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A Power-Law Upper Bound on the Correlations in the $2D$ Random Field Ising Model

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To the memory of Joe Imry (1939–2018);
insightful lover of physics and music.

Abstract: As first asserted by Y. Imry and S-K Ma, the famed discontinuity of the magnetization as function of the magnetic field in the two dimensional Ising model is eliminated, for all temperatures, through the addition of quenched random magnetic field of uniform variance, even if that is small. This statement is quantified here by a power-law upper bound on the decay rate of the effect of boundary conditions on the magnetization in finite systems, as function of the distance to the boundary. Unlike exponential decay which is only proven for strong disorder or high temperature, the power-law upper bound is established here for all field strengths and at all temperatures, including zero, for the case of independent Gaussian random field. Our analysis proceeds through a streamlined and quantified version of the Aizenman–Wehr proof of the Imry–Ma rounding effect.

Contents

1. Introduction

1.1. The Imry–Ma phenomenon for the 2D RFIM and its quantification. A first-order phase transition is one associated with phase coexistence, in which an extensive system admits at least two thermal equilibrium states which differ in their bulk densities of an extensive quantity. The thermodynamic manifestation of such a transition is the discontinuity in the derivative of the extensive system's free energy with respect to one of the coupling constants which affect the system's energy. At zero temperature, this would correspond to the existence of two infinite-volume ground states which differ in the bulk average of a local quantity.

In what is known as the Imry–Ma [17] phenomenon, in two-dimensional systems any first-order transition is *rounded off* upon the introduction of arbitrarily weak static, or *quenched*, disorder in the parameter which is conjugate to the corresponding extensive quantity.

Our goal here is to present quantitative estimates of this effect, strengthening the previously proven infinite-volume statement [5] by: i) upper bounds on the dependence of the local density on a finite-volume's boundary conditions, and ii) related bounds on the correlations among local quenched expectations, which are asymptotically independent functions of the quenched disorder.

The present discussion takes place in the context of the random-field Ising model (RFIM). In this case the original discontinuity is in the bulk magnetization, i.e. volume average of the local spin σ_u, and it occurs at zero magnetic field ($h = 0$). Since h is the conjugate parameter to the magnetization, the relevant disorder for the Imry–Ma phenomenon is given by site-independent random field ($\varepsilon\eta_u$). More explicitly, the system consists of Ising spin variables $\{\sigma_u\}_{u\in\mathbb{Z}^d}$, associated with the vertices of the d-dimensional lattice $\mathbb{Z}^d$, with the Hamiltonian

$$H(\sigma) := -\sum_{\{u,v\}\subseteq\mathbb{Z}^d} J_{u,v}\,\sigma_u\sigma_v - \sum_{v\in\mathbb{Z}^d}(h+\varepsilon\,\eta_v)\sigma_v, \tag{1.1}$$

and ferromagnetic translation-invariant coupling constants $\mathcal{J} = \{J_{u,v}\}$ ($J_{u,v} = J_{v,u} = J_{u-v,\mathbf{0}} \geq 0$).

For convenience we focus on the case that the (η_v) are independent standard Gaussians. However it is expected, and for many of the key results proven true, that the model's essential features are similar among all independent, identically distributed (η_v) whose common distribution has a continuous component.

The main result presented here is the proof that in the two-dimensional case at any temperature $T \geq 0$, the effect on the local quenched magnetization of the boundary conditions at distance L away decays by at least a power law ($1/L^\gamma$). This may be viewed as a quantitative extension of the uniqueness of the Gibbs state theorem [4,5]. It also implies a similar bound on correlations within the infinite-volume Gibbs state. A weaker upper bound, at the rate $1/\sqrt{\log\log L}$, was recently presented in [10], derived there by other means.

More explicitly: as the first question it is natural to ask whether the addition of random field terms in the Hamiltonian (1.1) changes the Ising model's phase diagram, whose salient feature is the phase transition which for $d > 1$ occurs at $h = 0$ and low enough temperatures, $T < T_c$. The initial prediction of Imry and Ma [17] was challenged by other arguments, however it was eventually proven to be true: For $d \geq 3$ the RFIM continues to have a first-order phase transition at $h = 0$ [8,15], whereas in two dimensions at any $\varepsilon \neq 0$ the model's bulk mean magnetization has a unique value for each h, and by implication it varies continuously in h at any temperature, including $T = 0$ [4,5]. Through the FKG property [13] of the RFIM one may also deduce that in two dimensions, at any temperature $T \geq 0$ and for almost every realization of the random field $\eta = (\eta_v)_{v\in\mathbb{Z}^2}$, the system has a unique Gibbs state. For $T = 0$ this translates into uniqueness of the infinite-volume ground state configuration, i.e. configuration(s) for which no flip of a finite number of spins results in lower energy. Additional background and pedagogical review of the RFIM may be found in [7, Chapter 7].

Seeking quantitative refinements of the above statement, we consider here the dependence of the *finite-volume* quenched magnetization $\langle\sigma_v\rangle^{\Lambda,\tau}$ on the boundary conditions τ placed on the exterior of a domain Λ. We denote by $\langle-\rangle^{\Lambda,\tau}$ the finite volume "τ state" quenched thermal average and by $\mathbb{E}$ the further average over the random field (both defined explicitly in Sect. 2). Due to the model's FKG monotonicity property the finite volume Gibbs states at arbitrary boundary conditions are bracketed between the + and the − state. Hence the relevant order parameter is

$$m(L) \equiv m(L; T, \mathcal{J}, h, \epsilon) := \frac{1}{2}\left[\mathbb{E}[\langle\sigma_{\mathbf{0}}\rangle^{\Lambda(L),+}] - \mathbb{E}[\langle\sigma_{\mathbf{0}}\rangle^{\Lambda(L),-}]\right] \tag{1.2}$$

where

$$\Lambda_u(L) := \{v \in \mathbb{Z}^2 : d(u,v) \leq L\}, \quad \Lambda(L) = \Lambda_{\mathbf{0}}(L), \tag{1.3}$$

with $d(u,v)$ the graph distance on $\mathbb{Z}^2$ and $\mathbf{0} := (0,0)$.

Theorem 1.1. *In the two-dimensional random-field Ising model with a finite-range interaction $\mathcal{J}$ and independent standard Gaussian random field (η_v), for any temperature $T \geq 0$, uniform field $h \in \mathbb{R}$, and field intensity $\varepsilon > 0$ there exist $C = C(\mathcal{J}, T, \varepsilon) > 0$ and $\gamma = \gamma(\mathcal{J}, T, \varepsilon) > 0$ such that for all large enough L*

$$m(L; T, \mathcal{J}, h, \epsilon) \leq \frac{C}{L^\gamma}. \tag{1.4}$$

For the nearest-neighbor interaction

$$J_{u,v} = J\,\delta_{d(u,v),1} \tag{1.5}$$

the proof yields

$$\gamma := 2^{-10}\cdot\chi\left(\frac{50J}{\varepsilon}\right) \tag{1.6}$$

in terms of the tail of the Gaussian distribution function:

$$\chi(t) := 2\int_t^\infty \phi(s)\,ds, \qquad \phi(s) = \frac{1}{\sqrt{2\pi}}e^{-s^2/2}. \tag{1.7}$$

The phenomenon and the arguments discussed in the proof are somewhat simpler to present in the limit of zero temperature, where the quenched random field is the only source of disorder. We therefore start by proving Theorem 1.1 for this case, emphasizing the setting of nearest-neighbor interaction. Then, in Sect. 4 we present the changes by which the argument extends to $T > 0$. With minor adjustments of the constants, discussed in Sect. 5, the natural extension of the statement to translation-invariant pair interactions of finite range is also valid.

1.2. Direct implications. By the FKG inequality (see Sect. 2.2), the difference whose mean is the order parameter is non-negative for any η (and all $T \geq 0$, $h \in \mathbb{R}$),

$$\langle \sigma_{\mathbf{0}} \rangle^{\Lambda(L),+} - \langle \sigma_{\mathbf{0}} \rangle^{\Lambda(L),-} \ \geq \ 0. \tag{1.8}$$

Hence the bound on the mean (1.4) implies (through Markov's inequality) that this quantifier of sensitivity to boundary condition is similarly small with high probability.

The order parameter $m(L)$ controls also the covariances of: *i)* the spins under the infinite-volume quenched Gibbs states $\langle - \rangle \equiv \langle - \rangle_{T,\mathcal{J},h,\varepsilon\eta}$, and *ii)* of the infinite-volume quenched Gibbs state magnetization $\langle \sigma_u \rangle$ under the random field fluctuations, over which the average is denoted by $\mathbb{E}(-)$. To express these statements we denote

$$\begin{aligned} \langle \sigma_u ; \sigma_v \rangle &:= \langle \sigma_u \sigma_v \rangle - \langle \sigma_u \rangle \, \langle \sigma_v \rangle \\ \mathbb{E}(\langle \sigma_u \rangle ; \langle \sigma_v \rangle) &:= \mathbb{E}\Big(\langle \sigma_u \rangle \, \langle \sigma_v \rangle \Big) - \mathbb{E}(\langle \sigma_u \rangle) \; \mathbb{E}(\langle \sigma_v \rangle) \\ &= \mathbb{E}\Big(\big[\langle \sigma_u \rangle - \mathbb{E}(\langle \sigma_0 \rangle) \big] \big[\langle \sigma_v \rangle - \mathbb{E}(\langle \sigma_0 \rangle) \big] \Big). \end{aligned} \tag{1.9}$$

Each of these truncated correlations is non-negative: in the former case due to the FKG property of the RFIM, and in the latter due to monotonicity of $\langle \sigma_u \rangle$ in η and the Harris/FKG inequality for product measures.

As we prove below (Lemma 6.1), for pairs $\{u, v\} \in \mathbb{Z}^2$, if $d(u, v) > \ell$ then

$$\mathbb{E}(\langle \sigma_u ; \sigma_v \rangle) \leq 2\, m(\ell; T, \mathcal{J}, h, \epsilon) \tag{1.10}$$

while if $d(u, v) \geq 2\ell + R(\mathcal{J})$, with $R(\mathcal{J}) := \max\{d(u, v) \ : \ J_{u,v} \neq 0\}$ (the interaction's range) then

$$\mathbb{E}(\langle \sigma_u \rangle ; \langle \sigma_v \rangle) \ \leq \ 4\, m(\ell; T, \mathcal{J}, h, \epsilon). \tag{1.11}$$

The comment made above in relation to (1.2), applies also here: The non-negativity of $\langle \sigma_u ; \sigma_v \rangle$, together with (1.10), implies that with high probability it does not exceed $m(\ell; T, \mathcal{J}, h, \epsilon)$ by a large multiple. The proof of (1.10) and (1.11) does not require the analysis which is developed in this paper. It is therefore postponed to Sect. 6.

For (1.11) of particular interest is $h = 0$ and $T = 0$. In this case $\langle \sigma_u \rangle$ coincides with the infinite-volume ground state configuration $\widehat{\sigma}_u(\eta)$ which, as is already known, is unique for almost all η. By the spin-flip symmetry $\mathbb{E}(\widehat{\sigma}_u) = 0$, and the bound (1.11) translates into:

$$0 \leq \mathbb{E}(\widehat{\sigma}_u \widehat{\sigma}_v) \ \leq \ 4\, m(\ell; 0, \mathcal{J}, 0, \epsilon). \tag{1.12}$$

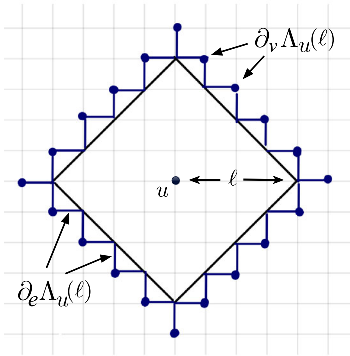

Fig. 1. A subset of $\mathbb{Z}^2$ of the form of $\Lambda_u(\ell)$ and its two boundary sets: the edge boundary $\partial_{\mathrm{e}}\Lambda_u$ and the vertex (external) boundary $\partial_{\mathrm{v}}\Lambda_u$, both drawn for the case of the nearest-neighbor interaction

1.3. A remaining question. As we shall discuss in greater detail in Appendix A, at high enough disorder, i.e. large enough ε, the order parameter $m(L)$ decays exponentially fast in L. Our results do not resolve the question of whether the two-dimensional model exhibits a disorder-driven phase transition, at which the decay rate changes from exponential to a power law, as the disorder is lowered (possibly even at $T = 0$). This remains among the interesting open problems concerning the Imry–Ma phenomenon in two dimensions, on which more is said in the open problem Sect. 7.

