PANEITZ OPERATOR FOR METRICS NEAR S³

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ABSTRACT. We derive the first and second variation formula for the Green's function pole's value of Paneitz operator on the standard three sphere. In particular it is shown that the first variation vanishes and the second variation is nonpositively definite. Moreover, the second variation vanishes only at the direction of conformal deformation. We also introduce a new invariant of the Paneitz operator and illustrate its close relation with the second eigenvalue and Sobolev inequality of Paneitz operator.

1. INTRODUCTION

The fourth order Q curvature equation ([Br, P]) has attracted interest due to the successful study in dimension four and its application to conformal geometry in dimension four ([CGY]). We are interested to possibly extend this analysis to dimension three. The effort to understand Q curvature in dimension three motivates many intriguing and challenging problems. In this dimension, the functions in H^2 are actually $\frac{1}{2}$ -Holder continuous, and hence the Green's function has well defined value at its pole. The sign of this value turns out to be an important issue. For the standard sphere, the Green's function is nonpositive everywhere but vanishes exactly at the pole. Our purpose in this article is to study this question for the conformal structures near the standard sphere.

Recall on a three manifold, the Q curvature is given by

$$Q = -\frac{1}{4}\Delta R - 2\left|Rc\right|^2 + \frac{23}{32}R^2,$$
(1.1)

and the fourth order Paneitz operator is defined as

$$P\varphi = \Delta^2 \varphi + 4 \operatorname{div} \left(\operatorname{Rc} \left(\nabla \varphi, e_i \right) e_i \right) - \frac{5}{4} \operatorname{div} \left(\operatorname{R} \nabla \varphi \right) - \frac{1}{2} Q \varphi.$$
 (1.2)

Here Rc is the Ricci curvature, R is the scalar curvature and e_1, e_2, e_3 is a local orthonormal frame with respect to the metric. For any positive smooth function ρ , the operator satisfies

$$P_{\rho^{-4}g}\varphi = \rho^7 P_g\left(\rho\varphi\right). \tag{1.3}$$

As a consequence we know ker $P_g = 0 \Leftrightarrow \ker P_{\rho^{-4}g} = 0$ and under this assumption, the Green's functions of P satisfy the transformation law

$$G_{\rho^{-4}g}(p,q) = \rho(p)^{-1} \rho(q)^{-1} G_g(p,q).$$
(1.4)

Let (M, q) be a smooth compact three dimensional Riemannian manifold, for $u, v \in C^{\infty}(M)$, we write

$$E(u,v) = \int_{M} Pu \cdot v d\mu$$

$$= \int_{M} \left(\Delta u \Delta v - 4Rc \left(\nabla u, \nabla v \right) + \frac{5}{4} R \nabla u \cdot \nabla v - \frac{1}{2} Quv \right) d\mu$$
(1.5)

and E(u) = E(u, u). Here μ is the measure associated with metric g. It is clear that E(u, v) makes sense for $u, v \in H^2(M)$.

The scaling invariant quantity $\mu\left(M\right)^{\frac{1}{3}}\int_{M}Qd\mu$ satisfies

$$\mu_{\rho^{-4}g}\left(M\right)^{\frac{1}{3}}\int_{M}Q_{\rho^{-4}g}d\mu_{\rho^{-4}g} = -2E_{g}\left(\rho\right)\left\|\rho^{-1}\right\|_{L^{6}(M,g)}^{2}.$$

Hence

$$\begin{split} \sup_{\widetilde{g} \in [g]} \widetilde{\mu} \left(M \right)^{\frac{1}{3}} \int_{M} \widetilde{Q} d\widetilde{\mu} &= -2 \inf_{\rho \in C^{\infty}, \rho > 0} E\left(\rho \right) \left\| \rho^{-1} \right\|_{L^{6}}^{2} \\ &= -2 \inf_{u \in H^{2}(M), u > 0} E\left(u \right) \left\| u^{-1} \right\|_{L^{6}}^{2}. \end{split}$$

Here [g] is the conformal class of metrics associated with g. As in [HY1], we write

$$I_4(u) = E(u) \left\| u^{-1} \right\|_{L^6}^2$$
(1.6)

and

$$Y_4(g) = \inf_{u \in H^2(M), u > 0} E(u) \left\| u^{-1} \right\|_{L^6}^2.$$
(1.7)

From above discussion we see $Y_4(\tilde{g}) = Y_4(g)$ for $\tilde{g} \in [g]$.

The question of whether $Y_4(g)$ is finite and achieved by some particular metrics was considered in [HY1, YZ]. This inequality is analytically different from the one of Yamabe invariant Y(g) (see [LP]) due to the negative power involved.

[HY1] shows that when ker P = 0, the value of the Green's function at pole plays a crucial role. In particular based on explicit calculation of this value on Berger's sphere, we were able to show $Y_4(g)$ is achieved on all Berger spheres. In general such an explicit formula is not available. On the other hand, properties of Paneitz operator on the standard three sphere are well understood.

On standard S^3 , we have

$$Pu = \Delta^2 u + \frac{1}{2}\Delta u - \frac{15}{16}u.$$
 (1.8)

Let N be the north pole, $\pi_N : S^3 \setminus \{N\} \to \mathbb{R}^3$ be the stereographic projection, using $x = \pi_N$ as the coordinates, the Green's function of P with pole at N is given by

$$G_N = -\frac{1}{4\pi} \frac{1}{\sqrt{|x|^2 + 1}}.$$
(1.9)

In particular $G_N(N) = 0$.

Proposition 1.1. Let g be the standard metric on S^3 , then for any $p \in S^3$ and any smooth symmetric (0,2) tensor h,

$$\partial_t|_{t=0} G_{g+th}(p,p) = 0.$$

Here G_{g+th} is the Green's function of the Paneitz operator P_{g+th} .

This calculation leads one to ask about the second variation. We have

Theorem 1.1. Using the stereographic projection π_N as the coordinate, the standard metric g on S^3 is written as

$$g = \frac{4}{\left(|x|^2 + 1\right)^2} |dx|^2 = \tau^{-4} |dx|^2, \qquad (1.10)$$

here

$$\tau = \sqrt{\frac{|x|^2 + 1}{2}}.$$
(1.11)

For any smooth symmetric (0,2) tensor h, denote

$$\theta = \tau^4 h, \tag{1.12}$$

then we have

$$\left. \partial_t^2 \right|_{t=0} G_{g+th}\left(N,N\right) \tag{1.13}$$

$$= -\frac{1}{64\pi^2} \int_{\mathbb{R}^3} \left(\sum_{ij} \left(\theta_{ikjk} + \theta_{jkik} - (tr\theta)_{ij} - \Delta\theta_{ij} \right)^2 - \frac{3}{2} \left(\theta_{ijij} - \Delta tr\theta \right)^2 \right) dx.$$

Here the derivatives θ_{ikjk} etc are taken with respect to the standard metric on \mathbb{R}^3 .

Using formula (1.13), we will show the second variation is always nonpositive and it vanishes only in the direction generated by conformal diffeomorphism. More precisely,

Proposition 1.2. For any smooth symmetric (0,2) tensor h on S^3 and $p \in S^3$,

$$\partial_t^2 \big|_{t=0} G_{g+th}(p,p) \le 0$$
 (1.14)

Moreover, $\partial_t^2|_{t=0} G_{g+th}(p,p) = 0$ if and only if $h = L_X g + f \cdot g$ for some smooth vector fields X and smooth function f on S^3 .

It is worth pointing out that in [HY2], motivated from recent works [GM, HR] for Q curvature in dimension five or higher and Proposition 1.1 and 1.2 above, it was shown that for smooth compact three manifold (M, g) with positive scalar and Q curvature, the Paneitz operator must have zero kernel and the Green's function pole's value is strictly negative except when (M, g) is conformal diffeomorphic to the standard S^3 . Further developments can be found in [HY3, HY4].

The Sobolev inequality of Paneitz operator on S^3 was first verified in [YZ]. Different proofs were given in [H, HY1]. The new approach motivates the condition NN and condition P for a Paneitz operator (see [HY1, section 5]). Here we will introduce a quantity for the Paneitz operator whose sign corresponds to condition NN and P. Let (M, g) be a smooth compact three dimensional Riemannian manifold without boundary. For any $p \in M$, denote (recall functions in $H^2(M)$ are $\frac{1}{2}$ -Holder continuous)

$$\nu(M, g, p) = \inf\left\{\frac{E(u)}{\int_{M} u^{2} d\mu} : u \in H^{2}(M) \setminus \{0\}, u(p) = 0\right\}.$$
(1.15)

When no confusion could arise we denote it as $\nu(g, p)$ or ν_p . We also write

$$\nu(M,g) = \inf_{p \in M} \nu(M,g,p)$$
(1.16)
= $\inf \left\{ \frac{E(u)}{\int_M u^2 d\mu} : u \in H^2(M) \setminus \{0\}, u(p) = 0 \text{ for some } p \right\}.$

The importance of $\nu(M,g)$ lies in that (M,g) satisfies condition P if and only if $\nu(M,g) > 0$ and it satisfies condition NN if and only if $\nu(M,g) \ge 0$. For the standard metric g on S^3 , $\nu(S^3,g) = 0$, in fact we have (see Example 4.1) $\nu(S^3,g,p) = 0$ for all $p \in S^3$. **Theorem 1.2.** Let g be the standard metric on S^3 , then for any $p \in S^3$ and any smooth symmetric (0,2) tensor h,

$$\partial_t|_{t=0} \,\nu \,(g+th,p) = 0 \tag{1.17}$$

and

$$\partial_t^2 \big|_{t=0} \nu \left(g + th, p \right) = -16 \left. \partial_t^2 \right|_{t=0} G_{g+th} \left(p, p \right). \tag{1.18}$$

In particular, $\partial_t^2 \big|_{t=0} \nu (g+th, p) \ge 0$ and it vanishes if and only if $h = L_X g + f \cdot g$ for some smooth vector fields X and smooth functions f on S^3 .

