SEMINORMAL LOG CENTERS AND DEFORMATIONS OF PAIRS

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The philosophy of Shokurov [Sho92] stresses the importance of understanding the log canonical centers of an lc pair (X, Δ) (see Definition 1). After the initial work of [Kaw98], a systematic study was started by [Amb03]. For extensions, surveys and comprehensive treatments see [Amb06] and [Fuj09]. The following are two of their principal results.

- Any union of log canonical centers is seminormal (see Definition 15).
- Any intersection of log canonical centers is also a union of log canonical centers.

The aim of this note is to extend these results to certain subvarieties of an lc pair (X, Δ) that are close to being a log canonical center. To state our results, we need a definition. (See [KM98] for basic concepts and results related to MMP. As in the above papers, we also work over a field of characteristic 0.)

Definition 1. Let (X, Δ) be lc and $Z \subset X$ an irreducible subvariety. Following Shokurov and Ambro, the *minimal log discrepancy* of Z is the infimum of the numbers $1 + a(E, X, \Delta)$ as E runs through all divisors over X whose center is Z [Amb99]. It is denoted by $\operatorname{mld}(Z, X, \Delta)$.

An irreducible subvariety $Z \subset X$ is called a *log center* of (X, Δ) if $\operatorname{mld}(Z, X, \Delta) < 1$. If $Z \subset X$ is a divisor, then Z is a log center iff it is an irreducible component of Δ and then its coefficient is $1 - \operatorname{mld}(Z, X, \Delta)$.

A *log canonical center* is a log center whose minimal log discrepancy equals 0.

Our first aim is to prove the following. (See Definition 15 for seminormality.)

Theorem 2. *Let (X, Δ) be an lc pair and $Z_i \subset X$ log centers for $i = 1, \dots, m$.*

(1) *If $\operatorname{mld}(Z_i, X, \Delta) < \frac{1}{6}$ for every i then $Z_1 \cup \cdots \cup Z_m$ is seminormal.*
(2) *If $\sum_{i=1}^m \operatorname{mld}(Z_i, X, \Delta) < 1$ then every irreducible component of $Z_1 \cap \cdots \cap Z_m$ is a log center with minimal log discrepancy $\leq \sum_{i=1}^m \operatorname{mld}(Z_i, X, \Delta)$.*

A result of this type is not entirely surprising. By Shokurov's conjecture on the boundedness of complements [Sho92], if $(X, \sum a_i D_i)$ is lc and the a_i are close enough to 1, then there is another lc pair $(X, \Delta' + \sum D_i)$ where the D_i all appear with coefficient 1. Thus the D_i are log canonical centers of $(X, \Delta' + \sum D_i)$ hence their union is seminormal and Du Bois [KK10]. In particular, there should be a function $\epsilon(n) > 0$ such that the union of the D_i with $a_i > 1 - \epsilon(\dim X)$ is seminormal and Du Bois. The function $\epsilon(n)$ is not known, but it must converge to 0 at least doubly exponentially. (See [Kol97, Sec.8] for the conjectured optimal value of $\epsilon(n)$ and for examples.)

Thus it is somewhat unexpected that, at least for seminormality, the bound in Theorem 2.1 is independent of the dimension.

Note that we do not assert that these Z_i are log canonical centers of some other lc pair (X, Δ'); this is actually not true. In particular, unlike log canonical centers, the Z_i are not Du Bois in general; see Example 5.5.

As Examples 5.1–3 show, the value $\frac{1}{6}$ is optimal. There is, however, one important special case when it can be improved to $\frac{1}{2}$. The precise statement is given in Theorem 16; here we mention a consequence which was the main reason of this project. The result implies that if we consider the moduli of lc pairs (X, Δ) where all the coefficients in Δ are $> \frac{1}{2}$, then we do not have to worry about embedded points on Δ. (Examples of Hassett show that embedded points do appear when the coefficients in Δ are $\leq \frac{1}{2}$. See [Kol10, Sec.6] for an overview and the forthcoming [Kol12] for details.)

Corollary 3. *Let $(X, \Delta = \sum_{i\in I} b_i B_i)$ be lc. Let $f : X \to C$ be a morphism to a smooth curve such that $(X, X_c + \Delta)$ is lc for every fiber $X_c := f^{-1}(c)$. Let $J \subset I$ be any subset such that $b_j > \frac{1}{2}$ for every $j \in J$ and set $B_J := \cup_{j\in J} B_j$.*

Then $B_J \to C$ is flat with reduced fibers.