2. Gibbs Equilibrium States

2.1. The Gibbs measure. Discussing the RFIM on $\mathbb{Z}^2$ we shall use the following terminology. Two vertices are deemed adjacent, $u \sim v$, if they differ by a unit vector. The graph distance on $\mathbb{Z}^2$ is denoted $d(u, v)$ and the graph ball of radius L around u is denoted $\Lambda_u(L)$, with $\Lambda(L)$ standing for $\Lambda_{\mathbf{0}}(L)$, as before Theorem 1.1 (Fig. 1). The *edge boundary* of a subset $\Lambda \subset \mathbb{Z}^2$ (which is used in decoupling estimates) is denoted

$$\partial_{\mathrm{e}}\Lambda := \{(u, v) : \ u \in \Lambda, \ v \in \mathbb{Z}^2 \setminus \Lambda, \ J_{u,v} \neq 0\} \tag{2.1}$$

and the *external boundary* (which is used when imposing boundary conditions) is

$$\partial_{\mathrm{v}}\Lambda := \{v \in \mathbb{Z}^2 \setminus \Lambda : \ \exists u \in \Lambda, J_{u,v} \neq 0\}. \tag{2.2}$$

The RFIM Gibbs equilibrium state in the finite subset $\Lambda \subset \mathbb{Z}^2$, at specified values of the parameters $(T, \mathcal{J}, h, \varepsilon)$, the random field η, and a configuration of boundary spin values $\tau : \partial_{\mathrm{v}}\Lambda \to \{-1, 1\}$, is the probability measure over $\Omega_\Lambda = \{-1, 1\}^\Lambda$ given by

$$\mathbb{P}^{\Lambda,\tau}(\sigma) := \frac{1}{Z^{\Lambda,\tau}} e^{-\frac{1}{T} H^{\Lambda,\tau}(\sigma)}, \tag{2.3}$$

where

$$H^{\Lambda,\tau}(\sigma) := -\sum_{u,v\in\Lambda} J_{u,v}\sigma_u\sigma_v - \sum_{(u,v)\in\partial_{\mathrm{e}}\Lambda} J_{u,v}\sigma_u\tau_v - \sum_{v\in\Lambda}(h + \varepsilon\eta_v)\sigma_v \tag{2.4}$$

and $Z^{\Lambda,\tau}$ is the corresponding normalizing factor (the "partition function"). The associated expectation operator is denoted $\langle - \rangle^{\Lambda,\tau}$. The notation $\mathbb{P}^{\Lambda,\pm}$ or $\langle - \rangle^{\Lambda,\pm}$ indicates that τ is the corresponding uniform configuration $\tau \equiv +1$ or $\tau \equiv -1$. The notation $\mathbb{P}$ and $\mathbb{E}$ is used for the probability and expectation operators, respectively, of the further average over the random field.

At $T = 0$, the measure $\mathbb{P}^{\Lambda,\tau}$ is supported on the almost-surely unique configuration which minimizes $H^{\Lambda,\tau}$. These ground-state configurations, which depend on $\varepsilon\eta$ and $(\mathcal{J}, h)$, are denoted here by $\sigma^{\Lambda,\tau} = (\sigma_v^{\Lambda,\tau})_{v\in\Lambda}$ (The dependence on η is not displayed, but it is in the focus of the discussion.)

2.2. Monotonicity properties. In our discussion we shall take advantage of the known monotonicity property of the ferromagnetic Ising model, which is that its Gibbs equilibrium states as well as the ground-state configurations, at given $\mathcal{J}$, h and ε, are increasing functions of the local field variables η and of the boundary spin configuration τ. The statement is a known consequence of the FKG inequality [13]. The $T = 0$ version can also be seen through a more direct argument.

Thus, for any region Λ and pairs of boundary conditions $\tau^-, \tau^+ : \partial_{\mathrm{v}}\Lambda \to \{-1, 1\}$:

$$\tau^- \leq \tau^+ \quad \Longrightarrow \quad \mathbb{P}^{\Lambda,\tau^+} \text{ stochastically dominates } \mathbb{P}^{\Lambda,\tau^-} \tag{2.5}$$

where an inequality between configurations is to be interpreted as holding pointwise. (Unlike $\mathbb{R}$, the configuration space is only partially ordered, but that suffices for our purpose.) The following special case is noted for later reference

$$\tau^- \leq \tau^+ \quad \Longrightarrow \quad \langle\sigma_v\rangle^{\Lambda,\tau^-} \leq \langle\sigma_v\rangle^{\Lambda,\tau^+} \text{ for each } v \in \Lambda. \tag{2.6}$$

By related reasoning, the Gibbs state at + (or $-$) boundary conditions is stochastically decreasing (and correspondingly increasing) in its dependence on Λ. In particular, for each $v \in \Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^2$:

$$\langle\sigma_v\rangle^{\Lambda_1,+} \geq \langle\sigma_v\rangle^{\Lambda_2,+} \quad \text{and} \quad \langle\sigma_v\rangle^{\Lambda_1,-} \leq \langle\sigma_v\rangle^{\Lambda_2,-}. \tag{2.7}$$

The above inequalities hold also at $T = 0$, where $\sigma_v^{\Lambda,\tau}$ substitutes for $\langle\sigma_v\rangle^{\Lambda,\tau}$. It is convenient to note this explicitly for later reference:

$$\tau^- \leq \tau^+ \quad \Longrightarrow \quad \sigma^{\Lambda,\tau^-} \leq \sigma^{\Lambda,\tau^+}, \tag{2.8}$$

$$\sigma_v^{\Lambda_1,+} \geq \sigma_v^{\Lambda_2,+} \quad \text{and} \quad \sigma_v^{\Lambda_1,-} \leq \sigma_v^{\Lambda_2,-}, \tag{2.9}$$

$$\sigma^{\Lambda_1,+} - \sigma^{\Lambda_1,-} \;\geq\; \sigma^{\Lambda_2,+} - \sigma^{\Lambda_2,-} \;\geq\; 0, \tag{2.10}$$

with the second and third assertions holding for $v \in \Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^2$.

3. Proof of the Main Result for $T = 0$

We start with the zero-temperature case of Theorem 1.1 as it already contains the main features of the problem while being technically simpler. For a further simplification, we consider first the nearest-neighbor interaction (1.5). The extension to finite-range interactions will follow in Sect. 5.

3.1. Influence/disagreement percolation. Due to the monotonicity of the ground state in the boundary conditions, the order parameter $m(L)$ which is defined in (1.2) can be viewed as the probability that the difference of the boundary conditions at distance L from a site v "percolates" to v:

$$m(L) = m(L; 0, \mathcal{J}, h, \epsilon) \;=\; \mathbb{P}\left(\sigma_{\mathbf{0}}^{\Lambda(L),+} > \sigma_{\mathbf{0}}^{\Lambda(L),-}\right). \tag{3.1}$$

Remark. Disagreement percolation provides a concrete manifestation of the influence of the boundary condition. The terms *disagreement percolation* and *influence percolation* are almost interchangeable: the former referring to specific manifestations of the latter. The term percolation is called for since the influence/disagreement spreads only along connected sets.

To learn about $m(L)$ we find it useful to consider the following functions of the disorder:

$$D_\ell(\eta) := \sum_{v\in\Lambda(\ell)} \mathbb{1}[\sigma_v^{\Lambda(3\ell),+} \neq \sigma_v^{\Lambda(3\ell),-}], \tag{3.2}$$

the number of sites in $\Lambda(\ell)$ to which the difference of the boundary conditions imposed on the boundary of $\Lambda(3\ell)$ has "percolated", and

$$\begin{aligned} B_\ell(\eta) := \sum_{(u,v)\in\partial_{\mathrm{e}}\Lambda(2\ell)} & J_{u,v}\, \mathbb{1}[\{\sigma_u^{\Lambda(3\ell)\setminus\Lambda(\ell),+} \neq \sigma_u^{\Lambda(3\ell)\setminus\Lambda(\ell),-}\} \\ & \cap \{\sigma_v^{\Lambda(3\ell)\setminus\Lambda(\ell),+} \neq \sigma_v^{\Lambda(3\ell)\setminus\Lambda(\ell),-}\}]\,. \end{aligned} \tag{3.3}$$

The latter is the combined strength of the edges crossing a separating surface at half the distance of $\Lambda(\ell)$ to the boundary of $\Lambda(3\ell)$, which contribute to the surface tension.

3.2. The surface tension. One may learn about the probability distribution of the disagreement set whose size D_ℓ measures through consideration of the surface tension, which for scale ℓ (always a positive integer) is defined as

$$\begin{aligned} \mathcal{T}_\ell(\eta) := -\big[&\mathcal{E}^{+,+}(\Lambda(3\ell)\setminus\Lambda(\ell)) + \mathcal{E}^{-,-}(\Lambda(3\ell)\setminus\Lambda(\ell)) - \mathcal{E}^{+,-}(\Lambda(3\ell)\setminus\Lambda(\ell)) \\ &-\mathcal{E}^{-,+}(\Lambda(3\ell)\setminus\Lambda(\ell))\big]. \end{aligned} \tag{3.4}$$

Here $\mathcal{E}^{s,s'}(\Lambda(3\ell)\setminus\Lambda(\ell))$ denotes the minimal value of the Hamiltonian $H^{\Lambda(3\ell)\setminus\Lambda(\ell),\tilde{\tau}}$ (see (2.4)) over spin configurations satisfying the boundary conditions

$$\tilde{\tau}_v \;=\; \begin{cases} s & v\in\partial_{\mathrm{v}}\Lambda(3\ell) \\ s' & v\in\partial_{\mathrm{v}}(\mathbb{Z}^2\setminus\Lambda(\ell)) \end{cases}. \tag{3.5}$$

Our analysis proceeds by contrasting a natural upper bound on the surface tension, with the analysis of the not-improbable fluctuations of $\mathcal{T}_\ell(\eta)$. For the upper bound we have:

Theorem 3.1. *In the RFIM with nearest-neighbor interaction, for each configuration of the random field:*

$$\mathcal{T}_\ell(\eta) \;\le\; 4\, B_\ell(\eta) \;\le\; 8J\, |\partial_{\mathrm{v}}\Lambda(2\ell)|. \tag{3.6}$$

Proof. Let A be the set of vertices in $\Lambda(3\ell)\setminus\Lambda(\ell)$ on which there is equality between the ground-state configurations with ++ and −− boundary conditions. The monotonicity property (2.8) implies that all ground states on $\Lambda(3\ell)\setminus\Lambda(\ell)$ must coincide on A. Consider making two modifications to the Hamiltonian in the domain $\Lambda(3\ell)\setminus\Lambda(\ell)$: First, rigidly restrict the spin values at all vertices in A to their common value in these ground states. This clearly has no effect on the energies of the ground-state configurations considered above. Second, remove the energy terms corresponding to bonds in $\partial_e\Lambda(2\ell)$ whose endpoints do not intersect A. This change may affect the energy of each of the four ground states by at most $B_\ell(\eta)$. Once both changes are made, the Hamiltonian decomposes into a sum of two terms, in whose minimization there is no interaction between the effects of the two components of the boundary. Thus the surface tension based on the modified Hamiltonian vanishes.

