomials of degree $\leq d$.

It is best to think of $k[x_1, \dots, x_n]_{(d)}$ as the vector space of degree $\leq d$ Taylor polynomials on $0 \in \mathbb{A}^n$ and $k[t_i]_{(d)}$ as the vector space of degree $\leq d$ Taylor polynomials on $0 \in L_i$.

Fix a line L_1 . For every other line L_i pick a linear form ℓ_i that vanishes on L_i but not on L_1 . Set $g_1 = \prod_{i>1} \ell_i$. Thus the image of g_1 under $\oplus q_i^*$ is zero in the summands $k[t_i]$ for $i > 1$ and equals (non-zero constant) $\cdot t_1^{r-1}$ in $k[t_1]$. Therefore the cokernel of $\oplus q_i^*$ stabilizes for $d \geq r - 1$. This gives the local genus formula

$$\delta(0 \in C_r^n) := \dim \text{coker} \left(k[x_1, \dots, x_n]_{(d)} \xrightarrow{\oplus q_i^*} \oplus_{i=1}^r k[t_i]_{(d)} \cong k^{r(d+1)} \right) \quad (45.2)$$

which holds for all $d \geq r - 1$.

Since the q_i^* preserve the degree, we can compute the cokernel one degree at a time. In degree 0 there are just the constants in $k[x_1, \dots, x_n]$ but r copies of the constants in the target in (45.2). Thus

$$\delta(0 \in C_r^n) \geq r - 1. \quad (45.3)$$

This leads to the weakest estimate (14.2).

In degree j we have $\binom{j+n-1}{n-1}$ monomials of degree i in $k[x_1, \dots, x_n]$ and r copies of t_i^j in the target. Therefore

$$\delta(0 \in C_r^n) \geq \sum_j \left[r - \binom{j+n-1}{n-1} \right] \quad (45.4)$$

where we sum over those $j \geq 0$ for which the quantity in the brackets is positive. (It is not hard to see that equality holds if the lines are in general position, but this is not important for us.) For $n = 2$ this sum can be easily computed and we get that

$$\delta(0 \in C_r^2) = \binom{r}{2}. \quad (45.5)$$

(See [Sha74, Sec.IV.4.1] for a different way of computing this.) This leads to the strongest estimate (14.4).

If $n = 3$ then there is no convenient closed form and the precise values depend on the position of the lines. For small values of r we get $\delta(0 \in C_2^3) \geq 1$, $\delta(0 \in C_3^3) \geq 2$, $\delta(0 \in C_4^3) \geq 4$ and $\delta(0 \in C_5^3) \geq 6$. It is easy to show that

$$\delta(0 \in C_r^3) \geq \frac{1}{\sqrt{2}} (r - 1)^{3/2}, \quad (45.6)$$

with equality holding only for $r = 3$.

46 (Arithmetic genus of a union of lines II). Continuing the discussion of (44), pick any singular point $p \in C$. Let $C(p) \subset C$ denote the union of the lines passing

though p . As in (45), after choosing an affine chart and coordinates we get maps between the spaces of Taylor polynomials

$$\text{Taylor}_d(p \in \mathbb{A}^n) \xrightarrow{\oplus q_i^*} \oplus_i \text{Taylor}_d(p \in L_i). \quad (46.1)$$

whose cokernel has dimension $\delta(p \in C(p))$. We can sum these over all singular points $\text{Sing } C$ of C to get maps

$$\text{LocRest}_d : W_{\mathbb{P}^n}(d) \rightarrow \bigoplus_{p \in \text{Sing } C} \oplus_i \text{Taylor}_d(p \in L_i). \quad (46.2)$$

Note that LocRest_d factors through rest_d . Indeed, the map from $W_{L_i}(d)$ to the right hand side of (46.2) is obtained by starting with a degree d homogeneous polynomial h on the line L_i and for each singular point $p \in L_i$ sending it to the degree $\leq d$ part of its Taylor expansion at p . Each line contains at most $r - 1$ singular points thus these maps are surjective for $d > (r - 1)^2$. This shows that

$$\dim \text{coker LocRest}_d \geq \sum_{p \in \text{Sing } C} \delta(p \in C(p)). \quad (46.3)$$

Combining (46.3) with the local bounds (45.3–6) completes the proof of Proposition 14 for unions of lines once we prove Proposition 43.

