

Multi-User Privacy: The Gray-Wyner System and Generalized Common Information

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Abstract—The problem of preserving privacy when a multi-variate source is required to be revealed partially to multiple users is modeled as a Gray-Wyner source coding problem with K correlated sources at the encoder and K decoders in which the k^{th} decoder, $k = 1, 2, \dots, K$, losslessly reconstructs the k^{th} source via a common link of rate R_0 and a private link of rate R_k . The privacy requirement of keeping each decoder oblivious of all sources other than the one intended for it is introduced via an equivocation constraint E_k at decoder k such that the total equivocation summed over all decoders $E \geq \Delta$. The set of achievable $(\{R_k\}_{k=1}^K, R_0, \Delta)$ rates-equivocation $(K + 2)$ -tuples is completely characterized. Using this characterization, two different definitions of common information are presented and are shown to be equivalent.

I. INTRODUCTION

Information sources often need to be made accessible to multiple legitimate users simultaneously. However, not all data from the source should be accessible to all users. For example, a computer retailer may need to share the annual revenue of all computers sold with all the vendors but share vendor-specific sale information only with a particular vendor. Similarly, a business consulting firm may share general data about a specific market with all clients associated with that market but share client-specific strategies with only that client. In both cases, one can view sharing the public (shared by all) information via a common link and the private information via a dedicated link. Maximizing the rate over the common link allows the information source (retailer/consulting firm) to share the most allowed publicly with all clients; however, the privacy guarantee requires that no client has access to private data of the other clients. This paper develops an abstract model and a methodology to study this problem.

We model the problem of revealing partial source information to multiple users while keeping the data specific to each user private from other users as a Gray-Wyner source coding problem with K correlated sources at the encoder and K decoders in which the k^{th} decoder, $k = 1, 2, \dots, K$, losslessly reconstructs the k^{th} source via a common link of rate R_0 and a private link of rate R_k . We model the privacy requirement

of keeping each decoder oblivious of all sources other than the one intended for it via an equivocation constraint E_k at decoder k such that the total equivocation summed over all decoders $E \geq \Delta$.

Since privacy is an important aspect of this problem, it is natural to understand the maximal total equivocation that is achievable if the rate on the common link is set to the maximum achievable. On the other hand, imposing the constraint of maximal total equivocation may lead to perhaps a different limit on the maximal rate on the common link. In this paper, we show that both requirements, which are formally different definitions, yield the same formulation for the maximal rate on the common link. In keeping with the literature, this common rate is defined as the *common information*.

The common information of two correlated random variables has been defined independently by Wyner [1] and Gács-Körner [2]. Wyner's definition of common information as applied to the two-user Gray-Wyner system (without privacy constraints) is the minimum rate on the common link such that the total information shared across all three links (one common and two private) does not exceed the source entropy. On the other hand, the Gács-Körner common information is the maximal entropy of a random variable that two non-interacting terminals can agree upon when one terminal has access to X^n and the other to Y^n where X and Y are correlated random variables. For two correlated variables X and Y , the Wyner common information C_W , the Gács-Körner common information C_{GK} , and the mutual information of the two variables are related as $C_{GK} \leq I(X; Y) \leq C_W$. Recently, the authors in [3] have generalized Wyner's definition of common information to K variables, henceforth referred to as $B(X_1, X_2, \dots, X_K)$ for K correlated variables. While the definition naturally generalizes the two variable common information, the resulting common information does not satisfy a non-increasing property with K as expected.

In this paper, we present two different definitions of common information: the first is the maximal rate on the common link for which the total equivocation is maximized, and the second is the maximal rate on the common link such that each user losslessly reconstructs its intended source at its entropy.

