

Sufficient Forecasting Using Factor Models

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Abstract

We consider forecasting a single time series when there is a large number of predictors and a possible nonlinear effect. The dimensionality was first reduced via a high-dimensional (approximate) factor model implemented by the principal component analysis. Using the extracted factors, we develop a novel forecasting method called the sufficient forecasting, which provides a set of sufficient predictive indices, inferred from high-dimensional predictors, to deliver additional predictive power. The projected principal component analysis will be employed to enhance the accuracy of inferred factors when a semi-parametric (approximate) factor model is assumed. Our method is also applicable to cross-sectional sufficient regression using extracted factors. The connection between the sufficient forecasting and the deep learning architecture is explicitly stated. The sufficient forecasting correctly estimates projection indices of the underlying factors even in the presence of a nonparametric forecasting function. The proposed method extends the sufficient dimension reduction to high-dimensional regimes by condensing the cross-sectional information through factor models. We derive asymptotic properties for the estimate of the central subspace spanned by these projection directions as well as the estimates of the sufficient predictive indices. We further show that the natural method of running multiple regression of target on estimated factors yields a linear estimate that actually falls into this central subspace. Our method and theory allow the number of predictors to be larger than the number of observations. We finally demonstrate that the sufficient forecasting improves upon the linear forecasting in both simulation studies and an empirical study of forecasting macroeconomic variables.

Key Words: Regression; forecasting; deep learning; factor model; semi-parametric factor model; principal components; learning indices; sliced inverse regression; dimension reduction.

1 Introduction

Regression and forecasting in a data-rich environment have been an important research topic in statistics, economics and finance. Typical examples include forecasts of a macroeconomic output using a large number of employment and production variables (Stock and Watson, 1989; Bernanke et al., 2005), forecasts of the values of market prices and dividends using cross-sectional asset returns (Sharpe, 1964; Lintner, 1965), and studies of association between clinical outcomes and high throughput genomics and genetic information such as microarray gene expressions. The predominant framework to harness the vast predictive information is via the *factor model*, which proves effective in simultaneously modeling the commonality and cross-sectional dependence of the observed data. It is natural to think that the latent factors drive simultaneously the vast covariate information as well as the outcome. This achieves the dimensionality reduction in the regression and predictive models.

Turning the curse of dimensionality into blessing, the latent factors can be accurately extracted from the vast predictive variables and hence they can be reliably used to build models for response variables. For the same reason, factor models have been widely employed in many applications, such as portfolio management (Fama and French, 1992; Carhart, 1997; Fama and French, 2015; Hou et al., 2015), large-scale multiple testing (Leek and Storey, 2008; Fan et al., 2012), high-dimensional covariance matrix estimation (Fan et al., 2008, 2013), and forecasting using many predictors (Stock and Watson, 2002*a,b*). Leveraging on the recent developments on the factor models, in this paper, we are particularly interested in the problem of forecasting. Our techniques can also be applied to regression problems, resulting in sufficient regression.

With little knowledge of the relationship between the forecast target and the latent factors, most research focuses on a linear model and its refinement. Motivated by the classic principal component regression (Kendall, 1957; Hotelling, 1957), Stock and Watson (2002*a,b*) employed a similar idea to forecast a single time series from a large number of predictors: first use the principal component analysis (PCA) to estimate the underlying common factors, followed by a linear regression of the target on the estimated factors. The key insight here is to condense information from many cross-sectional predictors into several predictive indices. Boivin and Ng (2006) pointed out that the relevance of factors is important to the forecasting power, and may lead to improved forecast. As an improvement to this procedure, Bair et al. (2006) used correlation screening to remove irrelevant predictors before performing the PCA. In a similar fashion, Bai and Ng (2008) applied boosting and employed thresholding rules to select “targeted predictors”, and Stock and Watson (2012) used shrinkage methods to downweight unrelated principal components. Kelly and Pruitt (2015) took into account the

covariance with the forecast target in linear forecasting, and they proposed a three-pass regression filter that generalizes partial least squares to forecast a single time series.

As mentioned, most of these approaches are fundamentally limited to linear forecasting. This yields only one index of extracted factors for forecasting. It does not provide sufficient predictive power when the predicting function is nonlinear and depends on multiple indices of extracted factors. Yet, nonlinear models usually outperform linear models in time series analysis (Tjøstheim, 1994), especially in multi-step prediction. When the link function between the target and the factors is arbitrary and unknown, a thorough exploration of the factor space often leads to additional gains. To the best of our knowledge, there are only a few papers in forecasting a nonlinear time series using factor models. Bai and Ng (2008) discussed the use of squared factors (i.e., volatility of the factors) in augmenting forecasting equation. Ludvigson and Ng (2007) found that the square of the first factor estimated from a set of financial factors is significant in the regression model for mean excess returns. This naturally leads to the question of which effective factor, or more precisely, which effective direction of factor space to include for higher moments. Cunha et al. (2010) used a class of nonlinear factor models with shape constraints to study cognitive and noncognitive skill formation. The nonparametric forecasting function would pose a significant challenge in estimating effective predictive indices.

In this work, we shall address this issue from a completely different perspective to existing forecasting methods. We introduce a favorable alternative method called the *sufficient forecasting*. Our proposed forecasting method springs from the idea of the sliced inverse regression, which was first introduced in the seminal work by Li (1991). In the presence of some arbitrary and unknown (possibly nonlinear) forecasting function, we are interested in estimating a set of *sufficient predictive indices*, given which the forecast target is independent of unobserved common factors. To put it another way, the forecast target relates to the unobserved common factors only through these sufficient predictive indices. Such a goal is closely related to the estimation of the *central subspace* in the dimension reduction literature (Cook, 2009). In a linear forecasting model, such a central subspace consists of only one dimension. In contrast, when a nonlinear forecasting function is present, the central subspace can go beyond one dimension, and our proposed method can effectively reveal multiple sufficient predictive indices to enhance the prediction power. This procedure therefore greatly enlarges the scope of forecasting using factor models. As demonstrated in numerical studies, the sufficient forecasting has improved performance over benchmark methods, especially under a nonlinear forecasting equation.

In summary, the contribution of this work is at least twofold. On the one hand, our proposed sufficient forecasting advances existing forecasting methods, and fills the impor-

tant gap between incorporating target information and dealing with nonlinear forecasting. Our work identifies effective factors that influence the forecast target without knowing the nonlinear dependence. On the other hand, we present a promising dimension reduction technique through factor models. It is well-known that existing dimension reduction methods are limited to either a fixed dimension or a diverging dimension that is smaller than the sample size (Zhu et al., 2006). With the aid of factor models, our work alleviates what plagues sufficient dimension reduction in high-dimensional regimes by condensing the cross-sectional information, whose dimension can be much higher than the sample size.

The rest of this paper is organized as follows. Section 2 first proposes the sufficient forecasting using factor models, and then extends the proposed method by using semi-parametric factor models. In Section 3, we establish the asymptotic properties for the sufficient forecasting, and we also prove that the simple linear estimate actually falls into the central subspace. Section 4 demonstrates the numerical performance of the sufficient forecasting in simulation studies and an empirical application. Section 5 includes a few concluding remarks. Technical proofs are rendered in the Appendix.

2 Sufficient forecasting

This section first presents a unified framework for forecasting using factor models. We then propose the sufficient forecasting procedure with unobserved factors, which estimates predictive indices without requiring estimating the unknown forecasting function.

2.1 Factor models and forecasting

Consider the following factor model with a target variable y_{t+1} which we wish to forecast:

$$y_{t+1} = h(\boldsymbol{\phi}'_1 \mathbf{f}_t, \dots, \boldsymbol{\phi}'_L \mathbf{f}_t, \epsilon_{t+1}), \quad (2.1)$$

$$x_{it} = \mathbf{b}'_i \mathbf{f}_t + u_{it}, \quad 1 \leq i \leq p, \quad 1 \leq t \leq T, \quad (2.2)$$

where x_{it} is the i -th predictor observed at time t , \mathbf{b}_i is a $K \times 1$ vector of factor loadings, $\mathbf{f}_t = (f_{1t}, \dots, f_{Kt})'$ is a $K \times 1$ vector of common factors driving both the predictors and the response, and u_{it} is the error term, or the idiosyncratic component. For ease of notation, we write $\mathbf{x}_t = (x_{1t}, \dots, x_{pt})'$, $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_p)'$ and $\mathbf{u}_t = (u_{1t}, \dots, u_{pt})'$. In (2.1), $h(\cdot)$ is an unknown link function, and ϵ_{t+1} is some stochastic error independent of \mathbf{f}_t and u_{it} . The vectors of linear combinations $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_L$ are K -dimensional orthonormal vectors. Clearly, the model is also applicable to the cross-sectional regression such as those mentioned at

the introduction of the paper. Note that the sufficient forecasting can be represented in a deep learning architecture (Bengio, 2009; Bengio et al., 2013), consisting of four layers of linear/nonlinear processes for dimension reduction and forecasting. The connection between sufficient forecasting and deep learning is illustrated in Figure 1. An advantage of our deep learning model is that it admits scalable and explicit computational algorithm.

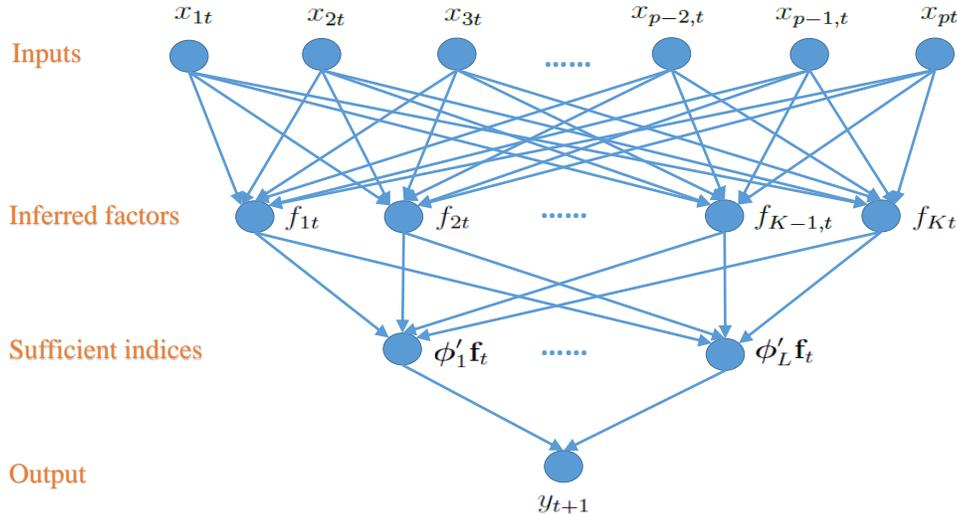


Figure 1: The sufficient forecasting is represented as a four-layer deep learning architecture. The factors f_{1t}, \dots, f_{Kt} are obtained from the inputs via principal component analysis.

Model (2.1) is a semiparametric multi-index model with latent covariates and unknown nonparametric function $h(\cdot)$. The target y_{t+1} depends on the factors \mathbf{f}_t only through L predictive indices $\phi'_1 \mathbf{f}_t, \dots, \phi'_L \mathbf{f}_t$, where L is no greater than K . In other words, these predictive indices are *sufficient* in forecasting y_{t+1} . Models (2.1) and (2.2) effectively reduce the dimension from the diverging p to a fixed L to estimate the nonparametric function $h(\cdot)$, and greatly alleviates the curse of dimensionality. Here, ϕ_i 's are also called the sufficient dimension reduction (SDR) directions, and their linear span forms the *central subspace* (Cook, 2009) denoted by $S_{y|\mathbf{f}}$. Note that the individual directions ϕ_1, \dots, ϕ_L are not identifiable without imposing any structural condition on $h(\cdot)$. However, the subspace $S_{y|\mathbf{f}}$ spanned by ϕ_1, \dots, ϕ_L can be identified. Therefore, throughout this paper, we refer any orthonormal basis ψ_1, \dots, ψ_L of the central subspace $S_{y|\mathbf{f}}$ as sufficient dimension reduction directions, and their corresponding predictive indices $\psi'_1 \mathbf{f}_t, \dots, \psi'_L \mathbf{f}_t$ as sufficient predictive indices.

Suppose that the factor model (2.2) has the following canonical normalization

$$\text{cov}(\mathbf{f}_t) = \mathbf{I}_K \text{ and } \mathbf{B}'\mathbf{B} \text{ is diagonal,} \quad (2.3)$$

where \mathbf{I}_K is a $K \times K$ identity matrix. This normalization serves as an identification condition of \mathbf{B} and \mathbf{f}_t because $\mathbf{B}\mathbf{f}_t = (\mathbf{B}\boldsymbol{\Omega})(\boldsymbol{\Omega}^{-1}\mathbf{f}_t)$ holds for any nonsingular matrix $\boldsymbol{\Omega}$. We assume for simplicity that the means of x_{it} 's and \mathbf{f}_t 's in (2.2) have already been removed, namely, $E(x_{it}) = 0$ and $E(\mathbf{f}_t) = 0$.

As a special case, suppose we have *a priori* knowledge that the link function $h(\cdot)$ in (2.1) is in fact linear. In this case, no matter what L is, there is only a single index of the factors \mathbf{f}_t in the prediction equation, i.e. $L = 1$. Now, (2.1) reduces to

$$y_{t+1} = \boldsymbol{\phi}'_1 \mathbf{f}_t + \epsilon_{t+1}.$$

Such linear forecasting problems using many predictors have been studied extensively in the econometrics literature, for example, Stock and Watson (2002*a,b*), Bai and Ng (2008), and Kelly and Pruitt (2015), among others. Existing methods are mainly based on the principal component regression (PCR), which is a multiple linear regression of the forecast target on the estimated principal components. The linear forecasting framework does not thoroughly explore the predictive power embedded in the underlying common factors. It can only create a single index of the unobserved factors to forecast the target. The presence of a nonlinear forecasting function with multiple forecasting indices would only deteriorate the situation. While extensive efforts have been made such as Boivin and Ng (2006), Bai and Ng (2009), and Stock and Watson (2012), these approaches are more tailored to handle issues of using too many factors, and they are fundamentally limited to the linear forecasting.