Remark 47. For any (proper, reduced) algebraic curve C there is a similar formula for the difference between the arithmetic genus of C and the arithmetic genus of its normalization \bar{C} in terms of local invariants computable from the singular points. (These local terms are denoted by $\ell(\bar{\mathcal{O}}_x/\mathcal{O}_x)$ in [Sha74, Vol.1,p.262].)

If $(p \in C)$ is an analytically irreducible curve singularity of multiplicity r in \mathbb{C}^3 then $\delta(p \in C) \geq \lfloor r^2/4 \rfloor$. Thus singularities with smooth branches have the smallest genus for fixed multiplicity.

6. ARITHMETIC GENUS OF SUBCURVES

The proof of Proposition 43 uses basic sheaf cohomology theory. Everything we need is in Sections III.1–5 of [Har77], though the key statements are exercises.

First we use the cohomological interpretation of the constant term of the Hilbert polynomial as the holomorphic Euler characteristic. This is a short argument.

Lemma 48. [Har77, Exrc.III.5.2] *Let $I \subset k[x_0, \dots, x_n]$ be a homogeneous ideal such that the corresponding scheme $C := V(I) \subset \mathbb{P}^n$ is 1-dimensional. Then*

- (1) $h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = H_C(0)$ and hence
- (2) $p_a(C) = h^1(C, \mathcal{O}_C) - h^0(C, \mathcal{O}_C) + 1.$ □

For complete intersection curves we need the following; this is a longer exercise.

Lemma 49. [Har77, Exrc.III.5.5] *For $i = 1, \dots, n-1$ let $H_i \subset \mathbb{P}^n$ be a hypersurface of degree a_i . Assume that the intersection $B := H_1 \cap \dots \cap H_{n-1}$ is 1-dimensional. Then*

- (1) $h^0(B, \mathcal{O}_B) = 1$ and hence
- (2) $p_a(B) = h^1(B, \mathcal{O}_B).$ □

50 (Proof of Proposition 43). We have a scheme theoretic intersection B and a reduced subcurve $C \subset B$ which is defined by an ideal sheaf $J_C \subset \mathcal{O}_B$. The exact sequence

$$0 \rightarrow J_C \rightarrow \mathcal{O}_B \rightarrow \mathcal{O}_C \rightarrow 0$$

gives

$$H^1(B, \mathcal{O}_B) \rightarrow H^1(C, \mathcal{O}_C) \rightarrow H^2(B, J_C) = 0;$$

the last vanishing holds since H^2 is always zero on a curve; cf. [Har77, III.2.7]. Thus $h^1(C, \mathcal{O}_C) \leq h^1(B, \mathcal{O}_B)$.

Since C is reduced, $h^0(C, \mathcal{O}_C)$ equals the number of connected components of C . Thus, by Lemma 48.2 and Lemma 49.2,

$$p_a(C) = h^1(C, \mathcal{O}_C) - h^0(C, \mathcal{O}_C) + 1 \leq h^1(C, \mathcal{O}_C) \leq h^1(B, \mathcal{O}_B) = p_a(B). \quad \square$$

Remark 51. I tried to find a more elementary proof of Proposition 43 but so far I have been unsuccessful. There is a vast classical literature on curves in \mathbb{P}^3 , but most of it studies smooth or only mildly singular curves.

Let X be a normal, projective variety of dimension n and H_1, \dots, H_{n-1} hyperplane sections such that $B := H_1 \cap \dots \cap H_{n-1}$ is 1-dimensional. There is a formula similar to (42.5) that computes the genus of B if B is smooth. However, when B is singular and $C \subset B$ is a reduced subcurve, it can happen that the arithmetic genus of C is bigger than the arithmetic genus of B . Thus Proposition 43 is a special property of \mathbb{P}^n . However, in all the examples that I computed, the arithmetic genus of C is not much bigger than the arithmetic genus of B .