Roughly speaking Theorem 1.2 tells us for Riemannian metrics near the standard metric on S^3 , as long as it is not conformal diffeomorphic to the standard sphere, condition P is satisfied and hence $Y_4(g)$ is achieved by [HY1]. More related results can be found in [HY2, HY3].

In section 2 below we will introduce technique simplifying various calculations. In section 3 we will derive the first and second variation formulas and justify its nonpositivity. In section 4, we will study the quantity $\nu(g)$ and its relations to $Y_4(g)$ and the second eigenvalue of Paneitz operator. Some of the lengthy calculations are collected in the appendix to streamline the discussions.

2. Some preparations

Because the formula of Q curvature and Paneitz operator are relatively complicated, it is crucial to take advantage of the conformal covariant property (1.3) to simplify the calculation of first and second variation of the Green's function pole's value. To achieve this we observe that the Paneitz operator gives us a sequence of fourth order conformal covariant operators. Indeed for smooth metric g and symmetric (0, 2) tensor h, we define the operator $P_{g,h}^{(k)}$ by the Taylor expansion

$$P_{g+th}\varphi \sim \sum_{k=0}^{\infty} t^k P_{g,h}^{(k)}\varphi.$$
 (2.1)

Here \sim means for any $m \geq 0$,

$$P_{g+th}\varphi = \sum_{k=0}^{m} t^k P_{g,h}^{(k)}\varphi + O\left(t^{m+1}\right)$$

as $t \to 0$.

Lemma 2.1. For any smooth function φ and positive smooth function ρ ,

$$P_{\rho^{-4}g,\rho^{-4}h}^{(k)}\varphi = \rho^7 P_{g,h}^{(k)}(\rho\varphi) \,. \tag{2.2}$$

This is the conformal covariant property of $P_{a,h}^{(k)}$. Indeed for t near 0,

$$P_{\rho^{-4}(g+th)}\varphi = \rho^7 P_{g+th} \left(\rho\varphi\right) = P_{\rho^{-4}g+t\rho^{-4}h}\varphi,$$

hence

$$\sum_{k=0}^{\infty} t^k P_{\rho^{-4}g,\rho^{-4}h}^{(k)} \varphi \sim \rho^7 \sum_{k=0}^{\infty} t^k P_{g,h}^{(k)}\left(\rho\varphi\right).$$

Equation (2.2) follows.

Careful calculation shows (see appendix)

$$P_{g,h}^{(1)}\varphi \qquad (2.3)$$

$$= -h_{ij} (\Delta\varphi)_{ij} - \Delta (h_{ij}\varphi_{ij}) - \frac{1}{2} (2h_{ijj} - (trh)_i) (\Delta\varphi)_i - \frac{1}{2} \Delta ((2h_{ijj} - (trh)_i) \varphi_i)$$

$$+ 2 (2h_{ikjk} - \Delta h_{ij} - (trh)_{ij}) \varphi_{ij} - 8Rc_{ij}h_{ik}\varphi_{jk} + \frac{5}{4}Rh_{ij}\varphi_{ij}$$

$$- \frac{5}{4} (h_{ijij} - \Delta trh - Rc_{ij}h_{ij}) \Delta\varphi - 2Rc_{ij} (2h_{ikj} - h_{ijk}) \varphi_k + \frac{5}{8}R (2h_{ijj} - (trh)_i) \varphi_i$$

$$+ \frac{3}{4} (h_{klkl} - \Delta trh - Rc_{kl}h_{kl})_i \varphi_i - \frac{3}{4}h_{ij}R_i\varphi_j + \frac{1}{8} \Delta (h_{ijij} - \Delta trh - Rc_{ij}h_{ij}) \varphi$$

$$- \frac{1}{8}h_{ij}R_{ij}\varphi - \frac{1}{16} (2h_{ijj} - (trh)_i) R_i\varphi + Rc_{ij} (2h_{ikjk} - \Delta h_{ij} - (trh)_{ij}) \varphi$$

$$- 2Rc_{ij}Rc_{ik}h_{jk}\varphi - \frac{23}{32}R (h_{ijij} - \Delta trh - Rc_{ij}h_{ij}) \varphi.$$

On the other hand the formula of $P_{g,h}^{(2)}\varphi$ is much more complicated, and we will not write it down here. Instead we observe that $P_{g,h}^{(2)}\varphi$ is a fourth order operator in φ and $P_{g,h}^{(2)}1$ can be written down in a reasonable way. Indeed (see appendix)

$$\begin{aligned} Q_{g+th} & (2.4) \\ &= Q - \frac{t}{4} \Delta \left(h_{ijij} - \Delta trh - Rc_{ij}h_{ij} \right) - 2tRc_{ij} \left(2h_{ikjk} - (trh)_{ij} - \Delta h_{ij} \right) \\ &+ \frac{23t}{16} R \left(h_{ijij} - \Delta trh - Rc_{ij}h_{ij} \right) + \frac{t}{8} \left(2h_{ijj} - (trh)_i \right) R_i + \frac{t}{4} h_{ij}R_{ij} + 4tRc_{ij}Rc_{ik}h_{jk} \\ &- \frac{t^2}{4} \Delta \left[\left(\Delta h_{ij} + (trh)_{ij} - h_{ikjk} - h_{ikkj} \right) h_{ij} \right] + \frac{t^2}{4} h_{ij} \left(h_{klkl} - \Delta trh - Rc_{kl}h_{kl} \right)_{ij} \\ &- \frac{t^2}{16} \Delta \sum_{ijk} \left(h_{ikj} + h_{jki} - h_{ijk} \right)^2 + \frac{t^2}{16} \Delta \sum_i \left(2h_{ijj} - (trh)_i \right)^2 \\ &+ \frac{t^2}{8} \left(2h_{ijj} - (trh)_i \right) \left(h_{klkl} - \Delta trh - Rc_{kl}h_{kl} \right)_i - \frac{t^2}{4} \Delta \left(Rc_{ij}h_{ij}^2 \right) \\ &- \frac{t^2}{2} \sum_{ij} \left(h_{ikjk} + h_{jkik} - (trh)_{ij} - \Delta h_{ij} \right)^2 + 2t^2 Rc_{ij}h_{kl} \left(2h_{ikjl} - h_{klij} - h_{ijkl} \right) \\ &+ 4t^2 Rc_{ij}h_{ik} \left(h_{jlkl} + h_{kljl} - (trh)_{ij} - \Delta h_{jk} \right) + \frac{23t^2}{32} \left(h_{ijij} - \Delta trh - Rc_{ij}h_{ij} \right)^2 \\ &+ \frac{23t^2}{16} R \left(\Delta h_{ij} + (trh)_{ij} - h_{ikjk} - h_{ikkj} \right) h_{ij} - t^2 Rc_{ij} \left(h_{ikl} + h_{kli} - h_{ilk} \right) \left(h_{jkl} + h_{klj} - h_{jlk} \right) \\ &+ t^2 Rc_{ij} \left(2h_{ikj} - h_{ijk} \right) \left(2h_{kll} - (trh)_k \right) + \frac{23t^2}{64} R \sum_{ijk} \left(h_{ikj} + h_{jki} - h_{ijk} \right) h_{ij} R_j - \frac{t^2}{8} \left(2h_{ikj} - h_{ijk} \right) h_{ij} R_k \\ &- \frac{t^2}{4} h_{ij}^2 R_{ij} - 4t^2 Rc_{ij} Rc_{ik} h_{jk}^2 - 2t^2 Rc_{ij} Rc_{kl} h_{ik} h_{jl} + \frac{23t^2}{16} R \cdot Rc_{ij} h_{ij}^2 + O \left(t^3 \right). \end{aligned}$$
Because

we deduce that

$$P_{g,h}^{(2)} 1 \qquad (2.5)$$

$$= -\frac{1}{8} h_{ij} (h_{klkl} - \Delta trh - Rc_{kl}h_{kl})_{ij} + \frac{1}{8} \Delta \left[\left(\Delta h_{ij} + (trh)_{ij} - h_{ikjk} - h_{ikkj} \right) h_{ij} \right] \\ -\frac{1}{16} (2h_{ijj} - (trh)_{i}) (h_{klkl} - \Delta trh - Rc_{kl}h_{kl})_{i} + \frac{1}{32} \Delta \sum_{ijk} (h_{ikj} + h_{jki} - h_{ijk})^{2} \\ -\frac{1}{32} \Delta \sum_{i} (2h_{ijj} - (trh)_{i})^{2} + \frac{1}{8} \Delta \left(Rc_{ij}h_{ij}^{2} \right) + \frac{1}{4} \sum_{ij} \left(h_{ikjk} + h_{jkik} - (trh)_{ij} - \Delta h_{ij} \right)^{2} \\ -Rc_{ij}h_{kl} (2h_{ikjl} - h_{klij} - h_{ijkl}) - 2Rc_{ij}h_{ik} \left(h_{jlkl} + h_{kljl} - (trh)_{jk} - \Delta h_{jk} \right) \\ -\frac{23}{64} (h_{ijij} - \Delta trh - Rc_{ij}h_{ij})^{2} - \frac{23}{32}R \left(\Delta h_{ij} + (trh)_{ij} - h_{ikjk} - h_{ikkj} \right) h_{ij} \\ + \frac{1}{2}Rc_{ij} (h_{ikl} + h_{kli} - h_{ilk}) (h_{jkl} + h_{klj} - h_{jlk}) - \frac{1}{2}Rc_{ij} (2h_{ikj} - h_{ijk}) (2h_{kll} - (trh)_{k}) \\ -\frac{23}{128}R \sum_{ijk} (h_{ikj} + h_{jki} - h_{ijk})^{2} + \frac{23}{128}R \sum_{i} (2h_{ijj} - (trh)_{i})^{2} \\ + \frac{1}{8} \left[h_{ij}^{2}R_{ij} + \frac{1}{2} (2h_{ikk} - (trh)_{i}) h_{ij}R_{j} + \frac{1}{2} (2h_{ikj} - h_{ijk}) h_{ij}R_{k} \right] + 2Rc_{ij}Rc_{ik}h_{jk}^{2} \\ + Rc_{ij}Rc_{kl}h_{ik}h_{jl} - \frac{23}{32}R \cdot Rc_{ij}h_{ij}^{2}.$$