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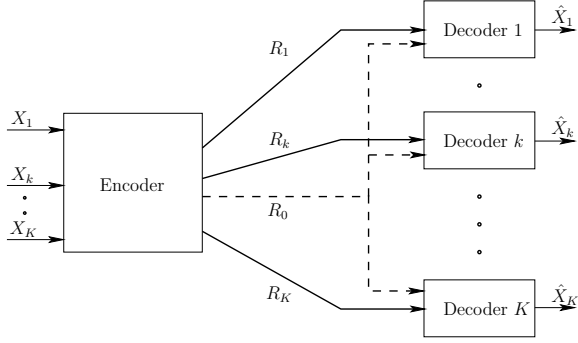


Fig. 1. The generalized Gray-Wyner source network.

We show that both definitions lead to the same formulation for common information $C(X_1, X_2, \dots, X_K)$. We present many properties of $C(X_1, X_2, \dots, X_K)$ and specifically show that $C(X_1, X_2, \dots, X_K) \leq B(X_1, X_2, \dots, X_K)$. To the best of our knowledge this is the first generalization of common information that preserves the non-increasing property and one whose form can be viewed as a natural generalization of the Gács-Körner common information to K variables.

The paper is organized as follows. In Section II, we present the system model. In Section III, we present the rate-equivocation region, develop a formulation for common information in two different ways, and present key properties. In Section IV, we compare our formulation with the K -variable generalization of Wyner's common information in [3] and illustrate with examples. We conclude in Section V.

II. SYSTEM MODEL

We consider the following source network. A centralized encoder observes K discrete, memoryless correlated sources, $\{X_k^n\}_{k=1}^K$ and is interested in communicating source X_k to decoder k in a lossless manner. The resources available at the encoder comprise two types of noiseless rate-limited links. There are K links of finite rate from the encoder to each of the K decoders and there is a common link of finite rate to all decoders. Figure 1 shows the source broadcasting network in consideration.

An $(n, \{M_k\}_{k=1}^K, M_0)$ code for this model is defined by $(K+1)$ encoding functions described as

$$f_0 : \mathcal{X}_1^n \times \dots \times \mathcal{X}_K^n \rightarrow \{1, \dots, M_0\}, \quad (1)$$

$$f_k : \mathcal{X}_1^n \times \dots \times \mathcal{X}_K^n \rightarrow \{1, \dots, M_k\}, \quad k = 1, \dots, K, \quad (2)$$

and K decoding functions,

$$g_k : \{1, \dots, M_0\} \times \{1, \dots, M_k\} \rightarrow \mathcal{X}_k^n, \quad k = 1, \dots, K.$$

We define the probability of error at decoder k as

$$P_{e,k} = \Pr(X_k^n \neq g_k(f_0(\overline{X}^n), f_k(\overline{X}^n))),$$

where $\overline{X}^n \triangleq \{X_k^n\}_{k=1}^K$. We define the equivocation at decoder k as

$$E_k = \frac{1}{n} H(\overline{X}^n \setminus X_k^n | f_0(\overline{X}^n), f_k(\overline{X}^n)),$$

and the total equivocation as $E = \sum_{k=1}^K E_k$.

Remark 1: Informally, E_k captures the average uncertainty, and hence privacy achievable, about the remaining $(K-1)$ unintended sources at decoder k .

An $(\{R_k\}_{k=1}^K, R_0, \Delta)$ rate-equivocation $(K+2)$ -tuple is achievable for the source network if there exists an $(n, \{M_k\}_{k=1}^K, M_0)$ code such that,

$$M_0 \leq 2^{nR_0}, \quad (3)$$

$$M_k \leq 2^{nR_k}, \quad k = 1, \dots, K \quad (4)$$

$$P_{e,k} \leq \epsilon_k, \quad k = 1, \dots, K \quad (5)$$

$$E \geq \Delta - \epsilon. \quad (6)$$

We denote by \mathcal{R} the region of all achievable $(\{R_k\}_{k=1}^K, R_0, \Delta)$ rate-equivocation $(K+2)$ -tuples.

III. MAIN CONTRIBUTIONS

A. Rate-Equivocation Region

We state our first result in the following theorem. The proof is presented in the appendix.