2.2 Sufficient forecasting

Traditional analysis of factor models typically focuses on the covariance with the forecast target $\text{cov}(\mathbf{x}_t, y_{t+1})$ and the covariance within the predictors \mathbf{x}_t , denoted by a $p \times p$ matrix

$$\boldsymbol{\Sigma}_x = \mathbf{B}\text{cov}(\mathbf{f}_t)\mathbf{B}' + \boldsymbol{\Sigma}_u, \quad (2.4)$$

where $\boldsymbol{\Sigma}_u$ is the error covariance matrix of \mathbf{u}_t . Usually $\text{cov}(\mathbf{x}_t, y_{t+1})$ and $\boldsymbol{\Sigma}_x$ are not enough to construct an optimal forecast, especially in the presence of a nonlinear forecasting function. To fully utilize the target information, our method considers the covariance matrix of the inverse regression curve, $E(\mathbf{x}_t|y_{t+1})$. By conditioning on the target y_{t+1} in model (2.2), we obtain

$$\text{cov}(E(\mathbf{x}_t|y_{t+1})) = \mathbf{B}\text{cov}(E(\mathbf{f}_t|y_{t+1}))\mathbf{B}',$$

where we used the assumption that $E(\mathbf{u}_t|y_{t+1}) = 0$. This salient feature does not impose any structure on the covariance matrix of \mathbf{u}_t . Under model (2.1), Li (1991) showed that

$E(\mathbf{f}_t|y_{t+1})$ is contained in the central subspace $S_{y|\mathbf{f}}$ spanned by ϕ_1, \dots, ϕ_L provided that $E(\mathbf{b}'\mathbf{f}_t|\phi_1'\mathbf{f}_t, \dots, \phi_L'\mathbf{f}_t)$ is a linear function of $\phi_1'\mathbf{f}_t, \dots, \phi_L'\mathbf{f}_t$ for any $\mathbf{b} \in \mathbb{R}^K$. This important result implies that $S_{y|\mathbf{f}}$ contains the linear span of $\text{cov}(E(\mathbf{f}_t|y_{t+1}))$. Thus, it is promising to estimate sufficient directions by investigating the top L eigenvectors of $\text{cov}(E(\mathbf{f}_t|y_{t+1}))$. To see this, let $\Phi = (\phi_1, \dots, \phi_L)$. Then, we can write

$$E(\mathbf{f}_t|y_{t+1}) = \Phi \mathbf{a}(y_{t+1}), \quad (2.5)$$

for some $L \times 1$ coefficient function $\mathbf{a}(\cdot)$, according to the aforementioned result by Li (1991). As a result,

$$\text{cov}(E(\mathbf{f}_t|y_{t+1})) = \Phi E[\mathbf{a}(y_{t+1})\mathbf{a}(y_{t+1})^T] \Phi'. \quad (2.6)$$

This matrix has L nonvanishing eigenvalues if $E[\mathbf{a}(y_{t+1})\mathbf{a}(y_{t+1})^T]$ is non-degenerate. Their corresponding eigenvectors have the same linear span as ϕ_1, \dots, ϕ_L .

It is difficult to directly estimate the covariance of the inverse regression curve $E(\mathbf{f}_t|y_{t+1})$ and obtain sufficient predictive indices. Instead of using the conventional nonparametric regression method, we follow Li (1991) to explore the conditional information of these underlying factors. To this end, we introduce the sliced covariance estimate

$$\Sigma_{f|y} = \frac{1}{H} \sum_{h=1}^H E(\mathbf{f}_t|y_{t+1} \in I_h) E(\mathbf{f}_t'|y_{t+1} \in I_h), \quad (2.7)$$

where the range of y_{t+1} is divided into H slices I_1, \dots, I_H such that $P(y_{t+1} \in I_h) = 1/H$. By (2.5), we have

$$\Sigma_{f|y} = \Phi \left[\frac{1}{H} \sum_{h=1}^H E(\mathbf{a}(y_{t+1})|y_{t+1} \in I_h) E(\mathbf{a}(y_{t+1})|y_{t+1} \in I_h)^T \right] \Phi', \quad (2.8)$$

This matrix has L nonvanishing eigenvalues as long as the matrix in the bracket is non-degenerate. In this case, the linear span created by the eigenvectors of the L largest eigenvalues of $\Sigma_{f|y}$ is the same as that spanned by ϕ_1, \dots, ϕ_L . Note that $H \geq \max\{L, 2\}$ is required in order for the matrix in the bracket to be non-degenerate.

To estimate $\Sigma_{f|y}$ in (2.7) with unobserved factors, a natural solution is to first consistently estimate factors \mathbf{f}_t from the factor model (2.2) and then use the estimated factors $\hat{\mathbf{f}}_t$ and the observed target y_{t+1} to approximate the sliced estimate $\Sigma_{f|y}$. Alternatively, we can start with the observed predictors \mathbf{x}_t . Since $E(\mathbf{u}_t|y_{t+1}) = 0$, we then have

$$E(\mathbf{x}_t|y_{t+1} \in I_h) = \mathbf{B} E(\mathbf{f}_t|y_{t+1} \in I_h),$$

and

$$E(\mathbf{f}_t | y_{t+1} \in I_h) = \mathbf{\Lambda}_b E(\mathbf{x}_t | y_{t+1} \in I_h),$$

where $\mathbf{\Lambda}_b = (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'$. Hence,

$$\mathbf{\Sigma}_{f|y} = \frac{1}{H} \sum_{h=1}^H \mathbf{\Lambda}_b E(\mathbf{x}_t | y_{t+1} \in I_h) E(\mathbf{x}'_t | y_{t+1} \in I_h) \mathbf{\Lambda}'_b, \quad (2.9)$$

Notice that $\mathbf{\Lambda}_b$ is also *unknown* and needs to be estimated. With a consistent estimator $\widehat{\mathbf{\Lambda}}_b$, we can use \mathbf{x}_t and y_{t+1} to approximate the sliced estimate $\mathbf{\Sigma}_{f|y}$.

In the sequel, we present two estimators, $\widehat{\mathbf{\Sigma}}_{f|y}^1$ in (2.12) and $\widehat{\mathbf{\Sigma}}_{f|y}^2$ in (2.13), estimated by using respectively (2.7) and (2.9). Interestingly, we show that these two estimators are exactly equivalent when factors and factor loadings are estimated from the constrained least squares. We denote two equivalent estimators $\widehat{\mathbf{\Sigma}}_{f|y}^1$ and $\widehat{\mathbf{\Sigma}}_{f|y}^2$ by $\widehat{\mathbf{\Sigma}}_{f|y}$. As in Remark 2.1, these two are in general not equivalent.

In Section 3, we shall show that under mild conditions, $\mathbf{\Sigma}_{f|y}$ is consistently estimated by $\widehat{\mathbf{\Sigma}}_{f|y}$ as $p, T \rightarrow \infty$. Furthermore, the eigenvectors of $\widehat{\mathbf{\Sigma}}_{f|y}$ corresponding to the L largest eigenvalues, denoted as $\widehat{\boldsymbol{\psi}}_j$ ($j = 1, \dots, L$), will converge to the corresponding eigenvectors of $\mathbf{\Sigma}_{f|y}$, which actually span the aforementioned central subspace $S_{y|\mathbf{f}}$. This will yield consistent estimates of sufficient predictive indices $\widehat{\boldsymbol{\psi}}_1' \widehat{\mathbf{f}}_t, \dots, \widehat{\boldsymbol{\psi}}_L' \widehat{\mathbf{f}}_t$. Given these estimated low-dimensional indices, we may employ one of the well-developed nonparametric regression techniques to estimate $h(\cdot)$ and make the forecast, including local polynomial modeling, regression splines, additive modeling, and many others (Green and Silverman, 1993; Fan and Gijbels, 1996; Li and Racine, 2007). We summarize the sufficient forecasting procedure in Algorithm 1.

Algorithm 1 Sufficient forecasting using factor models

- Step 1: Obtain the estimated factors $\{\widehat{\mathbf{f}}_t\}_{t=1, \dots, T}$ from (2.10) and (2.11);
 - Step 2: Construct $\widehat{\mathbf{\Sigma}}_{f|y}$ as in (2.12) or (2.13);
 - Step 3: Obtain $\widehat{\boldsymbol{\psi}}_1, \dots, \widehat{\boldsymbol{\psi}}_L$ from the L largest eigenvectors of $\widehat{\mathbf{\Sigma}}_{f|y}$;
 - Step 4: Construct the predictive indices $\widehat{\boldsymbol{\psi}}_1' \widehat{\mathbf{f}}_t, \dots, \widehat{\boldsymbol{\psi}}_L' \widehat{\mathbf{f}}_t$ and forecast y_{t+1} .
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To make the forecasting procedure concrete, we elucidate how factors and factor loadings are estimated. We temporarily assume that the number of underlying factors K is known to

us. Consider the following constrained least squares problem:

$$(\widehat{\mathbf{B}}_K, \widehat{\mathbf{F}}_K) = \arg \min_{(\mathbf{B}, \mathbf{F})} \|\mathbf{X} - \mathbf{B}\mathbf{F}'\|_F^2, \quad (2.10)$$

$$\text{subject to } T^{-1}\mathbf{F}'\mathbf{F} = \mathbf{I}_K, \quad \mathbf{B}'\mathbf{B} \text{ is diagonal}, \quad (2.11)$$

where $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_T)$, $\mathbf{F}' = (\mathbf{f}_1, \dots, \mathbf{f}_T)$, and $\|\cdot\|_F$ denotes the Frobenius norm. This is a classical principal components problem, and it has been widely used to extract underlying common factors (Stock and Watson, 2002a; Bai and Ng, 2002; Fan et al., 2013; Bai and Ng, 2013). The constraints in (2.11) correspond to the normalization (2.3). The minimizers $\widehat{\mathbf{F}}_K$ and $\widehat{\mathbf{B}}_K$ are such that the columns of $\widehat{\mathbf{F}}_K/\sqrt{T}$ are the eigenvectors corresponding to the K largest eigenvalues of the $T \times T$ matrix $\mathbf{X}'\mathbf{X}$ and

$$\widehat{\mathbf{B}}_K = T^{-1}\mathbf{X}\widehat{\mathbf{F}}_K.$$

To simplify notation, we use $\widehat{\mathbf{B}} = \widehat{\mathbf{B}}_K$, $\widehat{\mathbf{F}} = \widehat{\mathbf{F}}_K$ and $\widehat{\mathbf{F}}' = (\widehat{\mathbf{f}}_1, \dots, \widehat{\mathbf{f}}_T)$ throughout this paper.

To effectively estimate $\Sigma_{f|y}$ in (2.7) and (2.9), we shall replace the conditional expectations $E(\mathbf{f}_t|y_{t+1} \in I_h)$ and $E(\mathbf{x}_t|y_{t+1} \in I_h)$ by their sample counterparts. Denote the ordered statistics of $\{(y_{t+1}, \widehat{\mathbf{f}}_t)\}_{t=1, \dots, T-1}$ by $\{(y_{(t+1)}, \widehat{\mathbf{f}}_{(t)})\}_{t=1, \dots, T-1}$ according to the values of y , where $y_{(2)} \leq \dots \leq y_{(T)}$ and we only use information up to time T . We divide the range of y into H slices, where H is typically fixed. Each of the first $H-1$ slices contains the same number of observations $c > 0$, and the last slice may have less than c observations, which exerts little influence asymptotically. For ease of presentation, we introduce a double script (h, j) in which $h = 1, \dots, H$ refers to the slice number and $j = 1, \dots, c$ is the index of an observation in a given slice. Thus, we can write $\{(y_{(t+1)}, \widehat{\mathbf{f}}_{(t)})\}_{t=1, \dots, T-1}$ as

$$\{(y_{(h,j)}, \widehat{\mathbf{f}}_{(h,j)}) : y_{(h,j)} = y_{(c(h-1)+j+1)}, \widehat{\mathbf{f}}_{(h,j)} = \widehat{\mathbf{f}}_{(c(h-1)+j)}\}_{h=1, \dots, H; j=1, \dots, c}.$$

Based on the estimated factors $\widehat{\mathbf{f}}_t$, we have the estimate $\widehat{\Sigma}_{f|y}^1$ in the form of

$$\widehat{\Sigma}_{f|y}^1 = \frac{1}{H} \sum_{h=1}^H \left[\frac{1}{c} \sum_{l=1}^c \widehat{\mathbf{f}}_{(h,l)} \right] \left[\frac{1}{c} \sum_{l=1}^c \widehat{\mathbf{f}}_{(h,l)} \right]'. \quad (2.12)$$

Analogously, by using (2.9), we have an alternative estimator:

$$\widehat{\Sigma}_{f|y}^2 = \widehat{\Lambda}_b \left(\frac{1}{H} \sum_{h=1}^H \left[\frac{1}{c} \sum_{l=1}^c \mathbf{x}_{(h,l)} \right] \left[\frac{1}{c} \sum_{l=1}^c \mathbf{x}_{(h,l)} \right]' \right) \widehat{\Lambda}_b'. \quad (2.13)$$

where $\widehat{\Lambda}_b = (\widehat{\mathbf{B}}'\widehat{\mathbf{B}})^{-1}\widehat{\mathbf{B}}$ is the plug-in estimate of Λ_b . Two estimates of $\Sigma_{f|y}$ are seemingly different: $\widehat{\Sigma}_{f|y}^1$ based on estimated factors and $\widehat{\Sigma}_{f|y}^2$ based on estimated loadings. When factors and loadings are estimated using (2.10) and (2.11), the following proposition shows that $\widehat{\Sigma}_{f|y}^1$ and $\widehat{\Sigma}_{f|y}^2$ are the same.

