THE EULER–POISSON SYSTEM IN $2D$: GLOBAL STABILITY OF THE CONSTANT EQUILIBRIUM SOLUTION

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ABSTRACT. We consider the (repulsive) Euler-Poisson system for the electrons in two dimensions and prove that small smooth perturbations of a constant background exist for all time and remain smooth (never develop shocks). This extends to $2D$ the work of Guo [6].

CONTENTS

1. INTRODUCTION

In this paper we investigate the question of global existence for small perturbations of a constant background for the following *Euler–Poisson* system for the electrons in 2 dimensions:

$$\partial_t n_- + \nabla \cdot (n_- v_-) = 0$$
$$n_- m_- (\partial_t v_- + v_- \cdot \nabla v_-) + \nabla p(n_-) = e n_- \nabla \phi \tag{1.1}$$
$$\Delta \phi = 4\pi e (n_- - n_0).$$

Here the unknowns are $n_- \geq 0$, the density of electrons, and $v_- \in \mathbb{R}^2$, the velocity field of the electrons. These are functions defined for $(t, x) \in \mathbb{R} \times \mathbb{R}^2$. The positive constants m_-, e and n_0 correspond respectively to the mass of an electron, its charge and the average charge of an ion background. Finally, $p = p(n_-)$ is a pressure function, given by a constitutive relation which for simplicity we assume to be quadratic.

These equations model the behavior of a fluid of electrons in a warm adiabatic fully ionized plasma when the magnetic field and the motion of the ions is neglected. Neglecting the magnetic field is customary and reduces the number of unknowns. Neglecting the ion motion is relevant since the ratio of the masses of the electrons and the ions is typically very small[1]. We refer to [1] for more on the physical background.

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[1]It is no bigger than the ratio of the electron mass to the proton mass which equals 1/1836.

Equations (1.1) represent a coupling of a compressible (inviscid) fluid with an electrostatic field. For the pure compressible Euler equation, even small and smooth initial perturbations of a constant equilibrium can lead to formation of shocks in finite time [15].

However, here we show that the coupling with the self-consistent electric field stabilizes the system in the sense that small smooth perturbations of a constant background remain global and return to equilibrium. This phenomenon was first observed in $3D$ by Guo [6]. We also refer to Guo-Pausader [7] for a similar result for the ion equation and to the recent work of Germain-Masmoudi [3] for the complete Euler-Maxwell equation for the electrons in $3D$. On the other hand, large perturbations can lead to blow-up in finite time for the Euler-Poisson equation [8].

Previous work on the Euler-Poisson system in $2D$ also includes the work of Jang [11] and Wei-Tadmore-Bae [16] for radial data, and Jang-Li-Zhang [12] for the existence of wave operators.

For simplicity, we assume that the pressure law is quadratic $p(n_-) = T_-(n_-)^2/2$. The purpose of this is only to minimize the number of terms in the nonlinearity, but other powers could be treated similarly. After rescaling, we can then reduce to the following system[2]

$$\begin{aligned}
\partial_t n + \nabla \cdot (nv) &= 0, \\
\partial_t v + v \cdot \nabla v + a\nabla n &= b\nabla\phi, \\
\Delta\phi &= n - 1,
\end{aligned} \tag{1.2}$$

where $a, b \in (0, \infty)$, $a = T_-n_0/m_-$ is the square of the speed of sound and $b = 4\pi e^2 n_0/m_-$ is the electron plasma frequency. Our main Theorem asserts that small, neutral irrotational perturbations of a constant equilibrium $(n, v) = (1, 0)$ are global.

Theorem 1.1. *There exists $\varepsilon_0 > 0$ and a norm $\|\cdot\|_Y$ such that any initial data $(n(x, 0), v(x, 0))$ satisfying*

$$\|(n(\cdot, 0) - 1, v(\cdot, 0))\|_Y \le \varepsilon_0, \quad curl_x(v(\cdot, 0)) = 0$$

leads to a global solution (n, v) of (1.2) which returns to equilibrium in the sense that

$$\|n(\cdot, t) - 1\|_{L^\infty} + \|\nabla v(\cdot, t)\|_{L^\infty} \lesssim (1 + t)^{-1}.$$

Remark 1.2. *The precise nature of the norm is given in (2.4) below. It controls a finite number of derivatives and requires localization of the initial perturbations. Its most notable feature is that finiteness of the norm implies that the perturbation is electrically neutral:*

$$\int_{\mathbb{R}^2} (n(x, 0) - 1) dx = 0,$$

which is then conserved by the flow. This condition is also necessary in order to ensure finiteness of the physical (conserved) energy

$$E = \frac{1}{2} \int_{\mathbb{R}^2} \left[n|v|^2 + a(n - 1)^2 + b|\nabla\phi|^2 \right] dx, \tag{1.3}$$

hence we find it an acceptable assumption.

The irrotationality assumption is also propagated by the flow. It removes a component in the system that is only transported and does not obey nice decay estimates.

Theorem 1.1 follows from the more precise Theorem 2.1 below. In order to attack this problem, we restate it as a quasilinear dispersive equation. The main difficulties then come from the slow decay of the norms, the quadratic power of the nonlinearity and some nonlocal terms in the nonlinearity (Riesz transforms) that prevent good localization of the small frequencies and convenient use of the only almost integrable norm (L^∞).

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[2]We can even reduce further to $a = b = 1$; however, we have preferred to keep the constants a and b for their own physical interest.

The first two problems are classical in the study of quasilinear dispersive systems and several methods have been developed to address these difficulties, including normal form transformations [14] and commuting vector fields [13]. More recently refinements and new developments from Gustafson-Nakanishi-Tsai [9, 10] and Germain-Masmoudi-Shatah [3, 4, 5] have led to progress in dealing with various physical problems. Our analysis is in the framework of this general scheme, and is especially close to the analysis of the water-wave problem in [5].

The proof relies on two distinct components. The first component, *the energy method*, exploits the existence of a conserved physical energy given in (1.3), which comes from the subtle structure of the nonlinearity. This implies almost conservation of higher order energies, up to commutators that are *lower order* in derivative and are at least of *cubic order* and therefore can be controlled if the solution decays sufficiently fast. In dimension 3, these error terms are integrable and the higher order energies remain bounded (see e.g. [6, 14]). In dimension 2, all the norms are only almost integrable and the highest order energy is allowed to grow slowly.

The second component of the proof is concerned with proving the decay estimate needed to control the commutators above. It exploits the fact that we consider a perturbation of a *constant* solution, which allows us to use the *Fourier transform method*. The analysis here depends on the structure of the linearized problem (in particular the *dispersion relation $\omega(k)$*) and on delicate cancellation properties of the nonlinearity (the *null structure*), which are particularly important to bound the constribution of very low frequencies. We first integrate the quadratic terms using a normal form transformation[3]. We are then left with several cubic terms which oscillate along different phases. We account for the elliptic phases (which never vanish) using another iteration of the normal form transformation. This produces quartic terms which can be easily estimated. We account for the remaining non-elliptic phase by exploiting its more "hyperbolic" nature: it is somewhat local in space and "commutes[4]" at first order with the position operator $f \mapsto xf$.

As mentioned before, this is similar to the approach used in [5] for the water-wave problem, which has a similar structure. However, we introduce some different ideas, such as more involved space–frequency analysis, that we hope can find further applications.

First of all, we only work on the linear profile and derive all our estimates from information about it. This was already done in [10] in $3D$ but encounters difficulties in $2D$ due to the fact that we need an endpoint dispersion estimate (L^∞). This forces us to localize in space using the resolution of unity associated with the position operator in order to create a norm having the nice properties of L^2 but which is stronger than the L^1-norm. Controlling this norm leads us to make most of our analysis in the physical space rather than in the Fourier space. We believe this gives a clearer picture and accounts better for the finite speed of propagation inherent in dispersive equations[5].

In particular, once we introduce localization in physical space, the main problem becomes to bound uniformly the linear profile on a spatial dyadic ring in xL^2. Assume to simplify that the initial data is only concentrated in the ball of radius 1 and that we are looking at a distance $R >> 1$. The bound follows from three distinct ingredients:

(1) First, by variation on the finite speed of propagation principle, one can see that, for times $t \lesssim R$, the solution at R is only strongly influenced by the solution at nearby locations (at distance smaller than $R/2$). Since it starts small and it interacts nonlinearly, it remains so.

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[3]This transformation, introduced in [14], is always possible if the phase velocity $\omega(k)/|k|$ is decreasing and $\omega(k) \geq 0$ is increasing.

[4]in the sense that, as for "commuting vector fields", it obeys a variant of Leibniz rule: $xT[f, g, h] = T_1[xf, g, h] + T_2[f, xg, h] + T_3[f, g, xh]$ where T_1, T_2 and T_3 have similar properties as T

[5]Note that this finite speed of propagation is only for the linear flow. The nonlinear flow contains nonlocal operators (Riesz transforms) and does not satisfy exact analogues of the finite speed of propagation for solutions of Klein-Gordon equation, except in the radial case as pointed out in [11].

(2) Second, by decay property of the solutions, all the interactions after a time $t > R^{1+\delta}$ add up in L^2 to a size smaller than $R^{1-\delta/100}$ and thus are acceptable.

(3) Finally, one only needs to consider a small portion of space-time when $R \sim x$, $R \le t \le R^{1+\delta}$. This is of course the main interaction region. In this region, various decay estimates (in time or in space) become equivalent which gives more flexibility. Since for cubic nonlinearities, one only need a small improvement on the decay estimates, this region can be controlled by exploiting the "hyperbolic" structure of the phase: it cannot be stationary in all directions.

Our strategy is as follows: In Section 2, we reformulate (1.1) into a quasilinear scalar dispersive equation for a complex-valued unknown, (2.3) by diagonalizing the linearized system. We then prove our main theorem about the new equation, Theorem 2.1 assuming two propositions. This implies Theorem 1.1 In Section 3, we prove our first proposition, Proposition 2.2 which gives a good local existence theory with energy estimates suitable for our analysis. In Section 4 we prove the second proposition, Proposition 2.3, which implies the decay of solutions needed for the energy estimates. This is done by first using a normal form transformation and then following the strategy explained in points (1), (2) and (3) above. Finally in Section 5, we collect various technical estimates needed in the analysis.

2. Main definitions and propositions

We first remark that since we start with irrotational initial velocity, the velocity remains irrotational, since $\omega = \mathrm{curl}(v)$ satisfies a well-known transport equation. Hence we may assume that $v = \nabla h$ for some velocity potential h. We also set $n = 1 + \rho$ and (1.2) becomes

$$
\begin{aligned}
\partial_t \rho + \Delta h + \partial_\alpha(\rho \partial_\alpha h) &= 0, \\
\partial_t h + (1/2)(\partial_\alpha h \partial_\alpha h) + a\rho &= b\phi. \\
\Delta \phi &= \rho.
\end{aligned}
\tag{2.1}
$$

Let $|\nabla|$ denote the operator on $\mathbb{R}^2$ defined by the Fourier multiplier $\xi \mapsto |\xi|$, and let $g := |\nabla|^{-1}\rho$. In terms of g, h the system (2.1) becomes

$$
\begin{aligned}
\partial_t g - |\nabla| h + |\nabla|^{-1} \partial_j(|\nabla| g \cdot \partial_j h) &= 0, \\
\partial_t h + (a|\nabla| + b|\nabla|^{-1})g + (1/2)(\partial_j h \partial_j h) &= 0.
\end{aligned}
\tag{2.2}
$$

Letting

$$
U := \sqrt{a|\nabla|^2 + b}\, g + i|\nabla| h,
$$

we derive the equation

$$
(\partial_t + i\Lambda)U = \frac{i}{4}\sum_{j=1}^{2} R_j \Lambda \big[|\nabla|\Lambda^{-1}(U + \overline{U}) \cdot R_j(U - \overline{U}) \big] + \frac{i}{8}\sum_{j=1}^{2} |\nabla| \big[R_j(U - \overline{U}) \cdot R_j(U - \overline{U}) \big], \tag{2.3}
$$

where

$$
\Lambda := \sqrt{a|\nabla|^2 + b}, \qquad R_j := \partial_j/|\nabla|.
$$

We fix $\varphi : \mathbb{R} \to [0,1]$ an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. Let

$$
\varphi_k(x) := \varphi(|x|/2^k) - \varphi(|x|/2^{k-1}) \text{ for any } k \in \mathbb{Z}, \, x \in \mathbb{R}^2, \qquad \varphi_I := \sum_{m \in I \cap \mathbb{Z}} \varphi_m \text{ for any } I \subseteq \mathbb{R}.
$$

Let P_k, $k \in \mathbb{Z}$, denote the operator on $\mathbb{R}^2$ defined by the Fourier multiplier $\xi \to \varphi_k(\xi)$. Similarly, for any $I \subseteq \mathbb{R}$ let P_I denote the operator on $\mathbb{R}^2$ defined by the Fourier multiplier $\xi \to \varphi_I(\xi)$. For $T \ge 1$ and

integers $1 \leq N_0 < N$ we define

$$X_T^N(\mathbb{R}^2 \times [0,T]) := \{f \in C([0,T] : H^N(\mathbb{R}^2)) : e^{it\Lambda}f \in C([0,T] : Y^{N_0}(\mathbb{R}^2))$$

$$\text{and } \|f\|_{X_T^N} := \sup_{t \in [0,T]} (1+t)^{-\delta} \|f(t)\|_{H^N} + \sup_{t \in [0,T]} \|e^{it\Lambda}f(t)\|_{Y^{N_0}} < \infty\}, \qquad (2.4)$$

where

$$Y^{N_0}(\mathbb{R}^2) := \{\phi \in L^2(\mathbb{R}^2) : \|\phi\|_{Y^{N_0}} := \|\phi\|_{H^{N_0}} + \|\phi\|_Z < \infty\},$$

$$\|\phi\|_Z := \sup_{k \in \mathbb{Z}} (2^{k/10} + 2^{10k}) \Big[\|P_k\phi\|_{L^2} + \sum_{j \in \mathbb{Z}_+} 2^j \|\varphi_j(x) \cdot P_k\phi(x)\|_{L^2}\Big]. \qquad (2.5)$$

The spaces X_T^N are our main spaces, and we use them to control our "smooth" solutions. First we control the high energy norm H^N which is allowed to grow slowly in time; this growth appears to be necessary in 2 dimensions, due to the non-integrable factor $(1+t)^{-1}$ in the dispersive bound (2.7) below, but is not necessary in 3 dimensions (see [6]).

We also control an intermediate energy norm H^{N_0}, for some N_0 chosen smaller than N, uniformly in time. This intermediate norm is mostly for convenience and can be removed.

Finally, we control the key Z norm described in (2.5), which captures the dispersive nature of our flow. Variants of this norm are of course possible, see for example the similar norms used in [3, 4, 5]. We make the specific choice described in (2.5) in order to achieve two basic inequalities,

$$\|R_1 f\|_Z + \|R_2 f\|_Z \lesssim \|f\|_Z,$$
$$\sup_{k \in \mathbb{Z}} (2^{k/10} + 2^{10k}) \|P_k f\|_{L^1} \lesssim \|f\|_Z. \qquad (2.6)$$

The first inequality in (2.6) is a consequence of the proof of Lemma 5.2 and is necessary because our nonlinearity contains several Riesz transforms, see (2.3). The second inequality in (2.6) is an easy consequence of the definition and explains, in particular, the choice of the l^1 sum in j. It can be combined with the basic dispersive estimate

$$\|P_l e^{-it\Lambda} f\|_{L^\infty} \lesssim (1+t)^{-1}(1 + 2^{2l})\|f\|_{L^1}, \qquad t \in \mathbb{R}, \qquad (2.7)$$

to show that

$$\|P_k e^{-it\Lambda} f\|_{L^\infty} \lesssim (1+t)^{-1}(2^{k/10} + 2^{8k})^{-1}\|f\|_Z, \qquad t \in \mathbb{R}, k \in \mathbb{Z}. \qquad (2.8)$$

We can now state our main theorem in terms of the function U.

Theorem 2.1. *Assume $N = 30$, $N_0 = 20$, and $\delta = 1/100$. There is $\overline{\varepsilon}$ sufficiently small such that if*

$$\|U_0\|_{H^N} + \|U_0\|_{Y^{N_0}} \leq \overline{\varepsilon}$$

then there is a unique global solution $U \in C([0,\infty) : H^N)$ of the initial-value problem

$$\begin{cases} (\partial_t + i\Lambda)U = \frac{i}{4}\sum_{j=1}^2 \Lambda R_j\big[|\nabla|\Lambda^{-1}(U + \overline{U}) \cdot R_j(U - \overline{U})\big] + \frac{i}{8}\sum_{j=1}^2 |\nabla|\big[R_j(U - \overline{U}) \cdot R_j(U - \overline{U})\big], \\ U(0) = U_0. \end{cases}$$

$$(2.9)$$

In addition, $U \in X_T^N$ for any $T \geq 1$ and

$$\sup_{t \in [0,\infty)} \big[(1+t)^{-\delta}\|e^{it\Lambda}U(t)\|_{H^N} + \|e^{it\Lambda}U(t)\|_{Y^{N_0}}\big] \lesssim \|U_0\|_{H^N} + \|U_0\|_{Y^{N_0}}. \qquad (2.10)$$

It is easy to see that Theorem 2.1 implies Theorem 1.1, since

$$n - 1 = \Lambda^{-1}|\nabla|(\Re U), \qquad v_1 = R_1(\Im U), \qquad v_2 = R_2(\Im U).$$

On the other hand, Theorem 2.1 is a consequence of Proposition 2.2 and Proposition 2.3 below. We start with the local existence theory:

Proposition 2.2. *(i) There exists $\delta_0 > 0$ such that for any $U_0 \in H^3$ satisfying*

$$\|U_0\|_{H^3} \leq \delta_0, \tag{2.11}$$

there exists a unique solution $U \in C([0,1] : H^3)$ of (2.9) such that $U(0) = U_0$ and

$$\sup_{t \in [0,1]} \|U(t)\|_{H^3} \lesssim \|U_0\|_{H^3}.$$

(ii) Assume in addition that $U_0 \in H^M$, $M \in [3,40] \cap \mathbb{Z}$. Then $U \in C([0,1] : H^M)$ and for any $0 \leq s \leq t \leq 1$,

$$E_M(t) \leq E_M(s) + C \int_s^t \|U(t')\|_{Z'} \cdot E_M(t') dt' \tag{2.12}$$

where the M-th order energy, defined in (3.4) below, satisfies $E_M(t) \simeq \|U(t)\|_{H^M}^2$ uniformly in time and

$$\|f\|_{Z'} := \sup_{k \in \mathbb{Z}} (2^{k/2} + 2^{2k}) \|P_k f\|_{L^\infty}. \tag{2.13}$$

(iii) With N_0 as in Theorem 2.1, assume $T \geq 1$ and $U \in C([0,T] : H^{N_0})$ is a solution of (2.9) with the property that $U_0 \in Y^{N_0}$. Then

$$\sup_{t \in [0,T]} \|e^{it\Lambda} U(t)\|_{Y^{N_0}} \leq C\big(\|U_0\|_{Y^{N_0}}, \sup_{t \in [0,T]} \|U(t)\|_{H^{N_0}}, T\big),$$

$$\lim_{t' \to t} \|e^{it'\Lambda} U(t') - e^{it\Lambda} U(t)\|_{Y^{N_0}} = 0, \qquad \text{for any } t \in [0,T].$$

The main conclusion in the proposition is the energy inequality (2.12), which depends on the Z' norm defined in (2.13). This norm has to be chosen strong enough to allow for the energy inequality (2.12) to hold; in particular

$$\|\partial_i R_j f\|_{L^\infty} + \|R_j f\|_{Z'} \lesssim \|f\|_{Z'}, \qquad j,k \in \{1,2\}.$$

On the other hand, the Z' norm has to be chosen small enough, in such a way that the function $t \to \|U(t)\|_{Z'}$ is almost integrable in time; the main inequality we need is

$$\|e^{-it\Lambda} f\|_{Z'} \lesssim (1+t)^{-1} \|f\|_Z, \qquad \text{for any } t \in [0,\infty), \tag{2.14}$$

which is an easy consequence of (2.8).

To prove the global result in Theorem 2.1 we also need the following bootstrap estimate:

Proposition 2.3. *Assume that $U \in X_T^N$ is a solution of the equation*

$$(\partial_t + i\Lambda)U = \frac{i}{4} \sum_{j=1}^2 \Lambda R_j \big[|\nabla| \Lambda^{-1}(U + \overline{U}) \cdot R_j(U - \overline{U}) \big] + \frac{i}{8} \sum_{j=1}^2 |\nabla| \big[R_j(U - \overline{U}) \cdot R_j(U - \overline{U}) \big]$$

with the property that

$$\|U\|_{X_T^N} \leq \varepsilon_0 \leq 1. \tag{2.15}$$

Then

$$\sup_{t \in [0,T]} \|e^{it\Lambda} U(t) - U(0)\|_{Y^{N_0}} \lesssim \varepsilon_0^2. \tag{2.16}$$

2.1. **Proof of Theorem 2.1.** Theorem 2.1 is a consequence of Proposition 2.2 and Proposition 2.3. Indeed, assume that we start with data U_0 satisfying $\|U_0\|_{H^N} + \|U_0\|_{Y^{N_0}} = \epsilon \leq \overline{\varepsilon}$, for some $\overline{\varepsilon}$ sufficiently small relative to the value of δ_0 in Proposition 2.2 (i). Assume that we have constructed a solution $U \in C([0,T] : H^N)$ (using Proposition 2.2 (ii)), for some $T \geq 1$, such that

$$\sup_{t \in [0,T]} \|e^{it\Lambda} U(t)\|_{Y^{N_0}} \leq \epsilon^{3/4}. \tag{2.17}$$

Then, using (2.14),

$$\|U(t)\|_{Z'} \lesssim (1+t)^{-1} \epsilon^{3/4}, \qquad \text{for any } t \in [0,T].$$

Then, using (2.12),

$$E_N(t) - E_N(0) \leq C\epsilon^{3/4} \int_0^t (1+s)^{-1} E_N(s) \, ds, \qquad t \in [0,T],$$

which shows that $E_N(t) \leq E_N(0)(1+t)^\delta$, for any $t \in [0,T]$. Therefore $U \in X_T^N$ and

$$\|U\|_{X_T^N} \leq 2\epsilon^{3/4}.$$

We can apply now Proposition 2.3 to conclude that

$$\sup_{t \in [0,T]} \|e^{it\Lambda} U(t) - U(0)\|_{Y^{N_0}} \lesssim \epsilon^{3/2}.$$

In other words, if U satisfies $\sup_{t \in [0,T]} \|e^{it\Lambda} U(t)\|_{Y^{N_0}} \leq \epsilon^{3/4}$ as in (2.17) then U satisfies the stronger inequality $\sup_{t \in [0,T]} \|e^{it\Lambda} U(t)\|_{Y^{N_0}} \leq \|U(0)\|_{Y^{N_0}} + C\epsilon^{3/2}$. In view of the continuity of the norm Y^{N_0} (see Proposition 2.2 (iii)), it follows that

$$\sup_{t \in [0,T]} \|e^{it\Lambda} U(t)\|_{Y^{N_0}} \leq 2\|U(0)\|_{Y^{N_0}}$$

for any solution $U \in C([0,T] : H^N)$ of the initial-value problem (2.9) with $\|U_0\|_{H^N} + \|U_0\|_{Y^{N_0}} \leq \overline{\varepsilon}$. The global regularity part of the theorem and the bound (2.10) follow, using again the energy estimate (2.12).

3. Proof of Proposition 2.2

In this section we prove Proposition 2.2, using the energy method. We regularize the equation by parabolic regularization, get uniform estimates and then pass to the limit. The method is, of course, well known. We present all the details here for two reasons: to justify the key energy estimate (2.12), with the somewhat unusual Z' norm in the right-hand side of the inequality, and to justify the claim in Proposition 2.2 that the solution defines a continuous flow in the Z space. These facts are important in passing to the global result.

We start with the main lemma.