Theorem 1: The region \mathcal{R} of achievable rates-equivocation $(K+2)$ -tuples for the source network shown in Figure 1 is the union of all $(k+2)$ -tuples $(\{R_k\}_{k=1}^K, R_0, \Delta)$ that satisfy

$$R_0 \geq I(X_1, X_2, \dots, X_K; W), \quad (7)$$

$$R_k \geq H(X_k|W), \quad k = 1, 2, \dots, K, \quad (8)$$

$$\Delta \leq \sum_{k=1}^K H(\overline{X}|W, X_k) \quad (9)$$

where the union is over all auxiliary random variables W arbitrarily correlated with (X_1, X_2, \dots, X_K) , and where $\overline{X} \triangleq (X_1, X_2, \dots, X_K)$.

Remark 2: The rate region \mathcal{R}_{G-W} of the Gray-Wyner network without additional equivocation constraints is the region of $(K+1)$ rate tuples that satisfy (7) and (8).

B. Common Information of K Correlated Variables

We now present two definitions for the common information of K correlated random variables.

Definition 1: The common information of K correlated random variables, C_1 , is the maximal value of R_0 , such that $(\{R_k\}_{k=1}^K, R_0, \Delta_{\max}) \in \mathcal{R}$, where

$$\Delta_{\max} \triangleq \sum_{k=1}^K H(\overline{X}|X_k).$$

Definition 2: The common information of K correlated random variables, C_2 , is the maximal value of R_0 , such that $(\{H(X_k) - R_0\}_{k=1}^K, R_0) \in \mathcal{R}_{G-W}$.

We next state our second result.

Theorem 2: C_1 and C_2 are related as follows:

$$C_1 = C_2 = \max_{W-X_k-\overline{X} \setminus X_k, k=1,2,\dots,K} I(X_1 X_2 \dots X_K; W). \quad (10)$$

Proof: From Definition 1, the achievable equivocation E must satisfy

$$E \geq \Delta_{\max} = \sum_{k=1}^K H(\overline{X}|X_k)$$

On the other hand, any achievable $(\{R_k\}_{k=1}^K, R_0, E) \in \mathcal{R}$ also satisfies

$$E \leq \sum_{k=1}^K H(\bar{X}|W, X_k).$$

We therefore, have the following constraint:

$$\sum_{k=1}^K H(\bar{X}|W, X_k) \geq \sum_{k=1}^K H(\bar{X}|X_k)$$

which is equivalent to the following K constraints:

$$I(\bar{X} \setminus X_k; W|X_k) = 0, \quad k = 1, \dots, K. \quad (11)$$

Therefore, from Definition 1, C_1 is equal to the maximal R_0 subject to (11), which implies that

$$C_1 = \max_{W-X_k-\bar{X}\setminus X_k, k=1, \dots, K} I(X_1, \dots, X_K; W).$$

From Definition 2, C_2 is defined as the maximal R_0 such that $R_k + R_0 = H(X_k)$, for $k = 1, \dots, K$, and $(\{R_k\}_{k=1}^K, R_0) \in \mathcal{R}_{G-W}$. We therefore have the following constraints for $k = 1, \dots, K$:

$$H(X_k) = R_k + R_0 \quad (12)$$

$$\geq H(X_k|W) + I(X_1, \dots, X_K; W). \quad (13)$$

These constraints are equivalent to

$$I(\bar{X} \setminus X_k; W|X_k) = 0, \quad k = 1, \dots, K.$$

Therefore, C_2 can be written as follows:

$$C_2 = \max_{W-X_k-\bar{X}\setminus X_k, k=1, \dots, K} I(X_1, \dots, X_K; W).$$

C. Common Information: Properties

We will now develop some properties of common information of K correlated random variables defined in Theorem 2.

Proposition 1: The common information of K random variables, $C(X_1, X_2, \dots, X_K)$, is monotonically decreasing in K .