Proposition 2.1. *Let $\widehat{\mathbf{f}}_t$ and $\widehat{\mathbf{B}}$ be estimated by solving the constrained least square problem in (2.10) and (2.11). Then, the two estimators (2.12) and (2.13) of $\Sigma_{f|y}$ are equivalent, i.e.*

$$\widehat{\Sigma}_{f|y}^1 = \widehat{\Sigma}_{f|y}^2.$$

Remark 2.1. There are alternative ways for estimating factors \mathbf{F} and factor loadings \mathbf{B} . For example, Forni et al. (2000, 2015) studied factor estimation based on projection, Liu et al. (2012), Xue and Zou (2012) and Fan et al. (2015b) employed rank-based methods for robust estimation of the covariance matrices, and Connor et al. (2012) applied a weighted additive nonparametric estimation procedure to estimate characteristic-based factor models. These methods do not necessarily lead to the equivalence of $\widehat{\Sigma}_{f|y}^1$ and $\widehat{\Sigma}_{f|y}^2$ as in Proposition 2.1. However, the proposed sufficient forecasting will benefit from using generalized dynamic factor models (Forni et al., 2000, 2015) or more robust factor models. This would go beyond the scope of this paper, and is left for future research.

2.3 Sufficient forecasting with projected principal components

Often in econometrics and financial applications, there may exist explanatory variables to model the loading matrix of factor models (Connor et al., 2012; Fan et al., 2015a). In such cases, the loading coefficient \mathbf{b}_i in (2.15) can be modeled by nonparametric functions $\mathbf{b}_i = \mathbf{g}(\mathbf{z}_i) + \boldsymbol{\gamma}_i$, where \mathbf{z}_i denotes the vector of d subject-specific covariates, and $\boldsymbol{\gamma}_i$ is the part of \mathbf{b}_i that can not be explained by the covariates \mathbf{z}_i . Suppose that $\boldsymbol{\gamma}_i$'s are independently identically distributed with mean zero, and they are independent of \mathbf{z}_i , u_{it} , and ϵ_{t+1} .

Now we consider the following semi-parametric factor model to forecast y_{t+1} :

$$y_{t+1} = h(\boldsymbol{\phi}'_1 \mathbf{f}_t, \dots, \boldsymbol{\phi}'_L \mathbf{f}_t, \epsilon_{t+1}), \quad (2.14)$$

$$x_{it} = (\mathbf{g}(\mathbf{z}_i) + \boldsymbol{\gamma}_i)' \mathbf{f}_t + u_{it}, \quad 1 \leq i \leq p, \quad 1 \leq t \leq T, \quad (2.15)$$

where $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_L$, ϵ_{t+1} , \mathbf{f}_t , and u_{it} follow the same assumptions in Section 2.1. Note that (2.15) has the matrix form of $\mathbf{X} = (\mathbf{G}(\mathbf{Z}) + \boldsymbol{\Gamma})\mathbf{F}' + \mathbf{U}$, which decomposes the factor loading matrix \mathbf{B} into the subject-specific component $\mathbf{G}(\mathbf{Z})$ and the orthogonal residual component $\boldsymbol{\Gamma}$. Following Fan et al. (2015a), the unobserved latent vectors \mathbf{f}_t in the semi-parametric factor

model (2.15) can be better estimated by an order of magnitude, as long as the contributions of \mathbf{z}_i are genuine.

Given covariates \mathbf{z}_i , we first use the projected PCA (Fan et al., 2015a) to estimate latent factors. Projected PCA is implemented by projecting the predictors \mathbf{X} onto the sieve space spanned by $\{\mathbf{z}_1, \dots, \mathbf{z}_p\}$ and applying the conventional PCA on the projected predictors $\widehat{\mathbf{X}}$. As an example, we use additive spline model to approximate the function g in (2.15). Let $\varphi_1, \dots, \varphi_J$ is a set of J basis functions. Then, we define $\varphi(\mathbf{z}_i) = (\varphi_1(z_{i1}), \dots, \varphi_J(z_{i1}), \dots, \varphi_1(z_{id}), \dots, \varphi_J(z_{id}))'$, and $\varphi(\mathbf{Z}) = (\varphi(\mathbf{z}_1), \dots, \varphi(\mathbf{z}_p))'$ as a p by Jd design matrix, created by fitting additive spline models. Then, the projected predictors (or smoothed predictors by using additive model) is

$$\widehat{\mathbf{X}} = \mathbf{P}\mathbf{X}, \quad (2.16)$$

where $\mathbf{P} = \varphi(\mathbf{Z})(\varphi(\mathbf{Z})'\varphi(\mathbf{Z}))^{-1}\varphi(\mathbf{Z})'\mathbf{X}$ is the projection matrix onto the sieve space. The projected PCA is to run PCA on the projected matrix $\widehat{\mathbf{X}}$ to extract the latent factor $\widehat{\mathbf{F}}$. The advantage of this is that $\widehat{\mathbf{F}}$ is estimated far more accurately, according to Fan et al. (2015a). Note that the projection matrix \mathbf{P} was constructed by using the additive spline model. It can also be constructed by using the multivariate kernel function.

After obtaining $\widehat{\mathbf{F}}$, we simply follow Section 2.2 to obtain SDR directions $\widehat{\boldsymbol{\psi}}_1, \dots, \widehat{\boldsymbol{\psi}}_L$ and sufficient predictive indices $\widehat{\boldsymbol{\psi}}_1'\widehat{\mathbf{f}}_t, \dots, \widehat{\boldsymbol{\psi}}_L'\widehat{\mathbf{f}}_t$ to forecast forecast y_{t+1} .

2.4 Choices of tuning parameters

Sufficient forecasting includes determination of the number of slices H , the number of predictive indices L , and the number of factors K . In practice, H has little influence on the estimated directions, which was first pointed out in Li (1991) and can be seen from (2.8). The choice of H differs from that of a smoothing parameter in nonparametric regression, which may lead to a slower rate of convergence. As shown in Section 3, we always have the same rate of convergence for $\widehat{\boldsymbol{\phi}}_i$ no matter how H is specified in the sufficient forecasting. The reason is that (2.8) gives the same eigenspace as long as $H \geq \max\{L, 2\}$.

In terms of the choice of L , the first L eigenvalues of $\boldsymbol{\Sigma}_{f|y}$ must be significantly different from zero compared to the estimation error. Several methods such as Li (1991) and Schott (1994) have been proposed to determine L . For instance, the average of the smallest $K - L$ eigenvalues of $\boldsymbol{\Sigma}_{f|y}$ would follow a χ^2 -distribution if the underlying factors are normally distributed, where K is the number of underlying factors. Should that average be large, there would be at least $L + 1$ predictive indices in the sufficient forecasting model.

Pertaining to many factor analysis, the number of latent factors K might be unknown

to us. There are many existing approaches to determining K in the literature, e.g., Bai and Ng (2002), Hallin and Liška (2007), Alessi et al. (2010). Recently, Lam and Yao (2012) and Ahn and Horenstein (2013) proposed a ratio-based estimator by maximizing the ratio of two adjacent eigenvalues of $\mathbf{X}'\mathbf{X}$ arranged in descending order, i.e.

$$\hat{K} = \arg \max_{1 \leq i \leq kmax} \hat{\lambda}_i / \hat{\lambda}_{i+1},$$

where $\hat{\lambda}_1 \geq \dots \geq \hat{\lambda}_T$ are the eigenvalues. The estimator enjoys good finite-sample performances and was motivated by the following observation: the K largest eigenvalues of $\mathbf{X}'\mathbf{X}$ grow unboundedly as p increases, while the others remain bounded. Fan et al. (2015a) proposed a similar ratio-based estimator for estimating the number of factors in semiparametric factor models. We note here that once a consistent estimator of K is found, the asymptotic results in this paper hold true for the unknown K case by a conditioning argument. Unless otherwise specified, we shall assume a known K in the sequel.

3 Asymptotic properties

We define some necessary notation here. For a vector \mathbf{v} , $\|\mathbf{v}\|$ denotes its Euclidean norm. For a matrix \mathbf{M} , $\|\mathbf{M}\|$ and $\|\mathbf{M}\|_1$ represent its spectral norm and ℓ_1 norm respectively, defined as the largest singular value of \mathbf{M} and its maximum absolute column sum. $\|\mathbf{M}\|_\infty$ is its matrix ℓ_∞ norm defined as the maximum absolute row sum. For a symmetric matrix \mathbf{M} , $\|\mathbf{M}\|_1 = \|\mathbf{M}\|_\infty$. Denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and largest eigenvalue of a symmetric matrix respectively.

3.1 Assumptions

We first detail the assumptions on the forecasting model (2.1) and the associated factor model (2.2), in which common factors $\{\mathbf{f}_t\}_{t=1,\dots,T}$ and the forecasting function $h(\cdot)$ are unobserved and only $\{(\mathbf{x}_t, y_t)\}_{t=1,\dots,T}$ are observable.

Assumption 3.1 (Factors and Loadings). *For some $M > 0$,*

(1) *The loadings \mathbf{b}_i satisfy that $\|\mathbf{b}_i\| \leq M$ for $i = 1, \dots, p$. As $p \rightarrow \infty$, there exist two positive constants c_1 and c_2 such that*

$$c_1 < \lambda_{\min}\left(\frac{1}{p}\mathbf{B}'\mathbf{B}\right) < \lambda_{\max}\left(\frac{1}{p}\mathbf{B}'\mathbf{B}\right) < c_2.$$

(2) *Identification: $T^{-1}\mathbf{F}'\mathbf{F} = \mathbf{I}_K$, and $\mathbf{B}'\mathbf{B}$ is a diagonal matrix with distinct entries.*

(3) *Linearity*: $E(\mathbf{b}'\mathbf{f}_t|\phi'_1\mathbf{f}_t, \dots, \phi'_L\mathbf{f}_t)$ is a linear function of $\phi'_1\mathbf{f}_t, \dots, \phi'_L\mathbf{f}_t$ for any $\mathbf{b} \in \mathbb{R}^p$, where ϕ_i 's come from model (2.1).

Condition (1) is often known as the *pervasive* condition (Bai and Ng, 2002; Fan et al., 2013) such that the factors impact a non-vanishing portion of the predictors. Condition (2) corresponds to the PC1 condition in Bai and Ng (2013), which eliminates rotational indeterminacy in the individual columns of \mathbf{F} and \mathbf{B} . Condition (3) ensures that the (centered) inverse regression curve $E(\mathbf{f}_t|y_{t+1})$ is contained in the central subspace, and it is standard in the dimension reduction literature. The linearity condition is satisfied when the distribution of \mathbf{f}_t is elliptically symmetric (Eaton, 1983), and it is also asymptotically justified when the dimension of \mathbf{f}_t is large (Hall and Li, 1993).

We impose the strong mixing condition on the data generating process. Let \mathcal{F}_∞^0 and \mathcal{F}_T^∞ denote the σ -algebras generated by $\{(\mathbf{f}_t, \mathbf{u}_t, \epsilon_{t+1}) : t \leq 0\}$ and $\{(\mathbf{f}_t, \mathbf{u}_t, \epsilon_{t+1}) : t \geq T\}$ respectively. Define the mixing coefficient

$$\alpha(T) = \sup_{A \in \mathcal{F}_\infty^0, B \in \mathcal{F}_T^\infty} |P(A)P(B) - P(AB)|.$$

Assumption 3.2 (Data generating process). $\{\mathbf{f}_t\}_{t \geq 1}$, $\{\mathbf{u}_t\}_{t \geq 1}$ and $\{\epsilon_{t+1}\}_{t \geq 1}$ are three strictly stationary processes, and they are mutually independent. The factor process also satisfies that $E\|\mathbf{f}_t\|^4 < \infty$ and $E(\|\mathbf{f}_t\|^2|y_{t+1}) < \infty$. In addition, the mixing coefficient $\alpha(T) < c\rho^T$ for all $T \in \mathbb{Z}^+$ and some $\rho \in (0, 1)$.

The independence between $\{\mathbf{f}_t\}_{t \geq 1}$ and $\{\mathbf{u}_t\}_{t \geq 1}$ (or $\{\epsilon_{t+1}\}_{t \geq 1}$) is similar to Assumption A(d) in Bai and Ng (2013). The independence between $\{\mathbf{u}_t\}_{t \geq 1}$ and $\{\epsilon_{t+1}\}_{t \geq 1}$, however, can be relaxed, making the data generation process more realistic. For example, we just need to assume that $E(\mathbf{u}_t|y_{t+1}) = 0$ for $t \geq 1$ for the subsequent theories to hold. Nonetheless, we stick to this simplified assumption for clarity.

Moreover, we make the following assumption on the residuals and dependence of the factor model (2.2). Such conditions are similar to those in Bai (2003), which are needed to consistently estimate the common factors as well as the factor loadings.

Assumption 3.3 (Residuals and Dependence). *There exists a positive constant $M < \infty$ that does not depend on p and T , such that*

- (1) $E(\mathbf{u}_t) = \mathbf{0}$, and $E|u_{it}|^8 \leq M$.
- (2) $\|\Sigma_u\|_1 \leq M$, and for every $i, j, t, s > 0$, $(pT)^{-1} \sum_{i,j,t,s} |E(u_{it}u_{js})| \leq M$
- (3) For every (t, s) , $E|p^{-1/2}(\mathbf{u}'_s\mathbf{u}_t - E(\mathbf{u}_s\mathbf{u}_t))|^4 \leq M$.

