A GEOMETRIC APPROACH TO INELASTIC COLLAPSE*

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Abstract. We show in this note how to interpret logarithmic spiral tilings as one-dimensional particle systems undergoing inelastic collapse. By deforming the spirals appropriately, we can simulate collisions among particles with distinct or varying coefficients of restitution. Our geometric constructions provide a strikingly simple illustration of a widely studied phenomenon in the physics of dissipative gases: the collapse of inelastic particles.

1 Introduction

Collisions in a granular gas preserve momentum but not kinetic energy. Interactions are dissipative, with the velocities of two colliding particles governed by a nonnegative matrix $\left(\begin{smallmatrix} p & q \\ q & p \end{smallmatrix}\right)$, for $p \leq 1/2$ and $p + q = 1$. When the coefficient of restitution, defined as $r = 1 - 2p$, is less than 1, the collisions are inelastic and the particles may collapse to a single point in a finite amount of time: this intriguing phenomenon of *inelastic collapse* was first investigated in one dimension by Bernu & Mazighi [2] and McNamara & Young [6]. Further studies and extensions to a larger number n of particles were given in [1, 2, 3, 4, 5, 6, 7, 8]. In the case $n = 3$, inelastic collapse requires $r < 7 - 4\sqrt{3}$ [4, 6, 7], while in general the requirement is that $n \gtrsim 2(\ln 2)/(1 - r)$. Matching constructions for large n exist but entail intricate eigenvalue estimates [1, 2]. We rederive these bounds by simple geometric means, and we also extend them to other types of collisions. Our particle systems are derived from one-dimensional projections of spiral tilings of a disk (see §2). Using different spirals allows the presence of particles with different coefficients of restitution (see §3). The notable feature of our arguments is to be entirely geometric.

2 The Inelastic Collapse of Identical Particles

We describe the dynamics of n identical particles moving towards the center of a disk and colliding along the way. The one-dimensional system is derived by projection to a line. We begin with the geometry of the system, which is a quadrilaterial tiling of the complex unit disk by logarithmic spirals.

2.1 Spiral tilings

Fix $0 < \lambda_o < 1$ and let $\mathcal{C}_\alpha = \left\{ \lambda_o^{|\varphi - \alpha|} e^{i\varphi} \,\middle|\, \varphi \in \mathbb{R} \right\}$. The curve $\mathcal{C}_\alpha$ consists of two logarithmic spirals running clockwise and counterclockwise from the point $e^{i\alpha}$. The family $\{\mathcal{C}_\alpha\}_{0 \leq \alpha < 2\pi}$

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forms two foliations of the unit complex disk $\mathcal{D}$ (minus the origin). Whereas no pair of spirals going in the same direction meet, the other pairs intersect infinitely often along the diameter bisecting their starting points. Fix an integer $n > 2$ and write $\theta = \pi/n$. We rectify the spiral $\mathcal{C}_\alpha$ by creating the vertices $\lambda_o^{|k\theta-\alpha|}e^{ik\theta}$ for all $k \in \mathbb{Z}$; then we join consecutive pairs by straightline segments, which produces the polygonal spiral $\mathcal{C}_\alpha^R$ in Figure 1(i).

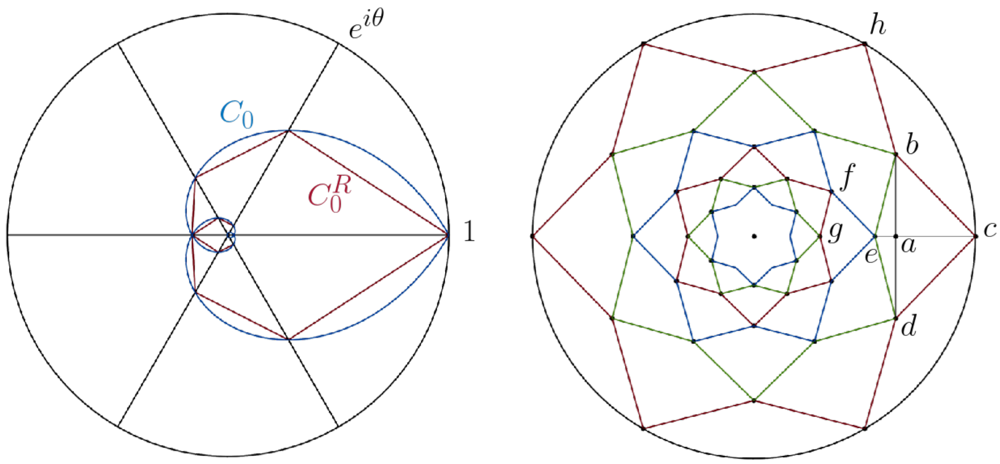

Figure 1: (i) The spirals $\mathcal{C}_\alpha$ and $\mathcal{C}_\alpha^R$, for $\alpha = 0$ and $\theta = \pi/3$; (ii) an (n, λ)-tiling for a system of $2n = 12$ colliding particles.