Proof: Consider an arbitrary W satisfying the Markov chain relationship

$$W - X_k - \bar{X} \setminus X_k, \quad k = 1, \dots, K. \quad (14)$$

First consider the following sequence of inequalities:

$$\begin{aligned} & I(X_1, \dots, X_{K-1}, X_K; W) \\ &= I(X_1, \dots, X_{K-1}; W) + I(X_K; W|X_1, \dots, X_{K-1}) \end{aligned} \quad (15)$$

$$\leq I(X_1, \dots, X_{K-1}; W) + I(X_2, \dots, X_K; W|X_1) \quad (16)$$

$$= I(X_1, \dots, X_{K-1}; W) \quad (17)$$

where (17) follows from the Markov chain relationship $W - X_1 - (X_2, \dots, X_K)$. Now consider the following sequence of

inequalities:

$$\begin{aligned} & C(X_1, \dots, X_K) \\ &= \max_{W-X_k-\bar{X}\setminus X_k, k=1, \dots, K} I(X_1, \dots, X_K; W) \end{aligned} \quad (18)$$

$$\leq \max_{W-X_k-\bar{X}\setminus X_k, k=1, \dots, K} I(X_1, \dots, X_{K-1}; W) \quad (19)$$

$$\leq \max_{W-X_k-\bar{X}\setminus (X_k, X_K), k=1, \dots, (K-1)} I(X_1, \dots, X_{K-1}; W) \quad (20)$$

$$= C(X_1, \dots, X_{K-1}) \quad (21)$$

where (19) follows from (17) and (20) follows from the fact that the Markov chain relationship $W - X_k - \bar{X} \setminus X_k$ implies the Markov chain relationship $W - X_k - \bar{X} \setminus (X_k, X_K)$. Since the random variable X_K could be chosen arbitrarily from the set (X_1, \dots, X_K) , (21) shows that the common information is monotonically decreasing in K . ■

Proposition 2: $C(X_1, X_2, \dots, X_K)$ is upper bounded as

$$C(X_1, X_2, \dots, X_K) \leq \min_{i \neq j, i, j=1, 2, \dots, K} I(X_i; X_j). \quad (22)$$

Proof: We consider an arbitrary W satisfying (14), and upper bound the following mutual information:

$$I(X_1, \dots, X_K; W) = I(X_i; W) + I(\bar{X} \setminus X_i; W|X_i) \quad (23)$$

$$= I(X_i; W) \quad (24)$$

$$\leq I(X_i; X_j, W) \quad (25)$$

$$= I(X_i; X_j) + I(X_i; W|X_j) \quad (26)$$

$$= I(X_i; X_j) \quad (27)$$

where (24) follows from the Markov chain condition $W - X_i - \bar{X} \setminus X_i$, and (27) follows from the Markov chain condition $W - X_j - X_i$. The choice of (i, j) was arbitrary, and therefore, the common information is upper bounded by the minimum of pairwise mutual information among all pairs, i.e.,

$$C(X_1, \dots, X_K) \leq \min_{i \neq j} I(X_i; X_j). \quad \blacksquare$$

IV. COMPARISON AND EXAMPLES

In [1] Wyner defines the common information of two correlated random variables (X_1, X_2) as

$$B(X_1, X_2) = \inf_{X_1 \rightarrow W \rightarrow X_2} I(X_1, X_2; W).$$

One interpretation of this common information can be obtained from the Gray-Wyner source network. The common information $B(X_1, X_2)$ of two random variables is given as the smallest value of R_0 such that $(R_1, R_2, R_0) \in \mathcal{R}_{G-W}$ and $R_0 + R_1 + R_2 \leq H(X_1, X_2)$. Recently, this notion of common information was generalized to K correlated random variables in [3]. The common information, $B(X_1, \dots, X_K)$, of K correlated random variables, as defined in [3], is given by smallest value of R_0 such that $(\{R_k\}_{k=1}^K, R_0) \in \mathcal{R}_{G-W}$ and $R_0 + \sum_{i=1}^K R_k \leq H(X_1, \dots, X_K)$. The common information $B(X_1, \dots, X_K)$ is given as