Chasing through the proofs of (49), the key property seems to be that Kodaira's vanishing theorem holds for \mathbb{P}^n .

7. RULED SURFACES

The referee pointed out that information about ruled surfaces is hard to extract from the current literature, so here I summarize the pertinent facts with proofs.

We are interested in the geometry of ruled surfaces, thus in this section we work over an algebraically closed field K , though almost everything works over any infinite field.

Definition 52. A *smooth minimal ruled surface* is a projective surface M with a morphism $g : M \rightarrow C$ to a smooth curve all of whose fibers, also called *rulings*, are (isomorphic to) lines.

A *ruled surface* is a projective surface $S \subset \mathbb{P}^n$ that is the image of a smooth ruled surface M by a morphism $\pi : M \rightarrow S$ that sends the rulings to lines. We call

$$C \xleftarrow{g} M \xrightarrow{\pi} S$$

a *presentation* of S . If π is birational, we call it a *birational presentation*. Any surface in \mathbb{P}^n can be birationally projected to \mathbb{P}^3 , so we focus on surfaces in \mathbb{P}^3 .

We will show that every ruled surface has a birational presentation, and, with two exceptions, the birational presentation is unique. Thus birationality is frequently part of the definition. (Note that the literature is inconsistent. Sometimes a ruled surface means a smooth minimal ruled surface, a ruled surface as above or any surface that is birational to a ruled surface.)

Proposition 53. *Let $S \subset \mathbb{P}^3$ be an irreducible surface of degree d . Then*

- (1) *either S contains at most d^4 lines*
- (2) *or S is ruled.*

Proof. First we use affine coordinates. A typical line on S can be given parametrically as $t \mapsto (a_1t + b_1, a_2t + b_2, t)$. If $f(x, y, z) = 0$ is an affine equation of S , such a line is contained in S iff

$$f(a_1t + b_1, a_2t + b_2, t) \equiv 0.$$

Expanding by the powers of t , we get a system of $d + 1$ equations of degree $\leq d$ in the variables a_1, b_1, a_2, b_2 . By Bézout, the system either has at most d^4 solutions (leading to the first case) or the solution set contains an algebraic curve $C \subset \mathbb{A}^4$ (with a_1, b_1, a_2, b_2 as coordinates). In this case

$$\pi : C \times \mathbb{A}^1 \rightarrow S \quad \text{given by} \quad (c, t) \mapsto (a_1t + b_1, a_2t + b_2, t)$$

is a rational map from an (affine) ruled surface to S . There could be several such curves C and the resulting map π need not be birational, but we do get at least 1 rational presentation of S .

A few details need to be ironed out. In general, C is neither smooth nor projective. Thus one should work with the Grassmannian parametrizing all lines in \mathbb{P}^3 ; see [Sha74, Vol.1,p.42]. Then we have to normalize C to get a smooth ruled surface mapping onto S . \square

54 (Special ruled surfaces). There are 3 types of ruled surfaces that are exceptional for many of the results. These are

- (1) planes,
- (2) smooth quadrics and
- (3) cones.

The plane has infinitely many birational presentations given by the family of all lines passing through a given point. Correspondingly, the plane can be viewed as a cone in infinitely many ways. A smooth quadric has 2 birational presentations. A cone (that is not a plane) has a unique birational presentation but all the lines pass through the unique vertex. Every other ruled surface will be called *non-special*.

Fix a presentation $C \xrightarrow{g} M \xrightarrow{\pi} S$ and let $Z \subset S$ be any subset. Then $g(\pi^{-1}(Z)) \subset C$ is the set of rulings that meet Z in at least 1 point.