3.2 Rate of convergence

The following theorem establishes the convergence rate of the sliced covariance estimate of inverse regression curve (i.e. $\widehat{\Sigma}_{f|y}$ as in (2.12) or (2.13)) under the spectral norm. It further implies the same convergence rate of the estimated SDR directions associated with sufficient predictive indices. For simplicity, we assume that K and H are fixed, though they can be extended to allow K and H to depend on T . This assumption enables us to obtain faster rate of convergence. Note that the regression indices $\{\phi_i\}_{i=1}^L$ and the PCA directions $\{\psi_i\}_{i=1}^L$ are typically different, but they can span the same linear space.

Theorem 3.1. *Suppose that Assumptions 3.1-3.3 hold and let $\omega_{p,T} = p^{-1/2} + T^{-1/2}$. Then under the forecasting model (2.1) and the associated factor model (2.2), we have*

$$\|\widehat{\Sigma}_{f|y} - \Sigma_{f|y}\| = O_p(\omega_{p,T}). \quad (3.1)$$

If the L largest eigenvalues of $\Sigma_{f|y}$ are positive and distinct, the eigenvectors $\widehat{\psi}_1, \dots, \widehat{\psi}_L$ associated with the L largest eigenvalues of $\widehat{\Sigma}_{f|y}$ give the consistent estimate of directions ψ_1, \dots, ψ_L respectively, with rates

$$\|\widehat{\psi}_j - \psi_j\| = O_p(\omega_{p,T}), \quad (3.2)$$

for $j = 1, \dots, L$, where ψ_1, \dots, ψ_L form an orthonormal basis for the central subspace $S_{y|f}$ spanned by ϕ_1, \dots, ϕ_L .

When $L = 1$ and the link function $h(\cdot)$ in model (2.1) is linear, Stock and Watson (2002a) concluded that having estimated factors as regressors in the linear forecasting does not affect consistency of the parameter estimates in a forecasting equation. Theorem 3.1 extends the theory, showing that it is also true for any unknown link function $h(\cdot)$ and multiple predictive indices by using the sufficient forecasting. The two estimates coincide in the linear forecasting case, but our procedure can also deal with the nonlinear forecasting cases.

Remark 3.1. Combining Theorem 3.1 and Fan et al. (2015a), we could obtain the rate of convergence for sufficient forecasting using semiparametric factor models (i.e., (2.14)-(2.15)). Under some conditions, when $J = (p \min\{T, p, v_p^{-1}\})^{1/\kappa}$ and $p, T \rightarrow \infty$, we have

$$\|\widehat{\Sigma}_{f|y} - \Sigma_{f|y}\| = O_p(p^{-1/2} \cdot (p \min\{T, p, v_p^{-1}\})^{\frac{1}{2\kappa} - \frac{1}{2}} + T^{-1/2}),$$

where $\kappa \geq 4$ is a constant characterizing the accuracy of sieve approximation (see Assumption

4.3 of Fan et al. (2015a)), and $v_p = \max_k \frac{1}{p} \sum_i \text{var}(\gamma_{ik})$. Then, for $j = 1, \dots, L$, we have

$$\|\widehat{\boldsymbol{\psi}}_j - \boldsymbol{\psi}_j\| = O_p(p^{-1/2} \cdot (p \min\{T, p, v_p^{-1}\})^{\frac{1}{2\kappa} - \frac{1}{2}} + T^{-1/2}),$$

where $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_L$ form an orthonormal basis for $S_{y|\mathbf{f}}$, when the L largest eigenvalues of $\boldsymbol{\Sigma}_{f|y}$ are positive and distinct. It is obvious that $(p \min\{T, p, v_p^{-1}\})^{\frac{1}{2\kappa} - \frac{1}{2}} \rightarrow 0$ even when T is finite. Thus, with covariates \mathbf{Z} , the sufficient forecasting achieves a faster rate of convergence by using the projected PCA (Fan et al., 2015a). For space consideration, we do not present technical details in this paper.

As a consequence of Theorem 3.1, the sufficient predictive indices can be consistently estimated as well. We present this result in the following corollary.

Corollary 3.1. *Under the same conditions of Theorem 3.1, for any $j = 1, \dots, L$, we have*

$$\widehat{\boldsymbol{\psi}}_j' \widehat{\mathbf{f}}_t \rightarrow_p \boldsymbol{\psi}_j' \mathbf{f}_t. \quad (3.3)$$

Translating into the original predictors \mathbf{x}_t and letting $\widehat{\boldsymbol{\xi}}_j = \widehat{\boldsymbol{\Lambda}}_b' \widehat{\boldsymbol{\psi}}_j$, we then have

$$\widehat{\boldsymbol{\xi}}_j' \mathbf{x}_t \rightarrow_p \boldsymbol{\xi}_j' \mathbf{f}_t. \quad (3.4)$$

Corollary 3.1 further implies the consistency of our proposed sufficient forecasting by using the classical nonparametric estimation techniques (Green and Silverman, 1993; Fan and Gijbels, 1996; Li and Racine, 2007).

Remark 3.2. In view of (3.3) and (3.4) in Corollary 3.1, the sufficient forecasting finds not only the sufficient dimension reduction directions $\widehat{\boldsymbol{\psi}}_1, \dots, \widehat{\boldsymbol{\psi}}_L$ for estimated factors $\widehat{\mathbf{f}}_t$ but also the sufficient dimension reduction directions $\widehat{\boldsymbol{\xi}}_1, \dots, \widehat{\boldsymbol{\xi}}_L$ for predictor \mathbf{x}_t . The sufficient forecasting effectively obtains the projection of the p -dimensional predictor \mathbf{x}_t onto the L -dimensional subspace $\widehat{\boldsymbol{\xi}}_1' \mathbf{x}_t, \dots, \widehat{\boldsymbol{\xi}}_L' \mathbf{x}_t$ by using the factor model, where the number of predictors p can be larger than the number of observations T . It is well-known that the traditional sliced inverse regression and many other dimension reduction methods are limited to either a fixed dimension or a diverging dimension that is smaller than the sample size (Zhu et al., 2006). By condensing the cross-sectional information into indices, our work alleviates what plagues the sufficient dimension reduction in high-dimensional regimes, and the factor structure in (2.1) and (2.2) effectively reduces the dimension of predictors and extends the methodology and applicability of the sufficient dimension reduction.

3.3 Linear forecasting under link violation

Having extracted common factors from the vast predictors, the PCR is a simple and natural method to run a multiple regression and forecast the outcome. Such a linear estimate is easy to construct and provides a benchmark for our analysis. When the underlying relationship between the response and the latent factors is nonlinear, directly applying linear forecasts would violate the link function $h(\cdot)$ in (2.2). But what does this method do in the case of model misspecification? In what follows, we shall see that asymptotically the linear forecast falls into the central subspace $S_{y|\mathbf{f}}$, namely, it is a weighted average of the sufficient predictive indices.

With the underlying factors $\{\widehat{\mathbf{f}}_t\}$ estimated as in (2.10) and (2.11), the response y_{t+1} is then regressed on $\widehat{\mathbf{f}}_t$ via a multiple regression. In this case, the design matrix is orthogonal due to the normalization (2.11). Therefore, the least-squares estimate of regression coefficients from regressing y_{t+1} on $\widehat{\mathbf{f}}_t$ is

$$\widehat{\boldsymbol{\phi}} = \frac{1}{T-1} \sum_{t=1}^{T-1} y_{t+1} \widehat{\mathbf{f}}_t. \quad (3.5)$$

To study the behavior of regression coefficient $\widehat{\boldsymbol{\phi}}$, we shall assume normality of the underlying factors \mathbf{f}_t . The following theorem shows that, regardless of the specification of the link function $h(\cdot)$, $\widehat{\boldsymbol{\phi}}$ falls into the central subspace $S_{y|\mathbf{f}}$ spanned by $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_L$ as $p, T \rightarrow \infty$. More specifically, it converges to the sufficient direction

$$\bar{\boldsymbol{\phi}} = \sum_{i=1}^L E((\boldsymbol{\phi}'_i \mathbf{f}_t) y_{t+1}) \boldsymbol{\phi}_i,$$

which is a weighted average of sufficient directions $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_L$. In particular, when the link function is linear and $L = 1$, the PCR yields an asymptotically consistent direction.

Theorem 3.2. *Consider model (2.1) and (2.2) under assumptions of Theorem 3.1. Suppose the factors \mathbf{f}_t are normally distributed and that $E(y_t^2) < \infty$. Then, we have*

$$\|\widehat{\boldsymbol{\phi}} - \bar{\boldsymbol{\phi}}\| = O_p(\omega_{p,T}). \quad (3.6)$$

As a consequence of Theorem 3.2, the linear forecast $\widehat{\boldsymbol{\phi}}' \widehat{\mathbf{f}}_t$ is a weighted average of the sufficient predictive indices $\boldsymbol{\phi}'_1 \mathbf{f}_t, \dots, \boldsymbol{\phi}'_L \mathbf{f}_t$.

Corollary 3.2. *Under the same conditions of Theorem 3.2, we have*

$$\widehat{\boldsymbol{\phi}}' \widehat{\mathbf{f}}_t \rightarrow_p \bar{\boldsymbol{\phi}}' \mathbf{f}_t = \sum_{i=1}^L E((\boldsymbol{\phi}'_i \mathbf{f}_t) y_{t+1}) \boldsymbol{\phi}'_i \mathbf{f}_t. \quad (3.7)$$

It is interesting to see that when $L = 1$, $\bar{\phi} = E((\phi_1' \mathbf{f}_t) y_{t+1}) \phi_1$. As a result, the least squares coefficient $\hat{\phi}$ in (3.5), using the estimated factors as covariates, delivers an asymptotically consistent estimate of the forecasting direction ϕ_1 up to a multiplicative scalar. In this case, we can first get a scatter plot for the data $\{(\hat{\phi}' \hat{\mathbf{f}}_t, y_{t+1})\}_{t=1}^{T-1}$ to check if the linear fit suffices and then employ a nonparametric smoothing to get a better fit if necessary. This can correct the bias of PCR, which delivers exactly one index of extracted factors for forecasting.

When there exist multiple sufficient predictive indices (i.e., $L \geq 2$), the above robustness no longer holds. The estimated coefficient $\hat{\phi}$ belongs to the central subspace, and it only reveals one dimension of the central subspace $S_{y|\mathbf{f}}$, leading to limited predictive power. The central subspace $S_{y|\mathbf{f}}$, in contrast, is entirely contained in $\Sigma_{f|y}$ as in (2.7). By estimating $\Sigma_{f|y}$, the sufficient forecasting tries to recover all the effective directions and would therefore capture more driving forces to forecast.

As will be demonstrated in Section 4, the proposed method is comparable to the linear forecasting when $L = 1$, but significantly outperforms the latter when $L \geq 2$ in both simulation studies and empirical applications.

4 Numerical studies

In this section, we conduct both Monte Carlo experiments and an empirical study to numerically assess the proposed sufficient forecasting. Sections 4.1-4.3 include three different simulation studies, and Section 4.4 shows the empirical study.

4.1 Linear forecasting

We first consider the case when the target is a linear function of the latent factors plus some noise. Prominent examples include asset return predictability, where we could use the cross section of book-to-market ratios to forecast aggregate market returns (Campbell and Shiller, 1988; Polk et al., 2006; Kelly and Pruitt, 2013). To this end, we specify our data generating process as

$$\begin{aligned} y_{t+1} &= \phi' \mathbf{f}_t + \sigma_y \epsilon_{t+1}, \\ x_{it} &= \mathbf{b}_i' \mathbf{f}_t + u_{it}, \end{aligned}$$

where we let $K = 5$ and $\phi = (0.8, 0.5, 0.3, 0, 0)'$, namely there are 5 common factors driving the predictors and a linear combination of them predicting the outcome. Factor loadings \mathbf{b}_i 's are drawn from the standard normal distribution. To account for serial correlation, we

set f_{jt} and u_{it} as two AR(1) processes respectively,

$$f_{jt} = \alpha_j f_{jt-1} + e_{jt}, \quad u_{it} = \rho_i u_{it-1} + \nu_{it}.$$

We draw α_j, ρ_i from $\sim U[0.2, 0.8]$ and fix them during simulations, while the noises e_{jt}, ν_{it} and ϵ_{t+1} are standard normal respectively. σ_y is taken as the variance of the factors, so that the infeasible best forecast using $\phi' \mathbf{f}_t$ has an R^2 of approximately 50%.

Table 1: Performance of forecasts using in-sample and out-sample R^2

p	T	In-sample R^2 (%)			Out-of-sample R^2 (%)		
		SF(1)	PCR	PC1	SF(1)	PCR	PC1
50	100	46.9	47.7	7.8	35.1	39.5	2.4
50	200	46.3	46.5	6.6	42.3	41.7	4.4
100	100	49.3	50.1	8.9	37.6	40.3	3.0
100	500	47.8	47.8	5.5	43.6	43.5	1.1
500	100	48.5	48.8	7.9	40.0	43.1	4.7
500	500	48.2	48.3	7.2	48.0	47.9	6.0

Notes: In-sample and out-of-sample median R^2 , recorded in percentage, over 1000 simulations. SF(1) denotes the sufficient forecast using one single predictive index, PCR denotes principal component regression, and PC1 uses only the first principal component.

We examine both the in-sample and out-of-sample performance between the principal component regression and the proposed sufficient forecasting. Akin to the in-sample R^2 typically used in linear regression, the out-of-sample R^2 , suggested by Campbell and Thompson (2008), is defined here as

$$R_{OS}^2 = 1 - \frac{\sum_{t=[T/2]}^T (y_t - \hat{y}_t)^2}{\sum_{t=[T/2]}^T (y_t - \bar{y}_t)^2}, \quad (4.1)$$

where in each replication we use the second half of the time series as the testing sample. \hat{y}_t is the fitted value from the predictive regression using all information before t and \bar{y}_t is the mean from the test sample. Note that R_{OS}^2 can be negative.