It follows that $\mathcal{T}_\ell(\eta) \le 4\, B_\ell(\eta)$ as claimed in the first inequality in (3.6). The second is its elementary consequence. □

The upper bound which (3.6) yields on $\mathcal{T}_\ell$ will be contrasted with the implications of the following representation.

Theorem 3.2. *For the RFIM with IID Gaussian random fields, the surface tension bears the following relation with disagreement percolation:*

$$\mathcal{T}_\ell(\eta) \;=\; 2\varepsilon \int_{\mathbb{R}} D_\ell(\eta^{(t)})\, dt \;=\; \frac{2\varepsilon}{\sqrt{|\Lambda(\ell)|}}\, \mathbb{E}_{\widehat{\eta}_\ell}\left(\frac{D_\ell(\eta)}{\phi(\widehat{\eta}_\ell)}\right), \tag{3.7}$$

where:

1) $\eta^{(t)}$ is defined by adding a uniform field of intensity t in $\Lambda(\ell)$

$$\eta_v^{(t)} := \begin{cases} \eta_v + t & v \in \Lambda(\ell) \\ \eta_v & \text{otherwise} \end{cases} \tag{3.8}$$

2) the subscript on $\mathbb{E}_{\widehat{\eta}_\ell}$ indicates the average over the variable

$$\widehat{\eta}_\ell := \frac{1}{\sqrt{|\Lambda(\ell)|}} \sum_{v\in\Lambda(\ell)} \eta_v \tag{3.9}$$

at fixed values of the other, orthogonal, Gaussian degrees of freedom which determine η

3) ϕ is the Gaussian density function (1.7).

Proof. To derive (3.7) we approach $\mathcal{T}_\ell(\eta)$ through another function, $G_\ell(\eta)$, which has already played a key role in the proof of the absence of symmetry breaking in the two-dimensional RFIM [4,5]. Its zero-temperature version corresponds to the difference in the ground-state energies in $\Lambda(3\ell)$ between the + and − boundary conditions:

$$G_\ell(\eta) := -\left[\mathcal{E}^+(\Lambda(3\ell)) - \mathcal{E}^-(\Lambda(3\ell))\right] \tag{3.10}$$

with $\mathcal{E}^\pm(\Lambda(3\ell)) := H^{\Lambda(3\ell),\pm}(\sigma^{\Lambda(3\ell),\pm})$.

The two functions are linked by the relation

$$\mathcal{T}_\ell(\eta) \;=\; \lim_{t\to\infty} G_\ell(\eta^{(t)}) - G_\ell(\eta^{(-t)}) \tag{3.11}$$

with $\eta^{(t)}$ defined by adding a uniform field of intensity t in $\Lambda(\ell)$, as described in (3.8). Equality (3.11) is based on the observation that if $|h+\varepsilon\eta_v| > 4J$ then $\sigma_v^{\Lambda(3\ell),\pm}$ are both given by $\text{sign}(h+\varepsilon\eta_v)$ (4 appears here as the number of neighbors of v in $\mathbb{Z}^2$).

The function $G_\ell(\eta)$ is Lipschitz continuous and non-decreasing in each of the coordinates of (η_v), $v \in \Lambda(3\ell)$, with

$$\frac{\partial}{\partial \eta_v} G_\ell(\eta) = \varepsilon \left[\sigma_v^{\Lambda(3\ell),+}(\eta) - \sigma_v^{\Lambda(3\ell),-}(\eta)\right] = 2\varepsilon\, \mathbb{1}_{\sigma_v^{\Lambda(3\ell),+}(\eta) \neq \sigma_v^{\Lambda(3\ell),-}(\eta)} \tag{3.12}$$

for Lebesgue-almost-every η. Combining this with (3.11) one gets

$$\mathcal{T}_\ell(\eta) = \varepsilon \int_{-\infty}^{\infty} \sum_{v\in\Lambda(\ell)} \left[\sigma_v^{\Lambda(3\ell),+}(\eta^{(t)}) - \sigma_v^{\Lambda(3\ell),-}(\eta^{(t)})\right] dt \;=\; 2\varepsilon \int_{-\infty}^{\infty} D_\ell(\eta^{(t)})\, dt. \tag{3.13}$$

Making use of the Gaussian structure, the shift by t affects the random field's normalized sum over $\Lambda(\ell)$, i.e., $\widehat{\eta}_\ell$ of (3.9), but it does not affect the independently distributed degrees of freedom which as Gaussian variables are orthogonal to it, $\widehat{\eta}_\ell^{(\perp)}$.

Writing $\eta = (\widehat{\eta}_\ell, \widehat{\eta}_\ell^{(\perp)})$ and $t\sqrt{|\Lambda(\ell)|} = s$, the change $\eta \mapsto \eta^{(t)}$ corresponds to the shift $(\widehat{\eta}, \widehat{\eta}^{(\perp)}) \mapsto (\widehat{\eta} + s, \widehat{\eta}^{(\perp)})$. Since the component $\widehat{\eta}_\ell$ has the standard Gaussian distribution, of density ϕ, the above integral can be rewritten as:

$$\begin{aligned}
\int_{-\infty}^{\infty} D_\ell(\eta^{(t)})\, dt &= \frac{1}{\sqrt{|\Lambda(\ell)|}} \int_{-\infty}^{\infty} D_\ell((\widehat{\eta}_\ell + s, \widehat{\eta}_\ell^{(\perp)}))\, ds \\
&= \frac{1}{\sqrt{|\Lambda(\ell)|}} \int_{-\infty}^{\infty} D_\ell((s, \widehat{\eta}_\ell^{(\perp)}))\, ds \\
&= \frac{1}{\sqrt{|\Lambda(\ell)|}} \int_{-\infty}^{\infty} D_\ell((s, \widehat{\eta}_\ell^{(\perp)}))\phi(s)^{-1} \cdot \phi(s)\, ds \\
&= \frac{1}{\sqrt{|\Lambda(\ell)|}}\, \mathbb{E}_{\widehat{\eta}_\ell} \left(D_\ell(\eta)\, \phi(\widehat{\eta}_\ell)^{-1}\right).
\end{aligned} \tag{3.14}$$

□

3.3. Proof outline for the RFIM ground states. Influence percolation quantities appear in both the surface tension formula (3.7) and the upper bound (3.6). The combination of these two yields the following relation, which underlies our analysis:

$$\frac{2\,\mathbb{E}(B_\ell)}{\varepsilon\sqrt{|\Lambda(\ell)|}} \geq \mathbb{E}\left(\frac{D_\ell}{|\Lambda(\ell)|}\, \frac{1}{\phi(\widehat{\eta}_\ell)}\right) \tag{3.15}$$

(Here and later we omit the explicit mention of η where it is clear from the context.)

To motivate the direction which the discussion is about to take, let us note that (3.15) allows a streamlined proof of the following statement, which is among the significant results established in [5].

Corollary 3.3. *In the two-dimensional RFIM with Gaussian random field, for any* $\varepsilon \neq 0$*, the system has a unique ground-state configuration.*

Proof. The monotonicity relations (2.9) imply that as the domains Λ_n increase to $\mathbb{Z}^2$, the ground state $\sigma^{\Lambda_n,+}$ converges pointwise to a limiting ground state σ^+, which is, moreover, independent of the choice of exhausting sequence Λ_n. The ground state σ^- is defined similarly with $-$ boundary conditions. The monotonicity relation (2.8) then shows that uniqueness of the ground state is equivalent to the vanishing of the quantity

$$m(\infty) := \lim_{\ell\to\infty} m(\ell) \;=\; \mathbb{P}\left(\sigma_v^+ \neq \sigma_v^-\right), \tag{3.16}$$

where v is an arbitrary point in $\mathbb{Z}^2$.

The monotonicity relation (2.10) further allows to deduce from (3.15) that

$$\frac{C\,J}{\varepsilon} \;\geq\; \mathbb{E}\left(\left[\frac{1}{|\Lambda(\ell)|}\sum_{v\in\Lambda(\ell)} \mathbb{1}[\sigma_v^+ \neq \sigma_v^-]\right]\frac{1}{\phi(\widehat{\eta}_\ell)}\right) \tag{3.17}$$

where $C > 0$ is an absolute constant.

The pair of ground states (σ^+, σ^-) form an ergodic process under translations (as a factor of the IID process η). Hence $\frac{1}{|\Lambda(\ell)|}\sum_{v\in\Lambda(\ell)} \mathbb{1}[\sigma_v^+ \neq \sigma_v^-]$ converges in distribution (and also almost surely) to a constant, given by its mean $m(\infty)$. At the same time, the probability distribution of $\widehat{\eta}_\ell$ does not depend on ℓ, being given by $N(0,1)$. By implication the probability distribution of the product of the two converges to that of $m(\infty)/\widehat{\eta}$. Combining that with Fatou's lemma and Skorokhod's representation theorem, one may deduce

$$\begin{aligned}\frac{\mathrm{C}\,J}{\varepsilon} \;&\geq\; \liminf_{\ell\to\infty}\,\mathbb{E}\left(\left[\frac{1}{|\Lambda(\ell)|}\sum_{v\in\Lambda(\ell)} \mathbb{1}[\sigma_v^+ \neq \sigma_v^-]\right]\frac{1}{\phi(\widehat{\eta}_\ell)}\right) \;\geq\; \mathbb{E}\left(\frac{m(\infty)}{\phi(\widehat{\eta})}\right)\\ &=\; m(\infty)\int_{-\infty}^{\infty} 1\,dx \;=\; m(\infty)\cdot\infty.\end{aligned} \tag{3.18}$$

This can hold true only if $m(\infty) = 0$. □

The ergodicity argument is not of much help for the finite-volume bounds which are sought here. It may however be substituted by more quantitative estimates, which are derived below under the assumption that $m(\ell) \to 0$ at only a sub-power slow rate. To produce a contradiction which replaces (3.18) we shall first show that (3.15) implies the following anti-concentration bound.

Proposition 3.4. *For each integer $\ell \geq 1$,*

$$\mathbb{P}\left(\frac{D_\ell}{\mathbb{E}(D_\ell)} < \frac{1}{2}\right) \;\geq\; \chi\left(\frac{4J}{\varepsilon}\cdot\frac{|\partial_{\mathrm{v}}\Lambda(2\ell)|}{\sqrt{|\Lambda(\ell)|}}\cdot\frac{m(\ell-1)}{m(4\ell)}\right), \tag{3.19}$$

where χ is the standard Gaussian distribution's two-sided tail (1.7).

It is worth keeping in mind that assuming regular power-law decay of $m(\ell)$, the right-hand side of (3.19) is uniformly positive (depending only on J and ε). The bound (3.19) will be contrasted with a conditional concentration-of-measure estimate, derived through the following two steps. For the convenience of presentation we summarize here the key statements, and postpone their proofs to the sections which follow.

I) Slow decay of a monotone sequence implies the existence of long stretches of somewhat comparable values:

Proposition 3.5. *For any monotone non-increasing sequence* (p_j) *satisfying* $0 \leq p_j \leq 1$, *and any* $\alpha > 0$*: if for some* $k \geq 1$ *it holds that*

$$p_k \geq k^{-\alpha} \tag{3.20}$$

then there exists an integer n *in the range* $\sqrt{k} \leq n \leq k$ *such that for all* $1 \leq j \leq n$,

$$p_n \leq p_j \leq p_n \left(\frac{n}{j}\right)^{2\alpha}. \tag{3.21}$$

The proposition will be employed with $(m(j))$ as the sequence (p_j).