The extension of these results to the semi log canonical case requires additional considerations; these will be treated in [Kol12, Chap 3].

4. The proof of Theorem 2 uses the following recently established result of Birkar [Bir11] and Hacon and Xu [HX11]. For $\dim X \leq 4$, it also follows from earlier results of Shokurov [Sho09].

Theorem 4.1. Let $g : X \to S$ be a projective, birational morphism and Δ', Δ'' effective $\mathbb{Q}$-divisors on X such that $\big(X, \Delta' + \Delta''\big)$ is dlt, $\mathbb{Q}$-factorial and $K_X + \Delta' + \Delta'' \sim_{\mathbb{Q},g} 0$. Then the $\big(X, \Delta''\big)$-MMP with scaling over S terminates with a $\mathbb{Q}$-factorial minimal model.

One of the difficulties in [Amb03, Fuj09] comes from making the proof independent of MMP assumptions. The proof in [HX11] uses several delicate properties of log canonical centers, including some of the theorems of [Amb03, Fuj09]. Thus, although the statement of Theorem 2 sharpens several of the theorems of [Amb03, Fuj09] on lc centers, it does not give a new proof.

Example 5. The following examples show that the numerical conditions of Theorem 2 are sharp.

1. $\big(\mathbb{A}^2, \frac{5}{6}(x^2 = y^3)\big)$ is lc, the curve $(x^2 = y^3)$ is a log center with mld $= \frac{1}{6}$ but it is not seminormal.

2. Consider $\big(\mathbb{A}^3, \frac{11}{12}(z - x^2 - y^3) + \frac{11}{12}(z + x^2 + y^3)\big)$. One can check that this is lc. The irreducible components of the boundary are smooth, but their intersection is a cuspidal curve, hence not seminormal. It is again a log center with mld $= \frac{1}{6}$.

3. The image of $\mathbb{C}^2_{uv}$ by the map $x = u, y = v^3, z = v^2, t = uv$ is a divisor $D_1 \subset X := (xy - zt) \subset \mathbb{C}^4$ and $\mathbb{C}^2 \to D_1$ is an isomorphism outside the origin. Note that the zero set of $(y^2 - z^3)$ is $D_1 + 2(y = z = 0)$. Let D_2, D_3 be 2 general members of the family of planes in the linear system $|(y = z = 0)|$. We claim that $\big(X, \frac{5}{6}D_1 + \frac{5}{6}D_2 + \frac{5}{6}D_3\big)$ is lc. Here D_1 is a log center with mld $= \frac{1}{6}$ but seminormality fails in codimension 3 on X.

In order to check the claim, blow up the ideal (x, z). On $\mathbb{C}^2_{uv}$ this corresponds to blowing up the ideal (u, v^2).

On one of the charts we have coordinates $x_1 := x/z, y, z$ and the birational transform D_1' of D_1 is given by $(y^2 = z^3)$. On the other chart we have coordinates

$x, z_1 := z/x, t$ and D_1' is given by $(z_1x^3 = t^2)$. Thus we see that $\bigl(B_{(x,z)}X, \frac{5}{6}D_1'\bigr)$ is lc. The linear system $|(y = z = 0)|$ becomes base point free on the blow-up, hence $\bigl(B_{(x,z)}X, \frac{5}{6}D_1' + \frac{5}{6}D_2' + \frac{5}{6}D_3'\bigr)$ is lc and so is $\bigl(X, \frac{5}{6}D_1 + \frac{5}{6}D_2 + \frac{5}{6}D_3\bigr)$.

4. Assume that $(X, \sum_{i\in I} a_iD_i)$ has simple normal crossing and $a_i \leq 1$ for every i. Let $J \subset I$ be a subset such that $a_j > 0$ for every $j \in J$ and $\sum_{j\in J} a_j > |J| - 1$. Then every irreducible component of $\cap_{j\in J} D_j$ is a log center of $(X, \sum_{i\in I} a_iD_i)$ with mld $= \sum_{j\in J}(1 - a_j) = |J| - \sum_{j\in J} a_j$. In particular, D_i is a log center of $(X, \sum_{i\in I} a_iD_i)$ with mld $= 1 - a_i$. Thus Theorem 2.2 is sharp. By [KM98, Sec.2.3], every log center of $(X, \sum_{i\in I} a_iD_i)$ arises this way.