The collection of polygonal curves $\{\mathcal{C}_{2j\theta}^R \,|\, 0 \le j < n\}$ forms an infinite sequence of nested concentric similar $2n$-gons $P_k := \lambda e^{i\theta}P_{k-1}$, where $\lambda = \lambda_o^\theta$ and P_0 is the outer "star" shown in Figure 1(ii): its vertices $e^{il\theta}\lambda^{(1-(-1)^l)/2}$ run in counterclockwise order $(0 \le l < 2n)$. To ensure that the shape is indeed a star, every other vertex of P_0 needs to be reflex, which requires that $\lambda < \cos\theta$. This partitions the polygon P_0 into an infinite collection of similar convex quadrilaterals, which forms an (n, λ)-*tiling*. We define the *fundamental ratio* $\rho := ae/ac$ of the (n, λ)-tiling and justify its name by noting that it is independent of the polygon P_k used to define it. Referring to Figure 1(ii), we observe that $ac = 1 - \lambda\cos\theta$ and $ae = \lambda\cos\theta - \lambda^2$ and that, for any $0 < \lambda < \cos\theta$,

$$\rho = \frac{\lambda(\cos\theta - \lambda)}{1 - \lambda\cos\theta} \qquad \text{and} \qquad 0 < \rho < 1\,. \tag{1}$$

2.2 Particles traveling in a disk

Place two particles at each one of the n outer vertices of P_0 and set them in motion along the two incident edges with a speed equal to bc. We show below that the particles will zigzag toward the center (as in the trajectory $c, b, e, f, g, \ldots$) provided that the coefficient of restitution r is equal to $\rho < 1$, where $r = 1 - 2p$; recall that, whenever two particles with velocities $u, v \in \mathbb{C}$ collide, they bounce away from each other and update their velocities as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} \leftarrow \begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}; \tag{2}$$

where $0 < p < q < 1$ and $p + q = 1$.

Lemma 1. The $2n$ particles travel along the edges of the tiling through pairwise collisions if and only if the fundamental ratio ρ is equal to the coefficient of restitution r. If each particle spends one unit of time on the boundary ∂P_0, then it travels on ∂P_k for a duration of δ^k, where $\delta = \lambda^2/\rho$. The total travel time is bounded if and only if $\lambda < \frac{1}{\cos\theta} - \tan\theta$, in which case it is equal to $1/(1-\delta)$.

Proof. For convenience, we tilt the tiling by θ to put b and f on the X-axis (Figure 2). Two particles travel from c and h to b with velocity u and v respectively. The first one bounces at b and proceeds with velocity $u' = pu + qv$. Since $u_x = v_x$ and $u_y = -v_y$, we have $u'_x = u_x$ and $u'_y = -ru_y$; therefore $|\mathit{slope}(u')| = r|\mathit{slope}(u)|$. By similarity, bc and ef are parallel; hence $|\mathit{slope}(u')| = r|\mathit{slope}(ef)|$. The consistency of the particle collision with the tiling means that u' should be parallel to the segment be. The condition thus becomes $|\mathit{slope}(be)| = r|\mathit{slope}(ef)|$; hence $r = mf/mb = \rho$.

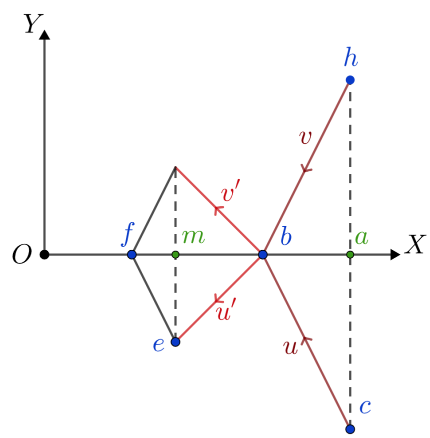

Figure 2: How colliding particles follow the edges of the (n, λ)-tiling. The coefficient of restitution must be equal to the ratio $\rho = mf/mb$.