Numerical results are summarized in Table 1, where SF(1) denotes the sufficient forecast using only one single predictive index, PCR denotes the principal component regression, and PC1 uses only the first principal component. Note that the first principal component is not necessarily the same as the sufficient regression direction ϕ . Hence, PC1 can perform poorly. As shown in Table 1, the sufficient forecasting SF(1), by finding one single projection of the

latent factors, yields comparable results as PCR, which has a total of five regressors. The good performance of SF(1) is consistent with Theorem 3.1, whereas the good performance of PCR is due to the fact that the true predictive model is linear (see Stock and Watson (2002a) and Theorem 3.2). In contrast, using the first principal component alone has very poor performance in general, as it is not the index for the regression target.

4.2 Forecasting with factor interaction

We next turn to the case when the interaction between factors is present. Interaction models have been employed by many researchers in both economics and statistics. For example, in an influential work, Rajan and Zingales (1998) examined the interaction between financial dependence and economic growth. Consider the model

$$y_{t+1} = f_{1t}(f_{2t} + f_{3t} + 1) + \epsilon_{t+1},$$

where ϵ_{t+1} is taken to be standard normal. The data generating process for the predictors x_{it} 's is set to be the same as that in the previous simulation, but we let $K=7$, i.e., 7 factors drive the predictors with only the first 3 factors associated with the response (in a nonlinear fashion). The true sufficient directions are the vectors in the plane $S_{f|y}$ generated by $\phi_1 = (1, 0, 0, 0, 0, 0, 0)'$ and $\phi_2 = (0, 1, 1, 0, 0, 0, 0)'/\sqrt{2}$. Following the procedure in Algorithm 1, we can find the estimated factors $\hat{\mathbf{f}}_t$ as well as the forecasting directions $\hat{\phi}_1$ and $\hat{\phi}_2$.

The forecasting model only involves a strict subset of the underlying factors. Had we made a linear forecast, all the estimated factors would likely be selected, resulting in issues such as use of irrelevant factors (i.e., only the first three factors suffice, but the first 7 factors are used). This is evident through the correlations between the response and the estimated factors, as in Table 2 and results in both bias (fitting a wrong model) and variance issue.

Table 2: $|\text{Corr}(y_{t+1}, \hat{\mathbf{f}}_{it})|$

p	T	$\hat{\mathbf{f}}_{1t}$	$\hat{\mathbf{f}}_{2t}$	$\hat{\mathbf{f}}_{3t}$	$\hat{\mathbf{f}}_{4t}$	$\hat{\mathbf{f}}_{5t}$	$\hat{\mathbf{f}}_{6t}$	$\hat{\mathbf{f}}_{7t}$
500	500	0.1819	0.1877	0.1832	0.1700	0.1737	0.1712	0.1663

Notes: Median correlation between the forecast target and the estimated factors in absolute values, based on 1000 replications.

Next, we compare the numerical performance between the principal component regression and the proposed sufficient forecasting. As in Section 4.1, we again examine their in-sample and out-of-sample forecasting performances using the in-sample R^2 and the out-of-sample R^2 respectively. Furthermore, we will measure the distance between any estimated regression

direction $\hat{\phi}$ and the central subspace $S_{f|y}$ spanned by ϕ_1 and ϕ_2 . We ensure that the true factors and loadings meet the identifiability conditions by calculating an invertible matrix \mathbf{H} such that $\frac{1}{T}\mathbf{H}\mathbf{F}'\mathbf{F}\mathbf{H}' = \mathbf{I}_K$ and $\mathbf{H}^{-1}\mathbf{B}'\mathbf{B}\mathbf{H}'^{-1}$ is diagonal. The space we are looking for is then spanned by $\mathbf{H}^{-1}\phi_1$ and $\mathbf{H}^{-1}\phi_2$. For notation simplicity, we still represent the rotated forecasting space as $S_{f|y}$. We employ the squared multiple correlation coefficient $R^2(\hat{\phi})$ to evaluate the effectiveness of $\hat{\phi}$, where

$$R^2(\hat{\phi}) = \max_{\phi \in S_{f|y}, \|\phi\|=1} (\hat{\phi}'\phi)^2.$$

This measure is commonly used in the dimension reduction literature; see Li (1991). We calculate $R^2(\hat{\phi})$ for the sufficient forecasting directions $\hat{\phi}_1, \hat{\phi}_2$ and the PCR direction $\hat{\phi}_{pcr}$.

Table 3: Performance of estimated index $\hat{\phi}$ using $R^2(\hat{\phi})$ (%)

p	T	Sufficient forecasting		PCR
		$R^2(\hat{\phi}_1)$	$R^2(\hat{\phi}_2)$	$R^2(\hat{\phi}_{pcr})$
100	100	84.5 (16.3)	64.4 (25.0)	91.4 (9.1)
100	200	92.9 (7.8)	83.0 (17.5)	95.7 (4.6)
100	500	96.9 (2.2)	94.1 (5.0)	96.7 (1.6)
500	100	85.4 (15.9)	57.9 (24.8)	91.9 (8.8)
500	200	92.9 (6.7)	82.9 (16.1)	95.9 (4.1)
500	500	97.0 (2.0)	94.5 (4.5)	98.2 (1.5)

Notes: Median squared multiple correlation coefficients in percentage over 1000 replications. The values in parentheses are the associated standard deviations. PCR denotes principal component regression.

Table 4: Performance of forecasts using in-sample and out-sample R^2

p	T	In-sample R^2 (%)			Out-of-sample R^2 (%)		
		SFi	PCR	PCRi	SFi	PCR	PCRi
100	100	46.2	38.5	42.4	20.8	12.7	13.5
100	200	57.7	35.1	38.6	41.6	24.0	24.7
100	500	77.0	31.9	34.9	69.7	29.1	31.5
500	100	49.5	38.7	22.7	21.9	16.6	19.2
500	200	58.9	34.7	39.0	40.2	22.2	24.0
500	500	79.8	32.5	35.6	72.3	26.9	28.2

Notes: In-sample and out-of-sample median R^2 in percentage over 1000 replications. SFi uses first two predictive indices and includes their interaction effect; PCR uses all principal components; PCRi extends PCR by including an extra interaction term built on the first two principal components of the covariance matrix of predictors.

In the sequel, we draw two conclusions from the simulation results summarized in Tables 3 and 4 based on 1000 independent replications.

Firstly, we observe from Table 3 that the length of time-series T has an important effect on $R^2(\hat{\phi}_1)$, $R^2(\hat{\phi}_2)$ and $R^2(\hat{\phi}_{pcr})$. As T increases from 100 to 500, all the squared multiple correlation coefficients increase too. In general, the PCR and the proposed sufficient forecasting perform very well in terms of the measure $R^2(\hat{\phi})$. The good performance of $R^2(\hat{\phi}_{pcr})$ is a testimony of the correctness of Theorem 3.2. However, the PCR is limited to the linear forecasting function, and it only picks up one effective sufficient direction. The sufficient forecasting successfully picks up both effective sufficient directions in the forecasting equation, which is consistent to Theorem 3.1.

Secondly, we evaluate the power of sufficient forecasting using the two predictive indices $\hat{\phi}'_1 \hat{\mathbf{f}}_t$ and $\hat{\phi}'_2 \hat{\mathbf{f}}_t$. Similar to the previous section, we build a simple linear regression to forecast, but include three terms, $\hat{\phi}'_1 \hat{\mathbf{f}}_1$, $\hat{\phi}'_2 \hat{\mathbf{f}}_2$ and $(\hat{\phi}'_1 \hat{\mathbf{f}}_1) \cdot (\hat{\phi}'_2 \hat{\mathbf{f}}_2)$, as regressors to account for interaction effects. In-sample and out-of-sample R^2 are reported. For comparison purposes, we also report results from principal component regression (PCR), which uses the 7 estimated factors extracted from predictors. In addition, we add the interaction between the first two principal components on top of the PCR, and denote this method as PCRi. Note that the principal component directions differ from the sufficient forecast directions, and it does not take into account the target information. Therefore, PCRi model is very different from the sufficient regression with interaction terms. As shown in Table 4, the in-sample R^2 's of PCR hover around 35%, and its out-of-sample R^2 's are relatively low. Including interaction between

the first two PCs of the covariance of the predictors does not help much, as the interaction term is incorrectly specified for this model. SFi picks up the correct form of interaction and exhibit better performance, especially when T gets reasonably large.

4.3 Forecasting using semiparametric factor models

When the factor loadings admit a semiparametric structure, this additional information often leads to improved estimation accuracy. In this section, we consider a simple design with one observed covariate and three latent factors. We set the three loading functions as $g_1(z) = z$, $g_2(z) = z^2 - 1$ and $g_3(z) = z^3 - 2z$, where the characteristic z is drawn from standard normal. We examine the same predictive model as outlined in Section 4.2, which involves factor interaction. The only difference now is that our loadings are covariate-dependent and we no longer have irrelevant factors.

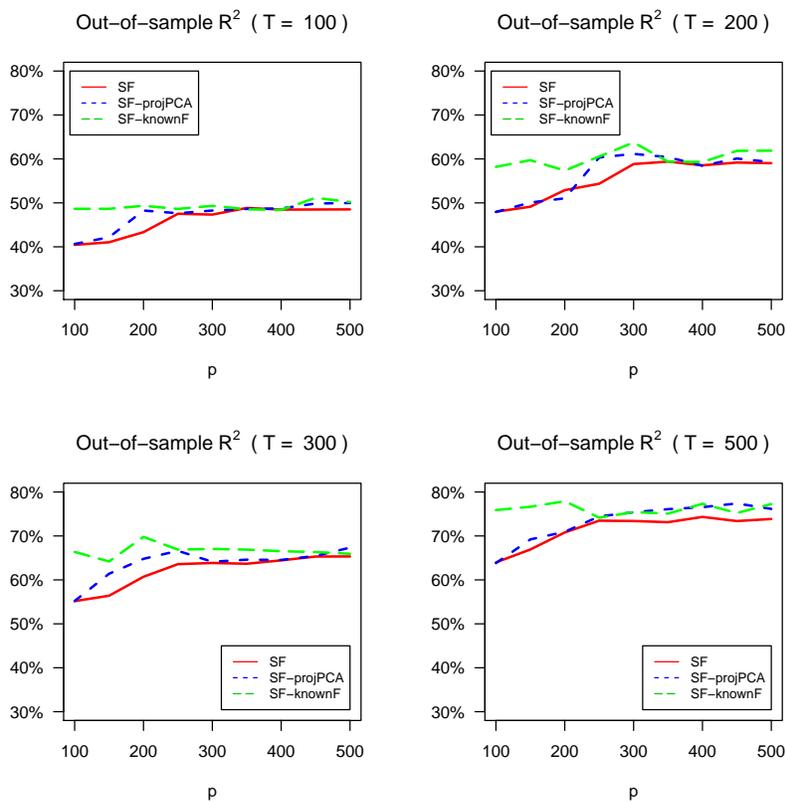


Figure 2: Out-of-sample R^2 over 1000 replications. SF denotes the sufficient forecasting method without using extra covariate information. SF-projPCA uses projected-PCA method to estimate the latent factors, and SF-knownF uses the true factors from the data generating process. All the methods are based on 2 predictive indices.

Figure 2 presents the out-of-sample performance of three methods for different combinations of (p, T) . The three methods differ in their ways of estimating the latent factors. In the first method, we simply use traditional PCA to extract underlying factors, whereas in the second method we resort to projected-PCA. For comparison, we also look at the performance when the true factors are used to forecast the target. In general, as T increases, the predictive power significantly increases, irrespective of the cross-sectional dimension p . For fixed T , the projected-PCA quickly picks up predictive power as p increases. Using extra covariate information helps up gain accuracy in estimating the underlying factors, and lifts up the out-of-sample R^2 very close to those obtained from using the true factors. This further demonstrates that with the presence of known characteristics in factor loadings, the projected-PCA would buy us more power in forecasting the factor-driven target.

4.4 An empirical example

As an empirical investigation, we use sufficient forecasting to forecast several macroeconomic variables. Our dataset is taken from Stock and Watson (2012), which consists of quarterly observations on 108 U.S. low-level disaggregated macroeconomic time series from 1959:I to 2008:IV. Similar datasets have been extensively studied in the forecasting literature (Bai and Ng, 2008; Ludvigson and Ng, 2009). These time series are transformed by taking logarithms and/or differencing to make them approximately stationary. We treat each of them as a forecast target y_t , with all the others forming the predictor set \mathbf{x}_t . The out-of-sample performance of each time series is examined, measured by the out-of-sample R^2 in (4.1). The procedure involves fully recursive factor estimation and parameter estimation starting half-way of the sample, using data from the beginning of the sample through quarter t for forecasting in quarter $t + 1$.

We present the forecasting results for a few representatives in each macroeconomic category. The time series are chosen such that the second eigenvalue of $\widehat{\Sigma}_{f|y}$ exceeds 50% of the first eigenvalue in the first half training sample. This implies that a single predictive index may not be enough to forecast the given target, so we could consider the effect of second predictive index. Note that in some categories, the second eigenvalues $\widehat{\Sigma}_{f|y}$ for all the time series are very small; in this case, we randomly pick a representative. As described in previous simulations, we use SF(1) to denote sufficient forecasting with one single predictive index, which corresponds to a linear forecasting equation with one regressor. SF(2) employs an additional predictive index on top of SF(1), and is fit by local linear regression. In all cases, we use 7 estimated factors extracted from 107 predictors, which are found to explain over 40 percent of the variation in the data; see Bai and Ng (2013).