5. Let X be a smooth variety and $D \subset X$ a reduced divisor. Then D is Du Bois iff (X, D) is lc. (See [KSS10, KK10] for much stronger results.) In particular, $D := (x^2 + y^3 + z^7 = 0) \subset \mathbb{A}^3$ is a log center of the lc pair $\bigl(\mathbb{A}^3, \frac{41}{42}D\bigr)$ with mld $= \frac{1}{42}$ but D is not Du Bois and it can not be an lc center of any lc pair (X, Δ). On the other hand, D is normal hence seminormal.

6 (Log centers and birational maps)**.** Let $g : (Y, \Delta_Y) \to (X, \Delta_X)$ be a proper birational morphism between lc pairs (with Δ_X, Δ_Y not necessarily effective) such that $K_Y + \Delta_Y \sim_{\mathbb{Q}} g^*\bigl(K_X + \Delta_X\bigr)$ and $g_*\Delta_Y = \Delta_X$.

If $Z \subset Y$ is a log center of (Y, Δ_Y) then $g(Z)$ is also a log center of (X, Δ_X) with the same mld. Moreover, every log center of (X, Δ_X) is the image of a log center of (Y, Δ_Y).

Thus, for any (X, Δ_X), we can use a log resolution $g : (Y, \Delta_Y) \to (X, \Delta_X)$ to reduce the computation of log centers to the simple normal crossing case considered in Example 5.4.

This implies that an lc pair (X, Δ) has only finitely many log centers and the union of all log centers of codimension ≥ 2 is the smallest closed subscheme $W \subset X$ such that $\bigl(X \setminus W, \Delta|_{X\setminus W}\bigr)$ is canonical.

7 (Proof of the divisorial case of Theorem 2)**.** We show Theorem 2 in the special case when (X, Δ') is dlt for some Δ' and the $Z_i =: D_i$ are $\mathbb{Q}$-Cartier divisors.

Since (X, Δ') is dlt, X is Cohen–Macaulay and so is $\sum D_i$ [KM98, 5.25]. In particular, $\sum D_i$ satisfies Serre's condition S_2. An S_2-scheme is seminormal iff it is seminormal at its codimension 1 points. By localization at codimension 1 points we are reduced to the case when $\dim X = 2$.

Then X has a quotient singularity at every point of $\sum D_i$, and Reid's covering method [K$^+$92, 20.3] reduces the claim to the smooth case. It is now an elementary exercise to see that if $(\mathbb{A}^2, \sum a_iD_i)$ is lc and $a_i > \frac{5}{6}$ then $\sum D_i$ has only ordinary nodes, hence it is seminormal.

Next we prove Theorem 2.2 assuming that $m = 2$ and $Z_i =: D_i$ are $\mathbb{Q}$-Cartier divisors. Then every irreducible component of $D_1 \cap D_2$ has codimension 2, thus it is again enough to check the smooth surface case. The exceptional divisor of the blow up of $x \in D_1 \cap D_2$ shows that x is a log center with mld $\leq (1 - a_1) + (1 - a_2)$.

Any argument along this line breaks down completely if we only assume that $(X, \sum a_iD_i)$ is lc. In general the D_i are not S_2, not even if $a_i = 1$. Thus seminormality at codimension 1 points does not imply seminormality.

Instead, we choose a suitable dlt model (Y, Δ_Y) of (X, Δ), use (7) on it and then descend seminormality from Y to X. The next two lemmas will be used to construct (Y, Δ_Y).

Lemma 8. *Let (X,Δ) be lc. Then there is a projective, birational morphism $g:(Y,\Delta_Y)\to(X,\Delta)$ such that*

1. *$\bigl(Y,\Delta_Y\bigr)$ is dlt, $\mathbb{Q}$-factorial (and Δ_Y is effective),*
2. *$K_Y+\Delta_Y\sim_{\mathbb{Q}} g^*\bigl(K_X+\Delta\bigr)$,*
3. *for every log center $Z\subset X$ of (X,Δ) there is a divisor $D_Z\subset Y$ such that $g(D_Z)=Z$ and D_Z appears in Δ_Y with coefficient $1-\operatorname{mld}(Z,X,\Delta)$.*

Proof. This is well known. Under suitable MMP assumptions, a proof is given in [K$^+$92, 17.10]. One can remove the MMP assumptions as follows.