If $B = Z$ is an irreducible curve then $\pi^{-1}(B) \subset M$ is also a curve hence

- (4) either $g(\pi^{-1}(B)) \subset C$ consist of finitely many points, B is a ruling and only finitely many rulings of the given presentation intersect B ,
- (5) or $g(\pi^{-1}(B)) = C$, hence every ruling intersects B .

A line $L \subset S$ is called *special* (for the given presentation) if it intersects every ruling.

A point $p \in S$ is called *special* if it is either singular or it lies on a special line. (We will show that a non-special ruled surface has non-special points.)

Proposition 55. *Let $S \subset \mathbb{P}^3$ be a non-special ruled surface of degree d . Then*

- (1) *there are at most d lines through any point of S ,*
- (2) *there is exactly one line through a non-special point of S ,*
- (3) *a non-special ruled surface admits a unique birational presentation,*
- (4) *there are at most 2 special lines and*
- (5) *there are at most $d - 2$ non-special lines intersecting a non-special line.*

Proof. Assume that $p \in S$ is a point with infinitely any lines through it. Choose affine coordinates such that $p = (0, 0, 0)$ and S has equation $f(x, y, z) = \sum_i f_i(x, y, z)$ where f_i is homogeneous of degree i . A parametrized line $t \mapsto (at, bt, ct)$ lies on S iff $f(at, bt, ct)$ is identically 0. This holds iff $f_i(a, b, c) = 0$

for every i . By Bézout, there are either finitely many (in fact $\leq d(d-1)$) solutions or the f_i have a common (homogeneous) factor $h(x, y, z)$. Then h divides f hence the cone ($h = 0$) is an irreducible component of S . This is a contradiction since S is irreducible and not a cone. Thus $\pi : M \rightarrow S$ is everywhere finite-to-one.

Next we claim that any 2 special lines L_1, L_2 are disjoint. If not then they span a plane P . As we noted, only finitely many rulings pass through the point $L_1 \cap L_2$, hence every other ruling meets P in 2 points. Thus every other ruling is contained in P hence $S = P$.

Assume next that S contains 3 special lines L_1, L_2, L_3 . For a quadric it is 3 conditions to contain a line, hence there is a quadric Q that contains all 3 lines. Thus every ruling meets Q in at least 3 points hence is contained in Q . Thus $S = Q$, proving (4).

Let $p \in S$ be a smooth point and $p \in B \subset S$ a line. Since p is smooth, B is locally defined by 1 equation at p [Sha74, Vol.1,p.108], thus $\pi^{-1}(B)$ is locally defined by 1 equation at $\pi^{-1}(p)$. Thus $\pi^{-1}(B)$ is 1-dimensional at $\pi^{-1}(p)$ [Sha74, Vol.1,p.71]. Thus either B is a ruling passing through p or B is special. Thus there is exactly one line through a non-special point of S , proving (2). Hence in the construction of Proposition 53 the curve C is unique and the resulting $\pi : M \rightarrow S$ is the unique birational ruling of S , proving (3). (Strictly speaking, we have only proved that $\pi : M \rightarrow S$ is injective on a dense open subset. This implies birationality in characteristic 0. In positive characteristic we still need to exclude purely inseparable maps. Since this has no bearing on curve counts, we do not pursue this issue.)

In order to get precise bounds on the number of lines, we use intersection theory (56) on the smooth surface M for the family $\{H_\lambda\}$ of pull-backs of plane sections $S \cap P_\lambda \subset S$ of S .

First choose planes $P_1, P_2 \subset \mathbb{P}^3$ such that the line $P_1 \cap P_2$ meets S in d distinct smooth points. Then H_1 and H_2 meet at the preimages of these points and $m_p(H_1, H_2) = 1$ at each of them. Thus $(H_1 \cdot H_2) = d$.