Lemma 3.1. *With $K = 50$, there is $\delta_1 > 0$ such that if*

$$U_0 \in H^\infty \text{ and } \|U_0\|_{H^3} \leq \delta_1$$

then, for any $\varepsilon > 0$, there is a solution $U = U^\varepsilon \in C([0,1] : H^{K+2})$ of the approximate equation

$$(\partial_t + i\Lambda - \varepsilon\Delta) U = \frac{i}{4} \sum_{j=1}^2 \Lambda R_j \left[|\nabla|\Lambda^{-1}(U + \overline{U}) \cdot R_j(U - \overline{U}) \right] + \frac{i}{8} \sum_{j=1}^2 |\nabla| \left[R_j(U - \overline{U}) \cdot R_j(U - \overline{U}) \right] \tag{3.1}$$

with initial data $U(0) = U_0$. Moreover,

$$\sup_{t \in [0,1]} \|U(t)\|_{H^3} \lesssim \delta_1 \tag{3.2}$$

and, for any $\sigma \in [3, K] \cap \mathbb{Z}$ and $0 \le s \le t \le 1$,

$$E_\sigma(t) \le E_\sigma(s) + C \int_s^t \|U(t')\|_{Z'} \cdot E_\sigma(t') dt', \tag{3.3}$$

where

$$E_P := \int_{\mathbb{R}^2} PU \cdot \overline{PU} dx - \frac{1}{8} \sum_{j=1}^2 \int_{\mathbb{R}^2} \left[|\nabla| \Lambda^{-1} (U + \overline{U}) \right] \cdot \left[R_j P(U - \overline{U}) \cdot R_j P(U - \overline{U}) \right] dx,$$

$$E_\sigma := \sum_{P = D^\alpha, |\alpha| \le \sigma} E_P. \tag{3.4}$$

Proof of Lemma 3.1. By standard parabolic theory, for any fixed $\varepsilon > 0$, the equation (3.1) admits a local solution $U = U^\varepsilon \in C([0, T^\varepsilon], H^{K+2})$, for some $T^\varepsilon > 0$, with

$$\sup_{t \in [0, T^\varepsilon]} \|U(t)\|_{H^3} \lesssim \delta_1. \tag{3.5}$$

In view of the definition (3.4) [6], it follows that

$$\|U(t)\|_{H^\sigma}^2 \lesssim E_\sigma(t) \lesssim \|U(t)\|_{H^\sigma}^2, \qquad t \in [0, T^\varepsilon], \ \sigma \in [0, K]. \tag{3.6}$$

We rewrite $U = X + iY$, thus

$$\partial_t X = \Lambda Y + \varepsilon \Delta X - \sum_{j=1}^2 \Lambda R_j(|\nabla| \Lambda^{-1} X \cdot R_j Y),$$

$$\partial_t Y = -\Lambda X + \varepsilon \Delta Y - (1/2) \sum_{j=1}^2 |\nabla|(R_j Y \cdot R_j Y), \tag{3.7}$$

$$E_P = \int_{\mathbb{R}^2} (PX)^2 + (PY)^2 \, dx + \sum_{j=1}^2 \int_{\mathbb{R}^2} |\nabla| \Lambda^{-1} X \cdot (R_j PY)^2 \, dx.$$

Therefore, for $P = D^\alpha$, $|\alpha| \le \sigma$,

$$\frac{d}{dt} E_P = I_P + II_P + III_P + IV_P + V_P + VI_P - \varepsilon VII_P,$$

where

$$I_P := \int_{\mathbb{R}^2} -2PX \cdot \sum_{j=1}^2 P\Lambda R_j(|\nabla| \Lambda^{-1} X \cdot R_j Y) \, dx,$$

$$II_P := \int_{\mathbb{R}^2} -PY \cdot \sum_{j=1}^2 P|\nabla|(R_j Y \cdot R_j Y) \, dx,$$

$$III_P := \sum_{j=1}^2 \int_{\mathbb{R}^2} |\nabla| Y \cdot (R_j PY)^2 \, dx,$$

$$IV_P := \sum_{j=1}^2 \int_{\mathbb{R}^2} -|\nabla| \Lambda^{-1} X \cdot 2 R_j PY \cdot \Lambda R_j PX \, dx,$$

$$V_P := \sum_{j=1}^2 \int_{\mathbb{R}^2} - \sum_{m=1}^2 |\nabla| R_m(|\nabla| \Lambda^{-1} X \cdot R_m Y) \cdot (R_j PY)^2 \, dx,$$

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[6]It is important to include the cubic terms in the definition of E_P, in order to obtain a key cancellation between the terms I_P and IV_P in the formulas below.

$$VI_P := \sum_{j=1}^{2} \int_{\mathbb{R}^2} -|\nabla|\Lambda^{-1}X \cdot R_j PY \cdot R_j P|\nabla|\Big(\sum_{m=1}^{2} R_m Y \cdot R_m Y\Big)\,dx$$

$$VII_P := 2\int_{\mathbb{R}^2} \sum_{m=1}^{2} (\partial_m PX)^2 + (\partial_m PY)^2\,dx$$

$$+ \sum_{j,m=1}^{2} \int_{\mathbb{R}^2} 4\partial_m|\nabla|\Lambda^{-1}X \cdot R_j PY \cdot \partial_m R_j PY + 2|\nabla|\Lambda^{-1}X \cdot (\partial_m R_j PY)^2\,dx.$$

We would like to bound the terms in the expression above in terms of $\|U\|_{H^\sigma}^2 \|U\|_{Z'}$. For this we need the bounds

$$\sum_{m,j=1}^{2} \|\partial_j R_m U\|_{L^\infty} + \sum_{m,j=1}^{2} \|\partial_j(|\nabla|\Lambda^{-1}X \cdot R_m Y)\|_{L^\infty} \lesssim \|U\|_{Z'},\tag{3.8}$$

$$\|\Lambda(D^\alpha f D^\beta g)\|_{L^2} \lesssim \|\nabla f\|_{L^\infty}\|g\|_{H^\sigma} + \|\nabla g\|_{L^\infty}\|f\|_{H^\sigma}, \qquad |\alpha|,|\beta| \geq 1,\ |\alpha|+|\beta| \leq \sigma.$$

which are proved in Lemma 5.3 (recall that $\|U(t)\|_{H^3} \lesssim \delta_1,\ t \in [0,T^\varepsilon])$.

Using (3.8)

$$|III_P| + |V_P| \lesssim \|U\|_{H^\sigma}^2 \|U\|_{Z'}.$$

Moreover, using also integration by parts and the formulas $-|\nabla| = \partial_1 R_1 + \partial_2 R_2$ and $\partial_j R_m = \partial_j R_m$,

$$|II_P| \lesssim \sum_{j,m=1}^{2} \Big|\int_{\mathbb{R}^2} PR_m Y \cdot P\partial_m(R_j Y \cdot R_j Y)\,dx\Big|$$

$$\lesssim \sum_{j,m=1}^{2} \Big[\Big|\int_{\mathbb{R}^2} PR_m Y \cdot \partial_m[P(R_j Y \cdot R_j Y) - bR_j Y \cdot PR_j Y]\,dx\Big| + \Big|\int_{\mathbb{R}^2} PR_m Y \cdot \partial_m(R_j Y \cdot PR_j Y)\,dx\Big|\Big]$$

$$\lesssim \|U\|_{H^\sigma}^2 \|U\|_{Z'} + \sum_{j,m=1}^{2} \Big|\int_{\mathbb{R}^2} PR_m Y \cdot \partial_j PR_m Y \cdot R_j Y\,dx\Big|$$

$$\lesssim \|U\|_{H^\sigma}^2 \|U\|_{Z'},$$

and similarly,

$$|VI_P| \lesssim \sum_{j,m=1}^{2} \Big|\int_{\mathbb{R}^2} |\nabla|\Lambda^{-1}X \cdot R_j PY \cdot \partial_j P(R_m Y \cdot R_m Y)\,dx\Big|$$

$$\lesssim \sum_{j,m=1}^{2} \Big|\int_{\mathbb{R}^2} |\nabla|\Lambda^{-1}X \cdot R_j PY \cdot \partial_j(PR_m Y \cdot R_m Y)\,dx\Big|$$

$$+ \sum_{j,m=1}^{2} \Big|\int_{\mathbb{R}^2} |\nabla|\Lambda^{-1}X \cdot R_j PY \cdot \partial_j\big[P(R_m Y \cdot R_m Y) - bPR_m Y \cdot R_m Y\big]\,dx\Big|$$

$$\lesssim \sum_{j,m=1}^{2} \Big|\int_{\mathbb{R}^2} \big(|\nabla|\Lambda^{-1}X \cdot R_m Y\big) \cdot R_j PY \cdot \partial_m PR_j Y\,dx\Big| + \|U\|_{H^\sigma}^2 \|U\|_{Z'}$$

$$\lesssim \|U\|_{H^\sigma}^2 \|U\|_{Z'},$$

where $b = 1$ if $P = \mathrm{Id}$ and $b = 2$ if $P = D^\alpha$, $|\alpha| \geq 1$. Therefore

$$\sum_{P=D^\alpha,\,|\alpha|\leq\sigma} \big[|II_P| + |III_P| + |V_P| + |VI_P|\big] \lesssim \|U\|_{H^\sigma}^2 \|U\|_{Z'}.\tag{3.9}$$

In addition, using (3.8) and the bound (5.11),

$$|I_P + IV_P| = 2 \Big| \sum_{j=1}^{2} \int_{\mathbb{R}^2} R_j PX \cdot \big[P\Lambda(|\nabla|\Lambda^{-1}X \cdot R_j Y) - \Lambda(|\nabla|\Lambda^{-1}X \cdot R_j PY) \big] \, dx \Big|$$

$$\lesssim \sum_{j=1}^{2} \Big| \int_{\mathbb{R}^2} R_j PX \cdot \Lambda \big[P|\nabla|\Lambda^{-1}X \cdot R_j Y \big] \, dx \Big| + \|U\|_{H^\sigma}^2 \|U\|_{Z'}$$

$$\lesssim \sum_{j=1}^{2} \Big| \int_{\mathbb{R}^2} R_j PX \cdot P|\nabla|X \cdot R_j Y \, dx \Big| + \|U\|_{H^\sigma}^2 \|U\|_{Z'}.$$

Since $|\nabla| = -\partial_1 R_1 - \partial_2 R_2$ and $\partial_j R_m = \partial_m R_j$ we can further estimate

$$|I_P + IV_P| \lesssim \sum_{j,m=1}^{2} \Big| \int_{\mathbb{R}^2} R_j PX \cdot P\partial_m R_m X \cdot R_j Y \, dx \Big| + \|U\|_{H^\sigma}^2 \|U\|_{Z'}$$

$$\lesssim \sum_{j,m=1}^{2} \Big| \int_{\mathbb{R}^2} \partial_j R_m PX \cdot PR_m X \cdot R_j Y \, dx \Big| + \|U\|_{H^\sigma}^2 \|U\|_{Z'} \qquad (3.10)$$

$$\lesssim \|U\|_{H^\sigma}^2 \|U\|_{Z'}.$$

Notice also that, for any $m, j \in \{1, 2\}$

$$\Big| \int_{\mathbb{R}^2} \partial_m |\nabla| \Lambda^{-1} X \cdot R_j PY \cdot \partial_m R_j PY \, dx \Big|$$

$$\lesssim \Big| \int_{\mathbb{R}^2} \partial_m |\nabla| \Lambda^{-1} X \cdot P_{(-\infty,0]} R_j PY \cdot \partial_m R_j PY \, dx \Big| + \Big| \int_{\mathbb{R}^2} \partial_m |\nabla| \Lambda^{-1} X \cdot P_{[1,\infty)} R_j PY \cdot \partial_m R_j PY \, dx \Big|$$

$$\lesssim \|\nabla U\|_{H^\sigma}^2 \big(\|P_{(-\infty,0]} R_j PY\|_{L^\infty} + \|\partial_m |\nabla| \Lambda^{-1} X\|_{L^\infty} \big)$$

$$\lesssim \|\nabla U\|_{H^\sigma}^2 \|U\|_{H^3}.$$

Since $\|U\|_{H^3}$ is small, it follows that

$$\sum_{P=D^\alpha, \, |\alpha| \leq \sigma} VII_P \geq 0.$$

The desired inequality (3.3) follows for $0 \leq s \leq t \leq T^\varepsilon$, using also (3.9) and (3.10). A simple continuity argument, using this inequality with $\sigma = 3$ and the bound $\|U(t)\|_{Z'} \lesssim \|U(t)\|_{H^3}$, shows that the solution U^ε can be extended up to time 1, which completes the proof of the lemma. $\qquad \square$

We estimate now differences of smooth solutions.

Lemma 3.2. *(i) Assume $K \in [10, 50]$, $I \subseteq \mathbb{R}$ is an interval, and $U, U' \in C(I : H^{K+2})$ satisfy*

$$(\partial_t + i\Lambda)U = F + \frac{i}{4} \sum_{j=1}^{2} \Lambda R_j \big[|\nabla|\Lambda^{-1}(U + \overline{U}) \cdot R_j(U - \overline{U}) \big] + \frac{i}{8} \sum_{j=1}^{2} |\nabla| \big[R_j(U - \overline{U}) \cdot R_j(U - \overline{U}) \big],$$

$$(\partial_t + i\Lambda)U' = F' + \frac{i}{4} \sum_{j=1}^{2} \Lambda R_j \big[|\nabla|\Lambda^{-1}(U' + \overline{U'}) \cdot R_j(U' - \overline{U'}) \big] + \frac{i}{8} \sum_{j=1}^{2} |\nabla| \big[R_j(U' - \overline{U'}) \cdot R_j(U' - \overline{U'}) \big]$$

$$(3.11)$$

on $\mathbb{R}^2 \times I$. Assume in addition the $F, F' \in C(I : H^K)$ and let

$$M_\sigma(t) := 1 + \|U(t)\|_{H^\sigma} + \|U'(t)\|_{H^\sigma}, \qquad \sigma \in [3, K] \cap \mathbb{Z}. \qquad (3.12)$$

Let

$$U = X + iY, \ U' = X' + iY', \ U^\delta = U' - U = X^\delta + iY^\delta,$$

$$F = G + iH, \ F' = G' + iH', \ F^\delta = F' - F = G^\delta + iH^\delta.$$

For any $P = D^\alpha$, $|\alpha| \le \sigma \in [3, K] \cap \mathbb{Z}$, let

$$E_P^\delta := \int_{\mathbb{R}^2} (PX^\delta)^2 + (PY^\delta)^2 \, dx + \sum_{j=1}^2 \int_{\mathbb{R}^2} |\nabla|\Lambda^{-1} X' \cdot (R_j PY^\delta)^2 \, dx. \tag{3.13}$$

Then, for any $t \in I$ for which $M_3(t) \le 2$, we have

$$|\partial_t E_P^\delta(t)| \lesssim (M_\sigma + \|F'(t)\|_{H^3}) \|U^\delta(t)\|_{H^\sigma}^2 + \|U\|_{H^{\sigma+1}} \|U^\delta\|_{H^{3/2}} \|U^\delta\|_{H^\sigma} + \|U^\delta\|_{H^\sigma} \|F^\delta\|_{H^\sigma}. \tag{3.14}$$

(ii) If $U, U' \in C(I : H^3)$ are solutions of (3.11) with $F = F' = 0$ and $M_3(t) = 1 + \|U(t)\|_{H^3} + \|U'(t)\|_{H^3} \le 2$ for some $t \in I$ then

$$|\partial_t E_{\mathrm{Id}}^\delta(t)| \lesssim \|U^\delta(t)\|_{L_x^2}^2. \tag{3.15}$$

Proof of Lemma 3.2. The real variables X^δ, Y^δ, X' satisfy the equations

$$\partial_t X^\delta = \Lambda Y^\delta + G^\delta - \sum_{j=1}^2 \Lambda R_j(|\nabla|\Lambda^{-1} X' \cdot R_j Y' - |\nabla|\Lambda^{-1} X \cdot R_j Y),$$

$$\partial_t Y^\delta = -\Lambda X^\delta + H^\delta - (1/2) \sum_{j=1}^2 |\nabla|(R_j Y' \cdot R_j Y' - R_j Y \cdot R_j Y), \tag{3.16}$$

$$\partial_t X' = \Lambda Y' + G' - \sum_{j=1}^2 \Lambda R_j(|\nabla|\Lambda^{-1} X' \cdot R_j Y').$$

Using the definition (3.13) we calculate

$$\partial_t E_P^\delta = I_P^\delta + II_P^\delta + III_P^\delta + IV_P^\delta + V_P^\delta + VI_P^\delta,$$

where

$$I_P^\delta := \int_{\mathbb{R}^2} -2PX^\delta \cdot \sum_{j=1}^2 P\Lambda R_j(|\nabla|\Lambda^{-1} X' \cdot R_j Y' - |\nabla|\Lambda^{-1} X \cdot R_j Y) \, dx,$$

$$II_P^\delta := \int_{\mathbb{R}^2} -PY^\delta \cdot \sum_{j=1}^2 P|\nabla|(R_j Y' \cdot R_j Y' - R_j Y \cdot R_j Y) \, dx,$$

$$III_P^\delta := \sum_{j=1}^2 \int_{\mathbb{R}^2} |\nabla|\Lambda^{-1} \partial_t X' \cdot (R_j PY^\delta)^2 \, dx,$$

$$IV_P^\delta := \sum_{j=1}^2 \int_{\mathbb{R}^2} -|\nabla|\Lambda^{-1} X' \cdot 2R_j PY^\delta \cdot \Lambda R_j PX^\delta \, dx,$$

$$V_P^\delta := \sum_{j=1}^2 \int_{\mathbb{R}^2} -|\nabla|\Lambda^{-1} X' \cdot R_j PY^\delta \cdot R_j P|\nabla|\Big(\sum_{m=1}^2 R_m Y' \cdot R_m Y' - R_m Y \cdot R_m Y \Big) \, dx$$

$$VI_P^\delta := 2 \int_{\mathbb{R}^2} PX^\delta \cdot PG^\delta + PY^\delta \cdot PH^\delta \, dx + 2 \sum_{j=1}^2 \int_{\mathbb{R}^2} |\nabla|\Lambda^{-1} X' \cdot R_j PY^\delta \cdot R_j PH^\delta \, dx.$$

Clearly,

$$|VI_P^\delta| \lesssim M_3 \|U^\delta\|_{H^\sigma} \|F^\delta\|_{H^\sigma}$$

and, using the last identity in (3.16),

$$|III_P^\delta| \lesssim (M_3^2 + \|F'\|_{H^3})\|U^\delta\|_{H^\sigma}^2.$$

Also, using Lemma 5.3 (i) and the identities $|\nabla| = -\partial_1 R_1 - \partial_2 R_2$ and $\partial_j R_m = \partial_m R_j$,

$$|II_P^\delta| \lesssim \sum_{j=1}^2 \left| \int_{\mathbb{R}^2} PY^\delta \cdot P|\nabla|(R_j Y^\delta \cdot R_j Y^\delta)\, dx \right| + \sum_{j=1}^2 \left| \int_{\mathbb{R}^2} PY^\delta \cdot P|\nabla|(R_j Y \cdot R_j Y^\delta)\, dx \right|$$

$$\lesssim M_3\|U^\delta\|_{H^\sigma}^2 + \|U\|_{H^\sigma}\|U^\delta\|_{H^\sigma}\|U^\delta\|_{H^3} + \sum_{j,m=1}^2 \left| \int_{\mathbb{R}^2} R_m PY^\delta \cdot \partial_m PR_j Y^\delta \cdot R_j Y^\delta\, dx \right|$$

$$+ \sum_{j,m=1}^2 \left[\left| \int_{\mathbb{R}^2} R_m PY^\delta \cdot \partial_m PR_j Y^\delta \cdot R_j Y\, dx \right| + \left| \int_{\mathbb{R}^2} R_m PY^\delta \cdot \partial_m PR_j Y \cdot R_j Y^\delta\, dx \right| \right]$$

$$\lesssim M_\sigma\|U^\delta\|_{H^\sigma}^2 + \|U\|_{H^{\sigma+1}}\|U^\delta\|_{H^\sigma}\|U^\delta\|_{H^{3/2}}.$$

Similarly,

$$|V_P^\delta| \lesssim \sum_{j,m=1}^2 \left| \int_{\mathbb{R}^2} |\nabla|\Lambda^{-1} X' \cdot R_j PY^\delta \cdot \partial_j P\left(R_m Y^\delta \cdot R_m Y^\delta + 2R_m Y \cdot R_m Y^\delta\right) dx \right|$$

$$\lesssim M_3^2\|U^\delta\|_{H^\sigma}^2 + M_3\|U\|_{H^\sigma}\|U^\delta\|_{H^\sigma}\|U^\delta\|_{H^3} + \sum_{j,m=1}^2 \left[\left| \int_{\mathbb{R}^2} |\nabla|\Lambda^{-1} X' \cdot R_j PY^\delta \cdot \partial_j PR_m Y^\delta \cdot R_m Y^\delta\, dx \right| \right.$$

$$\left. + \left| \int_{\mathbb{R}^2} |\nabla|\Lambda^{-1} X' \cdot R_j PY^\delta \cdot \partial_j PR_m Y^\delta \cdot R_m Y\, dx \right| + \left| \int_{\mathbb{R}^2} |\nabla|\Lambda^{-1} X' \cdot R_j PY^\delta \cdot \partial_j PR_m Y \cdot R_m Y^\delta\, dx \right| \right]$$

$$\lesssim M_3 M_\sigma\|U^\delta\|_{H^\sigma}^2 + M_3\|U\|_{H^{\sigma+1}}\|U^\delta\|_{H^\sigma}\|U^\delta\|_{H^{3/2}},$$

and

$$|I_P^\delta + IV_P^\delta| = 2\left| \sum_{j=1}^2 \int_{\mathbb{R}^2} R_j PX^\delta \cdot \Lambda\left[P(|\nabla|\Lambda^{-1} X' \cdot R_j Y^\delta + |\nabla|\Lambda^{-1} X^\delta \cdot R_j Y) - |\nabla|\Lambda^{-1} X' \cdot R_j PY^\delta\right] dx \right|$$

$$\lesssim M_3\|U^\delta\|_{H^\sigma}^2 + M_\sigma\|U^\delta\|_{H^\sigma}\|U^\delta\|_{H^3} + \sum_{j=1}^2 \left| \int_{\mathbb{R}^2} R_j PX^\delta \cdot \Lambda\left[P|\nabla|\Lambda^{-1} X' \cdot R_j Y^\delta\right] dx \right|$$

$$+ \sum_{j=1}^2 \left| \int_{\mathbb{R}^2} R_j PX^\delta \cdot \Lambda\left[P|\nabla|\Lambda^{-1} X^\delta \cdot R_j Y\right] dx \right| + \sum_{j=1}^2 \left| \int_{\mathbb{R}^2} R_j PX^\delta \cdot \Lambda\left[|\nabla|\Lambda^{-1} X^\delta \cdot PR_j Y\right] dx \right|.$$

Using also the estimate

$$\|\Lambda(f \cdot g) - \Lambda f \cdot g\|_{L^2} \lesssim \|f\|_{L^2}\|g\|_{H^3},$$

see the proof of Lemma 5.3 (iii), it follows that

$$\sum_{j=1}^2 \left| \int_{\mathbb{R}^2} R_j PX^\delta \cdot \Lambda\left[|\nabla|\Lambda^{-1} X^\delta \cdot PR_j Y\right] dx \right| \lesssim \|U^\delta\|_{H^\sigma}\|U^\delta\|_{H^{3/2}}\|U\|_{H^{\sigma+1}} + M_\sigma\|U^\delta\|_{H^\sigma}\|U^\delta\|_{H^3}.$$

In addition, for $\widetilde{Y} \in \{Y, Y^\delta\}$,

$$\sum_{j=1}^{2} \Big| \int_{\mathbb{R}^2} R_j P X^\delta \cdot \Lambda \big[P |\nabla| \Lambda^{-1} X^\delta \cdot R_j \widetilde{Y} \big] \, dx \Big| \lesssim M_3 \| U^\delta \|_{H^\sigma}^2 + \sum_{j=1}^{2} \Big| \int_{\mathbb{R}^2} R_j P X^\delta \cdot P |\nabla| X^\delta \cdot R_j \widetilde{Y} \, dx \Big|$$

$$\lesssim M_3 \| U^\delta \|_{H^\sigma}^2 + \sum_{j,m=1}^{2} \Big| \int_{\mathbb{R}^2} \partial_m R_j P X^\delta \cdot P R_m X^\delta \cdot R_j \widetilde{Y} \, dx \Big| \lesssim M_3 \| U^\delta \|_{H^\sigma}^2.$$

Finally

$$\sum_{j=1}^{2} \Big| \int_{\mathbb{R}^2} R_j P X^\delta \cdot \Lambda \big[P |\nabla| \Lambda^{-1} X \cdot R_j Y^\delta \big] \, dx \Big| \lesssim \| U^\delta \|_{H^\sigma} \| U^\delta \|_{H^{3/2}} \| U \|_{H^{\sigma+1}},$$

and it follows that

$$|I_P^\delta + IV_P^\delta| \lesssim M_\sigma \| U^\delta \|_{H^\sigma}^2 + \| U^\delta \|_{H^\sigma} \| U^\delta \|_{H^{3/2}} \| U \|_{H^{\sigma+1}}.$$

The inequality (3.14) follows from these estimates.