A method of Hacon (cf. [KK10, 3.1]) constructs a model satisfying (1–2). Since there are only finitely many log centers, it is enough to add the divisors D_Z one at a time. This is explained in [Kol08, 37]. A simplified proof is in [Fuj10, Sec.4]. □

Lemma 9. *Let $g:Y\to X$ be a projective, birational morphism and Δ_1,Δ_2 effective $\mathbb{Q}$-divisors on Y. Assume that $\bigl(Y,\Delta_1+\Delta_2\bigr)$ is dlt, $\mathbb{Q}$-factorial and $K_Y+\Delta_1+\Delta_2\sim_{\mathbb{Q},g}0$. By (4.1) a suitable (Y,Δ_2)-MMP over X terminates with a $\mathbb{Q}$-factorial minimal model $g^m:\bigl(Y^m,\Delta_2^m\bigr)\to X$. Then*

1. *$-\Delta_1^m$ is g^m-nef,*
2. *$g\bigl(\Delta_1\bigr)=g^m\bigl(\Delta_1^m\bigr)$ and*
3. *$\operatorname{Supp}(g^m)^{-1}\bigl(g^m\bigl(\Delta_1^m\bigr)\bigr)=\operatorname{Supp}\Delta_1^m$.*

Proof. Since $K_Y+\Delta_1+\Delta_2\sim_{\mathbb{Q},g}0$, we see that $K_{Y^m}+\Delta_1^m+\Delta_2^m\sim_{\mathbb{Q},g^m}0$. Thus $-\Delta_1^m\sim_{\mathbb{Q},g^m}K_{Y^m}+\Delta_2^m$ is g^m-nef. Since g^m has connected fibers and Δ_1^m is effective, every fiber of g^m is either contained in $\operatorname{Supp}\Delta_1^m$ or is disjoint from it. This proves (3).

In order to see (2), we prove by induction that, at every intermediate step $g^i:\bigl(Y^i,\Delta_2^i\bigr)\to X$ of the MMP, we have $g\bigl(\Delta_1\bigr)=g^i\bigl(\Delta_1^i\bigr)$. This is clear for $Y^0:=Y$. As we go from i to $i+1$, the image $g^i\bigl(\Delta_1^i\bigr)$ is unchanged if $Y^i\dashrightarrow Y^{i+1}$ is a flip. Thus we need to show that $g^{i+1}\bigl(\Delta_1^{i+1}\bigr)=g^i\bigl(\Delta_1^i\bigr)$ if $\pi_i:Y^i\to Y^{i+1}$ is a divisorial contraction with exceptional divisor E^i. Let $F^i\subset E^i$ be a general fiber of $E^i\to X$. It is clear that

$$g^{i+1}\bigl(\Delta_1^{i+1}\bigr)\subset g^i\bigl(\Delta_1^i\bigr)$$

and equality fails only if E^i is a component of Δ_1^i but no other component of Δ_1^i intersects F^i. Since π_i contracts a $\bigl(K_{Y^i}+\Delta_2^i\bigr)$-negative extremal ray, $-\Delta_1^i\sim_{\mathbb{Q},g^i}K_{Y^i}+\Delta_2^i$ shows that Δ_1^i is π_i-nef. However, an exceptional divisor has negative intersection with some contracted curve; a contradiction. □

10 (Proof of Theorem 2.1)**.** Set $\epsilon_i:=\operatorname{mld}(Z_i,X,\Delta)$. As in Lemma 8, let $g:(Y,\Delta_Y)\to(X,\Delta)$ be a $\mathbb{Q}$-factorial dlt model and $D_i\subset Y$ divisors such that $a(D_i,X,\Delta)=-1+\epsilon_i$ and $g(D_i)=Z_i$. Set $D:=\sum_{i=1}^m D_i$; then $g(D)=Z$.

Pick $1>c\geq 0$ such that $1-\epsilon_i\geq c$ for every i and write $\Delta_Y=cD+\Delta_2$ where Δ_2 is effective. (It may have common components with D.) Apply Lemma 9 to get a $\mathbb{Q}$-factorial model $g^m:Y^m\to X$ such that

1. $\bigl(Y^m,cD^m+\Delta_2^m\bigr)$ is lc,
2. $\bigl(Y^m,\Delta_2^m\bigr)$ is dlt,
3. $K_{Y^m}+cD^m+\Delta_2^m\sim_{\mathbb{Q},g^m}0$,
4. $-D^m$ is g^m-nef,
5. $\operatorname{Supp}D^m=\operatorname{Supp}(g^m)^{-1}(Z)$ and hence $g^m\bigl(D^m\bigr)=Z$.