II) A conditional variance bound:

Proposition 3.6. *For each* $0 < \alpha \leq \frac{1}{4}$ *there exists* $L_0 > 0$ *such that the following holds for all integer* $L \geq L_0$*. If*

$$m(L) \geq L^{-2\alpha} \tag{3.22}$$

and

$$m(L) \leq m(j) \leq m(L) \left(\frac{L}{j}\right)^{2\alpha}, \quad 1 \leq j \leq L \tag{3.23}$$

then

$$\mathrm{Var}\left(D_{\lfloor L/4 \rfloor}\right) \;\leq\; 241 \cdot \alpha \cdot \left(\mathbb{E}\left(D_{\lfloor L/4 \rfloor}\right)\right)^2. \tag{3.24}$$

Combining Propositions 3.5 and 3.6 with the assumption of sub-power decay of $(m(j))$ shows the existence of an infinite sequence of Ls for which (3.23) and (3.24) hold. With $\ell = \lfloor L/4 \rfloor$, Chebyshev's inequality and (3.24) imply that along this sequence

$$\mathbb{P}\left(\frac{D_\ell}{\mathbb{E}(D_\ell)} < \frac{1}{2}\right) \;\leq\; 1000\alpha. \tag{3.25}$$

At the same time, for $\alpha \to 0$, the ratio $m(\ell - 1)/m(4\ell)$ tends to 1 by (3.23), and the argument of χ in (3.19) is bounded by Const.J/ε, uniformly in ℓ and α. Hence, for small enough $\alpha > 0$, (3.25) is in contradiction with the anti-concentration bound (3.19).

The above line of reasoning allows to conclude that the initial assumption of sub-power decay is false. A quantitative version of the argument, proving the zero-temperature case of Theorem 1.1, is presented in Sect. 3.6 after the derivation of the above three propositions.

3.4. The anti-concentration estimate. In the proof of Proposition 3.4 we shall make use of the following variational principle.

Lemma 3.7. *Let* $w : \mathbb{R} \mapsto [0, \infty)$ *be a symmetric* ($w(-x) = w(x)$)*, non-increasing in* $|x|$*, probability density function on* $\mathbb{R}$*, i.e. satisfying* $\int_{\mathbb{R}} w(x)dx = 1$*. Then, for any* $p \in (0, 1]$,

$$\min\left\{ \int_{\mathbb{R}} f(x)\,dx \;\middle|\; 0 \leq f \leq 1,\ \int_{\mathbb{R}} f(x)\,w(x)\,dx \;=\; 1 - p \right\} = 2q \tag{3.26}$$

where the variation is over measurable functions satisfying the stated conditions, and q is the unique value related to p by

$$\int_{|x|>q} w(x)\, dx \;=\; p. \tag{3.27}$$

Proof. Among the functions satisfying the conditions spelled in (3.26) and (3.27) is $g(x) = \mathbb{1}_{[-q,q]}(x)$. A simple calculation shows that for any other variational function

$$\int f(x)dx - \int g(x)dx \;=\; \frac{1}{w(q)} \int [f(x) - g(x)]\, [w(q) - w(x)]\, dx \;\geq\; 0 \tag{3.28}$$

where the integrand is nonnegative since $w(x)$ is non-increasing in $|x|$ and $0 \leq f(x) \leq 1$ and hence $f(x) - g(x)$ and $w(q) - w(x)$ are of the same sign for every x.[1] □

One may note that the above assumptions of symmetry and monotonicity of the probability density $w(x)$ are not essential, and upon the natural reformulation of (3.27) can be omitted. They are however satisfied by the Gaussian density function $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$.

The above will next be used to prove the stated estimate.

Proof of Proposition 3.4. Let A be the event $\{\eta : D_\ell \geq \mathbb{E}(D_\ell)/2\}$ and let us denote its probability as $1-p$, i.e.

$$\mathbb{P}\Big(D_\ell \geq \frac{1}{2}\mathbb{E}(D_\ell)\Big) \;=\; \mathbb{P}(A) \;=\; 1-p. \tag{3.29}$$

From (3.15) one may deduce:

$$\frac{2}{\varepsilon}\, \frac{\mathbb{E}(B_\ell)}{\sqrt{|\Lambda(\ell)|}}\, \frac{|\Lambda(\ell)|}{\mathbb{E}(D_\ell)} \geq \frac{1}{2}\, \mathbb{E}\left(\mathbb{1}[A]\, \frac{1}{\phi(\widehat{\eta})}\right). \tag{3.30}$$

Expressed in terms of the conditional probability of A, conditioned on $\widehat{\eta}$, the term on the right is, by Lemma 3.7,

$$\begin{aligned} \mathbb{E}\left(\mathbb{1}[A]\, \frac{1}{\phi(\widehat{\eta})}\right) &= \int_{-\infty}^{\infty} \mathbb{P}\big(A \big| \widehat{\eta} = x\big)\, dx \\ &\geq \min\left\{ \int_{-\infty}^{\infty} f(x)\, dx \,\middle|\, 0 \leq f \leq 1,\ \int_{\mathbb{R}} f(x)\,\phi(x) dx = 1-p \right\} \\ &\geq\; 2\,q \end{aligned} \tag{3.31}$$

with q defined by:

$$\chi(q) \;\equiv\; \int_{|x|>q} \phi(x)\, dx \;=\; p. \tag{3.32}$$

Combining (3.31) with (3.30) we learn that

$$\frac{2}{\varepsilon}\, \frac{\mathbb{E}(B_\ell)}{\sqrt{|\Lambda(\ell)|}}\, \frac{|\Lambda(\ell)|}{\mathbb{E}(D_\ell)} \;\geq\; q. \tag{3.33}$$

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[1] We thank the anonymous referee for shortening to one line the previous three line argument.

Hence

$$\chi\Big(\frac{2}{\varepsilon}\,\frac{\mathbb{E}(B_\ell)}{\sqrt{|\Lambda(\ell)|}}\,\frac{|\Lambda(\ell)|}{\mathbb{E}(D_\ell)}\Big) \;\le\; \chi(q) \;=\; p \;=\; \mathbb{P}\Big(D_\ell < \frac{1}{2}\mathbb{E}(D_\ell)\Big). \tag{3.34}$$

To obtain the conclusion (3.19) of the proposition, it remains to note that, by the definitions (3.1), (3.2), (3.3) of $m(j)$, D_ℓ and B_ℓ, together with the monotonicity inequality (2.10),

$$\begin{aligned}\mathbb{E}(B_\ell) &\le 2J\,|\partial_{\mathrm{v}}\Lambda(2\ell)|\,m(\ell-1),\\ \mathbb{E}(D_\ell) &\ge |\Lambda(\ell)|\,m(4\ell).\end{aligned} \tag{3.35}$$

□

3.5. Implications of slow decay. We next show that slow decay of a monotone sequence implies the existence of long stretches of somewhat comparable values.

Proof of Proposition 3.5. Assume that for some k and $\alpha > 0$

$$p_k \ge k^{-\alpha}. \tag{3.36}$$

As (p_j) is non-increasing we need only prove the right-hand inequality in (3.21). Let n be the index at which $(j^{2\alpha} p_j)_{1\le j\le k}$ is maximized (the smallest index, if there is more than one possibility). By definition, $p_j \le p_n(n/j)^{2\alpha}$ for $1 \le j \le n$. Lastly, as $p_n \le 1$ we have $n^{2\alpha} \ge n^{2\alpha} p_n \ge k^{2\alpha} p_k \ge k^\alpha$ proving that $n \ge \sqrt{k}$. □

Next we turn to the implications of slow decay on the variance of the size of the disagreement set, $\mathrm{Var}(D_\ell)$.

Proof of Proposition 3.6. Assume that $m(L) \ge L^{-2\alpha}$, and that (3.23) holds for all $1 \le j \le L$. Throughout the proof we set

$$\ell := \lfloor L/4 \rfloor.$$

For $v \in \Lambda(\ell)$ let E_v denote the event $\{\eta \,:\, \sigma_v^{\Lambda(3\ell),+}(\eta) \ne \sigma_v^{\Lambda(3\ell),-}(\eta)\}$. In this notation:

$$\mathrm{Var}\,(D_\ell) = \sum_{v,w\in\Lambda(\ell)} \big[\mathbb{P}(E_v \cap E_w) - \mathbb{P}(E_v)\mathbb{P}(E_w)\big]. \tag{3.37}$$

We proceed to bound the terms in this sum.

By the FKG monotonicity (2.10) and the definition (3.1) of $(m(j))$, for any site $v \in \Lambda(\ell)$,

$$\mathbb{P}(E_v) \ge m(4\ell) \ge m(L) \tag{3.38}$$

and for any pair $v, w \in \Lambda(\ell)$, $v \ne w$,

$$\mathbb{P}(E_v \cap E_w) \le m(r(v,w))^2 \tag{3.39}$$

with

$$r(v,w) := \lfloor (d(v,w)-1)/2 \rfloor \tag{3.40}$$

and $d(v,w)$ the distance between the two sites. The bound (3.39) holds since if both v and w are affected by boundary conditions placed outside of $\Lambda(3\ell)$ then each spin is necessarily affected also by boundary conditions placed at distance $r(v,w)$ from the site. However, these two events are independent, since they depend only on the random fields in a pair of disjoint neighborhoods of v and w.

For pairs at distance $d(v, w) \leq 2$ we shall employ the simpler bound:

$$\mathbb{P}(E_v \cap E_w) - \mathbb{P}(E_v)\mathbb{P}(E_w) \leq \mathbb{P}(E_v) \leq m(2\ell). \tag{3.41}$$

Thus under the assumption (3.23) we get

$$\begin{aligned} \mathrm{Var}\,(D_\ell) &\leq |\Lambda(2)| \cdot |\Lambda(\ell)| m(2\ell) + \sum_{\substack{v,w\in\Lambda(\ell)\\ d(v,w)\geq 3}} \Big(m(r(v,w))^2 - m(L)^2 \Big) \\ &\leq |\Lambda(2)| \cdot |\Lambda(\ell)| m(2\ell) + m(L)^2 \sum_{\substack{v,w\in\Lambda(\ell)\\ d(v,w)\geq 3}} \left(\left(\frac{L}{r(v,w)} \right)^{4\alpha} - 1 \right). \end{aligned} \tag{3.42}$$

The sum in the last bound can be estimated through the observation that most pairs $v, w \in \Lambda(\ell)$ are at distance of order ℓ, in which case $\frac{L}{r(v,w)}$ is of order 1. As α is small, for such pairs $\left(\frac{L}{r(v,w)}\right)^{4\alpha} - 1$ is of order α. This leads to a bound of order $\alpha\, m(L)^2 L^4$ on the variance, which in light of (3.38) is of the order $\alpha \Big(\mathbb{E}(D_\ell)\Big)^2$.