In general, $P_{g,h}^{\left(1\right)}$ is not self adjoint, instead we have

Lemma 2.2. For every $\varphi, \psi \in C^{\infty}$,

$$\int_{M} P_{g,h}^{(1)} \varphi \cdot \psi d\mu = \int_{M} \varphi P_{g,h}^{(1)} \psi d\mu - \frac{1}{2} \int_{M} \left(P \varphi \cdot \psi - \varphi P \psi \right) tr h d\mu.$$
(2.6)

Indeed this follows from the Taylor expansion in t for

$$\int_{M} P_{g+th} \varphi \cdot \psi d\mu_{g+th} = \int_{M} \varphi P_{g+th} \psi d\mu_{g+th}.$$

To derive a variational formula for the Green's function pole's value we write

$$G_{g+th}(p,q) = G(p,q) + tI(p,q,h) + t^{2}II(p,q,h) + O(t^{3}).$$
(2.7)

Note that

$$\partial_t|_{t=0} G_{g+th}(p,p) = I(p,p,h)$$
(2.8)

and

$$\partial_t^2 \Big|_{t=0} G_{g+th}(p,p) = 2II(p,p,h).$$
(2.9)

We can write I and II in terms of $P_{g,h}^{(1)}$ and $P_{g,h}^{(2)}.$

Lemma 2.3. For any smooth symmetric (0,2) tensor h, let I and II be defined in (2.7), then

$$I(p,q,h) = -\int_{M} P_{g,h}^{(1)} G_q \cdot G_p d\mu - \frac{1}{2} G(p,q) trh(q), \qquad (2.10)$$

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and

$$II(p, p, h)$$

$$= -\int_{M} \left(P_{g,h}^{(2)} G_{p} \cdot G_{p} + P_{g,h}^{(1)} G_{p} \cdot I_{p} + \frac{1}{2} P_{g,h}^{(1)} G_{p} \cdot G_{p} trh \right) d\mu$$

$$-\frac{1}{2} I(p, p, h) trh(p) - \frac{1}{8} G(p, p) (trh(p))^{2} + \frac{1}{4} G(p, p) |h(p)|^{2}.$$
(2.11)

Here $G_p(q) = G(p,q)$, $I_p(q,h) = I(p,q,h)$. The integration should be understood in distribution sense.

Proof. For any smooth function φ we have

$$\varphi\left(p\right) = \int_{M} P_{g+th}\varphi \cdot G_{g+th,p}d\mu_{g+th},$$

expand everything into power series of t, using

$$d\mu_{g+th} = \left[1 + \frac{trh}{2} \cdot t + \left(\frac{(trh)^2}{8} - \frac{|h|^2}{4}\right)t^2 + O\left(t^3\right)\right]d\mu$$

we see

$$\int_{M} \left(I_{p} \cdot P\varphi + G_{p} P_{g,h}^{(1)} \varphi + \frac{1}{2} G_{p} trh \cdot P\varphi \right) d\mu = 0$$

and

$$0 = \int_{M} \left[P\varphi \cdot II_{p} + P_{g,h}^{(1)}\varphi \cdot I_{p} + P_{g,h}^{(2)}\varphi \cdot G_{p} + \frac{1}{2}P\varphi \cdot I_{p}trh + \frac{1}{2}P_{g,h}^{(1)}\varphi \cdot G_{p}trh + \frac{1}{8}P\varphi \cdot G_{p}\left(trh\right)^{2} - \frac{1}{4}P\varphi \cdot G_{p}\left|h\right|^{2} \right] d\mu.$$

By approximation we know the same formula remains true for $\varphi \in H^2(M)$. Let $\varphi = G_q$ or G_p , we get the lemma.

3. First and second variation of Green's function pole's value

Let N be the north pole on S^3 and $\pi_N : S^3 \setminus \{N\} \to \mathbb{R}^3$ be the stereographic projection. Using $x = \pi_N$ as the coordinate, we have the standard metric g on S^3 can be written as

$$g = \frac{4}{\left(\left|x\right|^{2} + 1\right)^{2}} \left|dx\right|^{2} = \tau^{-4} \left|dx\right|^{2}, \qquad (3.1)$$

here

$$\tau = \sqrt{\frac{|x|^2 + 1}{2}}.$$
(3.2)

By conformal invariance property (1.3), the Green's function of P with pole at N is given by

$$G_N = -\frac{1}{4\pi} \frac{1}{\sqrt{|x|^2 + 1}}.$$
(3.3)

More generally

$$G(x,y) = -\frac{1}{4\pi} \frac{|x-y|}{\sqrt{|x|^2 + 1}} \sqrt{|y|^2 + 1}.$$
(3.4)

We are ready to compute the first variation of Green's function pole value.

Proposition 3.1. For any $p \in S^3$ and smooth symmetric (0, 2) tensor h, I(p, p, h) = 0.

Proposition 1.1 follows from Proposition 3.1 and (2.8).

Proof. By symmetry we can assume p = N. For convenience we write

$$I(h) = I(N, N, h).$$

Because we need to discuss various function's behavior near N, we denote S as the south pole of S^3 , $\pi_S : S^3 \setminus \{S\} \to \mathbb{R}^3$ as the stereographic projection with respect to S. We can use $y = \pi_S$ as the coordinate. By Lemma 2.3 and the fact $G_N(N) = 0$,

$$I(h) = -\int_{S^3} P_{g,h}^{(1)} G_N \cdot G_N d\mu \text{ (in distribution sense).}$$

Let $\eta \in C^{\infty}(\mathbb{R}^3)$ such that $\eta|_{B_1} = 1$, $\eta|_{\mathbb{R}^3 \setminus B_2} = 0$ and $0 \le \eta \le 1$. For $\varepsilon > 0$, we write $\eta_{\varepsilon} = \eta\left(\frac{y}{\varepsilon}\right)$. By [HY1, Lemma 2.2] $\eta_{\varepsilon}G_N \to 0$ in $H^2(S^3)$, hence

$$I(h) = -\lim_{\varepsilon \to 0^+} \int_{S^3} P_{g,h}^{(1)} \left((1 - \eta_{\varepsilon}) G_N \right) \cdot G_N d\mu$$

= $-\int_{S^3} P_{g,h}^{(1)} G_N \cdot G_N d\mu$ (in pointwise product sense)

Note here we have used the dominated convergence theorem and the fact near N,

$$\left| P_{g,h}^{(1)} G_N \cdot G_N \right| \le c \left| y \right|^{-2}, \quad \left| P_{g,h}^{(1)} \left((1 - \eta_{\varepsilon}) G_N \right) \cdot G_N \right| \le c \left| y \right|^{-2}$$

here c is independent of ε . For convenience we denote $\theta = \tau^4 h$. By Lemma 2.1 we have

$$I(h)$$

$$= -\lim_{\varepsilon \to 0^+} \int_{S^3 \setminus B_\varepsilon(N)} P_{g,h}^{(1)} G_N \cdot G_N d\mu$$

$$= -\frac{1}{32\pi^2} \lim_{R \to \infty} \int_{|x| \le R} P_{|dx|^2,\theta}^{(1)} 1 dx$$

$$= -\frac{1}{256\pi^2} \lim_{R \to \infty} \int_{|x| \le R} \Delta \left(\theta_{ijij} - \Delta tr\theta\right) dx$$

$$= -\frac{1}{256\pi^2} \lim_{R \to \infty} \int_{|x| = R} \left(\theta_{ijij} - \Delta tr\theta\right)_k \frac{x_k}{R} dS$$

To understand the boundary term we use the following notation: let f be a smooth function defined outside a ball, we say $f = O^{(\infty)}(|x|^a)$ as $|x| \to \infty$ if for any $m, \partial_{i_1\cdots i_m} f(x) = O\left(|x|^{a-m}\right)$. In particular $h_{ij} = O^{(\infty)}\left(|x|^{-4}\right), \tau = O^{(\infty)}(|x|)$, hence $\theta_{ij} = O^{(\infty)}(1)$ and

$$\left(\theta_{ijij} - \Delta tr\theta\right)_k = O^{(\infty)}\left(|x|^{-3}\right),\,$$

this implies I(h) = 0.

It is worth pointing out that there are other ways to calculate I(N, N, h). For example one may do this by using the formula of $P_{g,h}^{(1)}$ on S^3 (see (5.16)). However the method in the above proof will be crucial for the calculation of second variation formula.

To continue we need the expression of I(N, q, h).