If $\epsilon_i < \frac{1}{6}$ for every i then we can assume that $c > \frac{5}{6}$. As we noted in (7), in this case D^m is seminormal and Lemma 11 shows that $g^m_* \mathcal{O}_{D^m} = \mathcal{O}_Z$. Thus Z is seminormal by Lemma 19. □

Lemma 11. *Let Y, X be normal varieties and $g : Y \to X$ a proper morphism such that $g_*\mathcal{O}_Y = \mathcal{O}_X$. Let D be a reduced divisor on Y and Δ'' an effective $\mathbb{Q}$-divisor on Y. Fix some $0 < c \leq 1$. Assume that*

1. *$\big(Y, cD + \Delta''\big)$ is lc,*
2. *$\big(Y, \Delta''\big)$ is dlt,*
3. *$K_Y + cD + \Delta'' \sim_{\mathbb{Q},g} 0$ and*
4. *$-D$ is g-nef (and hence $D = g^{-1}\big(g(D)\big)$.).*

Then $g_*\mathcal{O}_D = \mathcal{O}_{g(D)}$.

Proof. By pushing forward the exact sequence

$$0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$$

we obtain

$$\mathcal{O}_X = g_*\mathcal{O}_Y \to g_*\mathcal{O}_D \to R^1g_*\mathcal{O}_X(-D).$$

Note that

$$-D \sim_{\mathbb{Q},g} K_Y + \Delta'' + (1-c)(-D)$$

and the right hand side is of the form $K + \Delta + (g-\text{nef})$. Let $W \subset Y$ be an lc center of $\big(Y, \Delta''\big)$. Then W is not contained in D since then $\big(Y, cD + \Delta''\big)$ would not be lc along W. In particular, D is disjoint from the general fiber of $W \to X$ by (4). Thus from Theorem 13 we conclude that none of the associated primes of $R^1g_*\mathcal{O}_Y(-D)$ is contained in $g(D)$. On the other hand, $g_*\mathcal{O}_D$ is supported on $g(D)$, hence $g_*\mathcal{O}_D \to R^1g_*\mathcal{O}_Y(-D)$ is the zero map.

This implies that $\mathcal{O}_X \to g_*\mathcal{O}_D$ is surjective. This map factors through $\mathcal{O}_{g(D)}$, hence $g_*\mathcal{O}_D = \mathcal{O}_{g(D)}$. □

12 (A curious property of log centers)**.** Assume that (X, Δ) is klt and let $Z \subset X$ be a union of arbitrary log centers. As in (10) we construct $(Y, cD + \Delta'')$ which is klt. Thus, as we apply Lemma 11, the higher direct images $R^ig_*\mathcal{O}_Y$ and $R^ig_*\mathcal{O}_Y(-D)$ are zero for $i > 0$. Thus D is a reduced Cohen-Macaulay scheme D such that

$$g_*\mathcal{O}_D = \mathcal{O}_Z \quad \text{and} \quad R^ig_*\mathcal{O}_D = 0 \quad \text{for } i > 0.$$

Moreover, D is a divisor on a $\mathbb{Q}$-factorial klt pair.

This looks like a very strong property for a reduced scheme Z, but so far I have been unable to derive any useful consequences of it. In fact, I do not know how to prove that not every reduced scheme Z admits such a morphism $g : D \to Z$.

We have used the following form of [Amb03, 3.2, 7.4] and [Fuj09, 2.52].

Theorem 13. *Let $g : Y \to X$ be a projective morphism and M a line bundle on Y. Assume that $M \sim_{\mathbb{Q},g} K_Y + L + \Delta$ where (Y, Δ) is dlt and for every log canonical center $Z \subset Y$, the restriction of L to the general fiber of $Z \to X$ is semiample.*

Then every associated prime of R^ig_*M is the image of a log canonical center of (Y, Δ). □

14 (Proof of Theorem 2.2)**.** By induction on m, it is enough to prove Theorem 2.2 for the intersection of 2 log centers.