If the particle travels from c to b in one unit of time, then $u_y = ac$ and $u'_y = -ru_y = -rac$. It follows that the time δ for the particle to bounce from b to e is equal to $me/|u'_y| = \frac{1}{r}me/ac = \lambda^2/r$. More generally, δ is the ratio between the time spent on be and that spent on cb. By symmetry, the same ratio δ holds between the travel times along any two consecutive edges on the trajectory. This follows from the fact that the travel time along an edge is itself a ratio length/speed and that, from one boundary ∂P_k to the next, ∂P_{k+1}, the ratio between consecutive lengths is independent of k and the same is true of consecutive speeds. This implies a travel time of δ^k on ∂P_k. Convergence implies that $\delta < 1$, which, by (1), means that λ must be less than the smaller root of $\lambda^2\cos\theta - 2\lambda + \cos\theta$ (since the larger one exceeds 1). This gives us the inequality $\lambda < (1-\sin\theta)/\cos\theta$. Note that this condition is not implied by the previous requirement that $0 < \lambda < \cos\theta$. $\square$

By (1), setting $r = \rho$ for any $\lambda < \cos\theta$ produces a valid particle system traveling inward through the (n, λ)-tiling. Of course, the interesting question is whether this holds for *any* value of the coefficient of restitution. We address this issue below in the context of one-dimensional systems.

2.3 One-dimensional collapse

The real parts of the $2n$ particles' positions in the unit disk $\mathcal{D}$ describe a one-dimensional particle system. To see why, notice the linear transition (2) governing collisions in the complex plane applies to both the real and the complex parts of the particle velocities, and constant velocities in the complex plane result in constant velocities along the real axis. It is useful to distinguish between the *positive* particles, those numbered $1, \ldots, n$ counterclockwise around $\mathcal{D}$, from the others, the *negative* particles. The name comes from the fact that the positive (resp. negative) particles always remain in the upper (resp. lower) complex halfplane. Each positive particle j is naturally paired with the negative particle $2n + 1 - j$, since their trajectories are conjugate. Particles can only collide with other particles of the same sign or with their conjugates; in the latter case, the collision does not alter the motion along the real axis. This shows that the real-axis motion of the positive particles alone constitutes a bona fide one-dimensional collision system over n particles with the same coefficient of restitution.

Theorem 1. *Fix any integer $n > 2$, and write $\theta = \pi/n$ and $r_0 = (1 - \sin\theta)/(1 + \sin\theta)$. Given any positive coefficient of restitution $r \leq r_0$, there is a scaling factor λ such that the line projection of the (n, λ)-tiling forms the trajectory of a one-dimensional n-particle system exhibiting inelastic collapse. The collapse time is $r/\big(r - \lambda^2\big)$ for any $r < r_0$ and $\lambda = q\cos\theta - (q^2\cos^2\theta - r)^{1/2}$, where $q = (1 + r)/2$.*

Proof. Setting $r = \rho$ in (1) yields the quadratic equation

$$\lambda^2 - 2q(\cos\theta)\lambda + r = 0; \tag{3}$$

hence $\lambda = q\cos\theta \pm \sqrt{q^2\cos^2\theta - r}$. The roots need to be real; hence $\sin\theta \leq p/q$ or, equivalently, $r \leq r_0$. We verify that $0 < \lambda < \cos\theta$, as required of a valid (n, λ)-tiling, which is a consequence of $\sqrt{q^2\cos^2\theta - r} < p\cos\theta$. By Lemma 1, the collapse time is infinite if $\delta = \lambda^2/r \geq 1$ and equal to $\sum_{k\geq 0}\delta^k = 1/(1 - \delta) = r/(r - \lambda^2)$ if $\delta < 1$. The smaller root of (3), if strictly smaller, always satisfies the latter condition while the larger one never does. This follows from the fact that $\lambda_-\lambda_+ = r$, $q\cos\theta \geq \sqrt{r}$, and $\lambda_+ \geq q\cos\theta$; hence $\lambda_+^2 \geq r$. $\square$

In our construction, the upper bound on the coefficient of restitution is $(1-\sin\theta)/(1+\sin\theta)$. As n goes to infinity, this gives us $n \gtrsim 2\pi/(1 - r)$, which matches the bounds from [1, 2]. For $n = 3$, our construction rediscovers the classic bound of $7 - 4\sqrt{3}$ [4, 6, 7].

3 Distinct Coefficients of Restitution

Our construction does not require a fixed scaling λ. Instead of placing the vertices on circles of radius λ^k for $k \geq 0$, we can use an arbitrary decreasing radius sequence $(\lambda_k)_{k\geq 0}$, with $\lambda_0 = 1$. We assign a coefficient of restitution r_k for the collisions at radius λ_k; the dependency on k might reflect a gain or loss of elasticity after repeated collisions. For notational convenience, let $p = (1 - r_1)/2$, $\lambda = \lambda_1$, and $\mu = \lambda_2$. By reference to Figure 3, we now kick a particle from a to b with velocity $u = b - a$ (using complex numbers), and one from c to b with velocity $v = b - c$. Post-collision, the first particle travels from b to d with velocity $u' = pu + (1-p)v = \sigma_1(d - b)$, for some $\sigma_1 > 0$; hence $b - c + p(c - a) = \sigma_1(d - b)$. Since $a = 1$, $b = \lambda e^{i\theta}$, $c = e^{2i\theta}$, and $d = \mu$, we divide the equation by $e^{i\theta}$ and find that