Lemma 3.1. Let $\theta = \tau^4 h$, under the stereographic projection with respect to N, we denote the coordinate of q as y, then

$$I(N,q,h)$$

$$= -\frac{1}{256\pi^2} \int_{\mathbb{R}^3} (\theta_{ijij} - \Delta tr\theta) \cdot \frac{1}{\sqrt{|y|^2 + 1}} \left(\frac{2}{|x-y|} - \frac{2}{\sqrt{|x|^2 + 1}} - \frac{1}{\left(\sqrt{|x|^2 + 1}\right)^3} \right) dx$$

$$- \frac{G(N,q)}{8} \int_{S^3} \left(\Delta G_N - \frac{5}{2} G_N \right) h_{ijij} d\mu - \frac{G(N,q)}{8} \int_{S^3} \left(\Delta G_N + \frac{5}{16} G_N \right) trhd\mu$$

$$+ \frac{1}{8} G(N,q) trh(N).$$

$$(3.5)$$

Proof. Indeed it follows from Lemma 2.3 that

$$I(N,q,h) = I(q,N,h) = -\int_{S^3} P_{g,h}^{(1)} G_N \cdot G_q d\mu - \frac{1}{2} G(N,q) trh(N) = -\int_{S^3} P_{g,h}^{(1)} G_N \cdot (G_q - G_q(N)) d\mu - G(N,q) \int_{S^3} P_{g,h}^{(1)} G_N d\mu - \frac{1}{2} G(N,q) trh(N).$$

By Lemma 2.2 we have

$$\begin{aligned} &\int_{S^3} P_{g,h}^{(1)} G_N d\mu \\ &= \int_{S^3} G_N P_{g,h}^{(1)} 1 d\mu - \frac{1}{2} \int_{S^3} \left(P G_N - G_N \cdot P 1 \right) tr h d\mu \\ &= \int_{S^3} G_N \left(\frac{1}{8} \Delta \left(h_{ijij} \right) - \frac{5}{16} h_{ijij} \right) d\mu + \frac{1}{8} \int_{S^3} \left(\Delta G_N + \frac{5}{16} G_N \right) tr h d\mu - \frac{5}{8} tr h \left(N \right). \end{aligned}$$

Using the fact $G_q - G_q(N)$ vanishes at N, by the same method in the proof of Proposition 3.1,

$$\begin{split} & \int_{S^3} P_{g,h}^{(1)} G_N \cdot (G_q - G_q(N)) \, d\mu \\ &= \lim_{\varepsilon \to 0^+} \int_{S^3 \setminus B_\varepsilon(N)} P_{g,h}^{(1)} G_N \cdot (G_q - G_q(N)) \, d\mu \\ &= \frac{1}{32\pi^2} \lim_{R \to \infty} \int_{|x| \le R} P_{|dx|^2, \theta} 1 \cdot \frac{|x - y| - \sqrt{|x|^2 + 1}}{\sqrt{|y|^2 + 1}} dx \\ &= \frac{1}{256\pi^2} \int_{\mathbb{R}^3} \Delta \left(\theta_{ijij} - \Delta tr \theta \right) \cdot \frac{|x - y| - \sqrt{|x|^2 + 1}}{\sqrt{|y|^2 + 1}} dx \\ &= \frac{1}{256\pi^2} \int_{\mathbb{R}^3} \left(\theta_{ijij} - \Delta tr \theta \right) \cdot \frac{1}{\sqrt{|y|^2 + 1}} \left(\frac{2}{|x - y|} - \frac{2}{\sqrt{|x|^2 + 1}} - \frac{1}{\left(\sqrt{|x|^2 + 1}\right)^3} \right) dx. \end{split}$$

Equation (3.5) follows.

Theorem 3.1. For any smooth symmetric (0,2) tensor h, denote $\theta = \tau^4 h$, then

$$II(N, N, h)$$

$$= -\frac{1}{128\pi^2} \int_{\mathbb{R}^3} \left(\sum_{ij} \left(\theta_{ikjk} + \theta_{jkik} - (tr\theta)_{ij} - \Delta\theta_{ij} \right)^2 - \frac{3}{2} \left(\theta_{ijij} - \Delta tr\theta \right)^2 \right) dx.$$
(3.6)

Theorem 1.1 follows from Theorem 3.1 and (2.9).

Proof. By Lemma 2.3,

$$II(N, N, h)$$

$$= -\int_{S^3} \left(P_{g,h}^{(2)} G_N \cdot G_N + P_{g,h}^{(1)} G_N \cdot I_N + \frac{1}{2} P_{g,h}^{(1)} G_N \cdot G_N trh \right) d\mu.$$
(3.7)

First we note that because $G_N(N) = 0$, the same argument as in the proof of Proposition 3.1 shows

$$\int_{S^3} P_{g,h}^{(2)} G_N \cdot G_N d\mu$$

$$= \lim_{\varepsilon \to 0^+} \int_{S^3 \setminus B_\varepsilon(N)} P_{g,h}^{(2)} G_N \cdot G_N d\mu$$

$$= \frac{1}{32\pi^2} \lim_{R \to \infty} \int_{|x| \le R} P_{|dx|^2,\theta}^{(2)} 1 dx$$

$$= \frac{1}{128\pi^2} \int_{\mathbb{R}^3} \left[\sum_{ij} \left(\theta_{ikjk} + \theta_{jkik} - (tr\theta)_{ij} - \Delta \theta_{ij} \right)^2 - \frac{1}{16} \left(23\theta_{ijij} - 19\Delta tr\theta \right) \left(\theta_{klkl} - \Delta tr\theta \right) \right] dx,$$

Here we have used (2.5). Next using $I_N(N) = 0$ we have

$$\int_{S^3} P_{g,h}^{(1)} G_N \cdot I_N d\mu = \lim_{\varepsilon \to 0^+} \int_{S^3 \setminus B_\varepsilon(N)} P_{g,h}^{(1)} G_N \cdot I_N d\mu$$
$$= -\frac{1}{4\pi\sqrt{2}} \lim_{R \to \infty} \int_{|x| \le R} P_{|dx|^2,\theta}^{(1)} 1 \cdot \tau I_N dx$$

Since

$$\lim_{R \to \infty} \int_{|x| \le R} P^{(1)}_{|dx|^2, \theta} 1 \cdot \tau G_N dx = 0,$$

(see the proof of Proposition 3.1), by Lemma 3.1 we have

$$\begin{split} &\int_{S^3} P_{g,h}^{(1)} G_N \cdot I_N d\mu \\ &= \frac{1}{2048\pi^3} \int_{\mathbb{R}^3} P_{|dx|^2,\theta}^{(1)} 1 \left(\int_{\mathbb{R}^3} \left(\theta_{ijij} - \Delta tr\theta \right) (y) \left(\frac{2}{|x-y|} - \frac{2}{\sqrt{|y|^2 + 1}} - \frac{1}{\left(\sqrt{|y|^2 + 1} \right)^3} \right) dy \right) dx \\ &= \frac{1}{16384\pi^3} \int_{\mathbb{R}^3} dy \left(\theta_{ijij} - \Delta tr\theta \right) (y) \int_{\mathbb{R}^3} \Delta \left(\theta_{klkl} - \Delta tr\theta \right) (x) \left(\frac{2}{|x-y|} - \frac{2}{\sqrt{|y|^2 + 1}} - \frac{1}{\left(\sqrt{|y|^2 + 1} \right)^3} \right) dx \\ &= -\frac{1}{2048\pi^2} \int_{\mathbb{R}^3} \left(\theta_{ijij} - \Delta tr\theta \right)^2 dx. \end{split}$$

Similarly for the third term in (3.7) we have

$$\int_{S^3} \frac{1}{2} P_{g,h}^{(1)} G_N \cdot G_N trhd\mu = \frac{1}{2} \lim_{\varepsilon \to 0^+} \int_{S^3 \setminus B_\varepsilon(N)} P_{g,h}^{(1)} G_N \cdot G_N trhd\mu$$
$$= \frac{1}{64\pi^2} \lim_{R \to \infty} \int_{|x| \le R} P_{|dx|^2,\theta}^{(1)} 1 \cdot tr\theta dx$$
$$= \frac{1}{512\pi^2} \int_{\mathbb{R}^3} \Delta(\theta_{ijij} - \Delta tr\theta) \cdot tr\theta dx$$
$$= \frac{1}{512\pi^2} \int_{\mathbb{R}^3} \Delta tr\theta \cdot (\theta_{ijij} - \Delta tr\theta) dx.$$

Sum up we get (3.6).

Next we will study sign of the second variation. For convenience we write

$$II(h) = II(N, N, h).$$
(3.8)

First we observe that by conformal covariant property, for any smooth vector field X and function f,

$$II(L_X g + fg) = 0. (3.9)$$

Indeed let ϕ_t be the flow generated by X, then for t near 0,

$$\begin{aligned} G_{(1+tf)\phi_t^*g}\left(N,N\right) &= (1+tf\left(N\right))^{\frac{1}{2}} G_{\phi_t^*g}\left(N,N\right) \\ &= (1+tf\left(N\right))^{\frac{1}{2}} G_g\left(\phi_t\left(N\right),\phi_t\left(N\right)\right) = 0. \end{aligned}$$

Since $\partial_t|_{t=0} ((1+tf) \phi_t^* g) = L_X g + fg$, (3.9) follows. In fact we can say a little more: let II(h,k) be the symmetric form associated with II(h), if

$$\theta = \tau^4 h, \quad \kappa = \tau^4 k, \tag{3.10}$$

then it follows from Theorem 3.1 that

$$II(h,k)$$

$$= -\frac{1}{128\pi^2} \int_{\mathbb{R}^3} \left[\sum_{ij} \left(\theta_{ikjk} + \theta_{jkik} - (tr\theta)_{ij} - \Delta\theta_{ij} \right) \left(\kappa_{iljl} + \kappa_{jlil} - (tr\kappa)_{ij} - \Delta\kappa_{ij} \right) \right. \\ \left. -\frac{3}{2} \left(\theta_{ijij} - \Delta tr\theta \right) \left(\kappa_{klkl} - \Delta tr\kappa \right) \right] dx.$$

$$(3.11)$$

Lemma 3.2. Given smooth symmetric (0, 2) tensor h, vector field X and function f, we have

$$II(h, L_X g + fg) = 0. (3.12)$$

To achieve this we need the following technical fact:

Lemma 3.3. If h is a smooth symmetric (0,2) tensor on S^3 , then there exists a smooth vector field X such that

$$(h - L_X g)(N) = 0, \quad D(h - L_X g)(N) = 0$$

To derive this lemma, we start with the following linear algebra fact.