To prove (3.15) we notice that the previous estimates can be improved if $P = \mathrm{Id}$ and $F = F' = 0$. Indeed, similar arguments as before show that

$$|VI_{\mathrm{Id}}^\delta| = 0, \qquad |III_{\mathrm{Id}}^\delta| \lesssim M_3^2 \| U^\delta \|_{L^2}^2, \qquad |II_{\mathrm{Id}}^\delta| \lesssim M_3 \| U^\delta \|_{L^2}^2,$$

$$|V_{\mathrm{Id}}^\delta| \lesssim M_3^2 \| U^\delta \|_{L^2}^2, \qquad |I_{\mathrm{Id}}^\delta + IV_{\mathrm{Id}}^\delta| \lesssim M_3 \| U^\delta \|_{L^2}^2,$$

and the desired inequality follows. $\qquad\qquad\square$

We can now complete the proof of the proposition, using the definitions, Lemma 3.1, Lemma 3.2, and the Bona–Smith argument [2].

Proof of Proposition 2.2 (i), (ii). Assume $M \in [3, 40] \cap \mathbb{Z}$ is fixed and $U_0 \in H^M$ is a given data satisfying

$$\| U_0 \|_{H^3} \leq \delta_0 := \delta_1,$$

where δ_1 is the small constant in Lemma 3.1. Using Lemma 3.1, for any $k \in \mathbb{Z}_+$ and $\varepsilon > 0$ we construct a solution $U_k^\varepsilon \in C([0,1] : H^{K+2})$ of the equation (3.1), with initial condition $U_k^\varepsilon(0) = P_{(-\infty, k]} U_0$. It follows from (3.2) and (3.3) that

$$\sup_{t \in [0,1]} \| U_k^\varepsilon(t) \|_{H^3} \lesssim \| U_0 \|_{H^3}, \qquad \sup_{t \in [0,1]} \| U_k^\varepsilon(t) \|_{H^{M+d}} \lesssim 2^{dk} \| U_0 \|_{H^M}, \qquad d \in \{0, 1, 2\}. \qquad (3.17)$$

We fix k and apply now Lemma 3.2 to the solutions $U := U_k^\varepsilon$ and $U' := U_k^{\varepsilon'}$, with $F := \varepsilon \Delta U_k^\varepsilon$, $F' := \varepsilon' \Delta U_k^{\varepsilon'}$. It follows from (3.14) that

$$\| U_k^\varepsilon(t') - U_k^{\varepsilon'}(t') \|_{H^\sigma}^2 \lesssim \| U_k^\varepsilon(t) - U_k^{\varepsilon'}(t) \|_{H^\sigma}^2 + \int_t^{t'} \big(\| U_k^\varepsilon(s) - U_k^{\varepsilon'}(s) \|_{H^\sigma}^2 + (\varepsilon + \varepsilon') \| U_k^\varepsilon(s) - U_k^{\varepsilon'}(s) \|_{H^\sigma} \big)$$

$$\times \big(1 + \| U_k^\varepsilon(s) \|_{H^{\sigma+2}} + \| U_k^{\varepsilon'}(s) \|_{H^{\sigma+2}} \big) \, ds$$

for any $0 \leq t \leq t' \leq 1$ and any $\sigma \in [3, K] \cap \mathbb{Z}$. Therefore, using (3.17),

$$\sup_{t \in [0,1]} \| U_k^\varepsilon(t) - U_k^{\varepsilon'}(t) \|_{H^{M+2}}^2 \leq C(k)(\varepsilon + \varepsilon'),$$

which shows that $\lim_{\varepsilon \to 0} U_k^\varepsilon$ exists for any k fixed. To summarize, for any $k \in \mathbb{Z}_+$ we have constructed a solution $U_k \in C([0,1] : H^{M+2})$ of the equation

$$(\partial_t + i\Lambda) U_k = \frac{i}{4} \sum_{j=1}^{2} \Lambda R_j \big[|\nabla| \Lambda^{-1} (U_k + \overline{U_k}) \cdot R_j (U_k - \overline{U_k}) \big] + \frac{i}{8} \sum_{j=1}^{2} |\nabla| \big[R_j (U_k - \overline{U_k}) \cdot R_j (U_k - \overline{U_k}) \big],$$

with $U_k(0) = P_{(-\infty,k]}U_0$. Moreover, see (3.17),

$$\sup_{t\in[0,1]} \|U_k(t)\|_{H^3} \lesssim \|U_0\|_{H^3}, \qquad \sup_{t\in[0,1]} \|U_k(t)\|_{H^{M+d}} \lesssim 2^{dk}\|U_0\|_{H^M}, \qquad d \in \{0,1,2\}. \qquad (3.18)$$

We show now that the sequence $\{U_k\}_{k\in\mathbb{Z}_+}$ is Cauchy in $C([0,1] : H^M)$. For this we apply Lemma 3.2 to the solutions $U := U_k$ and $U' := U_{k'}$, $k+1 \leq k'$, with $F = F' = 0$. It follows from (3.15) that

$$\|U_{k'}(t') - U_k(t')\|_{L^2}^2 \lesssim \|U_{k'}(t) - U_k(t)\|_{L^2}^2 + \int_t^{t'} \|U_{k'}(s) - U_k(s)\|_{L^2}^2 \, ds,$$

for any $0 \leq t \leq t' \leq 1$. Since $\|U_{k'}(0) - U_k(0)\|_{L^2} \leq 2^{-3k}$ it follows that $\|U_{k'}(t) - U_k(t)\|_{L^2} \lesssim 2^{-3k}$ for any $t \in [0,1]$. By interpolation with (3.18),

$$\sup_{t\in[0,1]} \|U_{k'}(t) - U_k(t)\|_{H^{3/2}} \lesssim 2^{-3k/2}. \qquad (3.19)$$

It follows now from (3.14) and the bound (3.18) that

$$\|U_{k'}(t') - U_k(t')\|_{H^M}^2 \lesssim \|U_{k'}(t) - U_k(t)\|_{H^M}^2$$
$$+ (1 + \|U_0\|_{H^M}) \int_t^{t'} \|U_{k'}(s) - U_k(s)\|_{H^M}^2 + 2^{-k/2}\|U_{k'}(s) - U_k(s)\|_{H^M} \, ds,$$

for any $0 \leq t \leq t' \leq 1$. Therefore

$$\sup_{t\in[0,1]} \|U_{k'}(t) - U_k(t)\|_{H^M} \leq C(\|U_0\|_{H^M})[2^{-k/2} + \|U_{k'}(0) - U_k(0)\|_{H^M}].$$

This shows that the sequence $\{U_k\}_{k\in\mathbb{Z}_+}$ is Cauchy in $C([0,1] : H^M)$.

Letting $U := \lim_{k\to\infty} U_k$, we have constructed a solution $U \in C([0,1] : H^M)$ of (2.9) such that $U(0) = U_0$ and

$$\sup_{t\in[0,1]} \|U(t)\|_{H^3} \lesssim \|U_0\|_{H^3}.$$

The inequality (2.12) follows from the corresponding inequality (3.3) satisfied uniformly by the functions U_k^ε. The uniqueness of the solution U follows from (3.15). $\qquad \square$

Proof of Proposition 2.2 (iii). **Step 1.** For any integer $J \geq 0$ and $f \in H^{N_0}$ we define

$$\|f\|_{Z_J} := \sup_{k\in\mathbb{Z}}(2^{k/10} + 2^{10k})\Big[\|P_k f\|_{L^2} + \sum_{j\in\mathbb{Z}_+} 2^{\min(j,2J-j)}\|\varphi_j(x) \cdot P_k f(x)\|_{L^2}\Big],$$

and notice that

$$\|f\|_{Z_J} \leq \|f\|_Z, \qquad \|f\|_{Z_J} \lesssim_J \|f\|_{H^{N_0}}.$$

It suffices to prove that for any $t, t' \in [0,T]$ with $0 \leq t' - t \leq 1$, and any $J \in \mathbb{Z}_+$ we have

$$\|e^{it'\Lambda}U(t') - e^{it\Lambda}U(t)\|_{Z_J} \leq C\big(T, \sup_{t\in[0,T]} \|U(t)\|_{H^{N_0}}, \|U_0\|_Z\big) \cdot |t'-t|(1 + \sup_{s\in[t,t']} \|e^{is\Lambda}U(s)\|_{Z_J}). \qquad (3.20)$$

Indeed, assuming (3.20), it follows easily that

$$\sup_{t\in[0,T]} \|e^{it\Lambda}U(t)\|_{Z_J} \leq C\big(T, \sup_{t\in[0,T]} \|U(t)\|_{H^{N_0}}, \|U_0\|_Z\big),$$
$$\|e^{it'\Lambda}U(t') - e^{it\Lambda}U(t)\|_{Z_J} \leq C\big(T, \sup_{t\in[0,T]} \|U(t)\|_{H^{N_0}}, \|U_0\|_Z\big) \cdot |t'-t|, \qquad t, t' \in [0,T],$$

uniformly in J, and the desired conclusions follow by letting $J \to \infty$.

To prove (3.20) we define $V_\pm(t) = e^{\pm it\Lambda}U(t)$, as in section 4, and use the formula (4.9),

$$
\widehat{V_+}(\xi, t') - \widehat{V_+}(\xi, t)
$$
$$
= \sum_{(\mu,\nu)\in\{(+,+),(+,-),(-,-)\}} \int_t^{t'} \int_{\mathbb{R}^2} e^{is[\Lambda(\xi) - \mu\Lambda(\xi-\eta) - \nu\Lambda(\eta)]} m_{\mu\nu}(\xi,\eta) \widehat{V_\mu}(\xi-\eta, s) \widehat{V_\nu}(\eta, s)\, d\eta ds. \tag{3.21}
$$

Letting

$$
\widehat{A_{\mu\nu}}(\xi, s) := \int_{\mathbb{R}^2} e^{is[\Lambda(\xi) - \mu\Lambda(\xi-\eta) - \nu\Lambda(\eta)]} m_{\mu\nu}(\xi,\eta) \widehat{V_\mu}(\xi-\eta, s) \widehat{V_\nu}(\eta, s)\, d\eta,
$$

it suffices to prove that for any $s \in [0, T]$ and $(\mu, \nu) \in \{(+,+),(+,-),(-,-)\}$

$$
\|A_{\mu\nu}(s)\|_{Z_J} \leq C\big(s, \|V_+(s)\|_{H^{N_0}}, \|V_+(0)\|_Z\big) \cdot (1 + \|V_+(s)\|_{Z_J}), \qquad \text{uniformly in } J.
$$

In view of the definition, it suffices to prove that for any $k \in \mathbb{Z}$ and $s \in [0, T]$

$$
\|P_k A_{\mu\nu}(s)\|_{L^2} + \sum_{j\geq 0} 2^{\min(j, 2J-j)} \|\varphi_j \cdot P_k A_{\mu\nu}(s)\|_{L^2}
$$
$$
\leq C\big(s, \|V_+(s)\|_{H^{N_0}}, \|V_+(0)\|_Z\big) \cdot (1 + \|V_+(s)\|_{Z_J})(2^{k/10} + 2^{10k})^{-1}. \tag{3.22}
$$

We prove this bound in several steps: first we estimate the sum over small values of j, see (3.23). Then we estimate the contribution of the very low frequencies of the inputs V_μ, V_ν, see (3.26). Finally, we pass to the physical space to estimate the remaining contributions using localization, see (3.28) and (3.30).

For simplicity of notation, in the rest of the proof we let $\widetilde{C}$ denote constants that may depend only on $s, \|V_+(s)\|_{H^{N_0}}, \|V_+(0)\|_Z$, similar to the constant in the right-hand side of (3.22).

Step 2. Notice first that

$$
\|\widehat{P_k A_{\mu\nu}}(s)\|_{L^\infty} \lesssim 2^{-(N_0-1)k_+} \|V_+(s)\|_{H^{N_0}}^2.
$$

Therefore $\|P_k A_{\mu\nu}(s)\|_{L^2} \lesssim 2^k 2^{-(N_0-1)k_+} \|V_+(s)\|_{H^{N_0}}^2$ and it follows that

$$
\|P_k A_{\mu\nu}(s)\|_{L^2} + \sum_{j\leq \max(-41k/40, 2k) + C(s)} 2^j \|\varphi_j \cdot P_k A_{\mu\nu}(s)\|_{L^2} \leq \widetilde{C}(2^{k/10} + 2^{10k})^{-1}. \tag{3.23}
$$

Step 3. To estimate the remaining sum

$$
\sum_{j\geq \max(-41k/40, 2k) + C(s)} 2^{\min(j, 2J-j)} \|\varphi_j \cdot P_k A_{\mu\nu}(s)\|_{L^2},
$$

we decompose

$$
P_k A_{\mu\nu}(s) = \sum_{(k_1,k_2)\in\mathcal{X}_k} A^{\mu\nu}_{k,k_1,k_2}(s),
$$
$$
\widehat{A^{\mu\nu}_{k,k_1,k_2}}(\xi, s) := \int_{\mathbb{R}^2} e^{is[\Lambda(\xi) - \mu\Lambda(\xi-\eta) - \nu\Lambda(\eta)]} \varphi_k(\xi) m_{\mu\nu}(\xi,\eta) \widehat{P_{k_1} V_\mu}(\xi-\eta, s) \widehat{P_{k_2} V_\nu}(\eta, s)\, d\eta, \tag{3.24}
$$

where

$$
\mathcal{X}_k^1 := \{(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z} : |\max(k_1, k_2) - k| \leq 4\},
$$
$$
\mathcal{X}_k^2 := \{(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z} : \max(k_1, k_2) \geq k+4 \text{ and } |k_1 - k_2| \leq 4\},
$$
$$
\mathcal{X}_k := \mathcal{X}_k^1 \cup \mathcal{X}_k^2.
$$

Since

$$
|m_{\mu\nu}(\xi,\eta)| \lesssim |\xi| + |\eta| + |\xi - \eta|,
$$

see (4.6)–(4.8), it follows from (3.21) that

$$\|\widehat{P_l V_\pm}(s)\|_{L^\infty} \lesssim \|\widehat{P_l V_\pm}(0)\|_{L^\infty} + (1+s)\Big[2^l\|P_{[l-4,l+4]}V_+(s)\|_{L^2}\|P_{(-\infty,l+4]}V_+(s)\|_{L^2}$$

$$+ \sum_{(k_1,k_2)\in\mathcal{X}_l^2}(2^{k_1}+2^{k_2})\|P_{k_1}V_+(s)\|_{L^2}\|P_{k_2}V_+(s)\|_{L^2}\Big] \tag{3.25}$$

$$\leq \widetilde{C}(2^{l/10}+2^{10l})^{-1}.$$

Therefore

$$\|A_{k,k_1,k_2}^{\mu\nu}(s)\|_{L^2} \lesssim 2^{\max(k,k_1,k_2)}\min\big(\|\widehat{P_{k_1}V_\mu}(s)\|_{L^1}\|\widehat{P_{k_2}V_\nu}(s)\|_{L^2}, \|\widehat{P_{k_1}V_\mu}(s)\|_{L^2}\|\widehat{P_{k_2}V_\nu}(s)\|_{L^1}\big)$$

$$\leq \widetilde{C}2^{\max(k_1,k_2)}(1+2^{\max(k_1,k_2)})^{-N_0}2^{(19/10)\min(k_1,k_2)}.$$

Therefore

$$\sum_{j\geq\max(-41k/40,2k)}\sum_{(k_1,k_2)\in\mathcal{X}_k,\,\min(k_1,k_2)\leq-2j/3} 2^j\|A_{k,k_1,k_2}^{\mu\nu}(s)\|_{L^2} \leq \widetilde{C}2^{-10k_+}. \tag{3.26}$$

Step 4. It remains to bound the contribution

$$\sum_{j\geq\max(-41k/40,2k)+C(s)}\sum_{(k_1,k_2)\in\mathcal{X}_k,\,\min(k_1,k_2)\geq-2j/3} 2^{\min(j,2J-j)}\|\varphi_j\cdot A_{k,k_1,k_2}^{\mu\nu}(s)\|_{L^2}.$$

For this we write, using (3.24),

$$A_{k,k_1,k_2}^{\mu\nu}(x,s) = \int_{\mathbb{R}^2\times\mathbb{R}^2} P_{k_1}V_\mu(y,s)P_{k_2}V_\nu(z,s)K_{k,k_1,k_2}^{\mu\nu}(x,y,z)\,dydz,$$

$$K_{k,k_1,k_2}^{\mu\nu}(x,y,z) := c\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is[\Lambda(\xi)-\mu\Lambda(\xi-\eta)-\nu\Lambda(\eta)]}e^{i(x\cdot\xi-y\cdot(\xi-\eta)-z\cdot\eta)}m_{k,k_1,k_2}^{\mu\nu}(\xi,\eta)\,d\xi d\eta,$$

$$m_{k,k_1,k_2}^{\mu\nu}(\xi,\eta) := m_{\mu\nu}(\xi,\eta)\varphi_k(\xi)\varphi_{[k_1-1,k_1+1]}(\xi-\eta)\varphi_{[k_2-1,k_2+1]}(\eta).$$

Since

$$\sup_{\xi\in\mathbb{R}^2}|D^\alpha\Lambda(\xi)| \lesssim_{|\alpha|} 1, \qquad |\alpha|\geq 1,$$

it follows by integration by parts that

$$|K_{k,k_1,k_2}^{\mu\nu}(x,y,z)| \lesssim (1+2^{4k_1}+2^{4k_2})(|x-y|+|x-z|)^{-20}$$

$$\text{if } j\geq C(s),\ \min(k,k_1,k_2)\geq-40j/41,\ |x-y|+|x-z|\geq 2^{j-4}, \tag{3.27}$$

provided that the constant $C(s)$ is fixed sufficiently large. Letting

$$B_{j,k,k_1,k_2}^{\mu\nu}(x,s) := \int_{\mathbb{R}^2\times\mathbb{R}^2} V_{j,k_1}^\mu(y,s)V_{j,k_2}^\nu(z,s)K_{k,k_1,k_2}^{\mu\nu}(x,y,z)\,dydz,$$

$$V_{j,l}^\mu(x,s) := \varphi_{[j-4,j+4]}(x)\cdot P_l V_\mu(x,s),$$

it follows from (3.27) that

$$\|\varphi_j\cdot[A_{k,k_1,k_2}^{\mu\nu}(s)-B_{j,k,k_1,k_2}^{\mu\nu}(s)]\|_{L^2} \lesssim 2^{-10j}(1+2^{4k_1}+2^{4k_2})\|P_{k_1}V_+(s)\|_{L^2}\|P_{k_2}V_+(s)\|_{L^2}.$$

Since $\|P_l V_+(s)\|_{L^2} \lesssim 2^{-N_0 l_+}\|V_+(s)\|_{H^{N_0}}$, $l\in\mathbb{Z}$, it follows that

$$\sum_{j\geq\max(-41k/40,2k)+C(s)}\sum_{(k_1,k_2)\in\mathcal{X}_k,\,\min(k_1,k_2)\geq-2j/3} 2^j\|\varphi_j\cdot[A_{k,k_1,k_2}^{\mu\nu}(s)-B_{j,k,k_1,k_2}^{\mu\nu}(s)]\|_{L^2} \leq \widetilde{C}2^{-10k_+}. \tag{3.28}$$

Step 5. Finally we rewrite $B_{j,k,k_1,k_2}^{\mu\nu}(s)$ in the frequency space,

$$\widehat{B_{j,k,k_1,k_2}^{\mu\nu}}(\xi,s) = \int_{\mathbb{R}^2} e^{is[\Lambda(\xi)-\mu\Lambda(\xi-\eta)-\nu\Lambda(\eta)]}m_{k,k_1,k_2}^{\mu\nu}(\xi,\eta)\widehat{V_{j,k_1}^\mu}(\xi-\eta,s)\widehat{V_{j,k_2}^\nu}(\eta,s)\,d\eta,$$

and estimate

$$\|\widehat{B^{\mu\nu}_{j,k,k_1,k_2}}(s)\|_{L^2} \lesssim 2^{\max(k_1,k_2)}2^{\min(k_1,k_2)}\|\widehat{V^{\mu}_{j,k_1}}(s)\|_{L^2}\|\widehat{V^{\nu}_{j,k_2}}(s)\|_{L^2}.$$

Therefore

$$\sum_{j\geq\max(-41k/40,2k)+C(s)}\sum_{(k_1,k_2)\in\mathcal{X}_k,\,\min(k_1,k_2)\geq -2j/3}2^{\min(j,2J-j)}\|\varphi_j\cdot B^{\mu\nu}_{j,k,k_1,k_2}(s)\|_{L^2}$$
$$\lesssim \sum_{(k_1,k_2)\in\mathcal{X}_k,\,k_1\leq k_2}\sum_{j\geq 10}2^{\min(j,2J-j)}2^{k_1+k_2}\|V^{\pm}_{j,k_1}(s)\|_{L^2}\|V^{\pm}_{j,k_2}(s)\|_{L^2}. \tag{3.29}$$

We estimate the sum in the right-hand side of (3.29) in two different ways, depending on the relative sizes of k,k_1,k_2. If[7] $k_1=\min(k_1,k_2)\notin[100-10k,k-10]$ then we estimate $\|V^{\pm}_{j,k_1}(s)\|_{L^2}\leq\widetilde{C}2^{-N_0\max(k_1,0)}$, and the corresponding sum in the right-hand side of (3.29) is bounded by

$$\sum_{(k_1,k_2)\in\mathcal{X}_k,\,k_1\leq k_2,\,k_1\notin[100-10k,k-10]}\widetilde{C}2^{k_1+k_2}2^{-N_0\max(k_1,0)}(2^{k_2/10}+2^{10k_2})^{-1}\|V_+(s)\|_{Z_J}\leq\widetilde{C}\|V_+(s)\|_{Z_J}2^{-10k_+}.$$

On the other hand, if $k_1=\min(k_1,k_2)\in[100-10k,k-10]$ (in particular $k\geq 10$, $|k_2-k|\leq 4$) then we use $\|V^{\pm}_{j,k_2}(s)\|_{L^2}\leq\widetilde{C}2^{-N_0k}$, and the corresponding sum in the right-hand side of (3.29) is bounded by

$$\sum_{(k_1,k_2)\in\mathcal{X}_k,\,k_1\leq k_2,\,k_1\in[100-10k,k-10]}\widetilde{C}2^{k_1+k_2}2^{-N_0k}(2^{k_1/10}+2^{10k_1})^{-1}\|V_+(s)\|_{Z_J}\leq\widetilde{C}\|V_+(s)\|_{Z_J}2^{-10k}.$$

Therefore

$$\sum_{j\geq\max(-41k/40,2k)+C(s)}\sum_{(k_1,k_2)\in\mathcal{X}_k,\,\min(k_1,k_2)\geq -2j/3}2^{\min(j,2J-j)}\|\varphi_j\cdot B^{\mu\nu}_{j,k,k_1,k_2}(s)\|_{L^2}$$
$$\leq\widetilde{C}\|V_+(s)\|_{Z_J}2^{-10k_+}, \tag{3.30}$$

which completes the proof. □

4. Proof of Proposition 2.3

Recall first some of the notation from section 2: for $f\in H^1(\mathbb{R}^2)$ we define $|\nabla|f$, R_jf, Λf, and P_kf by multiplication with the Fourier multipliers

$$\xi\to|\xi|,\qquad \xi\to i\xi_j/|\xi|,\qquad \xi\to\Lambda(\xi)=\sqrt{a|\xi|^2+b},\qquad \xi\to\varphi_k(\xi)=\varphi(|\xi|/2^k)-\varphi(|\xi|/2^{k-1}) \tag{4.1}$$

respectively, where $a,b\in(0,\infty)$, $j\in\{1,2\}$, $k\in\mathbb{Z}$, and $\varphi:\mathbb{R}\to[0,1]$ is an even smooth function supported in $[-8/5,8/5]$ and equal to 1 in $[-5/4,5/4]$. For any set $I\in\mathbb{R}$ let

$$\varphi_I=\sum_{m\in I\cap\mathbb{Z}}\varphi_m.$$

For any $k\in\mathbb{Z}$ let

$$k_+:=\max(k,0).$$

Recall also the definition of our main spaces of "smooth" solutions $X^N_T(\mathbb{R}^2\times[0,T])$, $Y^{N_0}(\mathbb{R}^2)$, and $Z(\mathbb{R}^2)$ (see (2.4) and (2.5)), where $T\geq 1$, $N=30$ and $N_0=20$.