Next let $p \in S$ be any point and choose planes P_1, P_2 such that the line $P_1 \cap P_2$ meets S at p but is not contained in S . As we noted in (54), the rulings passing through p correspond to the set $g(\pi^{-1}(p))$, hence its cardinality is at most $|\pi^{-1}(p)| = |H_1 \cap H_2|$. By (55.6), $|H_1 \cap H_2| \leq (H_1 \cdot H_2) = d$. Thus there are at most d rulings passing through p . We can arrange that $P_1 \cap P_2$ meets S in at least one more point; this shows that there are at most $d-1$ rulings passing through p and at most 1 special line, proving (1).

Finally let $L \subset S$ be a ruling and choose P_1, P_2 such that $L = P_1 \cap P_2$. The corresponding H_1, H_2 are reducible curves of the form $H_i = B_i + C_i$ where $B_i = \sum_j a_{ij} F_j$, $a_{ij} > 0$ and $F_j \subset M$ are the rulings such that $L = \pi(F_j)$. The other rulings intersecting L correspond to the points $C_1 \cap C_2$. As before, we compute the intersection number

$$\begin{aligned} |C_1 \cap C_2| &\leq (C_1 \cdot C_2) \\ &= ((H_1 - B_1) \cdot (H_2 - B_2)) \\ &= (H_1 \cdot H_2) - (H_1 \cdot B_2) - (B_1 \cdot H_2) + (B_1 \cdot B_2) \\ &\leq d - 2. \end{aligned}$$

This proves (5). □

56 (Intersecting curves on smooth surfaces). For proofs see [Sha74, Sec.IV.1] or [Har77, V.1].

Let X be a smooth, projective surface. Given two curves $A, B \subset X$, there is an intersection number $(A \cdot B)$ attached to them. This number is symmetric, bilinear and unchanged if we vary the curves in families. Furthermore, if $A \cap B$ is finite then

$$(A \cdot B) = \sum_{p \in A \cap B} m_p(A, B) \quad (56.1)$$

where each $m_p(A, B)$ is a positive integer. Furthermore $m_p(A, B) = 1$ iff A, B are both smooth at p and are not tangent there.

8. SKETCH OF THE PROOF OF THE MONGE–SALMON–CAYLEY THEOREM

57 (Salmon's flecnodal equation). Let us start with 3 homogeneous forms in 3 variables

$$\sum_{1 \leq i \leq 3} a_i x_i, \quad \sum_{1 \leq i \leq j \leq 3} b_{ij} x_i x_j, \quad \sum_{1 \leq i \leq j \leq k \leq 3} c_{ijk} x_i x_j x_k. \quad (57.1)$$

We want to understand when they have a common zero. We eliminate x_3 from the linear equation and substitute into the others to get 2 homogeneous forms in 2 variables

$$\sum_{1 \leq i \leq j \leq 2} B_{ij} x_i x_j, \quad \sum_{1 \leq i \leq j \leq k \leq 2} C_{ijk} x_i x_j x_k. \quad (57.2)$$

They have a common zero iff their discriminant vanishes. After clearing the denominator (which is a power of a_3) this gives an equation in the original variables a_i, b_{ij}, c_{ijk} . After a short argument about the a -variables we get the following.

Claim 57.3. There is a polynomial $F(, ,)$ such that $F(a_i, b_{ij}, c_{ijk}) = 0$ iff the 3 forms in (57.1) have a common (nontrivial) zero. Furthermore, F has multidegree $(6, 3, 2)$. \square

Consider now a surface $S \subset \mathbb{C}^3$ given by an equation $f(x_1, x_2, x_3) = 0$. Fix a point $p = (p_1, p_2, p_3) \in S$ and write the Taylor expansion of f around p as

$$f = \sum_{i=0}^d f_i(x_1 - p_1, x_2 - p_2, x_3 - p_3) \quad (57.4)$$

where f_i is homogeneous of degree i . A parametric line

$$t \mapsto (p_1 + m_1 t, p_2 + m_2 t, p_3 + m_3 t)$$

is a triple tangent iff

$$f_1(m_1, m_2, m_3) = f_2(m_1, m_2, m_3) = f_3(m_1, m_2, m_3) = 0. \quad (57.5)$$

By (57.3) this translates into an equation $F(a_i, b_{ij}, c_{ijk}) = 0$ in the coefficients of the f_i , which are in turn given by the i th partial derivatives of f .