$$B(X_1, \dots, X_K) = \inf I(X_1, \dots, X_K; W)$$

where the infimum is over all distributions $p(w, x_1, \dots, x_K)$ that satisfy

$$\sum_{w \in \mathcal{W}} p(w, x_1, \dots, x_K) = p(x_1, \dots, x_K) \quad (28)$$

$$p(x_1, \dots, x_K | w) = \prod_{k=1}^K p(x_k | w). \quad (29)$$

It was shown in [3] that $B(X_1, \dots, X_K)$ is monotonically increasing in K . We believe that any intuitively satisfactory measure of common information should satisfy the property that the common information should decrease as the number of random variables increases. In Proposition 1, we showed that our measure of common information indeed satisfies this property.

We next prove a property of $B(X_1, \dots, X_K)$ that helps us in comparing it with our common information $C(X_1, \dots, X_K)$.

Proposition 3: $B(X_1, X_2, \dots, X_K)$ is lower bounded as follows:

$$\max_{i \neq j} I(X_i; X_j) \leq B(X_1, X_2, \dots, X_K). \quad (30)$$

Proof: To prove Proposition 3, consider an arbitrary W satisfying the constraints (28)-(29) and the following sequence of inequalities:

$$I(X_1, \dots, X_K; W) \geq I(X_i; W) \quad (31)$$

$$\geq I(X_i; X_j) \quad (32)$$

where (32) follows from the Markov chain relationship $X_i - W - X_j$, and from the data processing inequality. In arriving at (32), the choice of (i, j) was arbitrary, and therefore we can maximize over all pairs (i, j) such that $i \neq j$ to get the best possible lower bound in this manner. ■

Using Propositions 2 and 3, we have the following:

$$\begin{aligned} C(X_1, \dots, X_K) &\leq \min_{i \neq j} I(X_i; X_j) \\ &\leq \max_{i \neq j} I(X_i; X_j) \leq B(X_1, \dots, X_K). \end{aligned} \quad (33)$$

We will now give two examples to illustrate the usefulness of our definition $C(X_1, \dots, X_K)$ over $B(X_1, \dots, X_K)$.

Example 1: Consider $K = 3$ random variables (X_1, X_2, X_3) such that $X_1 \sim \text{Ber}(1/2)$, $X_2 = X_1 \oplus N$, where $N \sim \text{Ber}(\delta)$ and X_3 is independent of (X_1, X_2) . Since X_3 is independent of (X_1, X_2) , these sources have nothing in common and we should expect the ‘common information’ to be zero. Note that for these sources, $\min_{i \neq j} I(X_i; X_j) = 0$, whereas $\max_{i \neq j} I(X_i; X_j) = 1 - h(\delta)$. Therefore, from (33), we have

$$0 \leq C(X_1, X_2, X_3) \leq 0 \leq 1 - h(\delta) \leq B(X_1, X_2, X_3),$$

which implies that $C(X_1, X_2, X_3) = 0$, whereas $B(X_1, X_2, X_3) > 0$ for any $\delta \in (0, 1/2)$.

Example 2: Consider $K = 3$ random variables (X_1, X_2, X_3) such that $X_1 = (X_0, X_{1p})$, $X_2 = (X_0, X_{2p})$ and $X_3 = (X_0, X_{3p})$, where $(X_0, X_{1p}, X_{2p}, X_{3p})$ are all mutually independent. Since X_0 appears to be the only common part in all three sources, we

should expect the ‘common information’ to be equal to the entropy of X_0 . Note that for these sources, $\min_{i \neq j} I(X_i; X_j) = \max_{i \neq j} I(X_i; X_j) = H(X_0)$. Therefore, from (33), we have

$$0 \leq C(X_1, X_2, X_3) \leq H(X_0) \leq B(X_1, X_2, X_3),$$

It is straightforward to show that for these sources,

$$C(X_1, X_2, X_3) = B(X_1, X_2, X_3) = H(X_0).$$

Inspired by the above example, we show the following interesting property that in some sense relates $C(X_1, \dots, X_K)$ to $B(X_1, \dots, X_K)$.