Table 5: Out-of-sample Macroeconomic Forecasting

Category	Label	SF(1)	SF(2)	PCR	PC1
GDP components	GDP264	11.2	14.9	13.8	9.2
IP	IPS13	17.8	21.0	20.6	-2.9
Employment	CES048	24.1	26.4	23.4	25.7
Unemployment rate	LHU680	19.0	21.7	13.6	21.1
Housing	HSSOU	1.9	4.4	-0.8	7.4
Inventories	PMNO	19.8	20.2	18.6	6.3
Prices	GDP275_3	9.1	11.3	10.2	0.2
Wages	LBMNU	32.7	34.2	34.0	29.1
Interest rates	FYFF	3.6	2.4	-31.5	1.6
Money	CCINRV	4.8	16.0	1.3	-0.5
Exchange rates	EXRCAN	-4.7	4.5	-7.9	-1.3
Stock prices	FSDJ	-9.2	8.8	-14.5	-7.7
Consumer expectations	HHSNTN	-3.7	-5.0	-4.4	0.0

Notes: Out-of-sample R^2 for one-quarter ahead forecasts. SF(1) uses a single predictive index built on 7 estimated factors to make a linear forecast. SF(2) is fit by local linear regression using the first two predictive indices. PCR uses all 7 principal components as linear regressors. PC1 uses the first principal component.

Table 5 shows that SF(1) yields comparable performance as PCR. There are cases where SF(1) exhibits more predictability than PCR, e.g., Inventories and Interest rates. This is due to the fact that there are possible nonlinear effects from extracted factors on the target. In this case, the first predictive index obtained from our procedure accounts for such a nonlinear effect whereas the index created by PCR finds only the best linear approximation and can not find a second index. SF(2) improves predictability in many cases, where the target time series might not be linear in the latent factors. Taking CCINRV (consumer credit outstanding) for example, Figure 3 plots the eigenvalues of its corresponding $\hat{\Sigma}_{f|y}$, the estimated regression surface and the running out-of-sample R^2 's by different sample split date. As can be seen from the plot, there is a non-linear effect of the two underlying macro factors on the target. By taking such effect into account, SF(2) consistently outperforms the other methods.

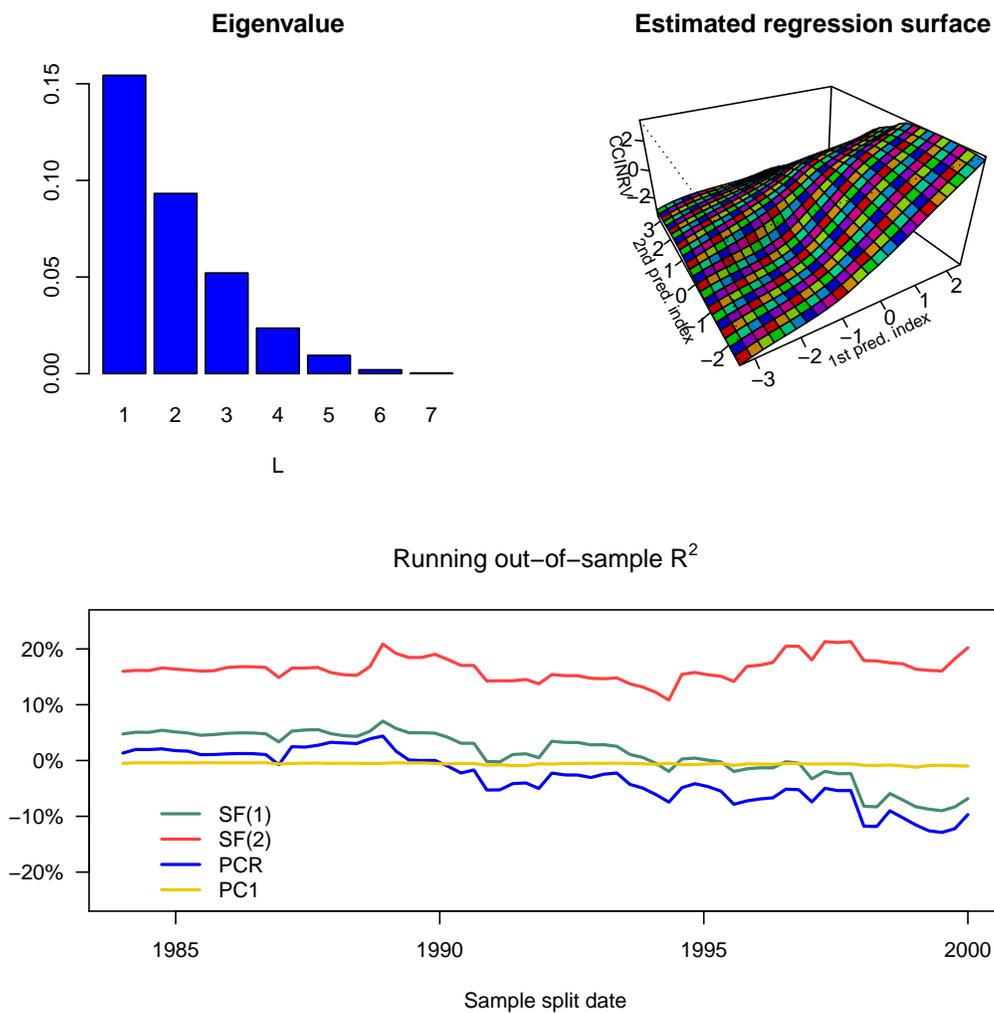


Figure 3: Forecasting results for CCINRV (consumer credit outstanding). The top left panel shows the eigenvalues of $\hat{\Sigma}_{f|y}$. The top right panel gives a 3-d plot of the estimated regression surface. The lower panel displays the running out-of-sample R^2 's for four methods described in Table 5.

5 Discussion

We have introduced the sufficient forecasting in a high-dimensional environment to forecast a single time series, which enlarges the scope of traditional factor forecasting. The key feature of the sufficient forecasting is its ability in extracting multiple predictive indices and providing additional predictive power beyond the simple linear forecasting, especially when the target is an unknown nonlinear function of underlying factors. We explicitly point out the interesting connection between the proposed sufficient forecasting and the deep learning (Bengio, 2009; Bengio et al., 2013). Furthermore, the proposed method extends the sufficient dimension reduction to high-dimensional regimes by condensing cross-sectional information through (semiparametric) factor models. We have demonstrated its efficacy through Monte Carlo experiments. Our empirical results on macroeconomic forecasting suggest that such procedure can contribute to substantial improvement beyond conventional linear models.

Acknowledgement

The authors are grateful to the co-editor and referees for their constructive comments. The authors also thank Stephane Bonhomme, A. Ronald Gallant, Yuan Liao, Rosa Matzkin, Jose Luis Montiel Olea, and participants at Penn State University (Econometrics), New York University (Stern), Peking University (Statistics), University of South Carolina (Statistics), and The Institute for Fiscal Studies for their helpful suggestions. This paper is supported by National Institutes of Health grants R01-GM072611 and R01GM100474-04 and National Science Foundation grants DMS-1206464, DMS-1308566, and DMS-1505256.

6 Appendix

We first cite two lemmas from Fan et al. (2013), which are needed subsequently in the proofs.

Lemma 6.1. *Suppose \mathbf{A} and \mathbf{B} are two semi-positive definite matrices, and that $\lambda_{\min}(\mathbf{A}) > c_{p,T}$ for some sequence $c_{p,T} > 0$. If $\|\mathbf{A} - \mathbf{B}\| = o_p(c_{p,T})$, then*

$$\|\mathbf{A}^{-1} - \mathbf{B}^{-1}\| = O_p(c_{p,T}^{-2})\|\mathbf{A} - \mathbf{B}\|.$$

Lemma 6.2. *Let $\{\lambda_i\}_{i=1}^p$ be the eigenvalues of Σ in descending order and $\{\boldsymbol{\xi}\}_{i=1}^p$ be their associated eigenvectors. Correspondingly, let $\{\hat{\lambda}_i\}_{i=1}^p$ be the eigenvalues of $\hat{\Sigma}$ in descending order and $\{\hat{\boldsymbol{\xi}}\}_{i=1}^p$ be their associated eigenvectors. Then,*

(a) *Weyl's Theorem:*

$$|\hat{\lambda}_i - \lambda_i| \leq \|\hat{\Sigma} - \Sigma\|.$$

(b) $\sin(\theta)$ Theorem (Davis and Kahan, 1970):

$$\|\widehat{\boldsymbol{\xi}}_i - \boldsymbol{\xi}_i\| \leq \frac{\sqrt{2} \cdot \|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}\|}{\min(|\widehat{\lambda}_{i-1} - \lambda_i|, |\lambda_i - \widehat{\lambda}_{i+1}|)}.$$

6.1 Proof of Proposition 2.1

Proof. It suffices to show that $\widehat{\mathbf{f}}_t = \widehat{\boldsymbol{\Lambda}}_b \mathbf{x}_t$, or equivalently, $\widehat{\mathbf{F}}' = \widehat{\boldsymbol{\Lambda}}_b \mathbf{X}$ in matrix form. First we let $\mathbf{M} = \text{diag}(\lambda_1, \dots, \lambda_K)$, where λ_i are the largest K eigenvalues of $\mathbf{X}'\mathbf{X}$. By construction, we have $(\mathbf{X}'\mathbf{X})\widehat{\mathbf{F}} = \widehat{\mathbf{F}}\mathbf{M}$. Thus,

$$\widehat{\mathbf{B}}'\widehat{\mathbf{B}} = T^{-2}\widehat{\mathbf{F}}'(\mathbf{X}'\mathbf{X})\widehat{\mathbf{F}} = T^{-2}\widehat{\mathbf{F}}'\widehat{\mathbf{F}}\mathbf{M} = T^{-1}\mathbf{M}.$$

Now, it is easy to see that

$$(\widehat{\mathbf{B}}'\widehat{\mathbf{B}})^{-1}\widehat{\mathbf{B}}'\mathbf{X} = T\mathbf{M}^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{X}')\mathbf{X} = \mathbf{M}^{-1}(\mathbf{X}'\mathbf{X}\widehat{\mathbf{F}})' = \widehat{\mathbf{F}}'.$$

This concludes the proof of Proposition 2.1. □

6.2 Proof of Theorem 3.1

Let \mathbf{V} denote the $K \times K$ diagonal matrix consisting of the K largest eigenvalues of the sample covariance matrix $T^{-1}\mathbf{X}'\mathbf{X}$ in the descending order. Define a $K \times K$ matrix

$$\mathbf{H} = (1/T)\mathbf{V}^{-1}\widehat{\mathbf{F}}'\mathbf{F}\mathbf{B}'\mathbf{B}, \tag{6.1}$$

where $\mathbf{F}' = (\mathbf{f}_1, \dots, \mathbf{f}_T)$. In factor models, the rotation matrix \mathbf{H} is typically used for identifiability. With identification in Assumptions 3.1 (2), Bai and Ng (2013) showed that \mathbf{H} is asymptotically an identity matrix. The result continues to hold under our setting.

Lemma 6.3. *Under Assumptions 3.1 (1)–(2), 3.2 and 3.3, we have*

$$\|\mathbf{H} - \mathbf{I}_K\| = O_p(\omega_{p,T}^2)$$

Proof. In view of Assumption A and Appendix B of Bai and Ng (2013), we only need to verify that Assumption A (c.ii) in Bai and Ng (2013) is satisfied under our models and assumptions, since all other assumptions of theirs are met here. To verify Assumption A (c.ii), it is enough to show that $\sum_{t=1}^T |E(u_{it}u_{js})| \leq M_1$ and $\sum_{i=1}^p |E(u_{it}u_{js})| \leq M_2$ for some positive constants $M_1, M_2 < \infty$.

Consider $E(u_{it}u_{js})$. Since $E|u_{it}|^8 \leq M$ for some constant M and any t by Assumption 3.3(1), by using Davydov's inequality (Corollary 16.2.4 in Athreya and Lahiri (2006)), there exist constants C_1 and C_2 such that for all (i, j) ,

$$|E(u_{it}u_{js})| \leq C_1 (\alpha(|t-s|))^{1-\frac{1}{8}-\frac{1}{8}} \leq C_2 \rho^{\frac{3}{4}|t-s|},$$

where $\alpha(\cdot)$ is the mixing coefficient, $\rho \in (0, 1)$, and $\alpha(|t-s|) < c\rho^{|t-s|}$ by Assumption 3.2. Hence, with $M_1 = 2C_2/(1 - \rho^{\frac{3}{4}})$, we have

$$\sum_{t=1}^T |E(u_{it}u_{js})| = \sum_{t=1}^T C_2 \rho^{\frac{3}{4}|t-s|} \leq 2C_2/(1 - \rho^{\frac{3}{4}}) = M_1.$$

In addition, it is easy to verify that there is a constant C_3 such that

$$|E(u_{it}u_{jt})| \leq C_3|E(u_{it}u_{jt})| = C_3|\boldsymbol{\Sigma}_u(i, j)|.$$

By assumption, we also have

$$\sum_{i=1}^p |E(u_{it}u_{jt})| \leq C_3 \sum_{i=1}^p |\boldsymbol{\Sigma}_u(i, j)| \leq C_3 \|\boldsymbol{\Sigma}_u\|_1 \leq C_3 M.$$

Thus, Assumption A (c.ii) in Bai and Ng (2013) is satisfied with $M_2 = C_3 M$. Therefore, with identification in Assumption 3.1(2), we have

$$\|\mathbf{H} - \mathbf{I}_K\| = O_p(\omega_{p,T}^2),$$

which completes the proof of Lemma 6.3. □

The next lemma shows that the normalization matrix $\boldsymbol{\Lambda}_b$ can be consistently estimated by the plug-in estimate $\widehat{\boldsymbol{\Lambda}}_b$ under the spectral norm, which is an important result to prove Theorem 3.1.