$$\lambda - e^{i\theta} + 2ip\sin\theta = \sigma_1\big(\mu e^{-i\theta} - \lambda\big);$$

therefore, $\lambda - \cos\theta = \sigma_1(\mu\cos\theta - \lambda)$ and $r_1 = \sigma_1\mu$. More generally, for $k > 0$, we replace λ and μ by λ_k and λ_{k+1}, respectively, and we scale the relations by λ_{k-1}:

$$\sigma_k = \frac{\lambda_{k-1}\cos\theta - \lambda_k}{\lambda_k - \lambda_{k+1}\cos\theta} \qquad \text{and} \qquad r_k = \frac{\cos\theta - \lambda_k/\lambda_{k-1}}{\lambda_k/\lambda_{k+1} - \cos\theta}\,. \tag{4}$$

Of course, we retrieve the relation $r = \rho$ in (1) in the case $\lambda_k = \lambda^k$ corresponding to having fixed coefficients of restitution.

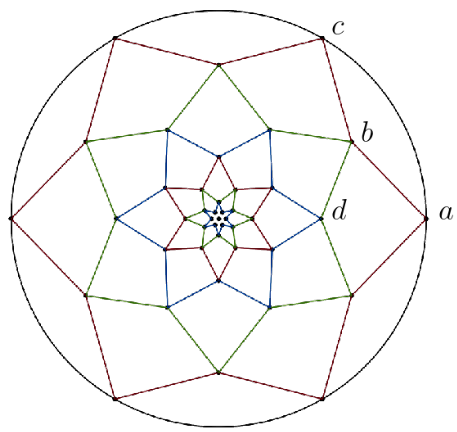

Figure 3: An irregular tiling with $\lambda_k = 0.95^k\lambda_0$.

3.1 Finite-time inelastic collapse

From the relation $u' = \sigma_1(d - b)$, we see that the time spent crossing bd is precisely $1/\sigma_1$. More generally, $1/\sigma_k$ is the time spent on the $(k+1)$-st star polygon, given a unit travel time on the previous polygon. It follows that the total travel duration is the sum of all the products of the form $1/\sigma_1 \cdots \sigma_k$, which is

$$1 + \sum_{k=1}^{\infty}\prod_{j=1}^{k}\frac{\lambda_j - \lambda_{j+1}\cos\theta}{\lambda_{j-1}\cos\theta - \lambda_j}\,. \tag{5}$$

By projection onto the real line, finite-time inelastic collapse is guaranteed if

$$\lambda_{k+1} \geq \frac{1+c}{\cos\theta}\,\lambda_k - c\lambda_{k-1},$$

for some fixed $c < 1$. Again, we can check that, if $\lambda_k = \lambda^k$, then bounded travel time means that $\lambda < \frac{1}{\cos\theta} - \tan\theta$, as claimed in Lemma 1.

3.2 Red-blue particles

Consider two species of particles, blue and red. The blue particles collide together with the coefficient of restitution r_1 and the same is true of the red ones. Particles of different colors, however, collide with the coefficient r_2. Arrange the particles as usual, with the sequence blue, blue, red, red, blue, blue, red, red, etc. Set the scaling factor $\lambda_k = \mu^j$ if $k = 2j$, and $\lambda_k = \lambda\mu^j$ if $k = 2j+1$. By (4), we choose

$$r_1 = \frac{\mu(\cos\theta - \lambda)}{\lambda - \mu\cos\theta} \qquad \text{and} \qquad r_2 = \frac{\lambda\cos\theta - \mu}{1 - \lambda\cos\theta}\,.$$

Each factor in (5) is of the form

$$\frac{\lambda_j - \lambda_{j+1}\cos\theta}{\lambda_{j-1}\cos\theta - \lambda_j} = \begin{cases} \mu(1-\lambda\cos\theta)/(\lambda\cos\theta - \mu) = \mu/r_2 & \text{if } j \text{ is even} \\ (\lambda - \mu\cos\theta)/(\cos\theta - \lambda) = \mu/r_1 & \text{else}\,. \end{cases}$$

The travel time is finite if $\mu^2 < r_1 r_2$, which is

$$\mu(\lambda - \mu\cos\theta)(1 - \lambda\cos\theta) < (\cos\theta - \lambda)(\lambda\cos\theta - \mu).$$

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