Proposition 4: For a set of sources X_1, X_2, \dots, X_K that satisfy

$$\min_{i \neq j} I(X_i; X_j) = \max_{i \neq j} I(X_i; X_j), \quad (34)$$

we have

$$C(X_1, X_2, \dots, X_K) = \min_{i \neq j} I(X_i; X_j) \quad (35)$$

$$\text{if } B(X_1, X_2, \dots, X_K) = \max_{i \neq j} I(X_i; X_j). \quad (36)$$

Proof: The constraint (34) implies that the mutual information $I(X_i; X_j)$ is the *same* for all $i, j \in \{1, \dots, K\}$, $i \neq j$. Let us start with a W^* that satisfies the infimization constraints for $B(X_1, \dots, X_K)$ and yields

$$B(X_1, \dots, X_K) = \max_{i \neq j} I(X_i; X_j) \quad (37)$$

$$= I(X_{i_0}; X_{j_0}), \quad (38)$$

for some $i_0 \neq j_0$. For this W^* , we have

$$I(X_{i_0}; X_{j_0}) = \max_{i \neq j} I(X_i; X_j) \quad (39)$$

$$= I(X_1, \dots, X_K; W^*) \quad (40)$$

$$= I(X_{i_0}; W^*) + I(\overline{X} \setminus X_{i_0}; W^* | X_{i_0}) \quad (41)$$

$$\geq I(X_{i_0}; X_{j_0}) + I(\overline{X} \setminus X_{i_0}; W^* | X_{i_0}) \quad (42)$$

where (42) follows from the fact that W^* satisfies the Markov relationship $X_{i_0} - W^* - X_{j_0}$, for all $i_0 \neq j_0$. In the derivation of (42), i_0 can be chosen arbitrarily due to (34). Therefore, (42) implies that this W^* also satisfies

$$I(\overline{X} \setminus X_i; W^* | X_i) = 0$$

for all $i = 1, \dots, K$. This in turn implies that W^* serves as a valid choice in the maximization for evaluation of $C(X_1, \dots, X_K)$. Therefore, we obtain the following lower bound for $C(X_1, \dots, X_K)$:

$$C(X_1, \dots, X_K) = \max_{W - X_k - \overline{X} \setminus X_k, k=1, \dots, K} I(X_1, \dots, X_K; W) \quad (43)$$

$$\geq I(X_1, \dots, X_K; W^*) \quad (44)$$

$$= \max_{i \neq j} I(X_i; X_j) \quad (45)$$

$$= \min_{i \neq j} I(X_i; X_j). \quad (46)$$

Hence, from Proposition 1, it now follows that if $B(X_1, \dots, X_K) = \max_{i \neq j} I(X_i; X_j)$, then

$C(X_1, \dots, X_K) = \min_{i \neq j} I(X_i; X_j)$. We remark here that a similar property has been shown for $K = 2$ by Ahlswede and Körner in [4]. ■

V. CONCLUDING REMARKS

We have abstracted the problem of privacy in a setting where a source interacts with multiple users via the Gray-Wyner source coding problem with additional equivocation constraints at each user and a total equivocation constraint. In addition to developing the rate-equivocation region, we have introduced two definitions of common information of K correlated variables and shown them both to have a form that can be viewed as a K -user generalization of the Gács-Körner common information (see also [4]).

VI. APPENDIX: PROOF OF THEOREM 1

The converse follows by minor modifications of the converse proof for the unconstrained Gray-Wyner problem [5] and is therefore omitted. We now outline the proof of achievability for Theorem 1.