Lemma 6.4. *Under Assumptions 3.1 (1)–(2), 3.2 and 3.3, we have*

- (a) $\|\widehat{\mathbf{B}} - \mathbf{B}\| = O_p(p^{1/2}\omega_{p,T})$,
- (b) $\|\widehat{\boldsymbol{\Lambda}}_b - \boldsymbol{\Lambda}_b\| = O_p(p^{-1/2}\omega_{p,T})$.

Proof. (a) First of all,

$$\begin{aligned} \|\widehat{\mathbf{B}} - \mathbf{B}\| &= \|\widehat{\mathbf{B}} - \mathbf{B}\mathbf{H}' + \mathbf{B}(\mathbf{H}' - \mathbf{I}_K)\| \\ &\leq \|\widehat{\mathbf{B}} - \mathbf{B}\mathbf{H}'\| + \|\mathbf{B}(\mathbf{H}' - \mathbf{I}_K)\|, \end{aligned}$$

where \mathbf{H} is defined in (6.1). The second term is bounded by $\|\mathbf{B}\| \cdot \|\mathbf{H}' - \mathbf{I}_K\| = O_p(p^{1/2}\omega_{p,T}^2) = o_p(p^{1/2}\omega_{p,T})$ by Lemma 6.3 and using the fact $\|\mathbf{B}\|^2 = \lambda_{\max}(\mathbf{B}'\mathbf{B}) < c_2 p$ from Assumption 3.1. It remains to bound the first term. Using the fact that $\|\cdot\| \leq \|\cdot\|_F$, we have

$$\|\widehat{\mathbf{B}} - \mathbf{B}\mathbf{H}'\|^2 \leq \|\widehat{\mathbf{B}} - \mathbf{B}\mathbf{H}'\|_F^2 = \sum_{i=1}^p \|\widehat{\mathbf{b}}_i - \mathbf{H}\mathbf{b}_i\|^2 \quad (6.2)$$

Next, note that both $\widehat{\mathbf{b}}_i = (1/T) \sum_{t=1}^T x_{it} \widehat{\mathbf{f}}_t$ and $(1/T) \sum_{t=1}^T \widehat{\mathbf{f}}_t \widehat{\mathbf{f}}_t' = \mathbf{I}_K$ hold as a result of the constrained least-squares problem (2.10) and (2.11). Then, we have

$$\begin{aligned} \widehat{\mathbf{b}}_i - \mathbf{H}\mathbf{b}_i &= \frac{1}{T} \sum_{t=1}^T (\widehat{\mathbf{f}}_t x_{it} - \mathbf{H}\mathbf{f}_t x_{it}) + \frac{1}{T} \sum_{t=1}^T \mathbf{H}\mathbf{f}_t (\mathbf{f}_t' \mathbf{b}_i + u_{it}) - \mathbf{H}\mathbf{b}_i \\ &= \frac{1}{T} \sum_{t=1}^T x_{it} (\widehat{\mathbf{f}}_t - \mathbf{H}\mathbf{f}_t) + \frac{1}{T} \sum_{t=1}^T \mathbf{H}\mathbf{f}_t u_{it} \end{aligned} \quad (6.3)$$

where we have used the fact that $\frac{1}{T} \sum_{t=1}^T \mathbf{f}_t \mathbf{f}_t' - \mathbf{I}_K = 0$. The remaining two terms on the right-hand side can be separately bounded as follows.

Under Assumptions 3.1 (1)–(2), 3.2 and 3.3, we have the following rate of convergence of the estimated factors,

$$\frac{1}{T} \sum_{t=1}^T \|\widehat{\mathbf{f}}_t - \mathbf{H}\mathbf{f}_t\|^2 = O_p(\omega_{p,T}^2).$$

This result is proved in Theorem 1 of Bai and Ng (2002).

For the first term in (6.3), it is easy to see that $E(x_{it}^2) = O(1)$ and thus $T^{-1} \sum_{t=1}^T x_{it}^2 = O_p(1)$. Hence, we use the Cauchy–Schwarz inequality to yield

$$\left\| \frac{1}{T} \sum_{t=1}^T x_{it} (\widehat{\mathbf{f}}_t - \mathbf{H}\mathbf{f}_t) \right\| \leq \left(\frac{1}{T} \sum_{t=1}^T x_{it}^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t=1}^T \|\widehat{\mathbf{f}}_t - \mathbf{H}\mathbf{f}_t\|^2 \right)^{1/2} = O_p(\omega_{p,T}).$$

For the second term, by using the Cauchy–Schwarz inequality again, we have

$$\left\| \frac{1}{T} \sum_{t=1}^T \mathbf{H}\mathbf{f}_t u_{it} \right\| \leq \|\mathbf{H}\| \cdot \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{f}_t u_{it} \right\|.$$

To bound $\|\widehat{\mathbf{b}}_i - \mathbf{H}\mathbf{b}_i\|^2$, we use the simple fact that $(a+b)^2 \leq 2(a^2+b^2)$. Therefore we combine two upper bounds to obtain that

$$\begin{aligned} \|\widehat{\mathbf{B}} - \mathbf{B}\mathbf{H}'\|^2 &\leq 2 \sum_{i=1}^p \left(\left\| \frac{1}{T} \sum_{t=1}^T x_{it} (\widehat{\mathbf{f}}_t - \mathbf{H}\mathbf{f}_t) \right\|^2 + \left\| \frac{1}{T} \sum_{t=1}^T \mathbf{H}\mathbf{f}_t u_{it} \right\|^2 \right) \\ &\leq 2 \sum_{i=1}^p \left(O_p(\omega_{p,T}^2) + O_p(1) \cdot (T^{-1} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T \mathbf{f}_t u_{it} \right\|^2) \right). \end{aligned}$$

To bound $\sum_{i=1}^p \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T \mathbf{f}_t u_{it} \right\|^2$, as pointed out in the proof of Lemma 4 of Bai and Ng (2002), we only need to prove that $E\left(\frac{1}{p} \sum_{i=1}^p \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T \mathbf{f}_t u_{it} \right\|^2\right) \leq M_1$ for some positive constant M_1 . Next, we use the Cauchy–Schwarz inequality and the independence between $\{\mathbf{f}_t\}_{t \geq 1}$ and $\{\mathbf{u}_t\}_{t \geq 1}$ to obtain

$$\begin{aligned} E\left(\frac{1}{p} \sum_{i=1}^p \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T \mathbf{f}_t u_{it} \right\|^2\right) &= \frac{1}{p} \sum_{i=1}^p \frac{1}{T} \sum_{s=1}^T \sum_{t=1}^T E(u_{is} u_{it}) E(\mathbf{f}'_s \mathbf{f}_t) \\ &\leq \frac{1}{pT} \sum_{i=1}^p \sum_{s=1}^T \sum_{t=1}^T |E(u_{is} u_{it})| \cdot \sqrt{E\|\mathbf{f}_s\|^2 E\|\mathbf{f}_t\|^2} \\ &\leq M \cdot C \quad (\equiv M_1) \end{aligned}$$

where $E\|\mathbf{f}_s\|^2 \leq C$ holds due to Assumption 3.2, and $\frac{1}{pT} \sum_{i=1}^p \sum_{s=1}^T \sum_{t=1}^T |E(u_{is} u_{it})| \leq M$ holds due to Assumption 3.3(2). Then, we can follow Bai and Ng (2002) to show that

$$\sum_{i=1}^p \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T \mathbf{f}_t u_{it} \right\|^2 = O_p(p).$$

Now, it immediately follows from (6.2) that

$$\|\widehat{\mathbf{B}} - \mathbf{B}\mathbf{H}'\|^2 = O_p(p\omega_{p,T}^2) + O_p(p/T) = O_p(p\omega_{p,T}^2).$$

(b) Begin by noting that

$$\|\widehat{\mathbf{B}}\| \leq \|\widehat{\mathbf{B}} - \mathbf{B}\| + \|\mathbf{B}\| = O_p(p^{\frac{1}{2}}\omega_{p,T}) + O_p(p^{1/2}) = O_p(p^{1/2}).$$

We use the triangle inequality to bound

$$\begin{aligned} \|\widehat{\mathbf{B}}'\widehat{\mathbf{B}} - \mathbf{B}'\mathbf{B}\| &\leq \|\widehat{\mathbf{B}}'\widehat{\mathbf{B}} - \mathbf{B}'\widehat{\mathbf{B}}\| + \|\mathbf{B}'\widehat{\mathbf{B}} - \mathbf{B}'\mathbf{B}\| \\ &\leq \|\widehat{\mathbf{B}} - \mathbf{B}\| \cdot \|\widehat{\mathbf{B}}\| + \|\mathbf{B}'\| \cdot \|\widehat{\mathbf{B}} - \mathbf{B}\| \end{aligned}$$

Note that $\|\widehat{\mathbf{B}}'\widehat{\mathbf{B}} - \mathbf{B}'\mathbf{B}\| = O_p(p\omega_{p,T}) = o_p(p)$ and $\lambda_{\min}(\mathbf{B}'\mathbf{B}) > c_1p$ hold by Assumption 3.1. Now, we use Lemma 6.1 to obtain

$$\|(\widehat{\mathbf{B}}'\widehat{\mathbf{B}})^{-1} - (\mathbf{B}'\mathbf{B})^{-1}\| = O_p(p^{-2}) \cdot \|\widehat{\mathbf{B}}'\widehat{\mathbf{B}} - \mathbf{B}'\mathbf{B}\| = O_p(p^{-1}\omega_{p,T}).$$

It follows that

$$\begin{aligned} \|\widehat{\Lambda}_b - \Lambda_b\| &= \|(\widehat{\mathbf{B}}'\widehat{\mathbf{B}})^{-1}\widehat{\mathbf{B}}' - (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\| \\ &= \|(\widehat{\mathbf{B}}'\widehat{\mathbf{B}})^{-1}\widehat{\mathbf{B}}' - (\mathbf{B}'\mathbf{B})^{-1}\widehat{\mathbf{B}}'\| + \|(\mathbf{B}'\mathbf{B})^{-1}\widehat{\mathbf{B}}' - (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\| \\ &\leq \|(\widehat{\mathbf{B}}'\widehat{\mathbf{B}})^{-1} - (\mathbf{B}'\mathbf{B})^{-1}\| \cdot \|\widehat{\mathbf{B}}\| + \|(\mathbf{B}'\mathbf{B})^{-1}\| \cdot \|\widehat{\mathbf{B}}' - \mathbf{B}'\| \\ &= O_p(p^{-1}\omega_{p,T}) \cdot O_p(p^{1/2}) + O_p(p^{-1}) \cdot O_p(p^{1/2}\omega_{p,T}) \\ &= O_p(p^{-1/2}\omega_{p,T}). \end{aligned}$$

which completes the proof of Lemma 6.4. \square

The following lemma lays the foundation of inverse regression, which can be found in Li (1991).

Lemma 6.5. *Under model (2.1) and Assumption 3.1 (3), the centered inverse regression curve $E(\mathbf{f}_t|y_{t+1}) - E(\mathbf{f}_t)$ is contained in the linear subspace spanned by $\phi'_k \text{cov}(\mathbf{f}_t)$, $k = 1, \dots, L$.*

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. In the first part, we derive the probability bound for $\|\widehat{\Sigma}_{f|y} - \Sigma_{f|y}\|$. Let $\widehat{\mathbf{m}}_h = \frac{1}{c} \sum_{l=1}^c \mathbf{x}_{(h,l)}$ denote the average of the predictors within the corresponding slice I_h , and $\mathbf{m}_h = E(\mathbf{x}_t|y_{t+1} \in I_h)$ represents its population version. By definition, $\Sigma_{f|y} = H^{-1} \sum_{h=1}^H (\Lambda_b \mathbf{m}_h)(\Lambda_b \mathbf{m}_h)'$. For fixed H , note that

$$\widehat{\Sigma}_{f|y} - \Sigma_{f|y} = H^{-1} \sum_{h=1}^H [(\widehat{\Lambda}_b \widehat{\mathbf{m}}_h)(\widehat{\Lambda}_b \widehat{\mathbf{m}}_h)' - (\Lambda_b \mathbf{m}_h)(\Lambda_b \mathbf{m}_h)'].$$

Thus,

$$\|\widehat{\Sigma}_{f|y} - \Sigma_{f|y}\| \leq H^{-1} \sum_{h=1}^H \left(\|\widehat{\Lambda}_b \widehat{\mathbf{m}}_h - \Lambda_b \mathbf{m}_h\| \cdot \|\widehat{\Lambda}_b \widehat{\mathbf{m}}_h\| + \|\Lambda_b \mathbf{m}_h\| \cdot \|\widehat{\Lambda}_b \widehat{\mathbf{m}}_h - \Lambda_b \mathbf{m}_h\| \right) \quad (6.4)$$

We now bound each of the above terms.