Lemma 3.4. Denote

 $\mathcal{P}_m = \left\{ \text{homogeneous degree } m \text{ polynomials on } \mathbb{R}^3 \right\}.$ If for $1 \leq i, j \leq 3$, $H_{ij} \in \mathcal{P}_1$, $H_{ij} = H_{ji}$, then there exists unique $A_i \in \mathcal{P}_2$ such that

$$\partial_i A_j + \partial_j A_i = H_{ij}$$

Proof. Let

$$\mathcal{X} = \left\{ \left[\begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \right] : A_i \in \mathcal{P}_2 \right\}$$

and

$$\mathcal{Y} = \left\{ \left[H_{ij} \right]_{1 \le i, j \le 3} : H_{ij} \in \mathcal{P}_1, H_{ij} = H_{ji} \right\}.$$

Note dim $\mathcal{X} = \dim \mathcal{Y} = 18$. Let

$$\phi: \mathcal{X} \to \mathcal{Y}: \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \mapsto [H_{ij}]_{1 \le i,j \le 3}$$

be given by $H_{ij} = \partial_i A_j + \partial_j A_i$. We need to show ϕ is a linear isomorphism. We only need to prove ker $\phi = 0$. Indeed if $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \in \ker \phi$, then

$$\partial_i A_j + \partial_j A_i = 0$$

This implies

$$x_i x_j \partial_i A_j + x_i x_j \partial_j A_i = 0.$$

Since $x_i \partial_i A_j = 2A_j$, we see $x_i A_i = 0$. Hence

$$0 = \partial_j \left(x_i A_i \right) = A_j + x_i \partial_j A_i = A_j - x_i \partial_i A_j = -A_j$$

The lemma follows.

Now we will use Taylor expansion to prove Lemma 3.3.

Proof of Lemma 3.3. By standard cut-off argument we see the conclusion is in fact a local statement. We choose a local coordinate near N, say y_1, y_2, y_3 such that $y_i(N) = 0$. Assume $X = X^i \frac{\partial}{\partial y_i}$, let α be the associated 1-form i.e. $\alpha_i = g_{ij}X^j$, then we only need to find α with

$$\alpha_{ij} + \alpha_{ji} = h_{ij} + O\left(|y|^2\right)$$

as $y \to 0$. In another way the equation is

$$\partial_i \alpha_j + \partial_j \alpha_i - 2\Gamma_{ij}^k \alpha_k = h_{ij} + O\left(|y|^2\right).$$

We will look for $\alpha_i = \alpha_i^{(1)} + \alpha_i^{(2)}$, here $\alpha_i^{(l)} \in \mathcal{P}_l$. We have the Taylor expansion of h_{ij} as $h_{ij} = h_{ij}(0) + h_{ij}^{(1)} + O\left(|y|^2\right)$. So the equation becomes

$$\partial_i \alpha_j^{(1)} + \partial_j \alpha_i^{(1)} = h_{ij} \left(0 \right) \tag{3.13}$$

and

$$\partial_i \alpha_j^{(2)} + \partial_j \alpha_i^{(2)} - 2\Gamma_{ij}^k (0) \,\alpha_k^{(1)} = h_{ij}^{(1)}. \tag{3.14}$$

For (3.13), we can simply choose $\alpha_i^{(1)} = \frac{1}{2}h_{ik}(0) y_k$. Using Lemma 3.4 we see (3.14) also has a solution. Lemma 3.3 follows.

With Lemma 3.3 at hand, we can proceed to prove Lemma 3.2.

Proof of Lemma 3.2. Note that

$$f = O^{(\infty)}(1), \quad \theta_{ij} = O^{(\infty)}(1),$$

as $|x| \to \infty$. By (3.10),

$$II(h, fg) = -\frac{1}{128\pi^2} \int_{\mathbb{R}^3} \left[-\left(\theta_{ikjk} + \theta_{jkik} - (tr\theta)_{ij} - \Delta\theta_{ij}\right) (f_{ij} + \Delta f \cdot \delta_{ij}) + 3\left(\theta_{ijij} - \Delta tr\theta\right) \Delta f \right] dx.$$

By integration by parts, we have

$$\int_{\mathbb{R}^3} -\left(\theta_{ikjk} + \theta_{jkik} - (tr\theta)_{ij} - \Delta\theta_{ij}\right) f_{ij}dx = \int_{\mathbb{R}^3} \left(-\theta_{ijij} + \Delta tr\theta\right) \Delta f dx$$

and

$$\int_{\mathbb{R}^3} -\left(\theta_{ikjk} + \theta_{jkik} - (tr\theta)_{ij} - \Delta\theta_{ij}\right) \Delta f \cdot \delta_{ij} dx = \int_{\mathbb{R}^3} \left(-2\theta_{ijij} + 2\Delta tr\theta\right) \Delta f dx.$$

Sum up we get II(h, fg) = 0.

Next we will show $II(h, L_Xg) = 0$. First we note that it follows from (3.9) that for any smooth vector fields X', X and smooth functions f', f, we have

$$II\left(L_{X'}g + f'g, L_Xg + fg\right) = 0.$$

To continue we can assume h satisfies

$$h(N) = 0, \quad Dh(N) = 0.$$

Indeed given any smooth h, by Lemma 3.3, we can find a smooth vector field X^\prime such that

$$(h - L_{X'}g)(N) = 0, \quad D(h - L_{X'}g)(N) = 0.$$

It follows that

$$II(h, L_Xg) = II(h - L_{X'}g, L_Xg).$$

Under the additional assumption on h, we have

$$\theta_{ij} = O^{(\infty)} \left(|x|^{-2} \right), \quad X_i = O^{(\infty)} \left(|x|^2 \right)$$

as $x \to \infty$. Here $X = X_i \frac{\partial}{\partial x_i}$. Let $\kappa = \tau^4 L_X g$, then

$$\kappa_{ij} = -4\tau^{-1}X\tau \cdot \delta_{ij} + X_{ij} + X_{ji}.$$

Hence

$$\kappa_{ikjk} + \kappa_{jkik} - (tr\kappa)_{ij} - \Delta\kappa_{ij} = -4\left(\tau^{-1}X\tau\right)_{ij} - 4\Delta\left(\tau^{-1}X\tau\right) \cdot \delta_{ij}$$

and

$$\kappa_{ijij} - \Delta tr\kappa = 8\Delta \left(\tau^{-1} X \tau\right).$$

It implies

$$= \frac{II(h, L_Xg)}{32\pi^2} \int_{\mathbb{R}^3} \left[-\left(\theta_{ikjk} + \theta_{jkik} - (tr\theta)_{ij} - \Delta\theta_{ij}\right) \left(\left(\tau^{-1}X\tau\right)_{ij} + \Delta\left(\tau^{-1}X\tau\right) \cdot \delta_{ij} \right) \right. \\ \left. + 3\left(\theta_{ijij} - \Delta tr\theta\right) \Delta\left(\tau^{-1}X\tau\right) \right] dx.$$

Note that $\tau^{-1}X\tau = O^{(\infty)}(|x|)$ and $\theta_{ij} = O^{(\infty)}(|x|^{-2})$, the same integration by parts argument as the beginning shows $II(h, L_Xg) = 0$.

Proposition 3.2. For any smooth symmetric (0, 2) tensor h, $II(h) \leq 0$. Moreover, II(h) = 0 if and only if $h = L_X g + f \cdot g$ for some smooth vector fields X and smooth functions f on S^3 .