Let $g : (Y, \Delta_Y) \to (X, \Delta)$ be a $\mathbb{Q}$-factorial dlt model and $D_1, D_2 \subset Y$ divisors such that $a(D_i, X, \Delta) = -1 + \operatorname{mld}(Z_i, X, \Delta)$ and $g(D_i) = Z_i$. Set $D := D_1 + D_2$.

Pick any $c > 0$ such that $\Delta_Y = cD + \Delta_2$ where Δ_2 is effective and apply Lemma 9. Thus we get a $\mathbb{Q}$-factorial model $g^m : Y^m \to X$ such that

(1) $K_{Y^m} + cD^m + \Delta_2^m \sim_{\mathbb{Q},g^m} 0$,
(2) $\operatorname{Supp} D^m = \operatorname{Supp}(g^m)^{-1}(Z_1 \cup Z_2)$.

By (2), every irreducible component $V_j \subset Z_1 \cap Z_2$ is dominated by an irreducible component of $W_j \subset D_1^m \cap D_2^m$. By (7), each W_j is a log center of $\big(Y^m, cD^m + \Delta_2^m\big)$ with mld $\leq \operatorname{mld}(Z_1, X, \Delta) + \operatorname{mld}(Z_2, X, \Delta)$. Thus V_j is a log center of (X, Δ) with the same minimal log discrepancy. $\square$

Definition 15. Let X be a reduced scheme and $U \subset X$ an open subscheme. We say that X is *seminormal relative to* U if every finite, universal homeomorphism $\pi : X' \to X$ that is an isomorphism over U is an isomorphism.

If this holds with $U = \emptyset$, then X is called *seminormal*. For more details, see [Kol96, Sec.I.7.2].

If X satisfies Serre's condition S_2 then seminormality depends only on the codimension 1 points of X. That is, X is seminormal relative to U iff there is a closed subset $Z \subset X$ of codimension ≥ 2 such that $X \setminus Z$ is seminormal relative to U.

With this definition, we can state the theorem behind Corollary 3 as follows.

Theorem 16. *Let $(X, S + \Delta)$ be an lc pair where S is $\mathbb{Q}$-Cartier. Let $Z_i \subset X$ be log centers of (X, Δ) for $i = 1, \dots, m$.*

If $\operatorname{mld}(Z_i, X, \Delta) < \frac{1}{2}$ for every i then $S \cup Z_1 \cup \cdots \cup Z_m$ is seminormal relative to $X \setminus S$.

Proof. By passing to a cyclic cover and using Lemma 18 we may assume that S is Cartier.

Next we closely follow (10). Let $g : (Y, S_Y + \Delta_Y) \to (X, S + \Delta)$ be a $\mathbb{Q}$-factorial dlt model and $D_i \subset Y$ divisors such that $a(D_i, X, \Delta) = -1 + \operatorname{mld}(Z_i, X, \Delta)$ and $g(D_i) = Z_i$. Pick $c > \frac{1}{2}$ such that $1 - \operatorname{mld}(Z_i, X, \Delta) \geq c$ for every i. Set $D := S_Y + \sum_i D_i$ and write $\Delta_Y = cD + \Delta_2$ where Δ_2 is effective.

Apply Lemma 9 to get a $\mathbb{Q}$-factorial model $g^m : Y^m \to X$ such that

(1) $\big(Y^m, cD^m + \Delta_2^m\big)$ is lc,
(2) $\big(Y^m, \Delta_2^m\big)$ is dlt,
(3) $K_{Y^m} + cD^m + \Delta_2^m \sim_{\mathbb{Q},g^m} 0$,
(4) $-D^m$ is g^m-nef and
(5) $g^m\big(D^m\big) = S \cup Z_1 \cup \cdots \cup Z_m$.

Using Lemmas 11 and 19 we see that it is enough to prove that D^m is seminormal relative to $Y^m \setminus S_Y^m$.

Since Y^m is dlt, it is Cohen–Macaulay hence D^m is S_2. As we noted in Definition 15, it is sufficient to check seminormality at codimension 2 points of Y^m. As in (7), this reduces to the smooth surface case. We see that if F is a smooth surface, $(F, S + cD)$ is lc and $c > \frac{1}{2}$ then, at every point of $S \cap D$, D is smooth and intersects S transversally. Thus $S + D$ is seminormal at all points of $S \cap D$. $\square$

Again note that the bound $\frac{1}{2}$ is sharp; $\big(\mathbb{A}^2, (x = 0) + \frac{1}{2}(x + y = 0) + \frac{1}{2}(x - y = 0)\big)$ is lc but its boundary is not seminormal at the origin.