From the definition, we immediately have

$$\begin{aligned}\|\widehat{\mathbf{m}}_h - \mathbf{m}_h\| &= \left\| \frac{1}{c} \sum_{l=1}^c (\mathbf{B}\mathbf{f}_{(h,l)} + \mathbf{u}_{(h,l)}) - \mathbf{B}E(\mathbf{f}_t|y_{t+1} \in I_h) \right\| \\ &\leq \|\mathbf{B}\| \cdot \left\| \frac{1}{c} \sum_{l=1}^c \mathbf{f}_{(h,l)} - E(\mathbf{f}_t|y_{t+1} \in I_h) \right\| + \left\| \frac{1}{c} \sum_{l=1}^c \mathbf{u}_{(h,l)} \right\|.\end{aligned}$$

Under Assumption 3.2, the sample mean $\frac{1}{c} \sum_{l=1}^c \mathbf{f}_{(h,l)}$ converges to the population mean $E(\mathbf{f}_t|y_{t+1} \in I_h)$ at the rate of $O_p(T^{-1/2})$. This holds true as the random variable $\mathbf{f}_t|y_{t+1} \in I_h$ is still stationary with finite second moments, and the sum of the α -mixing coefficients converges (Theorem 19.2 in Billingsley (1999)). This applies to $\mathbf{u}_t|y_{t+1} \in I_h$ as well. Therefore,

$$\|\widehat{\mathbf{m}}_h - \mathbf{m}_h\| = O_p(p^{\frac{1}{2}}) \cdot O_p(T^{-\frac{1}{2}}) + O_p(T^{-\frac{1}{2}}) = O_p(\sqrt{p/T}).$$

In addition to the inequality above, we also have

$$\|\mathbf{m}_h\| = \|E(\mathbf{x}_t|y_{t+1} \in I_h)\| \leq \|\mathbf{B}\| \cdot \|E(\mathbf{f}_t|y_{t+1} \in I_h)\| = O_p(p^{1/2}).$$

Thus, it follows that

$$\|\widehat{\mathbf{m}}_h\| \leq \|\widehat{\mathbf{m}}_h - \mathbf{m}_h\| + \|\mathbf{m}_h\| = O_p(p^{1/2}).$$

Now, we use both the triangle inequality and the Cauchy–Schwarz inequality to obtain that

$$\begin{aligned}\|\widehat{\Lambda}_b \widehat{\mathbf{m}}_h - \Lambda_b \mathbf{m}_h\| &\leq \|\widehat{\Lambda}_b \widehat{\mathbf{m}}_h - \Lambda_b \widehat{\mathbf{m}}_h\| + \|\Lambda_b \widehat{\mathbf{m}}_h - \Lambda_b \mathbf{m}_h\| \\ &\leq \|\widehat{\Lambda}_b - \Lambda_b\| \cdot \|\widehat{\mathbf{m}}_h\| + \|\Lambda_b\| \cdot \|\widehat{\mathbf{m}}_h - \mathbf{m}_h\| \\ &= O_p(p^{-1/2} \omega_{p,T}) \cdot O_p(p^{1/2}) + O_p(p^{-1/2}) \cdot O_p(\sqrt{p/T}) \\ &= O_p(\omega_{p,T}),\end{aligned}$$

where we use $\|\widehat{\Lambda}_b - \mathbf{H}\Lambda_b\| = O_p(p^{-1/2} \omega_{p,T})$ by Lemma 6.4 in the second inequality, and that $\|\Lambda_b\| \leq \|(\mathbf{B}'\mathbf{B})^{-1}\| \cdot \|\mathbf{B}\| = O_p(p^{-1/2})$ in the third equality.

Also we have that

$$\|\widehat{\Lambda}_b \widehat{\mathbf{m}}_h\| \leq (\|\widehat{\Lambda}_b - \Lambda_b\| + \|\Lambda_b\|) \cdot \|\widehat{\mathbf{m}}_h\| = O_p(p^{-1/2}) \cdot O_p(p^{1/2}) = O_p(1)$$

and

$$\|\Lambda_b \mathbf{m}_h\| \leq \|\Lambda_b\| \cdot \|\mathbf{m}_h\| = O_p(1).$$

Hence, from (6.4), we conclude the desired result as follows:

$$\|\widehat{\Sigma}_{f|y} - \Sigma_{f|y}\| = H^{-1} \sum_{h=1}^H (O_p(\omega_{p,T}) \cdot O_p(1) + O_p(1) \cdot O_p(\omega_{p,T})) = O_p(\omega_{p,T}).$$

In the second part, we show that $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_L$ are the desired SDR directions and further derive the probability bound for $\|\widehat{\boldsymbol{\psi}}_j - \boldsymbol{\psi}_j\|$ with $j = 1, \dots, L$. Let $\{\lambda_i\}_{i=1}^K$ be the eigenvalues of $\Sigma_{f|y}$ in the descending order and $\{\widehat{\lambda}_i\}_{i=1}^K$ be the eigenvalues of $\widehat{\Sigma}_{f|y}$ in the descending order. Let $\widehat{\boldsymbol{\psi}}_j$ be the eigenvector associated with the eigenvalue $\widehat{\lambda}_j$, and $\boldsymbol{\psi}_j$ is the eigenvector associated with the eigenvalue λ_j .

In view of Lemma 6.5, $E(\mathbf{f}_t|y_{t+1})$ is contained in the central subspace $S_{y|f}$ spanned by $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_L$,

since we have the normalization $\text{cov}(\mathbf{f}_t) = \mathbf{I}_K$ and $E(\mathbf{f}_t) = \mathbf{0}$. Recall the fact that the eigenvectors of $\Sigma_{f|y}$ corresponding to its L largest eigenvalues would form another orthogonal basis for $S_{y|\mathbf{f}}$. Hence, the eigenvectors ψ_1, \dots, ψ_L of $\Sigma_{f|y}$ constitute the SDR directions for model (2.1).

Note that $|\widehat{\lambda}_{j-1} - \lambda_{j-1}| \leq \|\widehat{\Sigma}_{f|y} - \Sigma_{f|y}\|$ holds by Weyl's Theorem in Lemma 6.2 (a). Similarly, we have $|\lambda_j - \widehat{\lambda}_{j+1}| \geq |\lambda_j - \lambda_{j+1}| - O_p(\omega_{p,T})$. Then, we have

$$\begin{aligned} |\widehat{\lambda}_{j-1} - \lambda_j| &\geq |\lambda_{j-1} - \lambda_j| - |\widehat{\lambda}_{j-1} - \lambda_{j-1}| \\ &\geq |\lambda_{j-1} - \lambda_j| - O_p(\omega_{p,T}), \end{aligned}$$

where the fact that $|\widehat{\lambda}_{j-1} - \lambda_{j-1}| \leq \|\widehat{\Sigma}_{f|y} - \Sigma_{f|y}\|$ is used.

Since $\Sigma_{f|y}$ have distinct eigenvalues, both $|\widehat{\lambda}_{j-1} - \lambda_j|$ and $|\lambda_j - \widehat{\lambda}_{j+1}|$ are bounded away from zero with probability tending to one for $j = 1, \dots, L$. Now, a direct application of $\sin(\theta)$ Theorem (Davis and Kahan, 1970) in Lemma 6.2 (b) shows that

$$\|\widehat{\psi}_j - \psi_j\| \leq \frac{\sqrt{2} \cdot \|\widehat{\Sigma}_{f|y} - \Sigma_{f|y}\|}{\min(|\widehat{\lambda}_{j-1} - \lambda_j|, |\lambda_j - \widehat{\lambda}_{j+1}|)} = O_p(1) \cdot O_p(\omega_{p,T}) = O_p(\omega_{p,T}),$$

where $j = 1, \dots, L$,

Therefore, we complete the proof of Theorem 3.1. □

6.3 Proof of Theorem 3.2

Proof. First we write $\widehat{\phi} = \frac{1}{T-1} \sum_{t=1}^{T-1} y_{t+1} \widehat{\mathbf{f}}_t$ in terms of the true factors \mathbf{f}_t , i.e.,

$$\begin{aligned} \widehat{\phi} &= \frac{\widehat{\Lambda}_b}{T-1} \sum_{t=1}^{T-1} y_{t+1} \mathbf{x}_t \\ &= \frac{\widehat{\Lambda}_b}{T-1} \sum_{t=1}^{T-1} (\mathbf{B} \mathbf{f}_t + \mathbf{u}_t) y_{t+1}, \end{aligned}$$

where we used the fact that $\widehat{\mathbf{f}}_t = \widehat{\Lambda}_b \mathbf{x}_t$ as shown in Proposition 2.1.

Using the triangular inequality, we have

$$\begin{aligned} \|\widehat{\phi} - \bar{\phi}\| &\leq \|\widehat{\phi} - (\widehat{\Lambda}_b \mathbf{B}) \bar{\phi}\| + \|(\widehat{\Lambda}_b \mathbf{B} - \mathbf{I}_K) \bar{\phi}\| \\ &= \left\| \frac{(\widehat{\Lambda}_b \mathbf{B})}{T-1} \sum_{t=1}^{T-1} \mathbf{f}_t y_{t+1} + \frac{\widehat{\Lambda}_b}{T-1} \sum_{t=1}^{T-1} \mathbf{u}_t y_{t+1} - (\widehat{\Lambda}_b \mathbf{B}) \bar{\phi} \right\| + \|(\widehat{\Lambda}_b \mathbf{B} - \mathbf{I}_K) \bar{\phi}\| \\ &\leq \left\| \frac{(\widehat{\Lambda}_b \mathbf{B})}{T-1} \sum_{t=1}^{T-1} (y_{t+1} \mathbf{f}_t - \bar{\phi}) \right\| + \left\| \frac{1}{T-1} \widehat{\Lambda}_b \sum_{t=1}^{T-1} \mathbf{u}_t y_{t+1} \right\| + \|(\widehat{\Lambda}_b \mathbf{B} - \mathbf{I}_K) \bar{\phi}\|. \end{aligned}$$

In the sequel, we will bound three terms on the right hand side of this inequality respectively.

By Lemmas 6.3 and 6.4, we have

$$\|\widehat{\Lambda}_b \mathbf{B}\| \leq (\|\widehat{\Lambda}_b - \Lambda_b\| + \|\Lambda_b\|) \cdot \|\mathbf{B}\| = O_p(1)$$

and

$$\|\widehat{\Lambda}_b \mathbf{B} - \mathbf{I}_K\| = \|\widehat{\Lambda}_b - \Lambda_b\| \cdot \|\mathbf{B}\| = O_p(\omega_{p,T}).$$

Since $\|\bar{\phi}\| = O_p(1)$, the third term on the right hand side of the inequality is $O_p(\omega_{p,T})$, i.e.,

$$\|(\widehat{\Lambda}_b \mathbf{B} - \mathbf{I}_K)\bar{\phi}\| = O_p(\omega_{p,T}).$$

For the second term, note that \mathbf{u}_t is independent of y_{t+1} , hence $E(\mathbf{u}_t y_{t+1}) = 0$. By law of large numbers and $\|\widehat{\Lambda}_b\| = O_p(p^{-1/2})$, we have

$$\left\| \frac{1}{T-1} \widehat{\Lambda}_b \sum_{t=1}^{T-1} \mathbf{u}_t y_{t+1} \right\| = O_p(p^{-1/2}) \cdot O_p(T^{-1/2}) = O_p(\omega_{p,T}).$$

It remains to bound the first term $\left\| \frac{1}{T-1} \sum_{t=1}^{T-1} (y_{t+1} \mathbf{f}_t - \bar{\phi}) \right\|$. To this end, we express \mathbf{f}_t along the basis ϕ_1, \dots, ϕ_L and their orthogonal hyperplane, namely,

$$\mathbf{f}_t = \sum_{j=1}^L \langle \mathbf{f}_t, \phi_j \rangle \phi_j + \mathbf{f}_t^\perp.$$

By the orthogonal decomposition of normal distribution, $\langle \mathbf{f}_t, \phi_j \rangle$ and \mathbf{f}_t^\perp are independent. In addition, y_{t+1} depends on \mathbf{f}_t only through $\phi_1' \mathbf{f}_t, \dots, \phi_L' \mathbf{f}_t$, and then, it is conditionally independent of \mathbf{f}_t^\perp . It follows from contraction property that y_{t+1} and \mathbf{f}_t^\perp are independent, unconditionally. Thus,

$$E(y_{t+1} \mathbf{f}_t^\perp) = E(y_{t+1}) E(\mathbf{f}_t^\perp) = \mathbf{0}.$$

Recall that $\bar{\phi} = \sum_{j=1}^L E((\phi_j' \mathbf{f}_t) y_{t+1}) \phi_j$. Now, we use the triangle inequality to obtain

$$\begin{aligned} & \left\| \frac{1}{T-1} \sum_{t=1}^{T-1} (y_{t+1} \mathbf{f}_t - \bar{\phi}) \right\| \\ &= \left\| \frac{1}{T-1} \sum_{t=1}^{T-1} \left[\sum_{j=1}^L (\phi_j' \mathbf{f}_t) y_{t+1} \phi_j + y_{t+1} \mathbf{f}_t^\perp - \sum_{j=1}^L E((\phi_j' \mathbf{f}_t) y_{t+1}) \phi_j \right] \right\| \\ &\leq \sum_{j=1}^L \left\| \frac{1}{T-1} \sum_{t=1}^{T-1} [(\phi_j' \mathbf{f}_t) y_{t+1} - E((\phi_j' \mathbf{f}_t) y_{t+1})] \phi_j \right\| + \left\| \frac{1}{T-1} \sum_{t=1}^{T-1} y_{t+1} \mathbf{f}_t^\perp \right\| \\ &\leq \sum_{j=1}^L \left| \frac{1}{T-1} \sum_{t=1}^{T-1} [(\phi_j' \mathbf{f}_t) y_{t+1} - E((\phi_j' \mathbf{f}_t) y_{t+1})] \right| + \left\| \frac{1}{T-1} \sum_{t=1}^{T-1} y_{t+1} \mathbf{f}_t^\perp \right\|, \end{aligned}$$

where we used the fact that $\|\phi_j\| = 1$ for any j . Note that each term here is $O_p(T^{-1/2})$ by Law of Large Numbers. Then, we have,

$$\left\| \frac{(\widehat{\Lambda}_b \mathbf{B})}{T-1} \left(\sum_{t=1}^{T-1} y_{t+1} \mathbf{f}_t - \bar{\phi} \right) \right\| \leq \|\widehat{\Lambda}_b \mathbf{B}\| \cdot O_p(T^{-1/2}) = O_p(\omega_{p,T}).$$

This concludes the desired bound that $\|\widehat{\phi} - \bar{\phi}\| = O_p(\omega_{p,T})$.

Therefore, we complete the proof of Theorem 3.2. \square

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