We proceed to make this argument precise. We first note that

$$\begin{aligned} \sum_{\substack{v,w\in\Lambda(\ell)\\ d(v,w)\geq 3}} \left(\left(\frac{L}{r(v,w)} \right)^{4\alpha} - 1 \right) &= \sum_{j=1}^{\ell} |\{(v,w) \subseteq \Lambda(\ell) \,:\, r(v,w) = j\}| \left(\left(\frac{L}{j} \right)^{4\alpha} - 1 \right) \\ &\leq |\Lambda(\ell)| \sum_{j=1}^{\ell} 32 j \left(\left(\frac{L}{j} \right)^{4\alpha} - 1 \right). \end{aligned} \tag{3.43}$$

For large ℓ and $0 < \alpha \leq \frac{1}{4}$

$$\begin{aligned} \sum_{j=1}^{\ell} j \left(\left(\frac{L}{j} \right)^{4\alpha} - 1 \right) &\leq \int_1^{\ell+1} L^{4\alpha} x^{1-4\alpha} dx - \int_0^{\ell} x dx \\ &\leq \frac{\ell^2}{2} \left[\frac{2}{2-4\alpha} \cdot \left(\frac{L}{\ell+1} \right)^{4\alpha} \cdot \left(\frac{\ell+1}{\ell} \right)^2 - 1 \right] \leq 15\,\alpha\,\ell^2. \end{aligned}$$

Substituting this into (3.42), along with $|\Lambda(r)| \geq 2r^2$, we conclude that

$$\begin{aligned} \mathrm{Var}\,(D_\ell) &\leq |\Lambda(2)| \cdot |\Lambda(\ell)| m(2\ell) + m(L)^2 \cdot |\Lambda(\ell)| \cdot 32 \cdot 15\alpha\ell^2 \\ &\leq |\Lambda(2)| \cdot |\Lambda(\ell)| m(2\ell) + 240\alpha \left(m(L) |\Lambda(\ell)| \right)^2. \end{aligned} \tag{3.44}$$

It remains to observe that, by (3.38),

$$\mathbb{E}\big(D_\ell\big) \geq m(L)|\Lambda(\ell)|.$$

Moreover, by our assumptions that $m(L) \geq L^{-2\alpha}$ and that (3.23) holds,

$$|\Lambda(2)| \cdot |\Lambda(\ell)| m(2\ell) \leq |\Lambda(2)| \cdot |\Lambda(\ell)| m(L) \left(\frac{L}{2\ell}\right)^{2\alpha} \leq \alpha\Big(m(L)|\Lambda(\ell)|\Big)^2 \leq \alpha\Big(\mathbb{E}\big(D_\ell\big)\Big)^2$$

for ℓ sufficiently large (as a function of α). This allows to rewrite (3.44) in the simpler form stated in the proposition:

$$\mathrm{Var}\,(D_\ell) \leq 241 \cdot \alpha \cdot \Big(\mathbb{E}\big(D_\ell\big)\Big)^2.$$

□

3.6. Putting it all together: $T = 0$ for the nearest-neighbor case. We now have all the tools for proving the assertion made in Theorem 1.1 for zero temperature.

Proof of Theorem 1.1 at $T=0$. Recall from (1.6) that

$$\gamma = 2^{-10}\chi\left(\frac{50J}{\varepsilon}\right). \tag{3.45}$$

In view of (3.1), if Theorem 1.1 does not hold at zero temperature for J and ε then

$$\limsup_{L\to\infty} L^{\gamma} \cdot m(L) = \infty,$$

which implies that

$$m(M) \geq M^{-\gamma} \text{ for infinitely many } M. \tag{3.46}$$

We assume, in order to obtain a contradiction, that (3.46) holds. Let $M \geq 64$, later chosen sufficiently large, be such that $m(M) \geq M^{-\gamma}$. Applying Proposition 3.5 we see that there is an $8 \leq \sqrt{M} \leq L \leq M$ such that

$$m(L) \leq m(j) \leq m(L)\left(\frac{L}{j}\right)^{2\gamma}, \quad 1 \leq j \leq L. \tag{3.47}$$

Let now $\ell = \lfloor L/4 \rfloor$. Applying Proposition 3.4 we obtain the anti-concentration inequality

$$\mathbb{P}\left(\frac{D_\ell}{\mathbb{E}\big(D_\ell\big)} < \frac{1}{2}\right) \geq \chi\left(\frac{4J}{\varepsilon} \cdot \frac{|\partial_{\mathrm{v}}\Lambda(2\ell)|}{\sqrt{|\Lambda(\ell)|}} \cdot \frac{m(\ell-1)}{m(4\ell)}\right).$$

The right-hand side may be simplified, using (3.47) together with the fact that $\gamma < 2^{-10}$, and noting that the assumption $L \geq 8$ implies that $|\partial_{\mathrm{v}}\Lambda(2\ell)| = 4(2\ell+1) \leq 3L$ and $|\Lambda(\ell)| = 1 + 2\ell(\ell+1) \geq \frac{L^2}{16}$. This yields

$$\mathbb{P}\left(\frac{D_\ell}{\mathbb{E}\big(D_\ell\big)} < \frac{1}{2}\right) \geq \chi\left(\frac{4J}{\varepsilon} \cdot \frac{3L}{L/4} \cdot \left(\frac{L}{\ell-1}\right)^{2\gamma}\right) \geq \chi\left(\frac{50J}{\varepsilon}\right). \tag{3.48}$$

We shall now reach a contradiction by applying Proposition 3.6 with $\ell = \lfloor L/4 \rfloor$, noting that the assumptions of that proposition are verified by (3.47) and the fact that

$\sqrt{M} \le L \le M$ and $m(M) \ge M^{-\alpha}$. The proposition implies that for L sufficiently large (obtained by choosing M sufficiently large), we have the concentration bound,

$$\operatorname{Var}(D_\ell) \le 241 \cdot \gamma \cdot \Big(\mathbb{E}(D_\ell)\Big)^2.$$

Chebyshev's inequality then shows that

$$\mathbb{P}\left(\frac{D_\ell}{\mathbb{E}(D_\ell)} < \frac{1}{2}\right) \le 1000\gamma.$$

As this contradicts (3.48) for the choice (3.45) of γ, we conclude that our initial assumption (3.46) must be false, implying that Theorem 1.1 holds at zero temperature. □

4. Extension of the Power-Law Upper Bound to $T > 0$

In this section we adapt the zero-temperature proof of Theorem 1.1 to the positive temperature case. Again, for simplicity, we focus first on the case of nearest-neighbor interaction with the extension to finite-range interactions to follow in Sect. 5.

4.1. Adjustments in the terminology. At positive temperature the relevant function of the random field and of the boundary conditions is not the single ground-state configuration but the corresponding Gibbs probability measure. We proceed to explain how the definition of the order parameter and the proof presented in the zero-temperature case are modified to account for this difference.

Influence/disagreement percolation. The order parameter, which at $T = 0$ was the disagreement percolation of (3.1)

$$m(j; 0, \mathcal{J}, h, \epsilon) = \mathbb{P}\left(\sigma_{\mathbf{0}}^{\Lambda(j),+} > \sigma_{\mathbf{0}}^{\Lambda(j),-}\right) \tag{4.1}$$

is replaced by the difference in the expected magnetization

$$m(j; T, \mathcal{J}, h, \epsilon) = \frac{1}{2}\Big[\mathbb{E}[\langle\sigma_{\mathbf{0}}\rangle^{\Lambda(j),+}] - \mathbb{E}[\langle\sigma_{\mathbf{0}}\rangle^{\Lambda(j),-}]\Big]. \tag{4.2}$$

Let us comment in passing that the available monotone coupling of the + and − probability measures allows to present also the last expression as the probability of disagreement percolation. However, to keep the discussion simple, we shall not stress this point.

Correspondingly, as a measure of the disagreement in $\Lambda(\ell)$ due to the difference in boundary conditions placed on $\Lambda(3\ell)$ we take

$$D_\ell(\eta) := \frac{1}{2}\sum_{v \in \Lambda(\ell)} \Big[\langle\sigma_v\rangle^{\Lambda(3\ell),+} - \langle\sigma_v\rangle^{\Lambda(3\ell),-}\Big]. \tag{4.3}$$

Recall also that, at $T = 0$, $B_\ell(\eta)/J$ counted the number of edges in the separating surface $\partial_{\mathrm{e}}\Lambda(2\ell)$ which contribute to the surface tension. At $T > 0$, we find it more convenient to count vertices rather than edges, leading to the definition

$$\tilde{B}_\ell(\eta) := \frac{J}{2}\sum_{v \in \partial_{\mathrm{v}}\Lambda(2\ell)} \Big[\langle\sigma_v\rangle^{\Lambda(3\ell)\setminus\Lambda(\ell),+} - \langle\sigma_v\rangle^{\Lambda(3\ell)\setminus\Lambda(\ell),-}\Big]. \tag{4.4}$$

Surface tension. For $T > 0$, the role which is played by energy in the zero-temperature analysis is taken by the free energy, which for different combinations of the boundary conditions is defined as:

$$\mathcal{F}_\ell^{s,s'} := -T \cdot \log(Z^{\Lambda(3\ell)\setminus\Lambda(\ell);s,s'}) \tag{4.5}$$

where s and s' indicate the $(\pm)$ boundary conditions placed on the external boundary of $\Lambda(3\ell)$ and the internal boundary of $\Lambda(\ell)$, respectively, and the partition function is

$$Z^{\Lambda(3\ell)\setminus\Lambda(\ell);s,s'} = Z^{s,s'} = \sum_{\sigma:\Lambda(3\ell)\setminus\Lambda(\ell)\to\{-1,1\}} \exp\left(-\frac{1}{T} H^{\Lambda(3\ell)\setminus\Lambda(\ell);s,s'}(\sigma)\right) \tag{4.6}$$

with $H^{\Lambda(3\ell)\setminus\Lambda(\ell);s,s'} = H^{s,s'}$ the Hamiltonian incorporating the boundary conditions.

Following this prescription, the extension of the surface tension, of (3.4), to positive temperatures is

$$\mathcal{T}_\ell(\eta) \;=\; T\, \log\left(\frac{Z^{+,+}\cdot Z^{-,-}}{Z^{+,-}\cdot Z^{-,+}}\right). \tag{4.7}$$

A similar replacement takes place in the definition of the function $G_\ell(\eta)$ in (3.10) and it is straightforward to check that the relation (3.11) still holds.

4.2. Extension of the proof to $T > 0$. The zero-temperature bound of Theorem 3.1 is modified into the following statement, in which we replace the references to the ground-state spins by their quenched averages and where, for simplicity, we have upper bounded a sum over $(u, v) \in \partial_e\Lambda(2\ell)$ (analogous to the one in Theorem 3.1) by a sum over $v \in \partial_v\Lambda(2\ell)$.

Theorem 4.1. *In the RFIM with nearest-neighbor interaction, for any realization of the field η,*

$$\mathcal{T}_\ell(\eta) \le 8\tilde{B}_\ell(\eta). \tag{4.8}$$

Proof. As in the $T = 0$ case, the set $\partial_v\Lambda(2\ell)$ enters the discussion as a separating barrier between the inner and the outer boundary of $\Lambda(3\ell)\setminus\Lambda(\ell)$. Denoting the restriction of the spin configuration to this set by $\tau : \partial_v\Lambda(2\ell) \to \{-1, 1\}$, let ρ_+ and, correspondingly, ρ_- be the two probability measures induced on it by the $(+, +)$ and $(-, -)$ boundary conditions. More explicitly,

$$\rho_+(\tau) = \frac{Z_\tau^{+,+}}{Z^{+,+}}, \qquad \rho_-(\tau) = \frac{Z_\tau^{-,-}}{Z^{-,-}}, \tag{4.9}$$

with $Z_\tau^{s,s'}$ the restricted partition functions

$$Z_\tau^{s,s'} := \sum_{\substack{\sigma:\Lambda(3\ell)\setminus\Lambda(\ell)\to\{-1,1\}\\ \sigma|_{\partial_v\Lambda(2\ell)}=\tau}} \exp\left(-\frac{1}{T} H^{s,s'}(\sigma)\right).$$

Considering first the $(+)$ case, let us note that

$$\sum_{\tau:\partial_v\Lambda(2\ell)\to\{-1,1\}} \rho_+(\tau)\,\frac{Z_\tau^{+,-}}{Z_\tau^{+,+}} \;=\; \frac{Z^{+,-}}{Z^{+,+}}. \tag{4.10}$$

Hence, by Jensen's inequality (and the convexity of $-\log(X)$), for each specified η (which is omitted in the following expression)

$$\log\left(\frac{Z^{+,+}}{Z^{+,-}}\right) \le \sum_{\tau:\partial_v\Lambda(2\ell)\to\{-1,1\}} \rho_+(\tau)\,\log\left(\frac{Z^{+,+}_\tau}{Z^{+,-}_\tau}\right). \tag{4.11}$$