Putting all together we get a polynomial

$$\text{Flec}_f(x_1, x_2, x_3) := F\left(\frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\right) \quad (57.6)$$

such that

$$f(x_1, x_2, x_3) = \text{Flec}_f(x_1, x_2, x_3) = 0 \quad (57.7)$$

defines the set of points of S where there is a triple tangent line. Furthermore, Flec_f has degree $\leq 6(d-1) + 3(d-2) + 2(d-3) = 11d - 18$ in x, y, z .

Note that the coefficients of the different f_i are not independent, thus one could end up with a lower degree polynomial. Salmon claims that in fact one gets a polynomial of degree $11d - 24$. I have not checked this part; in our applications we have used only that the degree is $\leq 11d$.

Note that when $\deg f = 3$, the Salmon bound is $11 \cdot 3 - 24 = 9$. A smooth cubic surface S contains 27 lines and their union is the complete intersection of S with a surface T of degree 9. So, in this case, the Salmon bound is sharp.

If a line is contained in S , then it is triply tangent everywhere, thus Flec_f vanishes on every line contained in S . This is useful only if Flec_f does not vanish identically on S . That is, we need to understand surfaces where every point has a triple tangent line. Monge proved that these are exactly the ruled surfaces. Monge writes a surface locally as a graph, thus from now on we work with holomorphic functions (over \mathbb{C}) or with C^3 -functions (over \mathbb{R}).

58 (Monge's theorem). Consider a graph $S := (z = f(x, y)) \subset \mathbb{C}^3$. Fix a point (x_0, y_0, z_0) . The line

$$(x_0 + t, y_0 + mt, z_0 + nt) \tag{58.1}$$

is a double tangent line of S iff $n = f_x(x_0, y_0) + f_y(x_0, y_0)m$ and

$$f_{xx}(x_0, y_0) + 2f_{xy}(x_0, y_0)m + f_{yy}(x_0, y_0)m^2 = 0. \tag{58.2}$$

The double tangent lines are also called *asymptotic directions*. By working on a smaller open set, we may assume that the Hessian of f has constant rank and is not identically 0. Thus the asymptotic directions define 2 vector fields on S . (Only 1 vector field if the rank is always 1.) Integrating these vector fields we get the *asymptotic curves* of the surface S .

The line (58.1) is a triple tangent if, in addition

$$f_{xxx}(x_0, y_0) + 3f_{xxy}(x_0, y_0)m + 3f_{xyy}(x_0, y_0)m^2 + f_{yyy}(x_0, y_0)m^3 = 0. \tag{58.3}$$

Thus the graph has a triple tangent iff the equations (58.2–3) have a common solution.

Claim 58.4. An asymptotic curve is a straight line iff all the corresponding asymptotic directions are triple tangents.

Proof. Assume that we have $u = u(t)$ defined by $a(t) + 2b(t)u + c(t)u^2 = 0$. By implicit differentiation, $u(t)$ is constant iff $a_t + 2b_t u + c_t u^2 \equiv 0$.

Assume next that $u = u(x, y)$ is defined by

$$a(x, y) + 2b(x, y)u + c(x, y)u^2 = 0$$

and we work along a path $(x(t), y(t))$. Then the condition becomes

$$a_x x' + (a_y y' + 2b_x u x') + (2b_y u y' + c_x u^2 x') + c_y u^2 y' \equiv 0.$$

In our case $a = f_{xx}, b = f_{xy}, c = f_{yy}$ and $u = y'/x'$ along the asymptotic curve. Substituting $y' = ux'$ and dividing by x' we get the condition

$$f_{xxx} + 3f_{xxy}u + 3f_{xyy}u^2 + f_{yyy}u^3 = 0,$$

which is the same as (58.3). □

See [MS84, 2.10] or [Tao14] for other variants of this argument.

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