Codebook generation: Fix an input distribution $p(w|x_1, \dots, x_K)$. Generate $2^{nI(X_1, \dots, X_K; W)}$ sequences according to the distribution $\prod_{t=1}^n p(w_t)$, and index these sequences as $w^n(i)$, for $i = 1, \dots, 2^{nI(X_1, \dots, X_K; W)}$. Independently and uniformly bin the X_k^n -sequences in $2^{nH(X_k|W)}$ bins, and index these bins as $b_{k,1}, \dots, b_{k,2^{nH(X_k|W)}}$, for $k = 1, \dots, K$.

Encoding scheme: Upon observing the (x_1^n, \dots, x_K^n) sequences, the encoder searches for a w^n sequence that is jointly typical with these sequences. Using standard arguments (as in [6]), it can be shown that the encoder can succeed in finding one such w^n sequence. The encoder sends the index of the w^n sequence on the public link, for which we require $R_0 \geq I(X_1, \dots, X_K; W)$. It sends the bin index of the source sequence x_k^n on the private link to decoder k , for which we require $R_k \geq H(X_k|W)$.

Decoding: At decoder k , the decoder looks for a unique x^n in bin b_k (received from the private link), that is jointly typical with the w^n sequence received from the public link. It can be shown that decoder k can reconstruct X_k^n with a vanishingly small probability of error. We omit the probability of error calculation as it follows from the same arguments as in [5].

Equivocation: We show that this coding scheme yields the total equivocation stated in Theorem 1. Let J_0 denote the encoder output for the public link and let J_k denote the encoder output for the private link to decoder k , for $k = 1, \dots, K$. For E_k , we have the following sequence of

inequalities:

$$E_k = \frac{1}{n} H(X_1^n, \dots, X_{k-1}^n, X_{k+1}^n, \dots, X_K^n | J_0, J_k) \quad (47)$$

$$= \frac{1}{n} H(\bar{X}^n \setminus X_k^n | J_0, J_k) \quad (48)$$

$$\geq \frac{1}{n} H(\bar{X}^n | J_0, J_k) - \frac{1}{n} H(X_k^n | J_0, J_k) \quad (49)$$

$$\geq \frac{1}{n} H(\bar{X}^n | J_0, J_k) - \epsilon_{k,n} \quad (50)$$

$$= \frac{1}{n} H(\bar{X}^n, J_0, J_k) - \frac{1}{n} H(J_0, J_k) - \epsilon_{k,n} \quad (51)$$

$$\geq \frac{1}{n} H(\bar{X}^n) - \frac{1}{n} H(J_0, J_k) - \epsilon_{k,n} \quad (52)$$

$$\geq \frac{1}{n} H(\bar{X}^n) - \frac{1}{n} H(J_0) - \frac{1}{n} H(J_k) - \epsilon_{k,n} \quad (53)$$

$$\geq H(X_1, \dots, X_K) - I(X_1, \dots, X_K; W) - H(X_k|W) - \epsilon_{k,n} \quad (54)$$

$$= H(X_1, \dots, X_K | W, X_k) - \epsilon_{k,n} \quad (55)$$

$$= H(\bar{X} | W, X_k) - \epsilon_{k,n}, \quad (56)$$

where (50) follows from Fano's inequality, and (54) follows from the facts that $H(J_0) \leq \log(|\mathcal{J}_0|) = nI(X_1, \dots, X_K; W)$, and $H(J_k) \leq \log(|\mathcal{J}_k|) = nH(X_k|W)$, for $k = 1, \dots, K$. Therefore, we have that

$$E = \sum_{k=1}^K E_k \geq \sum_{k=1}^K H(\bar{X} | W, X_k) - \epsilon.$$

Hence, this coding scheme yields an equivocation of $\Delta = \sum_{k=1}^K H(\bar{X} | W, X_k)$.

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