Combining the above with the analogous statement for $\rho_-(\tau)$ we get:

$$\begin{aligned}\mathcal{T}_\ell(\eta) &= T\log\left(\frac{Z^{+,+}}{Z^{+,-}}\cdot\frac{Z^{-,-}}{Z^{-,+}}\right)\\ &\le T\left[\sum_{\tau:\partial_v\Lambda(2\ell)\to\{-1,1\}} \rho_+(\tau)\,\log\left(\frac{Z^{+,+}_\tau}{Z^{+,-}_\tau}\right) + \sum_{\tau:\partial_v\Lambda(2\ell)\to\{-1,1\}} \rho_-(\tau)\log\left(\frac{Z^{-,-}_\tau}{Z^{-,+}_\tau}\right)\right].\end{aligned} \tag{4.12}$$

We now use the fact that the measure $\mathbb{P}^{+,+}$ stochastically dominates $\mathbb{P}^{-,-}$, as in (2.5). In particular, there exists a probability measure $\rho(\tau^+,\tau^-)$ on pairs $\tau^+,\tau^- : \partial_v\Lambda(2\ell)\to\{-1,1\}$ such that $\tau^+\ge\tau^-$ pointwise, with probability 1, and the marginal distribution of each τ^s is given by ρ_s. This *coupling* of measures allows to express (4.12) in the form

$$\mathcal{T}_\ell(\eta) \le T\left[\sum_{\tau^+,\tau^-:\partial_v\Lambda(2\ell)\to\{-1,1\}} \rho(\tau^+,\tau^-)\log\left(\frac{Z^{+,+}_{\tau^+}}{Z^{+,-}_{\tau^+}}\cdot\frac{Z^{-,-}_{\tau^-}}{Z^{-,+}_{\tau^-}}\right)\right]. \tag{4.13}$$

The coupling of the measures allows to bound the quantity on the right in terms of the positive temperature version of the disagreement percolation. The estimate is motivated by the observation that for every configuration τ:

$$Z^{+,+}_\tau\cdot Z^{-,-}_\tau = Z^{+,-}_\tau\cdot Z^{-,+}_\tau. \tag{4.14}$$

The proof is through the bijection associating to each pair $(\sigma^{+,+},\sigma^{-,-})$ contributing to the double sum on the left the following pair $(\sigma^{+,-},\sigma^{-,+})$ contributing to the double sum on the right:

$$\sigma^{+,-}_v := \begin{cases}\sigma^{+,+}_v & v\in\Lambda(3\ell)\setminus\Lambda(2\ell)\\ \sigma^{-,-}_v & v\in\Lambda(2\ell)\setminus\Lambda(\ell)\end{cases},\quad \sigma^{-,+}_v := \begin{cases}\sigma^{-,-}_v & v\in\Lambda(3\ell)\setminus\Lambda(2\ell)\\ \sigma^{+,+}_v & v\in\Lambda(2\ell)\setminus\Lambda(\ell)\end{cases}. \tag{4.15}$$

At the common value of the configuration τ over the separating set $\partial_v\Lambda(2\ell)$, the sums of the corresponding energy terms in (4.14) match.

Thus terms with $\tau^+ = \tau^-$ make no contribution to the sum (4.13). For the more general case we note that when the restriction of $\sigma^{+,+}$ ($\sigma^{-,-}$) to $\partial_v\Lambda(2\ell)$ is τ^+ (τ^-) and $\sigma^{+,-},\sigma^{-,+}$ are given by (4.15) then, with $\tau^+\ge\tau^-$,

$$\begin{aligned}&-\frac{1}{T}\left(H^{+,+}(\sigma^{+,+}) + H^{-,-}(\sigma^{-,-}) - H^{+,-}(\sigma^{+,-}) - H^{-,+}(\sigma^{-,+})\right)\\ &= \frac{J}{T}\sum_{(u,v)\in\partial_e\Lambda(2\ell)} \left(\sigma^{+,+}_u\sigma^{+,+}_v + \sigma^{-,-}_u\sigma^{-,-}_v - \sigma^{+,+}_u\sigma^{-,-}_v - \sigma^{-,-}_u\sigma^{+,+}_v\right)\end{aligned}$$

$$= \frac{J}{T} \sum_{(u,v)\in\partial_{\mathrm{e}}\Lambda(2\ell)} \left(\sigma_u^{+,+} - \sigma_u^{-,-}\right) \cdot \left(\sigma_v^{+,+} - \sigma_v^{-,-}\right)$$

$$= \frac{J}{T} \sum_{(u,v)\in\partial_{\mathrm{e}}\Lambda(2\ell)} \left(\sigma_u^{+,+} - \sigma_u^{-,-}\right) \cdot \left(\tau_v^{+} - \tau_v^{-}\right) \le \frac{4J}{T} \sum_{v\in\partial_{\mathrm{v}}\Lambda(2\ell)} \left(\tau_v^{+} - \tau_v^{-}\right), \quad (4.16)$$

where the third equality uses the fact that if $(u, v) \in \partial_{\mathrm{e}}\Lambda(2\ell)$ then $v \in \partial_{\mathrm{v}}\Lambda(2\ell)$ and the inequality uses the fact that each vertex $v \in \partial_{\mathrm{v}}\Lambda(2\ell)$ is incident to at most two edges $(u, v) \in \partial_{\mathrm{e}}\Lambda(2\ell)$ and the fact that $\tau^+ \ge \tau^-$ pointwise. Thus

$$\frac{Z_{\tau^+}^{+,+}}{Z_{\tau^+}^{+,-}} \cdot \frac{Z_{\tau^-}^{-,-}}{Z_{\tau^-}^{-,+}} \le \exp\left(\frac{4J}{T} \sum_{v\in\partial_{\mathrm{v}}\Lambda(2\ell)} \left(\tau_v^{+} - \tau_v^{-}\right)\right).$$

Finally, inserting this estimate in (4.13) we get

$$\mathcal{T}_\ell(\eta) \le 4J \sum_{\tau^+,\tau^-:\partial_{\mathrm{v}}\Lambda(2\ell)\to\{-1,1\}} \rho(\tau^+,\tau^-) \sum_{v\in\partial_{\mathrm{v}}\Lambda(2\ell)} \left(\tau_v^{+} - \tau_v^{-}\right).$$

Through the definition of $\rho(\tau^+, \tau^-)$ the above reduces to the bound asserted in (4.8). □

The representation of the surface tension given by Theorem 3.2, which enables a lower bound on its expected value at zero temperature, continues to hold at positive temperature with the exact same statement. The proof also remains the same, upon replacing (3.12) and (3.13) with the analogous

$$\frac{\partial}{\partial\eta_v} G_\ell(\eta) = \varepsilon\left[\langle\sigma_v\rangle^{\Lambda(3\ell),+} - \langle\sigma_v\rangle^{\Lambda(3\ell),-}\right] \quad (4.17)$$

and

$$\mathcal{T}_\ell(\eta) = \varepsilon \int_{-\infty}^{\infty} \sum_{v\in\Lambda(\ell)} \left[\langle\sigma_v\rangle^{\Lambda(3\ell),+}(\eta^{(t)}) - \langle\sigma_v\rangle^{\Lambda(3\ell),-}(\eta^{(t)})\right] dt = 2\varepsilon \int_{-\infty}^{\infty} D_\ell(\eta^{(t)})\, dt. \quad (4.18)$$

Combining Theorem 4.1 and (3.7) we obtain

$$\frac{4\,\mathbb{E}(\tilde{B}_\ell(\eta))}{\varepsilon\sqrt{|\Lambda(\ell)|}} \ge \mathbb{E}\left(\frac{D_\ell(\eta)}{|\Lambda(\ell)|}\frac{1}{\phi(\widehat{\eta})}\right) \quad (4.19)$$

which replaces (3.15) when $T > 0$. The bound implies that Proposition 3.4 continues to hold at positive temperature, with the exact same statement and with $2\tilde{B}_\ell$ replacing B_ℓ throughout the proof (noting, in particular, that

$$2\mathbb{E}(\tilde{B}_\ell) \le 2\,J\,|\partial_{\mathrm{v}}\Lambda(2\ell)|\,m(\ell-1) \quad (4.20)$$

holds instead of (3.35).

The upper bound on the variance of D_ℓ, given for $T = 0$ by Proposition 3.6, continues to hold exactly as stated also when $T > 0$. In the proof, the indicator random variable of the event E_v is replaced with the random variable

$$X_v := \frac{1}{2}\left[\langle\sigma_v\rangle^{\Lambda(3\ell),+} - \langle\sigma_v\rangle^{\Lambda(3\ell),-}\right]. \quad (4.21)$$

This yields, e.g., the analogous equation to (3.37),

$$\mathrm{Var}\,(D_\ell) = \sum_{v,w\in\Lambda(\ell)} \left[\mathbb{E}(X_v\cdot X_w) - \mathbb{E}(X_v)\mathbb{E}(X_w)\right] \tag{4.22}$$

and the analogous equation to (3.39),

$$\mathbb{E}(X_v\cdot X_w) \;\leq\; m(r(v,w))^2 \tag{4.23}$$

with $r(v,w)$ defined in (3.40). The last inequality holds as, via the monotonicity property (2.7),

$$\mathbb{E}(X_v\cdot X_w) \leq \frac{1}{4}\mathbb{E}\Big[\Big(\langle\sigma_v\rangle^{\Lambda_v(r(v,w)),+} - \langle\sigma_v\rangle^{\Lambda_v(r(v,w)),-}\Big) \\ \times \Big(\langle\sigma_v\rangle^{\Lambda_w(r(v,w)),+} - \langle\sigma_v\rangle^{\Lambda_w(r(v,w)),-}\Big)\Big]$$

after which one may rely on independence.

The end of the proof of Theorem 1.1, detailed in Sect. 3.6 for the zero-temperature case, applies without change to prove the theorem at positive temperature.

5. Extension to Finite-Range Interactions

At $T=0$, the proof for general finite-range interactions $\mathcal{J}$ remains the same with the following minor changes, in which $C_k(\mathcal{J})$ denote positive constants depending only on $\mathcal{J}$ and $R(\mathcal{J}) = \max\{d(u,v) \,:\, J_{u,v}\neq 0\}$ (the interaction's range).

1) The statement of Theorem 3.1 is changed by replacing the bound $B_\ell(\eta)\leq 2J\,|\partial_{\mathrm{v}}\Lambda(2\ell)|$ by

$$B_\ell(\eta) \;\leq \sum_{(u,v)\in\partial_{\mathrm{e}}\Lambda(2\ell)} J_{u,v}. \tag{5.1}$$

2) The condition $|h+\varepsilon\eta_v| > 4J$ appearing in the proof of Theorem 3.2 is replaced by $|h+\varepsilon\eta_v| > \sum_v J_{\mathbf{0},v}$.
3) The bound (3.35) is replaced by

$$\mathbb{E}(B_\ell) \leq C_1(\mathcal{J})|\partial_{\mathrm{v}}\Lambda(2\ell)|\,m(\ell - R(\mathcal{J})).$$

Consequently in (3.19), $4J|\partial_{\mathrm{v}}\Lambda(2\ell)|\,m(\ell-1)$ becomes $C_2(\mathcal{J})|\partial_{\mathrm{v}}\Lambda(2\ell)|\,m(\ell - R(\mathcal{J}))$.
4) In the proof of Proposition 3.6, the definition of $r(v,w)$ in (3.40) is replaced by

$$r(v,w) := \lfloor (d(v,w) - R(\mathcal{J}))/2\rfloor.$$

The simple bound (3.41) is then used for pairs v,w at distance $d(v,w)\leq R(\mathcal{J})+1$, leading to the factor $|\Lambda(2)|$ appearing in the proof being replaced by $|\Lambda(R(\mathcal{J})+1)|$. The statement of Proposition 3.6 is changed to allow L_0 to depend on $\mathcal{J}$ (besides α).