The rest of the section is concerned with the proof of Proposition 2.3, which we recall below:

Proposition 4.1. *Assume that $U\in X^N_T$ is a solution of the equation*

$$(\partial_t+i\Lambda)U=\frac{i}{4}\sum_{j=1}^2\Lambda R_j\big[|\nabla|\Lambda^{-1}(U+\overline{U})\cdot R_j(U-\overline{U})\big]+\frac{i}{8}\sum_{j=1}^2|\nabla|\big[R_j(U-\overline{U})\cdot R_j(U-\overline{U})\big] \tag{4.2}$$

---

[7]By convention, here and in the rest of the paper, $[a,b]:=\emptyset$ if $a>b$.

with the property that

$$\|U\|_{X_T^N} \leq \varepsilon_0 \leq 1. \tag{4.3}$$

Then

$$\sup_{t \in [0,T]} \|e^{it\Lambda}U(t) - U(0)\|_{Y^{N_0}} \lesssim \varepsilon_0^2. \tag{4.4}$$

We use first the method of normal forms to derive several new formulas describing the solution U. Then we use these formulas to prove the desired estimate (4.4).

4.1. **Renormalizations.** Assume that $U \in X_T^N$ satisfies the hypothesis of Proposition 4.1 and let

$$U_+ := U, \qquad U_- := \overline{U}.$$

Let $\widehat{U_+}, \widehat{U_-}$ denote the spatial Fourier transforms,

$$\widehat{U_\mu}(\xi, t) := \int_{\mathbb{R}^2} U_\mu(x, t) e^{-ix \cdot \xi} \, dx, \qquad \mu \in \{+, -\}, \ (\xi, t) \in \mathbb{R}^2 \times [0, T].$$

The equation (4.2) gives

$$[\partial_t + i\Lambda(\xi)]\widehat{U_+}(\xi, t) = \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} \int_{\mathbb{R}^2} m_{\mu\nu}(\xi, \eta)\widehat{U_\mu}(\xi - \eta, t)\widehat{U_\nu}(\eta, t) \, d\eta, \tag{4.5}$$

where

$$m_{++}(\xi, \eta) := c_0 i \Big[\frac{-\Lambda(\xi)|\xi - \eta|(\xi \cdot \eta)}{4\Lambda(\xi - \eta)|\xi||\eta|} + \frac{-|\xi|(\eta \cdot (\xi - \eta))}{8|\eta||\xi - \eta|} \Big], \tag{4.6}$$

$$m_{+-}(\xi, \eta) := c_0 i \Big[\frac{-\Lambda(\xi)|\eta|(\xi \cdot (\xi - \eta))}{4\Lambda(\eta)|\xi||\xi - \eta|} + \frac{\Lambda(\xi)|\xi - \eta|(\xi \cdot \eta)}{4\Lambda(\xi - \eta)|\xi||\eta|} + \frac{|\xi|(\eta \cdot (\xi - \eta))}{4|\eta||\xi - \eta|} \Big], \tag{4.7}$$

$$m_{--}(\xi, \eta) := c_0 i \Big[\frac{\Lambda(\xi)|\xi - \eta|(\xi \cdot \eta)}{4\Lambda(\xi - \eta)|\xi||\eta|} + \frac{-|\xi|(\eta \cdot (\xi - \eta))}{8|\eta||\xi - \eta|} \Big]. \tag{4.8}$$

Letting

$$V_\mu(t) := e^{\mu it\Lambda}U_\mu(t), \qquad \widehat{U_\mu}(\xi, t) = \widehat{V_\mu}(\xi, t)e^{-\mu it\Lambda(\xi)},$$

for $\mu \in \{+, -\}$, the equation (4.5) is equivalent to

$$\frac{d}{dt}[\widehat{V_+}(\xi, t)] = \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} \int_{\mathbb{R}^2} e^{it[\Lambda(\xi) - \mu\Lambda(\xi - \eta) - \nu\Lambda(\eta)]} m_{\mu\nu}(\xi, \eta)\widehat{V_\mu}(\xi - \eta, t)\widehat{V_\nu}(\eta, t) \, d\eta. \tag{4.9}$$

Therefore, letting

$$\Phi_{\mu\nu}(\xi, \eta) := \Lambda(\xi) - \mu\Lambda(\xi - \eta) - \nu\Lambda(\eta), \qquad (\mu, \nu) \in \{(+,+),(+,-),(-,-)\}, \tag{4.10}$$

it follows that

$$\widehat{V_+}(\xi, t) - \widehat{V_+}(\xi, 0) = \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} \int_0^t \int_{\mathbb{R}^2} e^{is\Phi_{\mu\nu}(\xi,\eta)} m_{\mu\nu}(\xi, \eta)\widehat{V_\mu}(\xi - \eta, s)\widehat{V_\nu}(\eta, s) \, d\eta ds.$$

It follows from the definition that, for any $(\mu, \nu) \in \{(+,+),(+,-),(-,-)\}$,

$$|\Phi_{\mu\nu}(\xi, \eta)|^{-1} \lesssim 1 + \min(|\xi|, |\eta|, |\xi - \eta|).$$

Therefore, we can integrate by parts in s to conclude that, for any $t \in [0, T]$,

$$\widehat{V_+}(\xi, t) - \widehat{V_+}(\xi, 0) = \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} - \int_0^t \int_{\mathbb{R}^2} e^{is\Phi_{\mu\nu}(\xi,\eta)} \frac{m_{\mu\nu}(\xi,\eta)}{i\Phi_{\mu\nu}(\xi,\eta)} \frac{d}{ds} \left[\widehat{V_\mu}(\xi - \eta, s) \widehat{V_\nu}(\eta, s) \right] d\eta ds$$

$$+ \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} \int_{\mathbb{R}^2} e^{it\Phi_{\mu\nu}(\xi,\eta)} \frac{m_{\mu\nu}(\xi,\eta)}{i\Phi_{\mu\nu}(\xi,\eta)} \widehat{V_\mu}(\xi - \eta, t) \widehat{V_\nu}(\eta, t) \, d\eta$$

$$- \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} \int_{\mathbb{R}^2} \frac{m_{\mu\nu}(\xi,\eta)}{i\Phi_{\mu\nu}(\xi,\eta)} \widehat{V_\mu}(\xi - \eta, 0) \widehat{V_\nu}(\eta, 0) \, d\eta.$$

For $t \in [0, T]$ let

$$\widehat{W}(\xi, t) := \widehat{V_+}(\xi, t) - \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} \int_{\mathbb{R}^2} e^{it\Phi_{\mu\nu}(\xi,\eta)} \frac{m_{\mu\nu}(\xi,\eta)}{i\Phi_{\mu\nu}(\xi,\eta)} \widehat{V_\mu}(\xi - \eta, t) \widehat{V_\nu}(\eta, t) \, d\eta.$$

It follows that

$$\widehat{W}(\xi, t) - \widehat{W}(\xi, 0) = \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} i \int_0^t H_{\mu\nu}(\xi, s) \, ds,$$

$$H_{\mu\nu}(\xi, s) := \int_{\mathbb{R}^2} e^{is\Phi_{\mu\nu}(\xi,\eta)} \frac{m_{\mu\nu}(\xi,\eta)}{\Phi_{\mu\nu}(\xi,\eta)} \frac{d}{ds} \left[\widehat{V_\mu}(\xi - \eta, s) \widehat{V_\nu}(\eta, s) \right] d\eta. \tag{4.11}$$

To calculate the functions $H_{\mu\nu}$ we use the formulas, see (4.9),

$$\frac{d}{ds}[\widehat{V_+}(\xi, s)] = \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} e^{is\Lambda(\xi)} \int_{\mathbb{R}^2} m_{\mu\nu}(\xi, \chi) \widehat{U_\mu}(\xi - \chi, s) \widehat{U_\nu}(\chi, s) \, d\chi,$$

$$\frac{d}{ds}[\widehat{V_-}(\xi, s)] = \sum_{(\mu,\nu) \in \{(-,-),(-,+),(+,+)\}} e^{-is\Lambda(\xi)} \int_{\mathbb{R}^2} m'_{\mu\nu}(\xi, \chi) \widehat{U_\mu}(\xi - \chi, s) \widehat{U_\nu}(\chi, s) \, d\chi. \tag{4.12}$$

where

$$m'_{--}(v, w) := \overline{m_{++}(-v, -w)}, \qquad m'_{-+}(v, w) := \overline{m_{+-}(-v, -w)}, \qquad m'_{++}(v, w) := \overline{m_{--}(-v, -w)}.$$

Therefore

$$e^{-is\Lambda(\xi)} H_{++}(\xi, s)$$

$$= \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{m_{++}(\xi,\eta)}{\Phi_{++}(\xi,\eta)} \left[\widehat{U_+}(\eta, s) m_{\mu\nu}(\xi - \eta, \chi) \widehat{U_\mu}(\xi - \eta - \chi, s) \widehat{U_\nu}(\chi, s) \right] d\eta d\chi$$

$$+ \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{m_{++}(\xi,\eta)}{\Phi_{++}(\xi,\eta)} \left[\widehat{U_+}(\xi - \eta, s) m_{\mu\nu}(\eta, \chi) \widehat{U_\mu}(\eta - \chi, s) \widehat{U_\nu}(\chi, s) \right] d\eta d\chi,$$

$$e^{-is\Lambda(\xi)} H_{+-}(\xi, s)$$

$$= \sum_{(\mu,\nu) \in \{(+,+),(+,-),(-,-)\}} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{m_{+-}(\xi,\eta)}{\Phi_{+-}(\xi,\eta)} \left[\widehat{U_-}(\eta, s) m_{\mu\nu}(\xi - \eta, \chi) \widehat{U_\mu}(\xi - \eta - \chi, s) \widehat{U_\nu}(\chi, s) \right] d\eta d\chi$$

$$+ \sum_{(\mu,\nu) \in \{(-,-),(-,+),(+,+)\}} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{m_{+-}(\xi,\eta)}{\Phi_{+-}(\xi,\eta)} \left[\widehat{U_+}(\xi - \eta, s) m'_{\mu\nu}(\eta, \chi) \widehat{U_\mu}(\eta - \chi, s) \widehat{U_\nu}(\chi, s) \right] d\eta d\chi,$$

and

$$e^{-is\Lambda(\xi)}H_{--}(\xi,s)$$

$$= \sum_{(\mu,\nu)\in\{(-,-),(-,+),(+,+)\}} \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{m_{--}(\xi,\eta)}{\Phi_{--}(\xi,\eta)}\big[\widehat{U_-}(\eta,s)m'_{\mu\nu}(\xi-\eta,\chi)\widehat{U_\mu}(\xi-\eta-\chi,s)\widehat{U_\nu}(\chi,s)\big]\,d\eta d\chi$$

$$+ \sum_{(\mu,\nu)\in\{(-,-),(-,+),(+,+)\}} \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{m_{--}(\xi,\eta)}{\Phi_{--}(\xi,\eta)}\big[\widehat{U_-}(\xi-\eta,s)m'_{\mu\nu}(\eta,\chi)\widehat{U_\mu}(\eta-\chi,s)\widehat{U_\nu}(\chi,s)\big]\,d\eta d\chi.$$

After reorganizing the terms, it follows that

$$H_{++}(\xi,s) + H_{+-}(\xi,s) + H_{--}(\xi,s)$$

$$= e^{is\Lambda(\xi)}\int_{\mathbb{R}^2\times\mathbb{R}^2} m_{+++}(\xi,\eta,\chi)\widehat{U_+}(\xi-\eta,s)\widehat{U_+}(\eta-\chi,s)\widehat{U_+}(\chi,s)\,d\eta d\chi$$

$$+ e^{is\Lambda(\xi)}\int_{\mathbb{R}^2\times\mathbb{R}^2} m_{++-}(\xi,\eta,\chi)\widehat{U_+}(\xi-\eta,s)\widehat{U_+}(\eta-\chi,s)\widehat{U_-}(\chi,s)\,d\eta d\chi$$

$$+ e^{is\Lambda(\xi)}\int_{\mathbb{R}^2\times\mathbb{R}^2} m_{+--}(\xi,\eta,\chi)\widehat{U_+}(\xi-\eta,s)\widehat{U_-}(\eta-\chi,s)\widehat{U_-}(\chi,s)\,d\eta d\chi$$

$$+ e^{is\Lambda(\xi)}\int_{\mathbb{R}^2\times\mathbb{R}^2} m_{---}(\xi,\eta,\chi)\widehat{U_-}(\xi-\eta,s)\widehat{U_-}(\eta-\chi,s)\widehat{U_-}(\chi,s)\,d\eta d\chi,$$

where

$$m_{+++}(\xi,\eta,\chi) := \frac{m_{++}(\xi,\chi)m_{++}(\xi-\chi,\eta-\chi)}{\Phi_{++}(\xi,\chi)} + \frac{m_{++}(\xi,\eta)m_{++}(\eta,\chi)}{\Phi_{++}(\xi,\eta)} + \frac{m_{+-}(\xi,\eta)m'_{++}(\eta,\chi)}{\Phi_{+-}(\xi,\eta)}, \quad (4.13)$$

$$m_{++-}(\xi,\eta,\chi) := \frac{m_{++}(\xi,\xi-\eta)m_{+-}(\eta,\chi)}{\Phi_{++}(\xi,\xi-\eta)} + \frac{m_{++}(\xi,\eta)m_{+-}(\eta,\chi)}{\Phi_{++}(\xi,\eta)} + \frac{m_{+-}(\xi,\chi)m_{++}(\xi-\chi,\eta-\chi)}{\Phi_{+-}(\xi,\chi)}$$

$$+ \frac{m_{+-}(\xi,\eta)m'_{-+}(\eta,\eta-\chi)}{\Phi_{+-}(\xi,\eta)} + \frac{m_{--}(\xi,\chi)m'_{++}(\xi-\chi,\xi-\eta)}{\Phi_{--}(\xi,\chi)} + \frac{m_{--}(\xi,\xi-\chi)m'_{++}(\xi-\chi,\xi-\eta)}{\Phi_{--}(\xi,\xi-\chi)},$$

$$(4.14)$$

$$m_{+--}(\xi,\eta,\chi) := \frac{m_{++}(\xi,\xi-\eta)m_{--}(\eta,\chi)}{\Phi_{++}(\xi,\xi-\eta)} + \frac{m_{++}(\xi,\eta)m_{--}(\eta,\chi)}{\Phi_{++}(\xi,\eta)} + \frac{m_{+-}(\xi,\chi)m_{+-}(\xi-\chi,\eta-\chi)}{\Phi_{+-}(\xi,\chi)}$$

$$+ \frac{m_{+-}(\xi,\eta)m'_{--}(\eta,\eta-\chi)}{\Phi_{+-}(\xi,\eta)} + \frac{m_{--}(\xi,\chi)m'_{-+}(\xi-\chi,\xi-\eta)}{\Phi_{--}(\xi,\chi)} + \frac{m_{--}(\xi,\xi-\chi)m'_{-+}(\xi-\chi,\xi-\eta)}{\Phi_{--}(\xi,\xi-\chi)},$$

$$(4.15)$$

and

$$m_{---}(\xi,\eta,\chi) := \frac{m_{+-}(\xi,\chi)m_{--}(\xi-\chi,\eta-\chi)}{\Phi_{+-}(\xi,\chi)} + \frac{m_{--}(\xi,\chi)m'_{--}(\xi-\chi,\eta-\chi)}{\Phi_{--}(\xi,\chi)} + \frac{m_{--}(\xi,\eta)m'_{--}(\eta,\chi)}{\Phi_{--}(\xi,\eta)}.$$

$$(4.16)$$

To summarize, we proved the following:

Proposition 4.2. *For $t\in[0,T]$ let*

$$U_+(t) = U(t), \qquad U_-(t) = \overline{U(t)}, \qquad V_+(t) = e^{it\Lambda}U_+(t), \qquad V_-(t) = e^{-it\Lambda}U_-(t),$$

$$\widehat{W}(\xi,t) = e^{it\Lambda(\xi)}\widehat{U}(\xi,t) + ie^{it\Lambda(\xi)} \sum_{(\mu,\nu)\in\{(+,+),(+,-),(-,-)\}} \int_{\mathbb{R}^2} \frac{m_{\mu\nu}(\xi,\eta)}{\Phi_{\mu\nu}(\xi,\eta)}\widehat{U_\mu}(\xi-\eta,t)\widehat{U_\nu}(\eta,t)\,d\eta,$$

where m_{++}, m_{+-}, m_{--} are defined in (4.6)–(4.8) and

$$\Phi_{\mu\nu}(\xi,\eta) = \Lambda(\xi) - \mu\Lambda(\xi-\eta) - \nu\Lambda(\eta), \qquad (\mu,\nu)\in\{(+,+),(+,-),(-,-)\}.$$

Then

$$\frac{d}{ds}[\widehat{V_+}(\xi,s)] = \sum_{(\mu,\nu)\in\{(+,+),(+,-),(-,-)\}} e^{is\Lambda(\xi)} \int_{\mathbb{R}^2} m_{\mu\nu}(\xi,\chi)\widehat{U_\mu}(\xi-\chi,s)\widehat{U_\nu}(\chi,s)\,d\chi, \tag{4.17}$$

and

$$
\begin{aligned}
\widehat{W}(\xi,t) - \widehat{W}(\xi,0) &= i\int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Lambda(\xi)} m_{+++}(\xi,\eta,\chi)\widehat{U_+}(\xi-\eta,s)\widehat{U_+}(\eta-\chi,s)\widehat{U_+}(\chi,s)\,d\eta d\chi ds \\
&\quad + i\int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Lambda(\xi)} m_{++-}(\xi,\eta,\chi)\widehat{U_+}(\xi-\eta,s)\widehat{U_+}(\eta-\chi,s)\widehat{U_-}(\chi,s)\,d\eta d\chi ds \\
&\quad + i\int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Lambda(\xi)} m_{+--}(\xi,\eta,\chi)\widehat{U_+}(\xi-\eta,s)\widehat{U_-}(\eta-\chi,s)\widehat{U_-}(\chi,s)\,d\eta d\chi ds \\
&\quad + i\int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Lambda(\xi)} m_{---}(\xi,\eta,\chi)\widehat{U_-}(\xi-\eta,s)\widehat{U_-}(\eta-\chi,s)\widehat{U_-}(\chi,s)\,d\eta d\chi ds,
\end{aligned}
\tag{4.18}
$$

where the symbols $m_{+++}, m_{++-}, m_{+--}, m_{---}$ are defined in (4.13)–(4.16).

4.2. **Estimates.** In this subsection we prove the bound (4.4), as a consequence of Lemma 4.3 and Lemma 4.4 below. Using the dispersive estimate

$$\|P_l e^{-it\Lambda} f\|_{L^\infty} \lesssim (1+t)^{-1}(1+2^{2l})\|f\|_{L^1}, \qquad t\in\mathbb{R}, \tag{4.19}$$

and the definition of the Z-norm, it follows that for any $l\in\mathbb{Z}$ and $t\in[0,T]$

$$\|P_l U_\pm(t)\|_{L^\infty} \lesssim (1+t)^{-1}(1+2^{2l})\|P_l V_\pm\|_{L^1} \lesssim (1+t)^{-1}\frac{1+2^{2l}}{2^{l/10}+2^{10l}}\varepsilon_0.$$

Moreover

$$\|\widehat{P_l U_\pm}(t)\|_{L^\infty} = \|\widehat{P_l V_\pm}(t)\|_{L^\infty} \lesssim \|P_l V_\pm\|_{L^1} \lesssim (2^{l/10}+2^{10l})^{-1}\varepsilon_0.$$

Therefore, using the assumption $\|U\|_{X_T^N} \le \varepsilon_0$ and the definitions (2.4)–(2.5), for any $t\in[0,T]$

$$
\begin{aligned}
\|P_l U_\pm(t)\|_{L^2} + \|P_l V_\pm(t)\|_{L^2} &\lesssim \varepsilon_0 \min(2^{-N_0 l}, (1+t)^\delta 2^{-Nl}), \qquad l\ge 0, \\
\|P_l U_\pm(t)\|_{L^2} + \|P_l V_\pm(t)\|_{L^2} &\lesssim \varepsilon_0 2^{9l/10}, \qquad l\le 0, \\
(2^{l/10}+2^{8l})\|P_l U_\pm(t)\|_{L^\infty} &\lesssim \varepsilon_0 \min(2^{2l},(1+t)^{-1}), \qquad l\in\mathbb{Z}, \\
\|\widehat{P_l U_\pm}(t)\|_{L^\infty} + \|\widehat{P_l V_\pm}(t)\|_{L^\infty} &\lesssim \varepsilon_0(2^{l/10}+2^{10l})^{-1}, \qquad l\in\mathbb{Z}.
\end{aligned}
\tag{4.20}
$$

Lemma 4.3. *We have*

$$\sup_{t\in[0,T]} \|e^{it\Lambda}U(t) - W(t)\|_{Y^{N_0}} \lesssim \varepsilon_0^2.$$

Proof of Lemma 4.3. Using the definitions, see Lemma 4.2, it suffices to prove that for any $(\mu,\nu)\in\{(+,+),(+,-),(-,-)\}$

$$\sup_{t\in[0,T]} \big\|F_{\mu\nu}(t)\big\|_{H^{N_0}} + \sup_{t\in[0,T]} \big\|F_{\mu\nu}(t)\big\|_Z \lesssim \varepsilon_0^2, \tag{4.21}$$

where

$$\widehat{F_{\mu\nu}}(\xi,t) := e^{it\Lambda(\xi)} \int_{\mathbb{R}^2} \frac{m_{\mu\nu}(\xi,\eta)}{\Phi_{\mu\nu}(\xi,\eta)}\widehat{U_\mu}(\xi-\eta,t)\widehat{U_\nu}(\eta,t)\,d\eta.$$

Step 1. We prove first certain L^2 estimates. For any $k \in \mathbb{Z}$ and $t \in [0,T]$ we have

$$\widehat{P_k F_{\mu\nu}}(\xi, t) = \varphi_k(\xi) e^{it\Lambda(\xi)} \int_{\mathbb{R}^2} \frac{m_{\mu\nu}(\xi,\eta)}{\Phi_{\mu\nu}(\xi,\eta)} \widehat{U_\mu}(\xi - \eta, t) \widehat{U_\nu}(\eta, t) \, d\eta$$

$$= \sum_{(k_1,k_2) \in \mathcal{X}_k^1 \cup \mathcal{X}_k^2} \varphi_k(\xi) e^{it\Lambda(\xi)} \int_{\mathbb{R}^2} \frac{m_{\mu\nu}(\xi,\eta)}{\Phi_{\mu\nu}(\xi,\eta)} \varphi_{k_1}(\xi - \eta) \widehat{U_\mu}(\xi - \eta, t) \varphi_{k_2}(\eta) \widehat{U_\nu}(\eta, t) \, d\eta \qquad (4.22)$$

$$= \sum_{(k_1,k_2) \in \mathcal{X}_k^1 \cup \mathcal{X}_k^2} e^{it\Lambda(\xi)} \int_{\mathbb{R}^2} P_{k,k_1,k_2}^{\mu\nu}(\xi,\eta) \widehat{P_{k_1} U_\mu}(\xi - \eta, t) \widehat{P_{k_2} U_\nu}(\eta, t) \, d\eta$$

where, as before,

$$\mathcal{X}_k^1 = \{(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z} : |\max(k_1, k_2) - k| \le 4\},$$
$$\mathcal{X}_k^2 = \{(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z} : \max(k_1, k_2) \ge k + 4 \text{ and } |k_1 - k_2| \le 4\},$$

and

$$P_{k,k_1,k_2}^{\mu\nu}(\xi,\eta) := \varphi_k(\xi) \frac{m_{\mu\nu}(\xi,\eta)}{\Phi_{\mu\nu}(\xi,\eta)} \varphi_{[k_1-1,k_1+1]}(\xi - \eta) \varphi_{[k_2-1,k_2+1]}(\eta). \qquad (4.23)$$

We use the estimates (4.20) and (5.12): for any $k \le 0$ and $t \in [0,T]$ it follows that

$$\|P_k F_{\mu\nu}(t)\|_{L^2} \lesssim \sum_{(k_1,k_2) \in \mathcal{X}_k^2} (1 + 2^{3k_1}) 2^{k_1} \|P_{k_1} U_\pm(t)\|_{L^2} \min(2^k \|P_{k_2} U_\pm(t)\|_{L^2}, \|P_{k_2} U_\pm(t)\|_{L^\infty})$$

$$+ \sum_{(k_1,k_2) \in \mathcal{X}_k^1, \, k_1 \le k_2} 2^k \|P_{k_1} U_\pm(t)\|_{L^2} \|P_{k_2} U_\pm(t)\|_{L^\infty} \qquad (4.24)$$

$$\lesssim \varepsilon_0^2 \min((1 + t)^{-1}, 2^k).$$

Similarly, for any $k \ge 0$ and $t \in [0,T]$,

$$\|P_k F_{\mu\nu}(t)\|_{L^2} \lesssim \sum_{(k_1,k_2) \in \mathcal{X}_k^2} 2^{2k}(1 + 2^{3k_1}) 2^{k_1} \|P_{k_1} U_\pm(t)\|_{L^\infty} \|P_{k_2} U_\pm(t)\|_{L^2}$$

$$+ \sum_{(k_1,k_2) \in \mathcal{X}_k^1, \, -20k_2 \le k_1 \le k_2} 2^{2k}(1 + 2^{3k_1}) 2^k \|P_{k_1} U_\pm(t)\|_{L^\infty} \|P_{k_2} U_\pm(t)\|_{L^2} \qquad (4.25)$$

$$+ \sum_{(k_1,k_2) \in \mathcal{X}_k^1, \, k_1 \le -20k_2} 2^{2k}(1 + 2^{3k_1}) 2^k \|P_{k_1} U_\pm(t)\|_{L^2} \|P_{k_2} U_\pm(t)\|_{L^\infty}$$

$$\lesssim \varepsilon_0^2 \min(2^{-(N-3)k}, (1 + t)^{-1} 2^{-15k}).$$

The bound on the first term in (4.21) follows easily from the last two inequalities.