Proof. In view of Lemma 3.2 and 3.3 we can assume h(N) = 0 and Dh(N) = 0. Under such assumption we have

$$\theta_{ij} = O^{(\infty)} \left(|x|^{-2} \right)$$

as $x \to \infty$. In particular $\theta_{ij} \in L^2(\mathbb{R}^3)$. Let

$$a_{ij}\left(\xi\right) = \theta_{ij}\left(\xi\right),$$

then $a_{ij} \in L^2(\mathbb{R}^3)$. By Parseval relation we have

$$\int_{\mathbb{R}^{3}} \left(\sum_{ij} \left(\theta_{ikjk} + \theta_{jkik} - (tr\theta)_{ij} - \Delta\theta_{ij} \right)^{2} - \frac{3}{2} \left(\theta_{ijij} - \Delta tr\theta \right)^{2} \right) dx$$

=
$$\int_{\mathbb{R}^{3}} \left(-2a_{ij}\overline{a}_{ik}\xi_{j}\xi_{k} \left| \xi \right|^{2} + \frac{1}{2}a_{ij}\overline{a}_{kl}\xi_{i}\xi_{j}\xi_{k}\xi_{l} + \frac{1}{2}a_{ij}\xi_{i}\xi_{j}\overline{a}_{kk} \left| \xi \right|^{2} + \frac{1}{2}\overline{a}_{ij}\xi_{i}\xi_{j}a_{kk} \left| \xi \right|^{2} + \frac{1}{2}\overline{a}_{ij}\xi_{i}\xi_{j}a_{kk} \left| \xi \right|^{2} + \sum_{ij} |a_{ij}|^{2} \left| \xi \right|^{4} - \frac{1}{2} \left| \sum_{i} a_{ii} \right|^{2} |\xi|^{4} \right) d\xi.$$

We will show the integrand in nonnegative. Indeed, if we write

$$A = \left[a_{ij}\right]_{1 \le i,j \le 3},$$

then A is symmetric and the integrand is equal to

$$-2|A\xi|^{2}|\xi|^{2} + \frac{1}{2}\left|\xi^{T}A\xi\right|^{2} + \frac{1}{2}\xi^{T}A\xi \cdot \overline{trA}|\xi|^{2} + \frac{1}{2}\xi^{T}\overline{A\xi} \cdot trA|\xi|^{2} \quad (3.15)$$
$$-\frac{1}{2}|trA|^{2}|\xi|^{4} + |A|^{2}|\xi|^{4}.$$

Assume $\xi \neq 0$, then we may find an orthogonal matrix O such that

$$\xi = O \left[\begin{array}{c} |\xi| \\ 0 \\ 0 \end{array} \right].$$

Denote

$$B = O^T A O,$$

then B is symmetric and the integrand (3.15) is equal to

$$\left(\frac{1}{2}\left|b_{22} - b_{33}\right|^2 + 2\left|b_{23}\right|^2\right)\left|\xi\right|^4 \tag{3.16}$$

and hence nonnegative. This implies $II(h) \leq 0$.

If II(h) = 0, then we have $b_{22} = b_{33}$ and $b_{23} = 0$. this implies

$$a_{ij} = \alpha \delta_{ij} + \beta_i \xi_j + \beta_j \xi_i, \qquad (3.17)$$

here α and β_i depend on ξ . To continue we recall the orthogonal decomposition [B, p130, lemma 4.57],

$$\mathcal{S}^2 S^3 = \mathcal{A} \oplus \mathcal{B}. \tag{3.18}$$

Here

$$\begin{aligned} \mathcal{S}^2 S^3 &= \left\{ C^{\infty} \text{ symmetric } (0,2) \text{ tensors on } S^3 \right\}, \\ \mathcal{A} &= \left\{ L_X g + fg : X \text{ is a } C^{\infty} \text{ vector field, } f \text{ is a } C^{\infty} \text{ function} \right\}, \\ \mathcal{B} &= \left\{ k \in \mathcal{S}^2 S^3 : trk = 0, k_{ijj} = 0 \right\}. \end{aligned}$$

To show $h \in \mathcal{A}$, we only need to prove $h \perp \mathcal{B}$. Indeed if $k \in \mathcal{B}$, let $\kappa = \tau^4 k = \kappa_{ij} dx_i dx_j$, then

$$\kappa_{ii} = 0, \quad \left(\tau^{-6} \kappa_{ij}\right)_j = 0. \tag{3.19}$$

On the other hand we have

$$\theta_{ij} = O^{(\infty)}\left(|x|^{-2}\right), \quad \tau^{-6}\kappa_{ij} = O^{(\infty)}\left(|x|^{-6}\right).$$
(3.20)

By Fourier transform we have

$$\widehat{\tau^{-6}\kappa_{ii}} = 0, \quad \widehat{\tau^{-6}\kappa_{ij}}\xi_j = 0.$$
(3.21)

Hence

$$\begin{split} \int_{S^3} \langle h, k \rangle \, d\mu &= \int_{\mathbb{R}^3} \theta_{ij} \kappa_{ij} \tau^{-6} dx \\ &= \int_{\mathbb{R}^3} \widehat{\theta_{ij}} \overline{\tau^{-6} \kappa_{ij}} d\xi \\ &= \int_{\mathbb{R}^3} \left(\alpha \delta_{ij} + \beta_i \xi_j + \beta_j \xi_i \right) \overline{\tau^{-6} \kappa_{ij}} d\xi \\ &= 0, \end{split}$$

here we have used (3.21) in the last step. Hence $h = L_X g + fg$ for some smooth vector field X and smooth function f.

4. A NEW INVARIANT FOR PANEITZ OPERATOR

Let (M, g) be a smooth compact three dimensional Riemannian manifold. For any $p \in M$, we set

$$\nu_{p} = \inf\left\{\frac{E(u)}{\int_{M} u^{2} d\mu} : u \in H^{2}(M) \setminus \{0\}, u(p) = 0\right\}.$$
(4.1)

 ν_p is always finite and achieved. Indeed we let $u_i \in H^2(M)$ such that $u_i(p) = 0$, $||u_i||_{L^2} = 1$ and $E(u_i) \to \nu_p$. In view of the fact

$$E(u_i) \ge c_1 \|u_i\|_{H^2(M)}^2 - c_2 \|u_i\|_{L^2}^2$$

for some positive constants c_1 and c_2 , we see $\|u_i\|_{H^2(M)}^2 \leq c$, independent of *i*. After passing to a subsequence we can assume $u_i \rightarrow u$ weakly in $H^2(M)$. It follows that $u_i \rightarrow u$ uniformly and hence u(p) = 0 and $\|u\|_{L^2} = 1$. By lower semicontinuity we have

$$E(u) \le \lim \inf_{i \to \infty} E(u_i) = \nu_p.$$

Hence $E(u) = \nu_p$ and u is a minimizer.

Note u satisfies

$$E\left(u,\varphi\right) =\nu_{p}\int_{M}u\varphi d\mu$$

for any $\varphi \in H^{2}(M)$ with $\varphi(p) = 0$. For any $\psi \in H^{2}(M)$, let $\varphi = \psi - \psi(p)$ we see

$$E(u,\psi) = \nu_p \int_M u\psi d\mu + \alpha \psi(p) \, .$$

Here α is a constant. In another word, we have

$$Pu = \nu_p u + \alpha \delta_p \tag{4.2}$$

in distribution sense and

$$u \in H^{2}(M), \|u\|_{L^{2}} = 1, u(p) = 0.$$
 (4.3)

Sometime to avoid confusion we write $u = u_p$ and $\alpha = \alpha_p$.

Example 4.1. Using [HY1, Lemma 7.1 and Corollary 7.1], we see on standard S^3 , $\nu_N = 0$ and it is achieved on constant multiple of the Green's function G_N . Calculation shows $\|G_N\|_{L^2} = \frac{1}{4}$, hence $u_N = 4G_N$, $\alpha_N = 4$.

$$\nu(M,g) = \inf_{p \in M} \nu(M,g,p)$$

$$= \inf \left\{ \frac{E(u)}{\int_{M} u^{2} d\mu} : u \in H^{2}(M) \setminus \{0\}, u \text{ vanishes somewhere} \right\}.$$

$$(4.4)$$

We will write $\nu(g)$ when no confusion could happen. Same argument as before shows $\nu(g)$ is finite and achieved. It is clear that condition P is satisfied if and only if $\nu(g) > 0$, condition NN is satisfied if and only if $\nu(g) \ge 0$. By Example 4.1 and symmetry, we see $\nu(S^3, g_{S^3}) = 0$.

Here we make some general discussion about ν_p and $\nu(g)$. For convenience we write $\nu = \nu(g)$. Let

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdot$$

be eigenvalues and φ_i be the associated orthonormal eigenfunctions of Paneitz operator P, then

$$\lambda_1 \le \nu_p \le \lambda_2, \quad \lambda_1 \le \nu \le \lambda_2.$$
 (4.5)

Indeed, for given $p \in M$, there exists c_1, c_2 not all zeroes such that $c_1\varphi_1(p) + c_2\varphi_2(p) = 0$, then

$$\nu_p \leq \frac{E\left(c_1\varphi_1 + c_2\varphi_2\right)}{\|c_1\varphi_1 + c_2\varphi_2\|_{L^2}^2} = \frac{\lambda_1 c_1^2 + \lambda_2 c_2^2}{c_1^2 + c_2^2} \leq \lambda_2.$$

The inequality of ν follows.

Assume $u \in H^2(M)$, $||u||_{L^2} = 1$ and u(p) = 0 for some p with $E(u) = \nu$ i.e. u is a minimizer for the ν problem, then

$$Pu = \nu u + \alpha \delta_p. \tag{4.6}$$

If $\sharp u^{-1}(0) > 1$ i.e. u vanishes at two or more points, then

$$Pu = \nu u, \quad \lambda_1 = \nu. \tag{4.7}$$

Indeed assume $u(p_1) = 0$ and $u(p_2) = 0$ for $p_1 \neq p_2$, then for $\varphi \in H^2$ with either $\varphi(p_1) = 0$ or $\varphi(p_2) = 0$,

$$E(u,\varphi) = \nu \int_{M} u\varphi d\mu.$$
(4.8)

Hence (4.8) is valid for any $\varphi \in H^2$. In another word $Pu = \nu u$. If $\lambda_1 < \nu$, then $\lambda_1 < \lambda_2$ and φ_1 does not vanish anywhere. Using

$$\int_M \varphi_1 \varphi_2 d\mu = 0,$$

we see φ_2 must change sign. Hence for $\varepsilon > 0$ small

$$\lambda_{1} < \nu \leq \frac{E\left(\varepsilon\varphi_{1} + \varphi_{2}\right)}{\left\|\varepsilon\varphi_{1} + \varphi_{2}\right\|_{L^{2}}^{2}} = \frac{\varepsilon^{2}\lambda_{1} + \lambda_{2}}{\varepsilon^{2} + 1} < \lambda_{2}$$

A contradiction with the fact ν is an eigenvalue. Hence ν must be the first eigenvalue.