5) The proof of Theorem 1.1 given in Sect. 3.6 is modified by taking into account the change described in item 3 above in the constants appearing in Proposition 3.4. Correspondingly, inequality (3.48) is modified to

$$\mathbb{P}\left(\frac{D_\ell}{\mathbb{E}(D_\ell)} < \frac{1}{2}\right) \geq \chi\left(\frac{C_3(\mathcal{J})}{\varepsilon} \cdot \left(\frac{L}{\ell - R(\mathcal{J})}\right)^{2\gamma}\right) \geq \chi\left(\frac{C_4(\mathcal{J})}{\varepsilon}\right) \tag{5.2}$$

holding for L sufficiently large, and the power γ appearing in the theorem is modified from its value in (3.45) to

$$\gamma = 2^{-10}\chi\left(\frac{C_4(\mathcal{J})}{\varepsilon}\right). \tag{5.3}$$

At $T > 0$, the argument extends to general finite-range interactions by applying the following changes:

1) The definition of $\tilde{B}_\ell$ in (4.4) is modified to

$$\tilde{B}_\ell(\eta) := \frac{1}{4} \sum_{v \in \partial_{\mathrm{v}} \Lambda(2\ell)} J_v \left[\langle \sigma_v \rangle^{\Lambda(3\ell)\setminus\Lambda(\ell),+} - \langle \sigma_v \rangle^{\Lambda(3\ell)\setminus\Lambda(\ell),-} \right] \tag{5.4}$$

with

$$J_v := \sum_{u:\, (u,v) \in \partial_{\mathrm{e}} \Lambda(2\ell)} J_{u,v}. \tag{5.5}$$

2) The proof of Theorem 4.1 is modified by replacing the inequality (4.16) with

$$\begin{aligned} &-\frac{1}{T}\left(H^{+,+}(\sigma^{+,+}) + H^{-,-}(\sigma^{-,-}) - H^{+,-}(\sigma^{+,-}) - H^{-,+}(\sigma^{-,+})\right) \\ &= \frac{1}{T} \sum_{(u,v)\in\partial_{\mathrm{e}}\Lambda(2\ell)} J_{u,v}\left(\sigma_u^{+,+}\sigma_v^{+,+} + \sigma_u^{-,-}\sigma_v^{-,-} - \sigma_u^{+,+}\sigma_v^{-,-} - \sigma_u^{-,-}\sigma_v^{+,+}\right) \\ &= \frac{1}{T} \sum_{(u,v)\in\partial_{\mathrm{e}}\Lambda(2\ell)} J_{u,v}\left(\sigma_u^{+,+} - \sigma_u^{-,-}\right)\cdot\left(\sigma_v^{+,+} - \sigma_v^{-,-}\right) \\ &= \frac{1}{T} \sum_{(u,v)\in\partial_{\mathrm{e}}\Lambda(2\ell)} J_{u,v}\left(\sigma_u^{+,+} - \sigma_u^{-,-}\right)\cdot\left(\tau_v^{+} - \tau_v^{-}\right) \leq \frac{2}{T} \sum_{v\in\partial_{\mathrm{v}}\Lambda(2\ell)} J_v\left(\tau_v^{+} - \tau_v^{-}\right), \end{aligned} \tag{5.6}$$

with this change propagating to the next two displayed equations in the proof.

3) The changes analogous to those described for the $T = 0$ case.

6. Magnetization Decoupling Bounds

For completeness sake we enclose here proofs that the influence percolation probability $m(\ell, ...)$ provides bounds on both the covariance between the quenched local magnetizations at distant sites and the spin - spin covariance within the Gibbs states at typical configurations of the random field, as was asserted in (1.10) and (1.11). The arguments apply in the generality of the random-field Ising model on a general infinite transitive graph, in any of its infinite-volume Gibbs states.

Lemma 6.1. *In the random field Ising model on a transitive graph, with spin-spin coupling of a finite range $R(\mathcal{J})$ and any pair of vertices $\{u, v\}$ at distance $d(u, v)$. If $d(u, v) > \ell$ then*

$$\mathbb{E}(\langle \sigma_u; \sigma_v \rangle) \leq 2\, m(\ell; T, \mathcal{J}, h, \epsilon), \tag{6.1}$$

while if $d(u, v) \geq 2\ell + R(\mathcal{J})$ then

$$\mathrm{Cov}\Big(\langle \sigma_u \rangle; \langle \sigma_v \rangle \rangle \Big) := \mathbb{E}(\langle \sigma_u \rangle; \langle \sigma_v \rangle) \leq 4\, m(\ell; T, \mathcal{J}, h, \epsilon). \tag{6.2}$$

Proof. i) By the FKG monotonicity of the RFIM Gibbs states, the Gibbs conditional expectation of σ_u, conditioned on the configuration's restriction to the complement of the set $\Lambda_u(\ell)$, satisfies, for any configuration of the random field

$$\langle \sigma_u \rangle^{\Lambda_u(\ell),-} \leq \langle \sigma_u \rangle^{\Lambda_u(\ell), \sigma_{\Lambda_u(\ell)^c}} \leq \langle \sigma_u \rangle^{\Lambda_u(\ell),+}. \tag{6.3}$$

Averaging over $\sigma_{\Lambda_u(\ell)^c}$, one learns that also the infinite-volume expectation value is bracketed by $\langle \sigma_u \rangle^{\Lambda_u(\ell),\pm}$:

$$\langle \sigma_u \rangle^{\Lambda_u(\ell),-} \leq \langle \sigma_u \rangle \leq \langle \sigma_u \rangle^{\Lambda_u(\ell),+}. \tag{6.4}$$

The two equations imply:

$$\Big| \langle \sigma_u \rangle \; - \; \langle \sigma_u \rangle^{\Lambda_u(\ell), \sigma_{\Lambda_u(\ell)^c}} \Big| \leq \Big[\langle \sigma_u \rangle^{\Lambda_u(\ell),+} - \langle \sigma_u \rangle^{\Lambda_u(\ell),-} \Big] \tag{6.5}$$

The covariance of the spins within the infinite-volume Gibbs state can be written as

$$\begin{aligned} \langle \sigma_u; \sigma_v \rangle &= \langle \big(\sigma_u - \langle \sigma_u \rangle \big) \; \sigma_v \rangle \\ &= \langle \big(\langle \sigma_u \rangle^{\Lambda_u(\ell), \sigma_{\Lambda_u(\ell)^c}} - \langle \sigma_u \rangle \big) \; \sigma_v \rangle \end{aligned} \tag{6.6}$$

where the second equation is by the state's Dobrushin-Lanford-Ruelle property and the assumption that $d(u, v) > \ell$.

Combining (6.6) with (6.5) we learn that for any realization of the random field

$$\Big| \langle \sigma_u; \sigma_v \rangle \Big| \; \leq \; \Big[\langle \sigma_u \rangle^{\Lambda_u(\ell),+} - \langle \sigma_u \rangle^{\Lambda_u(\ell),-} \Big]. \tag{6.7}$$

Averaging this relation over the disorder one gets (6.1).

ii) For the second covariance bound let

$$\langle \sigma_u \rangle^{\Lambda_u(\ell),av} := \frac{1}{2} \Big[\langle \sigma_u \rangle^{\Lambda_u(\ell),+} + \langle \sigma_u \rangle^{\Lambda_u(\ell),-} \Big] \tag{6.8}$$

and observe that since the random fields on which $\langle \sigma_u \rangle^{\Lambda_u(\ell),av}$ and $\langle \sigma_v \rangle^{\Lambda_v(\ell),av}$ depend belong to disjoint sets, their covariance vanishes:

$$\mathrm{Cov}\Big(\langle \sigma_u \rangle^{\Lambda_u(\ell),av}, \langle \sigma_u \rangle^{\Lambda_u(\ell),av} \Big) \; = \; 0 \tag{6.9}$$

Furthermore, by (6.4),

$$\Big| \langle \sigma_u \rangle \; - \; \langle \sigma_u \rangle^{\Lambda_u(\ell),av} \Big| \; \leq \; \frac{1}{2} \Big[\langle \sigma_u \rangle^{\Lambda_u(\ell),+} - \langle \sigma_u \rangle^{\Lambda_u(\ell),-} \Big]. \tag{6.10}$$

The claimed (6.2) then follows by a simple application of the general covariance bound:

$$\begin{aligned}\left|\mathrm{Cov}(A,B)-\mathrm{Cov}(\widetilde{A},\widetilde{B})\right| &= \Big|\mathbb{E}[(A-\widetilde{A})B]+\mathbb{E}[\widetilde{A}(B-\widetilde{B})] \\ &\quad +\mathbb{E}[(\widetilde{A}-A)]\mathbb{E}[B]+\mathbb{E}[\widetilde{A}]\mathbb{E}[(\widetilde{B}-B)]\Big| \\ &\le 2\,\|A-\widetilde{A}\|_1\cdot\|B\|_\infty \;+\; 2\,\|B-\widetilde{B}\|_1\cdot\|\widetilde{A}\|_\infty \end{aligned} \tag{6.11}$$

applied to

$$\begin{aligned} A &= \langle\sigma_u\rangle\,, \qquad \widetilde{A} &= \langle\sigma_u\rangle^{\Lambda_u(\ell),av} \\ B &= \langle\sigma_v\rangle\,, \qquad \widetilde{B} &= \langle\sigma_v\rangle^{\Lambda_v(\ell),av} \end{aligned} \tag{6.12}$$

for which, by (6.10) and the definition of $m(\ell;T,\mathcal{J},h,\epsilon)$,

$$\|A-\widetilde{A}\|_1 = \|B-\widetilde{B}\|_1 \;\le\; m(\ell;T,\mathcal{J},h,\epsilon). \tag{6.13}$$

□

7. Discussion and Open Questions

In summary: our study quantifies the analysis of [4,5] that for each value of the external field the model's Hamiltonian almost surely has a unique infinite-volume ground state, and similarly unique positive-temperature Gibbs states. The upper bounds proven here establish that the probability that the ground-state configuration depends on the quenched disorder at distance ℓ away decays by at least an ε-dependent power, and exponentially fast if the disorder parameter is sufficiently large. However, our understanding of the model remains incomplete. Following is a selection of open questions, some with relevance for physics models and some as a challenge to probabilists of related interests.

Exponential vs. power-law decay. As mentioned above, an open question of enduring interest is whether as the disorder parameter (ε/J) is tuned down the ground state's dependence on the quenched disorder makes a transition from exponential decay to a power law. Tentative but admittedly weak arguments have appeared for each of these possibilities ([8,14] and [12]). Also of interest is the corresponding question for the $O(N)$ symmetric models in dimensions $d \le 4$, the latter being the critical dimension for the Imry–Ma phenomenon in the presence of continuous symmetry.

Cluster dynamics. Consider the RFIM dynamics in which a large system with a quenched random field is subject to a slowly varying uniform magnetic field h. For $|h|/\varepsilon$ large enough, the ground state configuration is close to being constant, coinciding with the sign of h. As the uniform field is increased, starting from the sufficiently negative value, the corresponding ground state configuration changes in a sequence of flips, in which a cluster of − spins flips to + spins. Thus the graph is partitioned into connected clusters of sites for which at the given random field η the spins flip at a common value of h. It can be shown that in two dimensions almost surely each flip involves only a finite number of sites, and the mean value of the size of the cluster which flips along with a preselected site is finite throughout the regime in which the ground state spins decorrelate exponentially fast. Does the mean stay finite for arbitrarily small $\varepsilon > 0$?