Step 2. We prove now the bound on the second term in (4.21), i.e. for any $k \in \mathbb{Z}$ and $t \in [0,T]$

$$\sum_{j \in \mathbb{Z}_+} 2^j \|\varphi_j \cdot P_k F_{\mu\nu}(t)\|_{L^2} \lesssim \varepsilon_0^2 (2^{k/10} + 2^{10k})^{-1}.$$

Using (4.24) and (4.25), it suffices to prove that

$$\sum_{j \ge J_{k,t}} 2^j \|\varphi_j \cdot P_k F_{\mu\nu}(t)\|_{L^2} \lesssim \varepsilon_0^2 2^{-10k_+},$$

where, for some sufficiently large constant D,

$$2^{J_{k,t}} = D(1 + t + 2^{-41k/40})(1 + 2^{2k}). \qquad (4.26)$$

Using (4.22) and the formulas $\widehat{U_\mu}(\xi, t) = e^{-\mu i t \Lambda(\xi)} \widehat{V_\mu}(\xi, t)$ we write

$$P_k F_{\mu\nu}(t) = \sum_{(k_1, k_2) \in \mathcal{X}_k} F^{\mu\nu}_{k, k_1, k_2}(t),$$

$$\widehat{F^{\mu\nu}_{k, k_1, k_2}}(\xi, t) := \int_{\mathbb{R}^2} e^{it[\Lambda(\xi) - \mu\Lambda(\xi - \eta) - \nu\Lambda(\eta)]} P^{\mu\nu}_{k, k_1, k_2}(\xi, \eta) \widehat{P_{k_1} V_\mu}(\xi - \eta, t) \widehat{P_{k_2} V_\nu}(\eta, t) \, d\eta,$$

(4.27)

where, as before, $\mathcal{X}_k = \mathcal{X}_k^1 \cup \mathcal{X}_k^2$. Using this decomposition, it suffices to prove that

$$\sum_{j \geq J_{k,t}} \sum_{(k_1, k_2) \in \mathcal{X}_k} 2^j \|\varphi_j \cdot F^{\mu\nu}_{k, k_1, k_2}(t)\|_{L^2} \lesssim \varepsilon_0^2 2^{-10k_+},$$

(4.28)

for any $k \in \mathbb{Z}$ and $t \in [0, T]$. This is similar to the proof of Proposition 2.2 (iii) in the previous section, and follows from the bounds (4.29), (4.31), and (4.33) below.

Step 3. Since

$$\sup_{\xi, \eta \in \mathbb{R}^2} |P^{\mu\nu}_{k, k_1, k_2}(\xi, \eta)| \lesssim (1 + 2^{\min(k_1, k_2)}) 2^{\max(k_1, k_2)},$$

it follows from (4.20) that

$$\|F^{\mu\nu}_{k, k_1, k_2}(s)\|_{L^2} \lesssim (2^{\max(k_1, k_2)} + 2^{k_1 + k_2}) \min \left(\|\widehat{P_{k_1} V_\mu}(s)\|_{L^1} \|\widehat{P_{k_2} V_\nu}(s)\|_{L^2}, \|\widehat{P_{k_1} V_\mu}(s)\|_{L^2} \|\widehat{P_{k_2} V_\nu}(s)\|_{L^1} \right)$$

$$\lesssim \varepsilon_0^2 (2^{\max(k_1, k_2)} + 2^{k_1 + k_2}) 2^{(19/10) \min(k_1, k_2)} (1 + 2^{\max(k_1, k_2)})^{-N_0}.$$

Therefore

$$\sum_{j \geq J_{k,t}} \sum_{(k_1, k_2) \in \mathcal{X}_k, \, \min(k_1, k_2) \leq -2j/3} 2^j \|F^{\mu\nu}_{k, k_1, k_2}(s)\|_{L^2} \lesssim \varepsilon_0^2 2^{-10k_+}.$$

(4.29)

Step 4. It remains to bound the contribution

$$\sum_{j \geq J_{k,t}} \sum_{(k_1, k_2) \in \mathcal{X}_k, \, \min(k_1, k_2) \geq -2j/3} 2^j \|\varphi_j \cdot F^{\mu\nu}_{k, k_1, k_2}(t)\|_{L^2}.$$

For this we write, using (4.27),

$$F^{\mu\nu}_{k, k_1, k_2}(x, t) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} P_{k_1} V_\mu(y, t) P_{k_2} V_\nu(z, t) L^{\mu\nu}_{k, k_1, k_2}(x, y, z, t) \, dy dz,$$

$$L^{\mu\nu}_{k, k_1, k_2}(x, y, z, t) := c \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{it[\Lambda(\xi) - \mu\Lambda(\xi - \eta) - \nu\Lambda(\eta)]} e^{i(x \cdot \xi - y \cdot (\xi - \eta) - z \cdot \eta)} P^{\mu\nu}_{k, k_1, k_2}(\xi, \eta) \, d\xi d\eta.$$

Since

$$\sup_{\xi \in \mathbb{R}^2} |D^\alpha \Lambda(\xi)| \lesssim_{|\alpha|} 1$$

it follows by integration by parts that

$$|L^{\mu\nu}_{k, k_1, k_2}(x, y, z, t)| \lesssim (1 + 2^{4k_1} + 2^{4k_2})(|x - y| + |x - z|)^{-20}$$

$$\text{if } 2^j \geq D(1 + t), \ \min(k, k_1, k_2) \geq -40j/41, \ |x - y| + |x - z| \geq 2^{j-4},$$

(4.30)

provided that the constant D is fixed sufficiently large. Letting

$$G^{\mu\nu}_{j, k, k_1, k_2}(x, t) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} V^\mu_{j, k_1}(y, t) V^\nu_{j, k_2}(z, t) L^{\mu\nu}_{k, k_1, k_2}(x, y, z, t) \, dy dz,$$

$$V^\mu_{j, l}(x, t) = \varphi_{[j-4, j+4]}(x) \cdot P_l V_\mu(x, t),$$

it follows from (3.27) that

$$\|\varphi_j \cdot [F^{\mu\nu}_{k, k_1, k_2}(t) - G^{\mu\nu}_{j, k, k_1, k_2}(t)]\|_{L^2} \lesssim 2^{-10j}(1 + 2^{4k_1} + 2^{4k_2}) \|P_{k_1} V_+(t)\|_{L^2} \|P_{k_2} V_+(t)\|_{L^2}.$$

Since $\|P_l V_+(t)\|_{L^2} \lesssim 2^{-N_0 l_+}\|V_+(t)\|_{H^{N_0}}$, $l \in \mathbb{Z}$, it follows that

$$\sum_{j \geq J_{k,t}} \sum_{(k_1,k_2) \in \mathcal{X}_k,\, \min(k_1,k_2) \geq -2j/3} 2^j \|\varphi_j \cdot [F^{\mu\nu}_{k,k_1,k_2}(t) - G^{\mu\nu}_{j,k,k_1,k_2}(t)]\|_{L^2} \leq \varepsilon_0^2 2^{-10k_+}. \qquad (4.31)$$

Step 5. Finally we rewrite $G^{\mu\nu}_{j,k,k_1,k_2}(t)$ in the frequency space,

$$\widehat{G^{\mu\nu}_{j,k,k_1,k_2}}(\xi,t) = \int_{\mathbb{R}^2} e^{it[\Lambda(\xi) - \mu\Lambda(\xi-\eta) - \nu\Lambda(\eta)]} P^{\mu\nu}_{k,k_1,k_2}(\xi,\eta) \widehat{V^{\mu}_{j,k_1}}(\xi-\eta,t) \widehat{V^{\nu}_{j,k_2}}(\eta,t)\, d\eta,$$

and estimate

$$\|\widehat{G^{\mu\nu}_{j,k,k_1,k_2}}(t)\|_{L^2} \lesssim \|P^{\mu\nu}_{k,k_1,k_2}\|_{L^\infty} 2^{\min(k_1,k_2)} \|\widehat{V^{\mu}_{j,k_1}}(t)\|_{L^2} \|\widehat{V^{\nu}_{j,k_2}}(t)\|_{L^2}.$$

Therefore

$$\begin{aligned}
\sum_{j \geq J_{k,t}} \sum_{(k_1,k_2) \in \mathcal{X}_k,\, \min(k_1,k_2) \geq -2j/3} & 2^j \|\varphi_j \cdot G^{\mu\nu}_{j,k,k_1,k_2}(t)\|_{L^2} \\
& \lesssim \sum_{(k_1,k_2) \in \mathcal{X}_k,\, k_1 \leq k_2} \sum_{j \geq 10} 2^j 2^{k_1+k_2}(1 + 2^{\min(k_1,k_2)}) \|V^{\pm}_{j,k_1}(t)\|_{L^2} \|V^{\pm}_{j,k_2}(t)\|_{L^2}.
\end{aligned} \qquad (4.32)$$

We estimate the sum in the right-hand side of (3.29) in two different ways, depending on the relative sizes of k, k_1, k_2. If $k_1 = \min(k_1,k_2) \notin [100-10k, k-10]$ then we estimate $\|V^{\pm}_{j,k_1}(t)\|_{L^2} \lesssim \varepsilon_0 2^{-N_0 \max(k_1,0)}$, and the corresponding sum in the right-hand side of (4.32) is bounded by

$$C \sum_{(k_1,k_2) \in \mathcal{X}_k,\, k_1 \leq k_2,\, k_1 \notin [100-10k, k-10]} \varepsilon_0 2^{k_1+k_2} 2^{(1-N_0)\max(k_1,0)} (2^{k_2/10} + 2^{10k_2})^{-1} \|V_+(t)\|_Z \lesssim \varepsilon_0^2 2^{-10k_+}.$$

On the other hand, if $k_1 = \min(k_1,k_2) \in [100-10k, k-10]$ (in particular $k \geq 10$, $|k_2 - k| \leq 4$) then we use $\|V^{\pm}_{j,k_2}(t)\|_{L^2} \lesssim \varepsilon_0 2^{-N_0 k}$, and the corresponding sum in the right-hand side of (3.29) is bounded by

$$C \sum_{(k_1,k_2) \in \mathcal{X}_k,\, k_1 \leq k_2,\, k_1 \in [100-10k, k-10]} \varepsilon_0 2^{k_1+k_2} 2^{(1-N_0)k} (2^{k_1/10} + 2^{10k_1})^{-1} \|V_+(t)\|_Z \lesssim \varepsilon_0^2 2^{-10k}.$$

Therefore

$$\sum_{j \geq J_{k,t}} \sum_{(k_1,k_2) \in \mathcal{X}_k,\, \min(k_1,k_2) \geq -2j/3} 2^j \|\varphi_j \cdot G^{\mu\nu}_{j,k,k_1,k_2}(t)\|_{L^2} \lesssim \varepsilon_0^2 2^{-10k_+}, \qquad (4.33)$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.4. *We have*

$$\sup_{t \in [0,T]} \|W(t) - W(0)\|_{Y^{N_0}} \lesssim \varepsilon_0^3.$$

Proof of Lemma 4.4. **Step 1.** We start again by proving L^2 estimates. Using (4.18), for any $t \in [0,T]$ and $k \in \mathbb{Z}$ we write

$$\mathcal{F}(P_k(W(t) - W(0)))(\xi) = \sum_{(\mu,\nu,\sigma) \in \{(+,+,+),(+,+,-),(+,-,-),(-,-,-)\}} i \int_0^t \widehat{P_k F_{\mu\nu\sigma}}(\xi,s)\, ds, \qquad (4.34)$$

where

$$\begin{aligned}
\widehat{P_k F_{\mu\nu\sigma}}(\xi,s) &:= \varphi_k(\xi) e^{is\Lambda(\xi)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} m_{\mu\nu\sigma}(\xi,\eta,\chi) \widehat{U_\mu}(\xi-\eta,s) \widehat{U_\nu}(\eta-\chi,s) \widehat{U_\sigma}(\chi,s)\, d\eta d\chi \\
&= \sum_{k_1,k_2,k_3 \in \mathbb{Z}} e^{is\Lambda(\xi)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi,\eta,\chi) \widehat{P_{k_1} U_\mu}(\xi-\eta,s) \widehat{P_{k_2} U_\nu}(\eta-\chi,s) \widehat{P_{k_3} U_\sigma}(\chi,s)\, d\eta d\chi,
\end{aligned} \qquad (4.35)$$

and

$$P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi,\eta,\chi) := \varphi_k(\xi) m_{\mu\nu\sigma}(\xi,\eta,\chi) \varphi_{[k_1-1,k_1+1]}(\xi-\eta) \varphi_{[k_2-1,k_2+1]}(\eta-\chi) \varphi_{[k_3-1,k_3+1]}(\chi). \qquad (4.36)$$

Let
$$\mathcal{Y}_k^1 := \{(k_1, k_2, k_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : |\max(k_1, k_2, k_3) - k| \leq 4\},$$
$$\mathcal{Y}_k^2 := \{(k_1, k_2, k_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : \max(k_1, k_2, k_3) \geq k + 4, \ \max(k_1, k_2, k_3) - \mathrm{med}(k_1, k_2, k_3) \leq 4\},$$
$$\mathcal{Y}_k := \mathcal{Y}_k^1 \cup \mathcal{Y}_k^2.$$

Using Lemma 5.5 and (4.20) we estimate for $k \leq 0$ and $s \in [0, T]$,

$$
\begin{aligned}
\|P_k F_{\mu\nu\sigma}(s)\|_{L^2} \lesssim & \sum_{(k_1, k_2, k_3) \in \mathcal{Y}_k^2, \, k_1 \leq k_2 \leq k_3} (2^{2k_3} + 2^{7k_3}) \\
& \Big[2^{2k_1 + 10k} (1+s)^{-10} \|P_{k_1} U_\pm(s)\|_{L^2} \|P_{k_2} U_\pm(s)\|_{L^2} \|P_{k_3} U_\pm(s)\|_{L^2} \\
& + (\log(2+s) + |k|) \|P_{k_1} U_\pm(s)\|_{L^2} \|P_{k_2} U_\pm(s)\|_{L^\infty} \min(2^k \|P_{k_3} U_\pm(s)\|_{L^2}, \|P_{k_3} U_\pm(s)\|_{L^\infty}) \Big] \\
& + \sum_{(k_1, k_2, k_3) \in \mathcal{Y}_k^1, \, k_1 \leq k_2 \leq k_3} 2^{2k} \Big[2^{2k_1} (1+s)^{-10} \|P_{k_1} U_\pm(s)\|_{L^2} \|P_{k_2} U_\pm(s)\|_{L^2} \|P_{k_3} U_\pm(s)\|_{L^2} \\
& + \log(2+s) \|P_{k_1} U_\pm(s)\|_{L^2} \|P_{k_2} U_\pm(s)\|_{L^\infty} \|P_{k_3} U_\pm(s)\|_{L^\infty} \Big] \\
\lesssim & \ \varepsilon_0^3 (1+s)^{-1} (\log(2+s) + |k|) \min((1+s)^{-1}, 2^k).
\end{aligned}
$$

Similarly, if $k \geq 0$ and $s \in [0, T]$,

$$
\begin{aligned}
\|P_k F_{\mu\nu\sigma}(s)\|_{L^2} \lesssim & \sum_{(k_1, k_2, k_3) \in \mathcal{Y}_k^2, \, k_1 \leq k_2 \leq k_3} 2^{7k_3} \Big[2^{2k_1} (1+s)^{-10} \|P_{k_1} U_\pm(s)\|_{L^2} \|P_{k_2} U_\pm(s)\|_{L^2} \|P_{k_3} U_\pm(s)\|_{L^2} \\
& + \log(2+s) \|P_{k_1} U_\pm(s)\|_{L^\infty} \|P_{k_2} U_\pm(s)\|_{L^\infty} \|P_{k_3} U_\pm(s)\|_{L^2} \Big] \\
& + \sum_{(k_1, k_2, k_3) \in \mathcal{Y}_k^1, \, k_1 \leq k_2 \leq k_3} 2^{4k}(1 + 2^{3k_2}) \Big[2^{2k_1}(1+s)^{-10} \|P_{k_1} U_\pm(s)\|_{L^2} \|P_{k_2} U_\pm(s)\|_{L^2} \|P_{k_3} U_\pm(s)\|_{L^2} \\
& + \log(2+s) \|P_{k_1} U_\pm(s)\|_{L^\infty} \|P_{k_2} U_\pm(s)\|_{L^\infty} \|P_{k_3} U_\pm(s)\|_{L^2} \Big] \\
\lesssim & \ \varepsilon_0^3 (1+s)^{-9/5} 2^{-(N-4)k}.
\end{aligned}
$$

Therefore
$$
\begin{aligned}
\|P_k F_{\mu\nu\sigma}(s)\|_{L^2} &\lesssim \varepsilon_0^3 (1+s)^{-1} (\log(2+s) + |k|) \min((1+s)^{-1}, 2^k), && \text{if } k \leq 0, \\
\|P_k F_{\mu\nu\sigma}(s)\|_{L^2} &\lesssim \varepsilon_0^3 (1+s)^{-9/5} 2^{-(N-4)k}, && \text{if } k \geq 0.
\end{aligned} \tag{4.37}
$$

In particular, using (4.34)
$$\|W(t) - W(0)\|_{H^{N_0}} \lesssim \varepsilon_0^3, \qquad t \in [0, T],$$
as desired.

Step 2. We prove now the remaining bound on the Z norm, i.e.
$$\sum_{j \in \mathbb{Z}_+} 2^j \|\varphi_j \cdot P_k(W(t) - W(0))\|_{L^2} \lesssim \varepsilon_0^3 (2^{k/10} + 2^{10k})^{-1},$$

for any $t \in [0, T]$ and $k \in \mathbb{Z}$ fixed. We fix a dyadic decomposition of the function $\mathbf{1}_{[0,t]}$, i.e. we fix functions $q_0, \ldots, q_{L+1} : \mathbb{R} \to [0, 1]$, $|L - \log_2(2 + t)| \leq 2$, with the properties

$$\sum_{m=0}^{L+1} q_l(s) = \mathbf{1}_{[0,t]}(s), \qquad \mathrm{supp}\, q_0 \subseteq [0, 2], \qquad \mathrm{supp}\, q_{L+1} \subseteq [t-2, t], \qquad \mathrm{supp}\, q_m \subseteq [2^{m-1}, 2^{m+1}],$$

$$q_m \in C^1(\mathbb{R}) \qquad \text{and} \qquad \int_0^t |q_m'(s)|\, ds \lesssim 1 \qquad \text{for } m = 1, \ldots, L. \tag{4.38}$$

Using the formula (4.34) and inserting the partition $\{q_m\}_{m=0,\ldots,L+1}$, it suffices to prove that

$$\sum_{j\in\mathbb{Z}_+} 2^j \Big\|\varphi_j \cdot \int_{\mathbb{R}} q_m(s) P_k F_{\mu\nu\sigma}(s)\, ds\Big\|_{L^2} \lesssim \varepsilon_0^3 (2^{k/10} + 2^{10k})^{-1} 2^{-m/100}$$

for any $k \in \mathbb{Z}$, $m \in \{0,\ldots,L+1\}$, and any $(\mu,\nu,\sigma) \in \{(+,+,+),(+,+,-),(+,-,-),(-,-,-)\}$. In view of (4.37) and the support properties of the functions q_m, it remains to prove that

$$\sum_{j\geq M(k,m)} 2^j \Big\|\varphi_j \cdot \int_{\mathbb{R}} q_m(s) P_k F_{\mu\nu\sigma}(s)\, ds\Big\|_{L^2} \lesssim \varepsilon_0^3 2^{-10k_+} 2^{-m/100}, \tag{4.39}$$

where, for some constant D sufficiently large,

$$M(k,m) := D + \max(7m/10, -21k/20, 10k). \tag{4.40}$$

Step 3. We examine now the formula (4.35) and estimate first the contributions of very low and very high frequencies. More precisely, assume $j \geq M(k,m)$ is fixed. Using Lemma 5.5 with $l = \min(k_1,k_2,k_3,-10j)-1$ and (4.20) we estimate, if $\min(k_1,k_2,k_3) \leq -2j/3$,

$$\Big\| \int_{\mathbb{R}^2\times\mathbb{R}^2} P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi,\eta,\chi) \widehat{P_{k_1}U_\mu}(\xi-\eta,s) \widehat{P_{k_2}U_\nu}(\eta-\chi,s) \widehat{P_{k_3}U_\sigma}(\chi,s)\, d\eta d\chi \Big\|_{L^2_\xi}$$

$$\lesssim \varepsilon_0^3 (1 + 2^{7c})\big(2^{-20j} 2^{9b/10} 2^{-N_0 c_+} + j2^{9b/5} 2^{-m} 2^{-N_0 c_+}\big)$$

$$\lesssim \varepsilon_0^3 2^{-(N_0-7)c_+} 2^{-m} 2^{9b/10} 2^{-j/2},$$

where $b = \min(k_1,k_2,k_3)$ and $c = \max(k_1,k_2,k_3)$.