17 (Proof of Corollary 3)**.** None of the irreducible components of Δ is contained in a fiber of f, hence $f : B_J \to C$ is flat. The main point is to show that its fibers are reduced.

If $b_j > \frac{1}{2}$ then the corresponding divisor B_j is a log center and $\operatorname{mld}(B_j, X, \Delta) = 1 - b_j < \frac{1}{2}$. Thus, by Theorem 16, $X_c + B_J$ is seminormal relative to $X \setminus X_c$ for every $c \in C$. By Lemma 20 this implies that $X_c \cap B_J$ is reduced. □

We have used three easy properties of seminormal schemes.

Lemma 18. *Let $g : Y \to X$ be a finite morphism of normal schemes. Let $Z \subset X$ be a closed, reduced subscheme and $U \subset X$ an open subscheme. If $\operatorname{red} g^{-1}(Z)$ is seminormal relative to $g^{-1}U$ then Z is seminormal relative to U.*

Proof. We may assume that X, Y are irreducible and affine. Let $\pi : Z' \to Z$ be a finite, universal homeomorphism that is an isomorphism over $Z \cap U$. Pick $\phi \in \mathcal{O}_{Z'}$. Since $\operatorname{red} g^{-1}(Z)$ is seminormal relative to $g^{-1}U$, the pull back $\phi \circ g$ is a regular function on $\operatorname{red} g^{-1}(Z)$. We can lift it to a regular function Φ_X on X. Since Y is normal,

$$\Phi_Y := \tfrac{1}{\deg X/Y} \operatorname{tr}_{X/Y} \Phi_X$$

is regular on Y and $\Phi_Y|_Z = \phi$. Thus Z is seminormal relative to U. □

Lemma 19. *Let $g : Y \to X$ be a proper morphism of reduced schemes such that $g_*\mathcal{O}_Y = \mathcal{O}_X$. Let $U \subset X$ be an open subscheme. If Y is seminormal relative to $g^{-1}U$ then X is seminormal relative to U.*

Proof. Let $\pi : X' \to X$ be a finite, universal homeomorphism that is an isomorphism over U. Set $Y' := \operatorname{red}(Y \times_X X')$ with projection $\pi_Y : Y' \to Y$. Then π_Y is a finite, universal homeomorphism that is an isomorphism over $g^{-1}U$. Thus π_Y is an isomorphism, so we can factor g as $Y \to X' \to X$. This implies that $\pi_*\mathcal{O}_{X'} \subset g_*\mathcal{O}_Y = \mathcal{O}_X$, hence π is an isomorphism. □

Lemma 20. *Let X be semi normal relative to U. Let $X_1, X_2 \subset X$ be closed, reduced subschemes such that $X = X_1 \cup X_2$. Then $\mathcal{O}_{X_1 \cap X_2}$ has no nilpotent elements whose support is in $X \setminus U$.*

Proof. Let $I \subset \mathcal{O}_{X_1 \cap X_2}$ be the ideal sheaf of nilpotent elements whose support is in $X \setminus U$ and $r(X_1 \cap X_2) \subset X_1 \cap X_2$ the corresponding subscheme.

Let $r_i : \mathcal{O}_{X_i} \to \mathcal{O}_{X_1 \cap X_2}$ and $\bar{r}_i : \mathcal{O}_{X_i} \to \mathcal{O}_{r(X_1 \cap X_2)}$ denote the restriction maps. Then $\mathcal{O}_X$ sits in an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X_1} + \mathcal{O}_{X_2} \stackrel{(r_1, -r_2)}{\longrightarrow} \mathcal{O}_{X_1 \cap X_2} \to 0.$$

The similar sequence

$$0 \to A \to \mathcal{O}_{X_1} + \mathcal{O}_{X_2} \stackrel{(\bar{r}_1, -\bar{r}_2)}{\longrightarrow} \mathcal{O}_{r(X_1 \cap X_2)} \to 0$$

defines a coherent sheaf of $\mathcal{O}_X$-algebras A and $\operatorname{Spec}_X A \to X$ is a finite, universal homeomorphism $\pi : X' \to X$ that is an isomorphism over U. Since X is semi normal relative to U, $A = \mathcal{O}_X$ hence $X_1 \cap X_2 = r(X_1 \cap X_2)$. □

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