RFIM with other random field distributions. Our analysis focused on IID Gaussian disorder. In contrast, the theorem of [4,5] applies to a wide class of random field distributions. The Gaussian structure allowed a short-cut in the proof of Theorem 3.2. While we expect the results to be valid also well beyond this case, that is not done here.

Among the other distributions of interest are:

(1) A dilute coercive field, with (η_v) given by independent random variables with $\mathbb{P}(\eta_v = -\infty) = \mathbb{P}(\eta_v = \infty) = \varepsilon$ and $\mathbb{P}(\eta_v = 0) = 1 - 2\varepsilon$. This distribution was considered in [12] where an observation was initially made suggesting the possibility of a transition from exponential to power-law decay of correlations at low ε. (However, subsequent considerations have weakened the case for that, cf. also the discussion in [8]).

(2) Bounded variables, e.g. with (η_v) independent and uniformly distributed in $\{-1, 1\}$ or $[-1, 1]$. The former is of particular relevance for the case of Q-state Potts models with random couplings, for which σ_v takes values in $\{1, ..., Q\}$ and the Hamiltonian is:

$$H_\eta(\sigma) = - \sum_{\{x,y\}\in E(\mathbb{Z}^2)} \left(J + \varepsilon \eta_{x,y}\right) \mathbb{1}[\sigma_x = \sigma_y] \,. \tag{7.1}$$

The uniform bound on $|\eta|$ allows to keep the discussion separate from that of frustration effects.

The more general Imry–Ma phenomenon. While the RFIM is a bellwether for the more general Imry–Ma phenomenon, the general case is a bit more complicated on two accounts. The first is the lack of a-priori obvious pair of opposing boundary conditions for the definition of the order parameter. That can be addressed, as was done in [5], by inducing the $\pm$ states not through boundary conditions but throughout a mild shift of the uniform field beyond the corresponding boundary of the region under study, $h \to h \pm \delta h$ with δh in the range

$$|\Lambda(\ell)|^{-1} \ll \delta h \ll 1 \quad (\text{as } \ell \to \infty). \tag{7.2}$$

(An alternative is to define the order parameter though a maximization of the difference induced by different boundary spin configurations.) A potentially more substantial difference with the RFIM, is that in the general case the natural order parameter does not control the difference in the configurations, or measures, just in their (generalized) magnetizations. The resolution of this complication may require some new technical ideas.

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Appendix A. Exponential Decay at High Disorder

As a rule of thumb it is generally expected that at high enough disorder, be it thermal or due to noisy environment, correlations decay exponentially fast. Results in this vein for systems related to the RFIM can be found in the works of Berretti [6], Imbrie and Fröhlich [16], and Camia et al. [9].

Let us present here an especially simple proof of such behavior for the $T = 0$ case, i.e. exponential decay of the correlations of the RFIM's ground state, and also of the principle that fast enough power-law decay implies exponential decay.

Theorem A.1. *For the RFIM on $\mathbb{Z}^d$ with the nearest-neighbor interaction (1.5) and random field given by IID random variables (η_u), if*

$$\mathbb{P}\big(|h + \varepsilon\eta_{\mathbf{0}}| \le 2dJ\big) \;<\; p_c(d) \tag{A.1}$$

with $p_c(d)$ the critical density for site percolation on $\mathbb{Z}^d$, then $m(L; 0, \mathcal{J}, h, \varepsilon)$ decays exponentially fast in L.

Proof. At sites where $|h + \varepsilon\eta_v| > 2dJ$ the ground-state configuration is dictated by the sign of the local field. Hence disagreement percolation can propagate only along the sites with $|h+\varepsilon\eta_v| \le 2dJ$. In the regime described by (A.1) the exceptional sites form a sub-percolating point process, for which the connectivity probability is known to decay exponentially in the distance [2,19]. □

A boosted version of the above simple argument allows to conclude that if on some scale ℓ the probability of influence propagation is small enough ($1/\ell^{d-1}$ power law with a small prefactor) then on larger scales the influence decays exponentially fast. An analogous statement holds also for $T > 0$, but for simplicity of presentation we present the proof for $T = 0$.

Theorem A.2. *For the RFIM on $\mathbb{Z}^d$ with the nearest-neighbor interaction (1.5), there is a finite constant c_0 (depending only on d) with which: if for some $\ell < \infty$*

$$m(\ell; 0, \mathcal{J}, h, \varepsilon) \;\le\; c_0/\ell^{d-1} \tag{A.2}$$

then for all $L < \infty$

$$m(L; 0, \mathcal{J}, h, \varepsilon) \;\le\; C_1\, e^{-bL/\ell} \tag{A.3}$$

with $C_1, b \in (0, \infty)$ which do not depend on J, h, ϵ and ℓ.

In particular, we learn that $m(L; 0, \mathcal{J}, h, \varepsilon)$ cannot decay by a power law faster than $1/L$ without decaying exponentially.

Proof. In the following we say that a site $v \in \mathbb{Z}^2$ is sensitive to boundary conditions at distance ℓ if

$$\sigma_v^{\Lambda_v(\ell),+} \neq \sigma_v^{\Lambda_v(\ell),-}. \tag{A.4}$$

For each $L > \ell$ the event whose probability defines $m(L)$,

$$\langle\sigma_{\mathbf{0}}\rangle^{\Lambda(L),+} \;\neq\; \langle\sigma_{\mathbf{0}}\rangle^{\Lambda(L),-}, \tag{A.5}$$

requires the existence of a path from $\mathbf{0}$ to the set $\partial_{\mathrm{v}}\Lambda(L-\ell)$ along sites $v \in \mathbb{Z}^2$ at which the condition (A.4) holds.

Let now $\mathcal{P}_\ell$ be a partition of the vertex set of $\mathbb{Z}^d$ into a $\mathbb{Z}^d$-like array of disjoint cubic blocks of side length 2ℓ, and consider the random set of blocks in this partition which contain at least one site for which (A.4) holds. These block events are 1-step independent, in the sense that they are jointly independent for any collection of blocks of which no two are touching.

The probability that an individual block contains a site at which the condition (A.4) holds is trivially dominated by $|\partial_{\mathrm{v}}\Lambda(\ell)| \times m(\ell)$. Adjusting the constant c_0 in assumption (A.2) the above probability can be made as small as convenient. The claim then follows through a standard exponentially-decaying bound on the connectivity probability in 1-step independent percolation of small enough density. □

Appendix B. The Mandelbrot Percolation Analogy

The results presented above do not answer the question whether in two dimensions the exponential decay of correlations persists into arbitrarily small values of the disorder parameter, or whether the exponential decay turns into a power-law decay at low enough (but still non zero) ε. Related to this is the question of what would be a sensible algorithm for the computation of the ground state $\widehat{\sigma}$ for a given random field, and how would it perform at very low disorder.

An intriguing perspective is provided by the following hierarchal algorithm. It has the virtue of simplicity but also the drawback of being potentially misleading through over simplification. It is formulated for the specific case $h = 0$ and nearest-neighbor interaction.

Let $(\mathcal{P}_n)$, $n \geq 0$, be a sequence of nested partitions of $\mathbb{Z}^2$ into square blocks, with the blocks in $\mathcal{P}_n$ having side-length 3^n and the square containing x denoted by $D_{n,x}$. For each n, x we define the following as a *large-field* event in $D_{n,x}$:

$$\mathcal{F}_{n,x} := \{\eta \ : \ \varepsilon \left|\eta(D_{n,x})\right| \ > \ J\, |\partial_{\mathrm{e}} D_{n,x}|\}. \tag{B.1}$$

where $\eta(D) := \sum_{x\in D} \eta_x$ is the total block field.

A relevant feature of two dimensions is that the probabilities of the large-field events are scale invariant:

$$\mathbb{P}(\mathcal{F}_{n,x}) \ = \ \chi(4J/\varepsilon) := p \ \approx \exp[-8J^2/\varepsilon^2]. \tag{B.2}$$

For a given $x \in \mathbb{Z}^2$ the events $\mathcal{F}_{n,x}$ are not strictly independent, however the sequence (in n) of the corresponding indicator functions is easily seen to be asymptotic, in probability, to a stationary and mixing sequence of random variables.

Let $n(x;\eta)$ be the first non-negative integer for which large field is exhibited in $D_{n,x}$. Due to the above properties of the events $\mathcal{F}_{n,x}$ for any $J, \varepsilon > 0$ almost surely $n(x;\eta) < \infty$ for all x.

Under $\mathcal{F}_{0,x}$, i.e. in case the large-field event occurs at x already on the smallest scale, the value of the ground-state configuration at x is predictably given by $\mathrm{sign}(\eta_x)$, i.e. the sign of the field. In case the η_x is itself not large enough to meet this criterion, but the site is separated from the boundary of a set Λ by a loop of sites for which the large-field events occur at scale $n = 0$, one may still conclude that the finite-volume ground state at x does not depend on the boundary spin configuration $\sigma_{\partial_{\mathrm{v}}\Lambda}$.

Scaling up these observations, though along the way departing from rigor, we arrive at the following somewhat over-simplified algorithm for the assignment of a spin configuration $\tau(\eta)$ which may mimic the infinite-volume ground state $\widehat{\sigma}(\eta)$.

For each $x \in \mathbb{Z}^2$ let $k(x;\eta)$ be defined as the smallest $0 \leq k < n(x;\eta)$ for which x is separated from infinity by a loop of sites with $n(x;\eta) \leq k$, if such a k exists, and otherwise set $k(x;\eta) = n(x;\eta)$.

In the first case, i.e. $n(x;\eta) = k(x;\eta)$, we let $\tau(\eta)_x = \text{sign}(\eta(D(x))$. If $k(x;\eta) < n(x;\eta)$, the value of $\tau(\eta)_x$ is determined by minimizing the RFIM energy over the interior of the corresponding x-encapsulating loop, with the previously constructed values serving as boundary conditions for $\tau(\eta)_x$.

For the finite-volume version of the construction, in $\Lambda \subset \mathbb{Z}^2$, the above construction is modified by limiting the considerations of large-field events to cubes contained in Λ. In the last step, unless $\tau^{\Lambda,\pm}(\eta)_x$ is defined already through such events, its calculation will incorporate the boundary conditions imposed at $\partial_v\Lambda$.

Under the above algorithm the influence of the boundary conditions on $\tau(\eta)_x$ percolates over sites for which the events $\mathcal{F}_{n,x}$ did not yet occur. For an idea on the probability that the influence percolates deep inside Λ one may take the further approximation in which the correlations between the indicator functions of nested events $\mathcal{F}_{n,x}$ are ignored.

Under the latter approximation, the collection of sites not covered by any of the large-field events, has the distribution of the random fractal set discussed in Mandelbrot's "canonical curdling" model [18]. In particular, the influence-percolation process coincides with the Mandelbrot-percolation process at density p given by (B.2).

Curiously, as was proven by Chayes–Chayes–Durrett [11], the Mandelbrot-percolation process does undergo a phase transition. Its manifestation in the lattice version of the model is that the connectivity function decays exponentially fast for p large enough, but at p small the decay changes to a power law. (The model is most appealing in its continuum, or "ultraviolet", limit while our discussion is focused on its infinite-volume, or "infrared", limit. However in the analysis there is a simple relation between the two).

It should however be noted that for the finite-volume version of the construction, the existence of a path connecting x to $\partial_v\Lambda$ in the complement of the set of sites covered by large-field events is only a necessary condition for the dependence of $\tau^{\Lambda,\pm}(\eta)_x$ on the boundary conditions. As its value is determined through the energy minimization conditioned on both the $\pm$ boundary conditions and the randomly determined values along the large-field sets, the $\pm$ boundary conditions may lose their effect on $\tau(\eta)_x$ even before the geometric disconnection of x from $\partial_v\Lambda$. Thus the Mandelbrot-percolation's phase transition does not preclude exponential decay of the τ-analog of our finite-volume order parameter at all $p > 0$.

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