Similarly, if $\max(k_1,k_2,k_3) \geq j/10$ and $\min(k_1,k_2,k_3) \geq -2j/3$ (in particular $(k_1,k_2,k_3) \in \mathcal{Y}_k^2$), we use Lemma 5.5 with $l = -10j$ and (4.20) to estimate

$$\Big\| \int_{\mathbb{R}^2\times\mathbb{R}^2} P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi,\eta,\chi) \widehat{P_{k_1}U_\mu}(\xi-\eta,s) \widehat{P_{k_2}U_\nu}(\eta-\chi,s) \widehat{P_{k_3}U_\sigma}(\chi,s)\, d\eta d\chi \Big\|_{L^2_\xi}$$

$$\lesssim \varepsilon_0^3 (1 + 2^{7c})\big(2^{-20j} 2^{9b/10} 2^{-2N_0 c_+} + j2^k 2^{19b/10} 2^{-2N_0 c_+}\big)$$

$$\lesssim \varepsilon_0^3 2^{-(2N_0-8)c} 2^{9b/10}.$$

It follows easily that the corresponding sums are controlled as claimed in (4.39). Therefore it remains to prove that for any $k \in \mathbb{Z}$, $m \in \{0,\ldots,L+1\}$, and $(\mu,\nu,\sigma) \in \{(+,+,+),(+,+,-),(+,-,-),(-,-,-)\}$,

$$\sum_{j\geq M(k,m)} 2^j \Big\|\varphi_j \cdot \int_{\mathbb{R}} q_m(s) P_k F^{\mu\nu\sigma}_j(s)\, ds\Big\|_{L^2} \lesssim \varepsilon_0^3 2^{-10k_+} 2^{-m/1000}, \tag{4.41}$$

where, with $Q_j := P_{[-2j/3,j/10]}$,

$$\widehat{P_k F^{\mu\nu\sigma}_j}(\xi,s) := \varphi_k(\xi) e^{is\Lambda(\xi)} \int_{\mathbb{R}^2\times\mathbb{R}^2} m_{\mu\nu\sigma}(\xi,\eta,\chi) \widehat{Q_j U_\mu}(\xi-\eta,s) \widehat{Q_j U_\nu}(\eta-\chi,s) \widehat{Q_j U_\sigma}(\chi,s)\, d\eta d\chi. \tag{4.42}$$

Step 4. The claim (4.41) is proved in Lemma 5.6, see (5.20), if $(\mu,\nu,\sigma) = (+,+,-)$. In the remaining cases, using first the bound (5.19), it remains to control the sum over $j \in (M(k,m), M'(k,m)]$. More precisely, we have to prove that

$$\sum_{M(k,m)<j\leq M'(k,m)} 2^j \Big\|\varphi_j \cdot \int_{\mathbb{R}} q_m(s) P_k F^{\mu\nu\sigma}_j(s)\, ds\Big\|_{L^2} \lesssim \varepsilon_0^3 2^{-10k_+} 2^{-m/1000} \tag{4.43}$$

for any $k \in \mathbb{Z}$, $m \in \{0,\ldots,L+1\}$, and $(\mu,\nu,\sigma) \in \{(+,+,+),(+,-,-),(-,-,-)\}$. Since $M(k,m) = M'(k,m)$ if $m \leq \max(-21k/20, 10k)$, we may assume that

$$m \geq \max(-21k/20, 10k), \qquad M'(k,m) = D + m.$$

The bound (4.43) follows easily if $m = 0$ or $m = L + 1$, due to the support properties of q_0 and q_{L+1}. Indeed, if $m \in \{0, L + 1\}$ then

$$\sum_{M(k,m) < j \leq D+m} 2^j \Big\| \varphi_j \cdot \int_{\mathbb{R}} q_m(s) P_k F_j^{\mu\nu\sigma}(s) \, ds \Big\|_{L^2} \lesssim 2^m \sup_{s \in [2^{m-1}, 2^{m+1}]} \sup_{M(k,m) < j \leq D+m} \|P_k F_j^{\mu\nu\sigma}(s)\|_{L^2}.$$

We use Lemma 5.5 with $r = 2$, $l = -100j$ and appropriate choices of p_1, p_2, p_3 to show that

$$\|P_k F_j^{\mu\nu\sigma}(s)\|_{L^2} \lesssim (1 + m)^6 2^{-2m} 2^{-11k_+}.$$

Therefore, for (4.43) it remains to prove that if

$$m \geq \max(-21k/20, 100k), \qquad k_1, k_2, k_3 \in [-2(m+D)/3, (m+D)/10], \qquad m \in \{D, \ldots, L\}, \quad (4.44)$$

and

$$P_k \widehat{F^{\mu\nu\sigma}_{k_1,k_2,k_3}}(\xi, s) := \varphi_k(\xi) e^{is\Lambda(\xi)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} m_{\mu\nu\sigma}(\xi, \eta, \chi) \widehat{P_{k_1} U_\mu}(\xi - \eta, s) \widehat{P_{k_2} U_\nu}(\eta - \chi, s) \widehat{P_{k_3} U_\sigma}(\chi, s) \, d\eta d\chi,$$

$(\mu, \nu, \sigma) \in \{(+, +, +), (+, -, -), (-, -, -)\}$, then

$$\Big\| \int_{\mathbb{R}} q_m(s) P_k \widehat{F^{\mu\nu\sigma}_{k_1,k_2,k_3}}(s) \, ds \Big\|_{L^2} \lesssim \varepsilon_0^3 2^{-10k_+} 2^{-101m/100}. \quad (4.45)$$

Step 5. We rewrite

$$P_k \widehat{F^{\mu\nu\sigma}_{k_1,k_2,k_3}}(\xi, s) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\Phi_{\mu\nu\sigma}(\xi,\eta,\chi)} P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi, \eta, \chi) \widehat{P_{k_1} V_\mu}(\xi - \eta, s) \widehat{P_{k_2} V_\nu}(\eta - \chi, s) \widehat{P_{k_3} V_\sigma}(\chi, s) \, d\eta d\chi,$$

where $P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}$ is as in (4.36), and

$$\Phi_{\mu\nu\sigma}(\xi, \eta, \chi) := \Lambda(\xi) - \mu\Lambda(\xi - \eta) - \nu\Lambda(\eta - \chi) - \sigma\Lambda(\chi). \quad (4.46)$$

To prove (4.45) we integrate by parts in s; the key observation is that

$$\Phi_{\mu\nu\sigma}(\xi, \eta, \chi) \gtrsim \frac{1}{1 + \min(|\xi|, |\xi - \eta|, |\eta - \chi|, |\chi|)} \qquad \text{for any } \xi, \eta, \chi \in \mathbb{R}^2, \quad (4.47)$$

for $(\mu, \nu, \sigma) \in \{(+, +, +), (+, -, -), (-, -, -)\}$. This follows easily from (5.2). Therefore, for any $\xi \in \mathbb{R}^2$,

$$\int_{\mathbb{R}} q_m(s) P_k \widehat{F^{\mu\nu\sigma}_{k_1,k_2,k_3}}(\xi, s) \, ds = c \int_{\mathbb{R}} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\Phi_{\mu\nu\sigma}(\xi,\eta,\chi)}$$

$$\frac{P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi, \eta, \chi)}{\Phi_{\mu\nu\sigma}(\xi, \eta, \chi)} \frac{d}{ds} \big[q_m(s) \widehat{P_{k_1} V_\mu}(\xi - \eta, s) \widehat{P_{k_2} V_\nu}(\eta - \chi, s) \widehat{P_{k_3} V_\sigma}(\chi, s) \big] \, d\eta d\chi ds.$$

Using the properties of the functions q_m in (4.38), it follows that the left-hand side of (4.45) is bounded by $I + II + III + IV$ where

$$I := \sup_{s \in \text{supp} \, q_m} \Big\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi, \eta, \chi)}{\Phi_{\mu\nu\sigma}(\xi, \eta, \chi)} \widehat{P_{k_1} U_\mu}(\xi - \eta, s) \widehat{P_{k_2} U_\nu}(\eta - \chi, s) \widehat{P_{k_3} U_\sigma}(\chi, s) \, d\eta d\chi \Big\|_{L^2_\xi},$$

$$II := 2^m \sup_{s \in \text{supp} \, q_m} \Big\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi, \eta, \chi)}{\Phi_{\mu\nu\sigma}(\xi, \eta, \chi)} \widehat{P_{k_1} Z_\mu}(\xi - \eta, s) \widehat{P_{k_2} U_\nu}(\eta - \chi, s) \widehat{P_{k_3} U_\sigma}(\chi, s) \, d\eta d\chi \Big\|_{L^2_\xi},$$

$$III := 2^m \sup_{s \in \text{supp} \, q_m} \Big\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi, \eta, \chi)}{\Phi_{\mu\nu\sigma}(\xi, \eta, \chi)} \widehat{P_{k_1} U_\mu}(\xi - \eta, s) \widehat{P_{k_2} Z_\nu}(\eta - \chi, s) \widehat{P_{k_3} U_\sigma}(\chi, s) \, d\eta d\chi \Big\|_{L^2_\xi},$$

$$IV := 2^m \sup_{s \in \text{supp} \, q_m} \Big\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{P^{\mu\nu\sigma}_{k,k_1,k_2,k_3}(\xi, \eta, \chi)}{\Phi_{\mu\nu\sigma}(\xi, \eta, \chi)} \widehat{P_{k_1} U_\mu}(\xi - \eta, s) \widehat{P_{k_2} U_\nu}(\eta - \chi, s) \widehat{P_{k_3} Z_\sigma}(\chi, s) \, d\eta d\chi \Big\|_{L^2_\xi},$$

and

$$\widehat{Z_\mu}(\zeta, s) := e^{-\mu is\Lambda(\zeta)} \cdot \frac{d}{ds} [\widehat{V_\mu}(\zeta, s)].$$

Using the identity (4.17), together with the formulas (4.6)–(4.8), and the bounds (4.20) we estimate, for any $l \in \mathbb{Z}$ and $s \in \operatorname{supp} q_m$,

$$
\begin{aligned}
\|P_l Z_+(s)\|_{L^2} &\lesssim \sum_{(l_1,l_2)\in\mathcal{X}_l} \sum_{(\mu,\nu)\in\{(+,+),(+,-),(-,-)\}} \Big\| \int_{\mathbb{R}^2} m_{\mu\nu}(\xi,\chi) \widehat{P_{l_1}U_\mu}(\xi-\chi,s) \widehat{P_{l_2}U_\nu}(\chi,s)\, d\chi \Big\|_{L^2_\xi} \\
&\lesssim \sum_{(l_1,l_2)\in\mathcal{X}_l,\, l_1\leq l_2} (2^{l_2/4} + 2^{3l_2}) \min(\|P_{l_1}U_+\|_{L^2}\|P_{l_2}U_+\|_{L^\infty}, \|P_{l_1}U_+\|_{L^\infty}\|P_{l_2}U_+\|_{L^2}) \quad (4.48) \\
&\lesssim \varepsilon_0^2 2^{-m} 2^{-8l_+}.
\end{aligned}
$$

We apply Lemma 5.7, and the bounds (4.20) and (4.48). Recalling (4.44) and (4.47),

$$
I + II + III + IV \lesssim \varepsilon_0^4 2^{-2m} 2^{20k_+} 2^{m/2},
$$

and the desired bound (4.45) follows. $\qquad\square$

5. Technical estimates

In this section we collect several technical estimates used in the rest of the paper. We start with estimates on the function Λ.

Lemma 5.1. *With* $\Lambda(x) = \sqrt{a|x|^2 + b}$, $\Lambda : \mathbb{R}^2 \to \mathbb{R}$, *we have*

$$
\sup_{x\in\mathbb{R}^2} (1+|x|)^{|\alpha|-1} |D^\alpha \Lambda(x)| \lesssim_{|\alpha|} 1, \tag{5.1}
$$

$$
\frac{1}{|\Lambda(x) \pm \Lambda(y) \pm \Lambda(x+y)|} \lesssim 1 + \min(|x|,|y|,|x+y|) \qquad \textit{for any } x,y \in \mathbb{R}^2, \tag{5.2}
$$

and

$$
\frac{|x-y|}{(1+|x|^3+|y|^3)} \lesssim |\nabla\Lambda(x) - \nabla\Lambda(y)| \lesssim |x-y| \qquad \textit{for any } x,y \in \mathbb{R}^2. \tag{5.3}
$$

Moreover, if $\Phi_{\mu\nu}(\xi,\eta) := \Lambda(\xi) - \mu\Lambda(\xi-\eta) - \nu\Lambda(\eta)$ *as in* (4.10), $(\mu,\nu)\in\{(+,+),(+,-),(-,-)\}$, *then*

$$
|\nabla_{\xi,\eta}\Phi_{\mu\nu}(\xi,\eta)| \lesssim |\Phi_{\mu\nu}(\xi,\eta)| \qquad \textit{for any } \xi,\eta \in \mathbb{R}^2. \tag{5.4}
$$

Proof of Lemma 5.1. The bound (5.1) follows from the definition. To prove (5.2) we estimate

$$
\Lambda(x) + \Lambda(y) - \Lambda(x+y) \geq \sqrt{b}
$$

if $|x+y| \leq \max(|x|,|y|)$, and

$$
\begin{aligned}
\Lambda(x) + \Lambda(y) - \Lambda(x+y) &\geq (\sqrt{a|x|^2+b} - \sqrt{a}|x|) + (\sqrt{a|y|^2+b} - \sqrt{a}|y|) - (\sqrt{a|x+y|^2+b} - \sqrt{a}|x+y|) \\
&\geq \max[(\sqrt{a|x|^2+b} - \sqrt{a}|x|), (\sqrt{a|y|^2+b} - \sqrt{a}|y|)] \\
&\gtrsim \max[(1+|x|)^{-1}, (1+|y|)^{-1}]
\end{aligned}
$$

if $|x+y| \geq \max(|x|,|y|)$. The bound (5.2) follows.

The upper bound in (5.3) follows from the uniform bound $\|D^2\Lambda\|_{L^\infty} \lesssim 1$, see (5.1). To prove the lower bound in (5.3) let $\lambda(r) = \sqrt{ar^2+b}$, $\Lambda(\xi) = \lambda(|\xi|)$. Then

$$
|\nabla\Lambda(x) - \nabla\Lambda(y)|^2 = [\lambda'(|x|) - \lambda'(|y|)]^2 + 2\lambda'(|x|)\lambda'(|y|)\Big(1 - \frac{x}{|x|}\cdot\frac{y}{|y|}\Big). \tag{5.5}
$$

Notice that, for any $u \leq v \in \mathbb{R}_+$,

$$
\lambda'(v) - \lambda'(u) = \int_u^v \frac{ds}{(b+as^2)^{3/2}} \gtrsim \frac{v-u}{1+v^3} \quad \text{and} \quad \lambda'(v)\lambda'(u) \gtrsim \frac{uv}{1+v^2}.
$$

Therefore, using (5.5),

$$|\nabla\Lambda(x) - \nabla\Lambda(y)|^2 \gtrsim \frac{(|x| - |y|)^2}{(1 + |x|^3 + |y|^3)^2} + \frac{2|x||y| - 2x \cdot y}{1 + |x|^2 + |y|^2} \gtrsim \frac{|x - y|^2}{(1 + |x|^3 + |y|^3)^2},$$

as desired.

To prove (5.4) we may assume that $\Phi_{\mu\nu}(\xi, \eta) \ll 1$ (otherwise the bound follows from (5.1)). In particular, we may assume that $(\mu, \nu) \in \{(+, +), (+, -)\}$. Since $\Phi_{+-}(\xi, \eta) = -\Phi_{++}(\xi - \eta, \xi)$, it suffices to consider the case $(\mu, \nu) = (+, +)$. Estimating as in the proof of (5.2),

$$|\Phi_{++}(\xi, \eta)| \gtrsim (1 + |\eta|)^{-1} + (1 + |\xi - \eta|)^{-1} + (|\eta| + |\xi - \eta| - |\xi|). \tag{5.6}$$

On the other hand, using (5.5), if

$$1 \le \min(|\eta|, |\xi - \eta|) \le \max(|\eta|, |\xi - \eta|) \le |\xi| \tag{5.7}$$

then

$$
\begin{aligned}
|\nabla_{\xi,\eta}\Phi_{++}(\xi, \eta)|^2 &\lesssim |\nabla\Lambda(\xi) - \nabla\Lambda(\xi - \eta)|^2 + |\nabla\Lambda(\xi - \eta) - \nabla\Lambda(\eta)|^2 \\
&\lesssim |\lambda'(\gamma) - \lambda'(\alpha)|^2 + |\lambda'(\beta) - \lambda'(\alpha)|^2 + \frac{(\alpha + \beta)^2 - \gamma^2}{\alpha\beta} \\
&\lesssim \alpha^{-2} + \beta^{-2} + (\alpha + \beta - \gamma)(1/\alpha + 1/\beta).
\end{aligned}
$$

where $\alpha := |\xi - \eta|$, $\beta := |\eta|$, $\gamma := |\xi|$. The desired bound (5.4) follows using also (5.6), in the range described in (5.7). In the remaining range $|\Phi_{++}(\xi, \eta)| \gtrsim 1$, see (5.6), and the desired bound (5.4) follows from (5.1). $\qquad\square$

We show now that Riesz transforms map our main space Y^{N_0} to itself.

Lemma 5.2. *Assume $m : \mathbb{R}^2 \to \mathbb{C}$ satisfies the symbol-type estimates*

$$\sup_{|a| \in [0,10]} \sup_{\xi \in \mathbb{R}^2} |\xi|^{|a|} |\partial^a m(\xi)| \le 1. \tag{5.8}$$

Then, for any $\phi \in Y^{N_0}$

$$\|\mathcal{F}^{-1}(m \cdot \widehat{f})\|_{Y^{N_0}} \lesssim \|f\|_{Y^{N_0}}. \tag{5.9}$$

Proof of Lemma 5.2. Using Plancherel's theorem

$$\|\mathcal{F}^{-1}(m \cdot \widehat{f})\|_{H^{N_0}} \lesssim \|f\|_{H^{N_0}}.$$

To control the Z-norm, we fix $k \in \mathbb{Z}$, let $g = \mathcal{F}^{-1}(m \cdot \widehat{f})$, and write

$$\varphi_j(x) P_k g(x) = \int_{\mathbb{R}^2} P_k f(y) \cdot \varphi_j(x) K_k(x - y)\, dy \qquad \text{where} \qquad K_k(z) = c\int_{\mathbb{R}^2} e^{iz\xi} m(\xi)\varphi_{[k-1,k+1]}(\xi)\, d\xi.$$

In view of (5.8), $|K_k(z)| \lesssim 2^{2k}(1 + 2^k|z|)^{-4}$, therefore

$$
\begin{aligned}
|\varphi_j(x) P_k g(x)| &\lesssim |\varphi_j(x)| \int_{|y| \le 2^{j-2}} |P_k f(y)| \cdot |K_k(x - y)|\, dy + \sum_{j' \ge j-4} \int_{\mathbb{R}^2} |\varphi_{j'}(y) P_k f(y)| \cdot |K_k(x - y)|\, dy \\
&\lesssim |\varphi_j(x)| \cdot 2^{2k}(1 + 2^{k+j})^{-4} \cdot \|P_k f\|_{L^1} + \sum_{j' \ge j-4} (|\varphi_{j'} \cdot P_k f| * |K_k|)(x).
\end{aligned}
$$

Therefore

$$(2^{k/10} + 2^{10k}) \sum_{j \in \mathbb{Z}_+} 2^j \|\varphi_j \cdot P_k g\|_{L^2}$$

$$\lesssim (2^{k/10} + 2^{10k}) \sum_{j \geq 0} 2^{2j+2k}(1 + 2^{k+j})^{-4} \|P_k f\|_{L^1} + (2^{k/10} + 2^{10k}) \sum_{j \geq 0} \sum_{j' \geq j-4} 2^j \|\varphi_{j'} \cdot P_k f\|_{L^2}$$

$$\lesssim \|f\|_Z,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Our next lemma is used several times in the proof of Proposition 2.2.

Lemma 5.3. *(i) For any multi-indices α, β with $|\alpha|, |\beta| \geq 1$ and any $f, g \in H^{|\alpha|+|\beta|}$ we have*

$$\|\Lambda(D^\alpha f D^\beta g)\|_{L^2} \lesssim_{|\alpha|+|\beta|} \|\nabla f\|_{L^\infty} \|g\|_{H^{|\alpha|+|\beta|}} + \|\nabla g\|_{L^\infty} \|f\|_{H^{|\alpha|+|\beta|}}.$$

(ii) With the norm Z' defined as in (2.13), we have

$$\sum_{j=1}^{2} \left[\|\partial_j f\|_{L^\infty} + \|R_j f\|_{Z'} \right] \lesssim \|f\|_{Z'}, \tag{5.10}$$

$$\sum_{k \in \mathbb{Z}} (2^k + 2^{k/10}) \|P_k(f \cdot g)\|_{L^\infty} \lesssim \|f\|_{Z'} \|g\|_{H^1} + \|f\|_{H^1} \|g\|_{Z'}.$$

(iii) If $f, g \in H^2$ then

$$\|\Lambda(f \cdot g) - \Lambda f \cdot g\|_{L^2} \lesssim \|f\|_{L^2} \|g\|_{Z'}. \tag{5.11}$$

Proof of Lemma 5.3. For part (i), we estimate, for any $k \in \mathbb{Z}_+$,

$$\|P_k[\Lambda(D^\alpha f D^\beta g)]\|_{L^2} \lesssim 2^k \|P_k(D^\alpha f D^\beta g)\|_{L^2}$$

$$\lesssim 2^k \|P_{\leq k-4} D^\alpha f\|_{L^\infty} \|P_{[k-4,k+4]} D^\beta g\|_{L^2} + 2^k \|P_{[k-4,k+4]} D^\alpha f\|_{L^2} \|P_{\leq k-4} D^\beta g\|_{L^\infty}$$

$$+ 2^k \sum_{k_1, k_2 \geq k-3, |k_1-k_2| \leq 8} \|P_{k_1} D^\alpha f\|_{L^2} \|P_{k_2} D^\beta g\|_{L^\infty}$$

$$\leq \sum_{k' \geq k-4} 2^{k-k'} \|\nabla f\|_{L^\infty} \|P_{k'} g\|_{H^{|\alpha|+|\beta|}} + \sum_{k' \geq k-4} 2^{k-k'} \|\nabla g\|_{L^\infty} \|P_{k'} f\|_{H^{|\alpha|+|\beta|}},$$

where $P_{\leq l} = P_{(-\infty,l]}$. A similar calculation shows that

$$\|P_{\leq 0}[\Lambda(D^\alpha f D^\beta g)]\|_{L^2} \lesssim_{|\alpha|+|\beta|} \|\nabla f\|_{L^\infty} \|g\|_{H^{|\alpha|+|\beta|}} + \|\nabla g\|_{L^\infty} \|f\|_{H^{|\alpha|+|\beta|}},$$

and the desired estimate follows.

The first inequality in (5.10) follows from the definitions, since

$$\|P_k R_j f\|_{L^\infty} + 2^{-k} \|P_k \partial_j f\|_{L^\infty} \lesssim \|P_k f\|_{L^\infty}, \qquad k \in \mathbb{Z}, j \in \{1, 2\}.$$

To prove the second inequality we estimate, for any $k \in \mathbb{Z}$

$$\|P_k(f \cdot g)\|_{L^\infty}$$

$$\lesssim \sum_{|k_2-k| \leq 3} \|P_{\leq k-4} f \cdot P_{k_2} g\|_{L^\infty} + \sum_{|k_1-k| \leq 3} \|P_{k_1} f \cdot P_{\leq k-4} g\|_{L^\infty} + \sum_{k_1, k_2 \geq k-3, |k_1-k_2| \leq 8} \|P_{k_1} f \cdot P_{k_2} g\|_{L^\infty}$$

$$\lesssim \sum_{k' \geq k-3} \min(2^{k'}, 2^{k'/2}) \|f\|_{H^1} (2^{k'/2} + 2^{2k'})^{-1} \|g\|_{Z'} + \min(2^k, 2^{k/2}) \|g\|_{H^1} (2^{k/2} + 2^{2k})^{-1} \|f\|_{Z'}$$

$$\lesssim \min(1, 2^{-3k/2}) \left[\|f\|_{Z'} \|g\|_{H^1} + \|f\|_{H^1} \|g\|_{Z'} \right].$$

The bound in the second line of (5.10) follows.