Now we can state the following interesting relation between condition NN and the sign of λ_2 .

Proposition 4.1. Assume the Yamabe invariant Y(g) > 0 and there exists a $\tilde{g} \in [g]$ such that $\tilde{Q} \geq 0$ and not identically zero, then the following statements are equivalent

(1) $Y_4(g) > -\infty$. (2) $\lambda_2(P) > 0$. (3) $\nu(g) \ge 0$ *i.e.* P satisfies condition NN.

Proof. It follows from the assumption that $\lambda_1 < 0$. By [HY3, Proposition 1.2] and (1.4) we have ker P = 0 and $G_P(p,q) < 0$ for $p \neq q$. Here G_P is the Green's function of the Paneitz operator. Let $m \geq 1$ be the natural number such that $\lambda_m < 0$ and $\lambda_{m+1} > 0$ i.e. λ_m is the largest negative eigenvalue. By applying the classical Krein-Rutman theorem to the operator

$$Tf(p) = -\int_{M} G_{P}(p,q) f(q) d\mu(q)$$

we know λ_m must be simple and φ_m can not touch zero (see [HY3, section 4]). Without losing of generality, we assume $\varphi_m > 0$.

(1) \Rightarrow (2): If $\lambda_2 < 0$, then $m \ge 2$ and the first eigenfunction φ_1 must change sign. Let

$$\kappa = -\min_{p \in M} \frac{\varphi_1\left(p\right)}{\varphi_m\left(p\right)} > 0,$$

then $\varphi_1 + \kappa \varphi_m \ge 0$ and it touches zero somewhere. On the other hand $E\left(\varphi_1 + \kappa \varphi_m\right) = \lambda_1 + \kappa^2 \lambda_m < 0$, hence

$$Y_4(g) \le \left\| (\varphi_1 + \kappa \varphi_m + \varepsilon)^{-1} \right\|_{L^6}^2 E(\varphi_1 + \kappa \varphi_m + \varepsilon) \to -\infty$$

as $\varepsilon \downarrow 0$, a contradiction.

(2) \Rightarrow (3): Since $\lambda_2 > 0$, we get m = 1. Let $u \in H^2(M)$ such that u touches zero somewhere, $||u||_{L^2} = 1$ and $E(u) = \nu$. We claim $\sharp u^{-1}(0) = 1$. Indeed if $\sharp u^{-1}(0) > 1$, then by the discussion before Proposition 4.1 we know $P(u) = \nu u$ and $\nu = \lambda_1$. Its eigenfunction u can not touch zero, a contradiction. The claim follows i.e. u touches 0 exactly once. Assume u(p) = 0 and u > 0 on $M \setminus \{p\}$, then

$$P\left(u\right) = \nu u + \alpha \delta_p.$$

Hence

$$\int_{M} P\left(u\right) G_{L,p}^{-1} d\mu = \nu \int_{M} u G_{L,p}^{-1} d\mu.$$

Here G_L is the Green's function of the conformal Laplacian operator $L = -8\Delta + R$. On the other hand it follows from [HY3, Proposition 2.1] that

$$\int_{M} P(u) G_{L,p}^{-1} d\mu = \int_{M} u G_{L,p}^{-1} \left| R c_{G_{L,p}^{4}} g \right|_{g}^{2} d\mu.$$

Combine the two equalities above we get

$$\nu \int_{M} u G_{L,p}^{-1} d\mu = \int_{M} u G_{L,p}^{-1} \left| R c_{G_{L,p}^{4}} g \right|_{g}^{2} d\mu.$$

Hence $\nu \geq 0$.

(3)⇒(1): If E(u) = 0, u is not identically zero but u(p) = 0, then $u = cG_p$ (see [HY1, section 5]). Hence $G_p(p) = 0$. It follows from [HY3, Proposition 1.2] that (M,g) is conformal diffeomorphic to standard S^3 . In this case we know $Y_4(g) > -\infty$ (see [HY1, YZ]). On the other hand if E(u) > 0 for any $u \in H^2 \setminus \{0\}$ and u touches zero somewhere, then the Paneitz operator satisfies condition P and $Y_4(g) > -\infty$ (see [HY1]).

Indeed the above proof gives us the following

Corollary 4.1. Assume the Yamabe invariant Y(g) > 0, (M, g) is not conformal diffeomorphic to the standard S^3 and there exists a $\tilde{g} \in [g]$ such that $\tilde{Q} \ge 0$ and not identically zero, then the following statements are equivalent

- (1) $Y_4(g) > -\infty$.
- (2) $\lambda_2(P) > 0.$
- (3) $\nu(g) > 0$ i.e. P satisfies condition P.

We remark that Proposition 4.1 provides another argument for the third conclusion of [HY2, Theorem 1.1]. This approach does not need the connecting path to Berger's sphere and [HY1, Theorem 1.3].

Let $\tilde{g} = g + th$, for quantities in (4.2) and (4.3) we write

$$\nu(g+th,p) = \nu(p) + \nu^{(1)}(p,h)t + \nu^{(2)}(p,h)t^{2} + O(t^{3}), \qquad (4.9)$$

$$\alpha (g + th, p) = \alpha (p) + \alpha^{(1)} (p, h) t + \alpha^{(2)} (p, h) t^2 + O(t^3), \quad (4.10)$$

$$u_p(g+th,q) = u_p(q) + u_p^{(1)}(q,h)t + u_p^{(2)}(q,h)t^2 + O(t^3).$$
(4.11)

Hence $u_{p}^{(i)}\left(p,h\right)=0$ for i=1,2. Note because

$$\int_M \widetilde{u}_p^2 d\widetilde{\mu} = 1,$$

we have

$$\int_{M} \left(2u_p \cdot u_p^{(1)} + \frac{1}{2} u_p^2 trh \right) d\mu = 0, \tag{4.12}$$

and

$$\int_{M} \left[2u_{p} \cdot u_{p}^{(2)} + \left(u_{p}^{(1)} \right)^{2} + u_{p} \cdot u_{p}^{(1)} trh + \frac{1}{8} u_{p}^{2} \left(trh \right)^{2} - \frac{1}{4} u_{p}^{2} \left| h \right|^{2} \right] d\mu = 0. \quad (4.13)$$

For any smooth function φ , it follows from (4.2) that

$$\int_{M} \widetilde{u}_{p} \left(\widetilde{P} - \widetilde{\nu}_{p} \right) \varphi d\widetilde{\mu} = \widetilde{\alpha}_{p} \varphi \left(p \right).$$

Hence

$$\alpha_{p}^{(1)}\varphi(p)$$

$$= \int_{M} \left(u_{p} \left(P_{g,h}^{(1)} - \nu_{p}^{(1)} \right) \varphi + u_{p}^{(1)} \left(P - \nu_{p} \right) \varphi + \frac{1}{2} u_{p} \left(P - \nu_{p} \right) \varphi \cdot trh \right) d\mu$$
(4.14)

and

$$\begin{split} &\int_{M} \left[u_{p}^{(2)} \left(P - \nu_{p} \right) \varphi + u_{p}^{(1)} \left(P_{g,h}^{(1)} - \nu_{p}^{(1)} \right) \varphi + u_{p} \left(P_{g,h}^{(2)} - \nu_{p}^{(2)} \right) \varphi \right. (4.15) \\ &\quad + \frac{1}{2} u_{p}^{(1)} \left(P - \nu_{p} \right) \varphi \cdot trh + \frac{1}{2} u_{p} \left(P_{g,h}^{(1)} - \nu_{p}^{(1)} \right) \varphi \cdot trh \\ &\quad + \frac{1}{8} u_{p} \left(P - \nu_{p} \right) \varphi \cdot (trh)^{2} - \frac{1}{4} u_{p} \left(P - \nu_{p} \right) \varphi \cdot |h|^{2} \right] d\mu \\ &= \alpha_{p}^{(2)} \varphi \left(p \right). \end{split}$$

By approximation it is also true for $\varphi \in H^2(M)$ too. Hence taking $\varphi = u_p$, we get

$$\nu_p^{(1)} = \int_M u_p P_{g,h}^{(1)} u_p d\mu.$$
(4.16)

Similar arguments show

$$\nu_{p}^{(2)} \qquad (4.17)$$

$$= \int_{M} u_{p} P_{g,h}^{(2)} u_{p} d\mu + \int_{M} u_{p}^{(1)} P_{g,h}^{(1)} u_{p} d\mu + \frac{1}{2} \int_{M} u_{p} P_{g,h}^{(1)} u_{p} \cdot trh d\mu - \frac{1}{4} \nu_{p}^{(1)} \int_{M} u_{p}^{2} trh d\mu$$

Proposition 4.2. Let (S^3, g) be the standard sphere, then for any $p \in S^3$ and smooth symmetric (0, 2) tensor h, we have

$$\nu^{(1)}(p,h) = 0,$$

and

$$\nu^{(2)}(p,h) = -16II(p,p,h)$$

Theorem 1.2 follows from Proposition 3.2 and 4.2. By symmetry we can assume p = N, then it follows from (4.16) that

$$\nu^{(1)}(N,h) = \int_{S^3} u_N P_{g,h}^{(1)} u_N d\mu$$

= $16 \int_{S^3} G_N P_{g,h}^{(1)} G_N d\mu$
= 0

by the proof of Proposition 3.1.