To prove part (iii), we decompose

$$\Lambda(f \cdot g) - \Lambda f \cdot g = F_1 + F_2 + F_3,$$

$$F_1 = \sum_{k_1, k_2 \in \mathbb{Z}, \, |k_1 - k_2| \leq 10} (\Lambda - \sqrt{b})(P_{k_1} f \cdot P_{k_2} g) - (\Lambda - \sqrt{b}) P_{k_1} f \cdot P_{k_2} g,$$

$$F_2 = \sum_{k_1, k_2 \in \mathbb{Z}, \, k_1 \leq k_2 - 11} (\Lambda - \sqrt{b})(P_{k_1} f \cdot P_{k_2} g) - (\Lambda - \sqrt{b}) P_{k_1} f \cdot P_{k_2} g,$$

$$F_3 = \sum_{k_1, k_2 \in \mathbb{Z}, \, k_2 \leq k_1 - 11} \Lambda(P_{k_1} f \cdot P_{k_2} g) - \Lambda P_{k_1} f \cdot P_{k_2} g.$$

Recalling that $\Lambda = \sqrt{a|\nabla|^2 + b}$, we estimate easily

$$\|F_1\|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, \, |k_1 - k_2| \leq 10} 2^{k_1} \|P_{k_1} f\|_{L^2} \cdot \|P_{k_2} g\|_{L^\infty} \lesssim \|f\|_{L^2} \|g\|_{Z'}$$

and

$$\|F_2\|_{L^2} \lesssim \sum_{k_2 \in \mathbb{Z}} 2^{k_2} \|P_{\leq k_2 - 11} f\|_{L^2} \cdot \|P_{k_2} g\|_{L^\infty} \lesssim \|f\|_{L^2} \|g\|_{Z'}.$$

To estimate $\|F_3\|_{L^2}$ we write $F_3 = c \sum_{k_1, k_2 \in \mathbb{Z}, \, k_2 \leq k_1 - 11} F^3_{k_1, k_2}$, where

$$F^3_{k_1, k_2}(x) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{ix \cdot \xi}(\Lambda(\xi) - \Lambda(\xi - \eta)) \widehat{P_{k_1} f}(\xi - \eta) \widehat{P_{k_2} g}(\eta) \, d\xi d\eta$$

$$= \int_{\mathbb{R}^2 \times \mathbb{R}^2} P_{k_1} f(y) P_{k_2} g(z) G_{k_1, k_2}(x, y, z) \, dy dz,$$

and

$$G_{k_1, k_2}(x, y, z) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{ix \cdot \xi} e^{-iy \cdot (\xi - \eta)} e^{-iz \cdot \eta}(\Lambda(\xi) - \Lambda(\xi - \eta)) \varphi_{[k_1 - 1, k_1 + 1]}(\xi - \eta) \varphi_{[k_2 - 1, k_2 + 1]}(\eta) \, d\xi d\eta$$

$$= \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i(x-y) \cdot \chi} e^{i(x-z) \cdot \eta}(\Lambda(\chi + \eta) - \Lambda(\chi)) \varphi_{[k_1 - 1, k_1 + 1]}(\chi) \varphi_{[k_2 - 1, k_2 + 1]}(\eta) \, d\chi d\eta.$$

Since $|D^\alpha \Lambda(\xi)| \lesssim_{|\alpha|} (1 + |\xi|)^{1 - |\alpha|}$ for any $\xi \in \mathbb{R}^2$, by integration by parts in both χ and η we estimate

$$|G_{k_1, k_2}(x, y, z)| \lesssim 2^{2k_1}(1 + 2^{k_1}|x - y|)^{-3} \cdot 2^{3k_2}(1 + 2^{k_2}|x - z|)^{-3},$$

provided that $k_2 \leq k_1 - 11$. Therefore

$$\|F^3_{k_1, k_2}\|_{L^2} \lesssim 2^{k_2} \|P_{k_1} f\|_{L^2} \|P_{k_2} g\|_{L^\infty},$$

and the desired bound (5.11) follows. $\qquad\square$

We prove now an important bilinear estimate, which is used repeatedly in section 4. Recall the definition $k_+ = \max(k, 0)$ for any $k \in \mathbb{Z}$.

Lemma 5.4. *For any $(\mu, \nu) \in \{(+, +), (+, -), (-, -)\}$, $k, k_1, k_2 \in \mathbb{Z}$, and $p, q, r \in [1, \infty]$, $r \geq 2$, satisfying $1/p + 1/q + 1/r = 1$ we have*

$$\left\| \int_{\mathbb{R}^2} P^{\mu\nu}_{k, k_1, k_2}(\xi, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) \, d\eta \right\|_{L^2_\xi} \tag{5.12}$$
$$\lesssim (1 + 2^{2k})(1 + 2^{3 \min(k_1, k_2)})(2^k + 2^{k_1} + 2^{k_2}) \|f\|_{L^p} \|g\|_{L^q} 2^{k(1 - 2/r)},$$

where $P^{\mu\nu}_{k, k_1, k_2}$ is as in (4.23).

Proof of Lemma 5.4. We estimate the left-hand side of (5.12) by

$$
\sup_{\|h\|_{L^2}=1} \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} h(x) e^{ix\cdot\xi} P_{k,k_1,k_2}^{\mu\nu}(\xi,\eta) \widehat{f}(\xi-\eta)\widehat{g}(\eta)\, d\eta d\xi dx \right|
$$

$$
\lesssim \sup_{\|h\|_{L^2}=1} \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} (P_{[k-1,k+1]}h)(x) f(y) g(z) Q_{k,k_1,k_2}^{\mu\nu}(x,y,z)\, dx dy dz \right|,
$$

(5.13)

where

$$
P_{k,k_1,k_2}^{\mu\nu}(\xi,\eta) = \varphi_k(\xi) \frac{m_{\mu\nu}(\xi,\eta)}{\Phi_{\mu\nu}(\xi,\eta)} \varphi_{[k_1-1,k_1+1]}(\xi-\eta)\varphi_{[k_2-1,k_2+1]}(\eta),
$$

$$
Q_{k,k_1,k_2}^{\mu\nu}(x,y,z) = \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i(x-y)\cdot\xi} e^{i(y-z)\cdot\eta} P_{k,k_1,k_2}^{\mu\nu}(\xi,\eta)\, d\xi d\eta.
$$

(5.14)

We assume that $k_2 \leq k_1$ (the case $k_1 \leq k_2$ is similar), thus $\max(k,k_2) \leq k_1 + 4$. Recall that $\Phi_{\mu\nu}(\xi,\eta) = \Lambda(\xi) - \mu\Lambda(\xi-\eta) - \nu\Lambda(\eta)$, and the inequalities, see Lemma 5.1,

$$
|\Phi_{\mu\nu}(\xi,\eta)|^{-1} \lesssim 1 + \min(|\xi|,|\eta|,|\xi-\eta|), \quad \left|D_\xi^m D_\eta^n \Phi_{\mu\nu}(\xi,\eta)\right| \lesssim \min(1,|\Phi_{\mu\nu}(\xi,\eta)|), \quad |m|+|n| \in [1,10].
$$

The functions $(\xi,\eta) \to m_{\mu\nu}(\xi,\eta)\varphi_k(\xi)\varphi_{[k_1-1,k_1+1]}(\xi-\eta)\varphi_{[k_2-1,k_2+1]}(\eta)$ are sums of symbols of the form

$$
(2^k + 2^{k_1} + 2^{k_2})a_1(\xi)a_2(\xi-\eta)a_3(\eta),
$$

$$
\sup_{|m|\in[0,10]} \sup_{v\in\mathbb{R}^2} \left[2^{k|m|}|\partial^m a_1(v)| + 2^{k_1|m|}|\partial^m a_2(v)| + 2^{k_2|m|}|\partial^m a_3(v)| \right] \lesssim 1,
$$

see (4.6)–(4.8). Integrating by parts ξ and η in (5.14) it follows that

$$
\left|Q_{k,k_1,k_2}^{\mu\nu}(x,y,z)\right| \lesssim (1+2^{k_2})(2^k + 2^{k_1} + 2^{k_2}) \cdot [2^{2k_2}(1+2^{\min(0,k_2)}|y-z|)^{-4}] \cdot [2^{2k}(1+2^{\min(0,k)}|x-y|)^{-4}].
$$

The desired bound (5.12) follows from (5.13) and the estimate

$$
\|P_{[k-1,k+1]}h\|_{L^r} \lesssim 2^{k(1-2/r)}\|h\|_{L^2}, \qquad r \in [2,\infty].
$$

$\square$

The last two lemmas, which concern trilinear estimates, are needed only in the proof of Lemma 4.4.

Lemma 5.5. *Assume $k,k_1,k_2,k_3 \in \mathbb{Z}$ and $l \leq \min(k,k_1,k_2,k_3) - 1$. Then*

$$
\left\| \int_{\mathbb{R}^2\times\mathbb{R}^2} P_{k,k_1,k_2,k_3}^{\mu\nu\sigma}(\xi,\eta,\chi) \widehat{f}_1(\xi-\eta)\widehat{f}_2(\eta-\chi)\widehat{f}_3(\chi)\, d\eta d\chi \right\|_{L_\xi^2} \lesssim (1+2^{3\widetilde{a}})(2^{2\widetilde{c}} + 2^{4\widetilde{c}})
$$

$$
\left[2^{2l}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2} + (\min(\widetilde{a},k)-l)\,2^{k(1-2/r)}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\|f_3\|_{L^{p_3}} \right],
$$

(5.15)

where $P_{k,k_1,k_2,k_3}^{\mu\nu\sigma}$ are as in (4.36), $\widetilde{a} = \mathrm{med}(k_1,k_2,k_3)$, $\widetilde{c} = \max(k_1,k_2,k_3)$, and $p_1,p_2,p_3,r \in [2,\infty]$ satisfy

$$
\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{r} = 1.
$$

Proof of Lemma 5.5. We make linear changes of variables and relabel the functions f_1,f_2,f_3. In view of the formulas (4.13)–(4.16) we consider operators of the form

$$
T[f_1,f_2,f_3](\xi) = \varphi_k(\xi) \int_{\mathbb{R}^2\times\mathbb{R}^2} m^0(\xi,\xi_2+\xi_3)m^1(\xi_2+\xi_3,\xi_3)\widehat{P_{k_1}f_1}(\xi-\xi_2-\xi_3)\widehat{P_{k_2}f_2}(\xi_2)\widehat{P_{k_3}f_3}(\xi_3)\, d\xi_2 d\xi_3
$$

where $m^0(x,y) = \frac{m_{\mu\nu}(x,y)}{\Phi_{\mu\nu}(x,y)}$ or $m^0(x,y) = \frac{m_{\mu\nu}(x,x-y)}{\Phi_{\mu\nu}(x,x-y)}$ and $m^1 = m_{\mu\nu}$ or $m^1 = m'_{\mu\nu}$, for suitable $\mu,\nu \in \{+,-\}$. It suffices to prove that

$$
\|T[f_1,f_2,f_3]\|_{L^2} \lesssim (1+2^{3\widetilde{a}})(2^{2\widetilde{c}} + 2^{4\widetilde{c}})
$$

$$
\left[2^{2l}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2} + (\min(\widetilde{a},k)-l)\,2^{k(1-2/r)}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\|f_3\|_{L^{p_3}} \right].
$$

(5.16)

We can further decompose

$$T = \sum_{n \in \mathbb{Z}} T_n,$$

$$T_n[f_1, f_2, f_3](\xi) = \varphi_k(\xi) \int_{\mathbb{R}^2} m^0(\xi, v) \varphi_n(v) \widehat{P_{k_1} f_1}(\xi - v) \widehat{G}(v) \, dv, \qquad (5.17)$$

$$\widehat{G}(v) = \int_{\mathbb{R}^2} m^1(v, \xi_3) \widehat{P_{k_2} f_2}(v - \xi_3) \widehat{P_{k_3} f_3}(\xi_3) \, d\xi_3.$$

Since $\|P_n G\|_{L^1} \lesssim (2^{k_2} + 2^{k_3}) \|f_2\|_{L^2} \|f_3\|_{L^2}$, we estimate

$$\sum_{n \le l} \|T_n[f_1, f_2, f_3]\|_{L^2} \lesssim \sum_{n \le l} (2^k + 2^{k_1})(1 + 2^n) \|f_1\|_{L^2} \|\widehat{P_n G}\|_{L^1}$$

$$\lesssim (2^{2l} + 2^{3l})(2^k + 2^{k_1})(2^{k_2} + 2^{k_3}) \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}.$$

Since $\|P_n G\|_{L^{p_2 p_3/(p_2+p_3)}} \lesssim (2^{k_2} + 2^{k_3}) \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}}$, we estimate using Lemma 5.4,

$$\|T_n[f_1, f_2, f_3]\|_{L^2} \lesssim 2^{k(1-2/r)}(1 + 2^{2k})(1 + 2^{3\min(k_1, n)})(2^k + 2^{k_1}) \|f_1\|_{L^{p_1}} \|G\|_{L_x^{p_2 p_3/(p_2+p_3)}}$$

$$\lesssim 2^{k(1-2/r)}(1 + 2^{2k})(1 + 2^{3\min(k_1, n)})(2^k + 2^{k_1})(2^{k_2} + 2^{k_3}) \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}}. \qquad (5.18)$$

It is easy to see that there are at most $C + |\min(\widetilde{a}, k) - l|$ values of n in $[l, \infty) \cap \mathbb{Z}$ for which $T_n[f_1, f_2, f_3]$ is nontrivial, and for all such values $2^{\min(k_1, n)} \lesssim 2^{\mathrm{med}(k_1, k_2, k_3)}$. The desired bound (5.16) follows.   □

Lemma 5.6. *Assume* $m \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $(\mu, \nu, \sigma) \in \{(+,+,+), (+,+,-), (+,-,-), (-,-,-)\}$, $s \in [2^{m-1}, 2^{m+1}]$. *Then, for any* $f_1, f_2, f_3 \in Y^{N_0}$,

$$\sum_{j \ge M'(k,m)} 2^j \left\| \varphi_j \cdot P_k \mathcal{F}^{-1} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\Phi_{\mu\nu\sigma}(\xi,\eta,\chi)} m_{\mu\nu\sigma}(\xi, \eta, \chi) \widehat{Q_j f_1}(\xi - \eta) \widehat{Q_j f_2}(\eta - \chi) \widehat{Q_j f_3}(\chi) d\eta d\chi \right\|_{L_x^2}$$

$$\lesssim 2^{-51m/50} 2^{-10k_+} \|f_1\|_{Y^{N_0}} \|f_2\|_{Y^{N_0}} \|f_3\|_{Y^{N_0}}, \qquad (5.19)$$

where, as in the proof of Lemma 4.4, $Q_j = P_{[-2j/3, j/10]}$, $\Phi_{\mu\nu\sigma}(\xi, \eta, \chi) = \Lambda(\xi) - \mu\Lambda(\xi - \eta) - \nu\Lambda(\eta - \chi) - \sigma\Lambda(\chi)$, *and for some sufficiently large constant* D,

$$M'(k, m) := D + \max(m, -21k/20, 10k).$$

In addition, with $M(k, m)$ *as in (4.40), for any* $f_1, f_2, f_3 \in Y^{N_0}$

$$\sum_{j \ge M(k,m)} 2^j \left\| \varphi_j \cdot P_k \mathcal{F}^{-1} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\Phi_{++-}(\xi,\eta,\chi)} m_{++-}(\xi, \eta, \chi) \widehat{Q_j f_1}(\xi - \eta) \widehat{Q_j f_2}(\eta - \chi) \widehat{Q_j f_3}(\chi) d\eta d\chi \right\|_{L_x^2}$$

$$\lesssim 2^{-1001m/1000} 2^{-10k_+} \|f_1\|_{Y^{N_0}} \|f_2\|_{Y^{N_0}} \|f_3\|_{Y^{N_0}}. \qquad (5.20)$$

Proof of Lemma 5.6. We may assume that $\|f_1\|_{Y^{N_0}} = \|f_2\|_{Y^{N_0}} = \|f_3\|_{Y^{N_0}} = 1$ and decompose the multipliers $m_{\mu\nu\sigma}(\xi, \eta, \chi)$ into multipliers of the form $m^0(\xi, \xi_2 + \xi_3) m^1(\xi_2 + \xi_3, \xi_3)$ where, for some $(\mu, \nu), (\mu', \nu') \in \{(+,+), (+,-), (-,-)\}$,

$$m^1(x, y) = m_{\mu\nu}(x, y) \text{ or } m^1(x, y) = m_{\mu\nu}(x, x - y) \text{ or } m^1(x, y) = m'_{\mu\nu}(x, y) \text{ or } m^1(x, y) = m'_{\mu\nu}(x, x - y),$$

$$m^0(x, y) = \frac{m_{\mu'\nu'}(x, y)}{\Phi_{\mu'\nu'}(x, y)} \quad \text{or} \quad m^0(x, y) = \frac{m_{\mu'\nu'}(x, x - y)}{\Phi_{\mu'\nu'}(x, x - y)}.$$

$$(5.21)$$

After relabelling the variables, as in the proof of Lemma 5.5, we consider the functions $G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}$, $\mu, \nu, \sigma \in \{+, -\}$, defined by

$$\widehat{G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}}(\xi) := \varphi_k(\xi) \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\tilde{\Phi}_{\mu\nu\sigma}(\xi,\xi_2,\xi_3)} m^0(\xi, \xi_2 + \xi_3) m^1(\xi_2 + \xi_3, \xi_3)$$
$$\widehat{P_{k_1}f_1}(\xi - \xi_2 - \xi_3)\widehat{P_{k_2}f_2}(\xi_2)\widehat{P_{k_3}f_3}(\xi_3)\, d\xi_2 d\xi_3, \tag{5.22}$$

where $j \geq M(k, m)$, $k_1, k_2, k_3 \in [-2j/3, j/10] \cap \mathbb{Z}$, m^0, m^1 are as in (5.21), and

$$\tilde{\Phi}_{\mu\nu\sigma}(\xi, \xi_2, \xi_3) := \Lambda(\xi) - \mu\Lambda(\xi - \xi_2 - \xi_3) - \nu\Lambda(\xi_2) - \sigma\Lambda(\xi_3). \tag{5.23}$$

Step 1. We estimate now $2^j \|\varphi_j(x) \cdot G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}(x)\|_{L^2}$ for fixed k_1, k_2, k_3, j. For this we write, after changes of variables

$$\widehat{G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}}(\xi) = \varphi_k(\xi) \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\tilde{\Phi}_{\mu\nu\sigma}(\xi,v-\xi_3,\xi_3)} m^0(\xi, v) m^1(v, \xi_3)\widehat{P_{k_1}f_1}(\xi - v)\widehat{P_{k_2}f_2}(v - \xi_3)\widehat{P_{k_3}f_3}(\xi_3)\, dv d\xi_3.$$

We decompose

$$G^{\mu\nu\sigma}_{j,k_1,k_2,k_3} = \sum_{n \in \mathbb{Z}} G^{\mu\nu\sigma}_{n,j,k_1,k_2,k_3}$$

where

$$\widehat{G^{\mu\nu\sigma}_{n,j,k_1,k_2,k_3}}(\xi) = \varphi_k(\xi) \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\tilde{\Phi}_{\mu\nu\sigma}(\xi,v-\xi_3,\xi_3)}$$
$$\times m^0(\xi, v) m^1(v, \xi_3)\varphi_n(v) \cdot \widehat{P_{k_1}f_1}(\xi - v)\widehat{P_{k_2}f_2}(v - \xi_3)\widehat{P_{k_3}f_3}(\xi_3)\, dv d\xi_3.$$

Let

$$Q(\xi, v, \xi_3) := \varphi_k(\xi)\varphi_{[k_1-1,k_1+1]}(\xi - v)\varphi_{[k_2-1,k_2+1]}(v - \xi_3)\varphi_{[k_3-1,k_3+1]}(\xi_3) \cdot m^0(\xi, v) m^1(v, \xi_3). \tag{5.24}$$

For simplicity of notation we drop the subscripts j, k_1, k_2, k_3 and rewrite

$$G^{\mu\nu\sigma}_{j,k_1,k_2,k_3} = \sum_{n \in \mathbb{Z}} T^{\mu\nu\sigma}_n[P_{k_1}f_1, P_{k_2}f_2, P_{k_3}f_3], \tag{5.25}$$

where, for any $g_1, g_2, g_3 \in L^2(\mathbb{R}^2)$,

$$\mathcal{F}\big[T^{\mu\nu\sigma}_n[g_1, g_2, g_3]\big](\xi) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\tilde{\Phi}_{\mu\nu\sigma}(\xi,v-\xi_3,\xi_3)}\varphi_n(v)Q(\xi, v, \xi_3)\widehat{g_1}(\xi - v)\widehat{g_2}(v - \xi_3)\widehat{g_3}(\xi_3)\, dv d\xi_3. \tag{5.26}$$

We will also need the representation of the operators $T^{\mu\nu\sigma}_n[g_1, g_2, g_3]$ in the physical space,

$$T^{\mu\nu\sigma}_n[g_1, g_2, g_3](x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} g_1(y_1)g_2(y_2)g_3(y_3) \cdot R^{\mu\nu\sigma}_n(x; y_1, y_2, y_3)\, dy_1 dy_2 dy_3, \tag{5.27}$$

where

$$R^{\mu\nu\sigma}_n(x; y_1, y_2, y_3) := c \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} e^{i(x-y_1)\cdot\xi} e^{i(y_1-y_2)\cdot v} e^{i(y_2-y_3)\cdot\xi_3}$$
$$\times e^{is[\Lambda(\xi) - \mu\Lambda(\xi-v) - \nu\Lambda(v-\xi_3) - \sigma\Lambda(\xi_3)]}\varphi_n(v)Q(\xi, v, \xi_3)\, d\xi dv d\xi_3. \tag{5.28}$$

Recall that $j \geq m + D$ for D sufficiently large, and

$$\sup_{\xi \in \mathbb{R}^2} |D^\alpha \Lambda(\xi)| \lesssim_{|\alpha|} 1.$$

We prove bounds on the kernel $R_n^{\mu\nu\sigma}$ using the formula (5.28) and integration by parts. More precisely, we use the general bound

$$\left|\int_{\mathbb{R}^d} e^{iKf} g\, dx\right| \lesssim_N (K\rho)^{-N}|\mathrm{supp}\, g|, \qquad 0 < \rho \leq 1/\rho \leq K, \qquad \text{provided that} \tag{5.29}$$

$$\|D^\alpha g\|_{L^\infty} \lesssim_{|\alpha|} \rho^{-|\alpha|}, |\alpha| \geq 0, \qquad \|D^\alpha f \cdot \mathbf{1}_{\mathrm{supp}\, g}\|_{L^\infty} \lesssim_{|\alpha|} \rho^{1-|\alpha|}, |\alpha| \geq 1, \qquad |\nabla f| \geq \mathbf{1}_{\mathrm{supp}\, g},$$

which follows easily by smooth localization to balls of radius $\approx \rho$ and integration by parts.

Recall that $|k|, |k_1|, |k_2|, |k_3|$ are all bounded by $99j/100$. Integrating by parts only in ξ, it follows that

$$\text{if } |x - y_1| \geq 2^{j-10} \text{ then } \left|R_n^{\mu\nu\sigma}(x; y_1, y_2, y_3)\right| \lesssim 2^{2n}(2^j + |x - y_1|)^{-100}. \tag{5.30}$$

Therefore, letting

$$f_{j,k_1}^1(y) := \varphi_{[j-4,j+4]}(y) \cdot P_{k_1} f_1(y),$$

and noticing that the operators $T_n^{\mu\nu\sigma}$ are nontrivial only if $n \leq \max(k_2, k_3) + 4$,

$$\sum_{n\in\mathbb{Z}} 2^j \left\|\varphi_j \cdot T_n^{\mu\nu\sigma}[P_{k_1} f_1 - f_{j,k_1}^1, P_{k_2} f_2, P_{k_3} f_3]\right\|_{L^2} \lesssim 2^{-50j}. \tag{5.31}$$

To estimate the contributions of the functions f_{j,k_1}^1 we analyze two cases: $n \leq -4j/5$ and $n \geq -4j/5$. If $n \leq -4j/5$ then we rewrite in the frequency space, using the representation (5.26),

$$\mathcal{F}\big[T_n^{\mu\nu\sigma}[f_{j,k_1}^1, P_{k_2} f_2, P_{k_3} f_3]\big](\xi) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\tilde\Phi_{\mu\nu\sigma}(\xi, v-\xi_3, \xi_3)}$$

$$\times \varphi_n(v) Q(\xi, v, \xi_3) \widehat{f_{j,k_1}^1}(\xi - v) \widehat{P_{k_2} f_2}(v - \xi_3) \widehat{P_{k_3} f_3}(\xi_3)\, dv d\xi_3.$$

Since $|Q(\xi, v, \xi_3)| \lesssim 1 + 2^{3\max(k_1, k_2, k_3)}$, we estimate

$$\left\|\mathcal{F}\big[T_n^{\mu\nu\sigma}[f_{j,k_1}^1, P_{k_2} f_2, P_{k_3} f_3]\big]\right\|_{L^2} \lesssim (1 + 2^{3\max(k_1, k_2, k_3)})2^{2n}\|\widehat{f_{j,k_1}^1}\|_{L^2}\|\widehat{P_{k_2} f_2}\|_{L^2}\|\widehat{P_{k_3} f_3}\|_{L^2}.$$

Recalling that $j \geq m$ and $\max(k, k_1, k_2, k_3) \leq j/10 + 4$, it follows that

$$\sum_{n\leq -4j/5} 2^j \left\|T_n^{\mu\nu\sigma}[f_{j,k_1}^1, P_{k_2} f_2, P_{k_3} f_3]\right\|_{L^2} \lesssim 2^{-11m/10} 2^{-7k_+} 2^{-2j/5}. \tag{5.32}$$

To estimate the contributions corresponding to $n \geq -4j/5$ we reexamine the formula (5.28) and notice that we can integrate by parts in either ξ or v or ξ_3. As a result, using (5.29), the bound (5.30) can be replaced with the stronger bound

$$\text{if } |x - y_1| + |x - y_2| + |x - y_3| \geq 2^{j-10} \text{ then } \left|R_n^{\mu\nu\sigma}(x; y_1, y_2, y_3)\right| \lesssim (|x - y_3| + |x - y_1| + |x - y_2|)^{-100}.$$

As a consequence, if

$$f_{j,k_2}^2(y) := \varphi_{[j-4,j+4]}(y) \cdot P_{k_2} f_2(y), \qquad f_{j,k_3}^3(y) := \varphi_{[j-4,j+4]}(y) \cdot P_{k_3} f_3(y),$$

it follows that

$$\sum_{n\geq -4j/5} 2^j\Big[\left\|\varphi_j \cdot T_n^{\mu\nu\sigma}[f_{j,k_1}^1, P_{k_2} f_2 - f_{j,k_2}^2, P_{k_3} f_3]\right\|_{L^2} + \left\|\varphi_j \cdot T_n^{\mu\nu\sigma}[f_{j,k_1}^1, f_{j,k_2}^2, P_{k_3} f_3 - f_{j,k_3}^3]\right\|_{L^2}\Big] \lesssim 2^{-50j}. \tag{5.33}$$

Finally, we estimate the contributions of $T_n^{\mu\nu\sigma}[f_{j,k_1}^1, f_{j,k_2}^2, f_{j,k_3}^3]$ in the Fourier space, using the formula (5.26). Since $|Q(\xi, v, \xi_3)| \lesssim 1 + 2^{3\max(k_1, k_2, k_3)}$, we estimate as before

$$\left\|\mathcal{F}\big[T_n^{\mu\nu\sigma}[f_{j,k_1}^1, f_{j,k_2}^2, f_{j,k_3}^3]\big]\right\|_{L^2} \lesssim (1 + 2^{3\max(k_1, k_2, k_3)})2^{2n}\|\widehat{f_{j,k_1}^1}\|_{L^2}\|\widehat{f_{j,k_2}^2}\|_{L^2}\|\widehat{f_{j,k_3}^3}\|_{L^2}.$$

Recalling that $j \geq m$ and that the operators $T_n^{\mu\nu\sigma}$ are nontrivial only if $n \leq \max(k_2, k_3) + 4$,

$$\sum_{-4j/5 \leq n} 2^j \left\|T_n^{\mu\nu\sigma}[f_{j,k_1}^1, f_{j,k_2}^2, f_{j,k_3}^3]\right\|_{L^2} \lesssim 2^{-11m/10} 2^{-5k_+} 2^{-3j/5}. \tag{5.34}$$

It follows from the formula (5.25) and the bounds (5.31), (5.32), (5.33), and (5.34) that

$$2^j \big\| \varphi_j \cdot G^{\mu\nu\sigma}_{j,k_1,k_2,k_3} \big\|_{L^2} \lesssim 2^{-11m/10} 2^{-7k_+} 2^{-2j/5}$$

and the desired bound (5.19) follows by summing over j, k_1, k_2, k_3 in the appropriate ranges.

Step 2. It remains to prove (5.20). We may assume that $M(k,m) < M'(k,m)$, i.e.

$$m \geq \max(-21k/20, 10k), \qquad M'(k,m) = D + m. \tag{5.35}$$

We define the functions $G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}$ as in (5.22) and we have to consider the cases $(\mu,\nu,\sigma) = (+,+,-)$ and $(\mu,\nu,\sigma) = (-,+,+)$. We use first L^2 estimates to further restrict the ranges of the parameters: if $j \in [1, m+D]$ and $k_1, k_2, k_3 \in [-2j/3, j/10]$ then we use Lemma 5.5 with $r = 2$, $l = -100j$ and appropriate choices of p_1, p_2, p_3 to derive the bounds

$$\|G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}\|_{L^2} \lesssim j 2^{-2m} 2^{\widetilde{b}/2} 2^{-\widetilde{c}_+} \qquad \text{if } k \leq 0,$$

$$\|G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}\|_{L^2} \lesssim j 2^{-2m} \min(2^{-\widetilde{b}/5} 2^{-15\widetilde{c}_+}, 2^{3\widetilde{b}/5}) \qquad \text{if } k \geq 0,$$

where $\widetilde{b} = \min(k_1, k_2, k_3)$ and $\widetilde{c} = \max(k_1, k_2, k_3)$. Therefore

$$\|G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}\|_{L^2} \lesssim j 2^{-2m} 2^{-10k_+} \min(2^{\widetilde{b}/15}, 2^{-\widetilde{c}_+/2}),$$

so the contribution of the L^2 norms of the functions $G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}$ when $j \leq 49m/50 + 2D$ or when $\widetilde{b} \leq -m/50$ or when $\widetilde{c} \geq m/50$ are bounded by $C 2^{-10k_+} 2^{-1001m/1000}$ as desired. It remains to prove that

$$2^j \|\varphi_j(x) \cdot G^{\mu\nu\sigma}_{j,k_1,k_2,k_3}(x)\|_{L^2_x} \lesssim 2^{-101m/100} 2^{-10k_+}, \tag{5.36}$$

where $(\mu,\nu,\sigma) \in \{(+,+,-),(-,+,+)\}$, for parameters k, m, j, k_1, k_2, k_3 fixed and satisfying

$$k_1, k_2, k_3 \in [-m/50, m/400], \qquad j \in [49m/50 + 2D, m + D], \qquad m \geq \max(-21k/20, 10k). \tag{5.37}$$

We use again the identities (5.24)–(5.28). Using (5.24), (5.26), and the bound $|Q(\xi, v, \xi_3)| \lesssim 1 + 2^{3\max(k_1,k_2,k_3)}$, we estimate as before

$$\big\| T^{\mu\nu\sigma}_n[g_1, g_2, g_3] \big\|_{L^2} \lesssim (1 + 2^{3\max(k_1,k_2,k_3)}) 2^{2n} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2}, \tag{5.38}$$

for any $g_1, g_2, g_3 \in L^2(\mathbb{R}^2)$ and any $n \in \mathbb{Z}$. For (5.36) it suffices to prove that

$$\sum_{n \in \mathbb{Z}} 2^j \big\| \varphi_j \cdot T^{\mu\nu\sigma}_n[P_{k_1}f_1, P_{k_2}f_2, P_{k_3}f_3] \big\|_{L^2} \lesssim 2^{-11m/10}, \tag{5.39}$$

where k, m, j, k_1, k_2, k_3 satisfy (5.37), $(\mu,\nu,\sigma) \in \{(+,+,-),(-,+,+)\}$, and $\|f_\alpha\|_{Y^{N_0}} \leq 1$, $\alpha = 1, 2, 3$. As before, let

$$\widetilde{c} = \max(k_1, k_2, k_3), \qquad \widetilde{a} = \mathrm{med}(k_1, k_2, k_3), \qquad \widetilde{b} = \min(k_1, k_2, k_3).$$

Step 3. We estimate first the contribution to the left-hand side of (5.39) coming from $n \leq -4j/5$. Notice that this contribution is trivial unless $|k - k_1| \leq 4$ and $|k_2 - k_3| \leq 4$. For any $\widetilde{j} \geq 1$ let

$$f^\alpha_{\leq \widetilde{j}, k_\alpha}(y) := \varphi_{(-\infty, \widetilde{j}]}(y) \cdot P_{k_\alpha} f_\alpha(y), \qquad \alpha = 1, 2, 3.$$

Since $\|f_\alpha\|_{Y^{N_0}} \leq 1$, $\alpha \in \{1, 2, 3\}$, for any $\widetilde{j} \geq 1$ we have

$$\|P_{k_\alpha} f_\alpha - f^\alpha_{\leq \widetilde{j}, k_\alpha}\|_{L^2} \lesssim 2^{-\widetilde{j}} (2^{k_\alpha/10} + 2^{10k_\alpha})^{-1},$$

$$\|P_{k_\alpha} f_\alpha - f^\alpha_{\leq \widetilde{j}, k_\alpha}\|_{L^2} + \|f^\alpha_{\leq \widetilde{j}, k_\alpha}\|_{L^2} \lesssim \min(2^{9k_\alpha/10}, 2^{-N_0 k_\alpha}), \tag{5.40}$$

$$\|P_{k_\alpha} f_\alpha - f^\alpha_{\leq \widetilde{j}, k_\alpha}\|_{L^1} + \|f^\alpha_{\leq \widetilde{j}, k_\alpha}\|_{L^1} \lesssim \min(2^{-k_\alpha/10}, 2^{-10k_\alpha}).$$

Using (5.37), (5.38), and (5.40) it follows that

$$\big\| T^{\mu\nu\sigma}_n[P_{k_1}f_1, P_{k_2}f_2, P_{k_3}f_3] - T^{\mu\nu\sigma}_n[f^1_{\leq 4j/5, k_1}, f^2_{\leq 4j/5, k_2}, f^3_{\leq 4j/5, k_3}] \big\|_{L^2} \lesssim 2^{m/50} 2^{2n} 2^{-4j/5}.$$

therefore, using (5.37),

$$\sum_{n \le -4j/5} 2^j \big\| T_n^{\mu\nu\sigma}[P_{k_1}f_1, P_{k_2}f_2, P_{k_3}f_3] - T_n^{\mu\nu\sigma}[f^1_{\le 4j/5, k_1}, f^2_{\le 4j/5, k_2}, f^3_{\le 4j/5, k_3}] \big\|_{L^2} \lesssim 2^{-11m/10}. \tag{5.41}$$

To estimate the contributions of $T_n^{\mu\nu\sigma}[f^1_{\le 4j/5, k_1}, f^2_{\le 4j/5, k_2}, f^3_{\le 4j/5, k_3}]$ we use the physical space formula (5.27) and Lemma 5.1. Let $\Psi^{\mu\nu\sigma}(\xi, v, \xi_3, x, y_1, y_2, y_3)$ denote the phase in the integral defining the kernels $R^{\mu\nu\sigma}$, i.e.

$$\begin{aligned}
\Psi^{\mu\nu\sigma}(\xi, v, \xi_3, x, y_1, y_2, y_3) := & (x - y_1) \cdot \xi + (y_1 - y_2) \cdot v + (y_2 - y_3) \cdot \xi_3 \\
& + s[\Lambda(\xi) - \mu\Lambda(\xi - v) - \nu\Lambda(v - \xi_3) - \sigma\Lambda(\xi_3)].
\end{aligned} \tag{5.42}$$

Recalling (5.37), it follows from (5.3) that if $|v| \in [2^{n-1}, 2^{n+1}]$ then

$$\Big|\nabla_\xi\big[s(\Lambda(\xi) - \Lambda(\xi - v) - \Lambda(v - \xi_3) + \Lambda(\xi_3))\big]\Big| = s\big|\nabla\Lambda(\xi) - \nabla\Lambda(\xi - v)\big| \lesssim 2^{m+n},$$

$$\Big|\nabla_{\xi_3}\big[s(\Lambda(\xi) + \Lambda(\xi - v) - \Lambda(v - \xi_3) - \Lambda(\xi_3))\big]\Big| = s\big|\nabla\Lambda(v - \xi_3) - \nabla\Lambda(\xi_3)\big| \gtrsim 2^{m+k_3 - 5c_+} \gtrsim 2^{9m/10}.$$

Using (5.29) and integrating by parts only in ξ in the formula (5.28),

$$\text{if } |x - y_1| \ge 2^{j-10} \text{ then } |R_n^{++-}(x; y_1, y_2, y_3)| \lesssim 2^{-100j}.$$

Similarly, using (5.29) and integrating by parts only in ξ_3 in the formula (5.28),

$$\text{if } |y_2 - y_3| \le 2^{4j/5 + 10} \text{ then } |R_n^{-++}(x; y_1, y_2, y_3)| \lesssim 2^{-100j},$$

provided that the constant D is sufficiently large. Thus

$$\sum_{n \le -4j/5} 2^j \big\| \varphi_j \cdot T_n^{\mu\nu\sigma}[f^1_{\le 4j/5, k_1}, f^2_{\le 4j/5, k_2}, f^3_{\le 4j/5, k_3}] \big\|_{L^2} \lesssim 2^{-50j},$$

for $(\mu, \nu, \sigma) \in \{(+, +, -), (-, +, +)\}$. Using also (5.41) it follows that

$$\sum_{n \le -4j/5} 2^j \big\| \varphi_j \cdot T_n^{\mu\nu\sigma}[P_{k_1}f_1, P_{k_2}f_2, P_{k_3}f_3] \big\|_{L^2} \lesssim 2^{-11m/10}, \tag{5.43}$$

as desired.

Step 4. We consider now the contribution to the left-hand side of (5.39) coming from $n \ge -4j/5$. Using Lemma (5.5), more precisely the bound (5.18), together with the bounds (5.40) and the dispersive estimate (4.19), it follows that

$$\begin{aligned}
\big\| T_n^{\mu\nu\sigma}[P_{k_1}f_1 - f^1_{\le j/2, k_1}, P_{k_2}f_2, P_{k_3}f_3] \big\|_{L^2} &\lesssim 2^{-j/2} 2^{-2m} 2^{m/4}, \\
\big\| T_n^{\mu\nu\sigma}[f^1_{\le j/2, k_1}, P_{k_2}f_2 - f^2_{\le j/2, k_2}, P_{k_3}f_3] \big\|_{L^2} &\lesssim 2^{-j/2} 2^{-2m} 2^{m/4}, \\
\big\| T_n^{\mu\nu\sigma}[f^1_{\le j/2, k_1}, f^2_{\le j/2, k_2}, P_{k_3}f_3 - f^3_{\le j, k_3}] \big\|_{L^2} &\lesssim 2^{-j/2} 2^{-2m} 2^{m/4}.
\end{aligned} \tag{5.44}$$

To estimate the contribution of the functions $f^1_{\le j/2, k_1}, f^2_{\le j/2, k_2}, f^3_{\le j/2, k_3}$ we recall the definitions (5.42) and notice that, in the support of the function $Q(\xi, v, \xi_3)\varphi_n(v)$,

$$\begin{aligned}
\big|\nabla_{\xi, \xi_3, v} \Psi^{++-}(\xi, v, \xi_3, x, y_1, y_2, y_3)\big| &\gtrsim 2^{9j/10}, \\
\big|\nabla_{\xi, \xi_3, v} \Psi^{-++}(\xi, v, \xi_3, x, y_1, y_2, y_3)\big| &\gtrsim 2^{9j/10},
\end{aligned} \tag{5.45}$$

provided that (5.37) holds, $|x| \approx 2^j$, and $|y_1| + |y_2| + |y_3| \lesssim 2^{j/2}$. Indeed, using (5.3), we notice that

$$\begin{aligned}
\big|\nabla_\xi \Psi^{++-}(\xi, v, \xi_3, x, y_1, y_2, y_3)\big| &\gtrsim 2^j & \text{unless } 2^j \lesssim s|v|, \\
\big|\nabla_{\xi_3} \Psi^{++-}(\xi, v, \xi_3, x, y_1, y_2, y_3)\big| &\gtrsim 2^{m+n} 2^{-10c_+} & \text{if } 2^j \lesssim s|v|.
\end{aligned}$$

The bound in the first line of (5.45) follows. In addition, we have

$$\text{if } \left|\nabla_\xi \Psi^{-++}(\xi, v, \xi_3, x, y_1, y_2, y_3)\right| \leq 2^{9j/10} \qquad \text{then } 2^j \lesssim s|v - 2\xi|,$$

$$\text{if } \left|\nabla_{\xi_3} \Psi^{-++}(\xi, v, \xi_3, x, y_1, y_2, y_3)\right| \leq 2^{9j/10} \qquad \text{then } s|v - 2\xi_3| \lesssim 2^{9j/10} 2^{10c_+},$$

$$\text{if } \left|\nabla_v \Psi^{-++}(\xi, v, \xi_3, x, y_1, y_2, y_3)\right| \leq 2^{9j/10} \qquad \text{then } s|\xi - \xi_3| \lesssim 2^{9j/10} 2^{10c_+}.$$

The inequality in the second line of (5.45) follows.

Given (5.45), we use (5.29) as before to conclude that if $n \geq -4j/5$ then

$$2^j \big\|\varphi_j \cdot T_n^{\mu\nu\sigma}[f_{\leq j/2,k_1}^1, f_{\leq j/2,k_2}^2, f_{\leq j/2,k_3}^3]\big\|_{L^2} \lesssim 2^{-50j}. \tag{5.46}$$

The desired bound (5.39) follows from (5.43), (5.44), and (5.46). This completes the proof of the lemma.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.7. *Assume that $k \in \mathbb{Z}$, $k_1, k_2, k_3 \in [-K, K/10] \cap \mathbb{Z}$ for some $K \geq 1$, $\widetilde{a} = \mathrm{med}(k_1, k_2, k_3)$, $\widetilde{c} = \max(k_1, k_2, k_3)$, $\rho \in [2^{-c_+}, 1]$, and $p_1, p_2, p_3 \in \{2, \infty\}$, $1/p_1 + 1/p_2 + 1/p_3 = 1/2$. Then*

$$\Big\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{P_{k,k_1,k_2,k_3}^{\mu\nu\sigma}(\xi, \eta, \chi)}{\Psi(\xi, \eta, \chi)} \widehat{f}_1(\xi - \eta)\widehat{f}_2(\eta - \chi)\widehat{f}_3(\chi) \, d\eta d\chi \Big\|_{L_\xi^2} \tag{5.47}$$
$$\lesssim \rho^{-6} 2^{6(\widetilde{a}_+ + \widetilde{c}_+)}\big[2^{-10K}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2} + K\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\|f_3\|_{L^{p_3}}\big],$$

where $P_{k,k_1,k_2,k_3}^{\mu\nu\sigma}$ is as in (4.36), and the smooth function $\Psi : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ satisfies the inequalities

$$\inf_{\xi,\eta,\chi \in \mathbb{R}^2} |\Psi(\xi, \eta, \chi)| \geq \rho, \qquad \sup_{|\alpha| \in [1,20]} \sup_{\xi,\eta,\chi \in \mathbb{R}^2} |D_{\xi,\eta,\chi}^\alpha \Psi(\xi, \eta, \chi)| \lesssim 1. \tag{5.48}$$

Proof of Lemma 5.7. We decompose the integral in the left-hand side of (5.47) as in the proof of Lemma 5.6, see (5.23). More precisely, with Q defined as in (5.24), it suffices to prove that

$$\Big\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{Q(\xi, v, \xi_3)}{\widetilde{\Psi}(\xi, v, \xi_3)} \widehat{f}_1(\xi - v)\widehat{f}_2(v - \xi_3)\widehat{f}_3(\xi_3) \, dv d\xi_3 \Big\|_{L_\xi^2} \tag{5.49}$$
$$\lesssim \rho^{-6} 2^{6(\widetilde{a}_+ + \widetilde{c}_+)}\big[2^{-10K}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2} + K\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\|f_3\|_{L^{p_3}}\big],$$

where $\widetilde{\Psi}$ satisfies the bounds

$$\inf_{\xi,v,\xi_3 \in \mathbb{R}^2} |\tilde{\Psi}(\xi, v, \xi_3)| \gtrsim \rho, \qquad \sup_{|\alpha| \in [1,20]} \sup_{\xi,v,\xi_3 \in \mathbb{R}^2} |D_{\xi,v,\xi_3}^\alpha \tilde{\Psi}(\xi, v, \xi_3)| \lesssim 1. \tag{5.50}$$

As in the proof of Lemma 5.6 we insert cutoff functions in v. Estimating the L^2 norms in the Fourier space and recalling the bound $|Q(\xi, v, \xi_3)| \lesssim 2^{3c_+}$, it follows that

$$\sum_{n \leq -10K} \Big\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{Q(\xi, v, \xi_3)}{\widetilde{\Psi}(\xi, v, \xi_3)} \varphi_n(v)\widehat{f}_1(\xi - v)\widehat{f}_2(v - \xi_3)\widehat{f}_3(\xi_3) \, dv d\xi_3 \Big\|_{L_\xi^2} \lesssim \sum_{n \leq -10K} \rho^{-1} 2^{3c_+} 2^{2n} \tag{5.51}$$
$$\lesssim 2^{3c_+} 2^{-10K}.$$

To estimate the remaining sum we pass to the physical space and estimate

$$\Big\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{Q(\xi, v, \xi_3)}{\widetilde{\Psi}(\xi, v, \xi_3)} \varphi_n(v)\widehat{f}_1(\xi - v)\widehat{f}_2(v - \xi_3)\widehat{f}_3(\xi_3) \, dv d\xi_3 \Big\|_{L_\xi^2} \tag{5.52}$$
$$\lesssim \sup_{\|g\|_{L^2} \leq 1} \Big| \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} g(x) f_1(y_1) f_2(y_2) f_3(y_3) R_n(x; y_1, y_2, y_3) \, dx dy_1 dy_2 dy_3 \Big|,$$

where

$$R_n(x; y_1, y_2, y_3) := \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} e^{i(x-y_1)\cdot\xi} e^{i(y_1-y_2)\cdot v} e^{i(y_2-y_3)\cdot\xi_3} \varphi_n(v) \frac{Q(\xi, v, \xi_3)}{\widetilde{\Psi}(\xi, v, \xi_3)} \, d\xi dv d\xi_3.$$

By integration by parts it is easy to see that

$$|R_n(x; y_1, y_2, y_3)| \lesssim |K_1(x - y_1)| \cdot |K_2(y_1 - y_2)| \cdot |K_3(y_2 - y_3)|$$

for some functions $K_1, K_2, K_3 : \mathbb{R}^2 \to [0, \infty)$ satisfying

$$\|K_1\|_{L^1} \cdot \|K_2\|_{L^1} \cdot \|K_3\|_{L^1} \lesssim \rho^{-6} 2^{6(\widetilde{a}_+ + \widetilde{c}_+)}.$$

Using (5.52), it follows that

$$\Big\| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{Q(\xi, v, \xi_3)}{\widetilde{\Psi}(\xi, v, \xi_3)} \varphi_n(v) \widehat{f_1}(\xi - v) \widehat{f_2}(v - \xi_3) \widehat{f_3}(\xi_3) \, dv d\xi_3 \Big\|_{L^2_\xi} \lesssim \rho^{-6} 2^{6(\widetilde{a}_+ + \widetilde{c}_+)} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}}. \tag{5.53}$$

The desired bound (5.49) follows from (5.51) and (5.53). $\square$

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