Next we note that (4.17) implies

$$\nu^{(2)}(N,h)$$

$$= \int_{S^3} u_N P_{g,h}^{(2)} u_N d\mu + \int_{S^3} u_N^{(1)} P_{g,h}^{(1)} u_N d\mu + \frac{1}{2} \int_{S^3} u_N P_{g,h}^{(1)} u_N \cdot trhd\mu.$$
(4.18)

To compute $u_N^{(1)}$, we observe that (4.14) implies

$$\int_{S^3} \left(u_N P_{g,h}^{(1)} \varphi + u_N^{(1)} P \varphi + \frac{1}{2} u_N P \varphi \cdot trh \right) d\mu = \alpha_N^{(1)} \varphi \left(N \right)$$

for any $\varphi \in H^2(S^3)$. Take $\varphi = u_p = 4G_p$, we see

$$\int_{S^3} u_N P_{g,h}^{(1)} u_p d\mu + 4u_N^{(1)}(p) + 2u_N(p) \operatorname{trh}(p) = \alpha_N^{(1)} u_p(N).$$
(4.19)

Since

$$u_{p}(N) = 4G_{p}(N) = 4G_{N}(p) = u_{N}(p),$$

it follows from Lemma 2.3 and (4.19) that

$$u_N^{(1)}(p) = 4I_N(p) + \frac{1}{4}\alpha_N^{(1)}u_N(p).$$
(4.20)

Hence

$$\nu^{(2)}(N,h) = 16 \int_{S^3} G_N P_{g,h}^{(2)} G_N d\mu + 16 \int_{S^3} P_{g,h}^{(1)} G_N \cdot I_N d\mu + 8 \int_{S^3} P_{g,h}^{(1)} G_N \cdot G_N trhd\mu$$

= -16II(N, N, h).

Here we have used $\int_{S^3} G_N P_{g,h}^{(1)} G_N d\mu = 0$ (which follows from the proof of Proposition 3.1) and (3.7). Proposition 4.2 follows.

5. Appendix: Taylor expansion formula for Q curvature of metrics depending on a parameter

Assume on a smooth three dimensional Riemannian manifold we have $\tilde{g} = g + th$, then under a local orthonormal frame with respect to g, we have

$$\widetilde{g}_{ij} = g_{ij} + th_{ij}, \tag{5.1}$$

and

$$\tilde{g}^{ij} = \delta_{ij} - th_{ij} + t^2 h_{ij}^2 + O\left(t^3\right).$$
(5.2)

Here h^2 is the tensor given by

$$h_{ij}^2 = h_{ik}h_{jk}.$$

The measure associated with \widetilde{g} is given by

$$d\tilde{\mu} = \left[1 + \frac{t}{2}trh + t^2\left(\frac{(trh)^2}{8} - \frac{|h|^2}{4}\right) + O\left(t^3\right)\right]d\mu.$$
 (5.3)

Here

$$\left|h\right|^{2} = \sum_{ij} \left(h_{ij}\right)^{2}.$$

The Christofel symbol satisfies

$$\widetilde{\Gamma}_{ij}^{k} - \Gamma_{ij}^{k}$$

$$= \frac{t}{2} \left(h_{ikj} + h_{jki} - h_{ijk} \right) - \frac{t^2}{2} \left(h_{i\alpha j} + h_{j\alpha i} - h_{ij\alpha} \right) h_{k\alpha} + O\left(t^3\right).$$
(5.4)

Note that the left hand side is a tensor. The Riemann curvature tensor satisfies

$$\widetilde{R}_{ijk}^{l} = R_{ijk}^{l} + \frac{t}{2} \left(h_{ilkj} + h_{jkli} + h_{klij} - h_{jlki} - h_{iklj} - h_{klji} \right)$$

$$- \frac{t^{2}}{2} \left(h_{i\alpha kj} + h_{jk\alpha i} + h_{k\alpha ij} - h_{j\alpha ki} - h_{ik\alpha j} - h_{k\alpha ji} \right) h_{l\alpha}$$

$$+ \frac{t^{2}}{4} \left(h_{i\alpha l} + h_{l\alpha i} - h_{il\alpha} \right) \left(h_{j\alpha k} + h_{k\alpha j} - h_{jk\alpha} \right)$$

$$- \frac{t^{2}}{4} \left(h_{i\alpha k} + h_{k\alpha i} - h_{ik\alpha} \right) \left(h_{j\alpha l} + h_{l\alpha j} - h_{jl\alpha} \right) + O\left(t^{3}\right).$$
(5.5)

In particular, the Ricci tensor is given by

$$\widetilde{Rc}_{ij}$$
(5.6)
$$= Rc_{ij} + \frac{t}{2} \left(h_{ikjk} + h_{jkik} - (trh)_{ij} - \Delta h_{ij} \right) \\
- \frac{t^2}{2} \left(h_{ikjl} + h_{jkil} - h_{klji} - h_{ijkl} \right) h_{kl} \\
+ \frac{t^2}{4} \left(h_{ikl} + h_{kli} - h_{ilk} \right) \left(h_{jkl} + h_{klj} - h_{jlk} \right) \\
- \frac{t^2}{4} \left(h_{ikj} + h_{jki} - h_{ijk} \right) \left(2h_{kll} - (trh)_k \right) + O(t^3)$$

here the tensor Δh is given by

$$\Delta h_{ij} = h_{ijkk}.$$

The scalar curvature is given by

$$\widetilde{R}$$
(5.7)
= $R + t (h_{ijij} - \Delta trh - Rc_{ij}h_{ij})$
+ $t^2 \left(\Delta h_{ij} + (trh)_{ij} - h_{ikjk} - h_{ikkj}\right) h_{ij} + \frac{t^2}{4} \sum_{ijk} (h_{ikj} + h_{jki} - h_{ijk})^2$
 $-\frac{t^2}{4} \sum_i (2h_{ijj} - (trh)_i)^2 + t^2 Rc_{ij}h_{ij}^2 + O(t^3).$

The Laplacian satisfies

$$\widetilde{\Delta}\varphi \qquad (5.8)$$

$$= \Delta\varphi - \frac{t}{2} \left(2h_{ijj} - (trh)_i\right)\varphi_i - th_{ij}\varphi_{ij}$$

$$+ \frac{t^2}{2} \left(2h_{ikk} - (trh)_i\right)h_{ij}\varphi_j + \frac{t^2}{2} \left(2h_{ikj} - h_{ijk}\right)h_{ij}\varphi_k + t^2h_{ij}^2\varphi_{ij} + O\left(t^3\right).$$

As a consequence we have

$$\widetilde{R}^{2}$$

$$= R^{2} + 2tR (h_{ijij} - \Delta trh - Rc_{ij}h_{ij})
+ t^{2} (h_{ijij} - \Delta trh - Rc_{ij}h_{ij})^{2} + 2t^{2}R (\Delta h_{ij} + (trh)_{ij} - h_{ikjk} - h_{ikkj}) h_{ij}
+ \frac{t^{2}R}{2} \sum_{ijk} (h_{ikj} + h_{jki} - h_{ijk})^{2} - \frac{t^{2}R}{2} \sum_{i} (2h_{ijj} - (trh)_{i})^{2} + 2t^{2}R \cdot Rc_{ij}h_{ij}^{2}
+ O (t^{3})$$
(5.9)

and

$$\begin{aligned} \left| \widetilde{Rc} \right|^{2} \tag{5.10} \\ &= \left| Rc \right|^{2} + tRc_{ij} \left(2h_{ikjk} - (trh)_{ij} - \Delta h_{ij} \right) - 2tRc_{ij}Rc_{ik}h_{jk} \\ &+ \frac{t^{2}}{4} \sum_{ij} \left(h_{ikjk} + h_{jkik} - (trh)_{ij} - \Delta h_{ij} \right)^{2} - t^{2}Rc_{ij}h_{kl} \left(2h_{ikjl} - h_{klij} - h_{ijkl} \right) \\ &- 2t^{2}Rc_{ij}h_{ik} \left(h_{jlkl} + h_{kljl} - (trh)_{jk} - \Delta h_{jk} \right) \\ &+ \frac{t^{2}}{2}Rc_{ij} \left(h_{ikl} + h_{kli} - h_{ilk} \right) \left(h_{jkl} + h_{klj} - h_{jlk} \right) \\ &- \frac{t^{2}}{2}Rc_{ij} \left(2h_{ikj} - h_{ijk} \right) \left(2h_{kll} - (trh)_{k} \right) + 2t^{2}Rc_{ij}Rc_{ik}h_{jk}^{2} + t^{2}Rc_{ij}Rc_{kl}h_{ik}h_{jl} \\ &+ O \left(t^{3} \right) \end{aligned}$$

$$\begin{split} \widetilde{\Delta}\widetilde{R} & (5.11) \\ = & \Delta R + t\Delta \left(h_{ijij} - \Delta trh - Rc_{ij}h_{ij}\right) - \frac{t}{2} \left(2h_{ijj} - (trh)_i\right) R_i - th_{ij}R_{ij} \\ & + t^2\Delta \left[\left(\Delta h_{ij} + (trh)_{ij} - h_{ikjk} - h_{ikkj}\right) h_{ij} \right] - t^2h_{ij} \left(h_{klkl} - \Delta trh - Rc_{kl}h_{kl}\right)_{ij} \\ & + \frac{t^2}{4}\Delta \sum_{ijk} \left(h_{ikj} + h_{jki} - h_{ijk}\right)^2 - \frac{t^2}{4}\Delta \sum_i \left(2h_{ijj} - (trh)_i\right)^2 \\ & - \frac{t^2}{2} \left(2h_{ijj} - (trh)_i\right) \left(h_{klkl} - \Delta trh - Rc_{kl}h_{kl}\right)_i + t^2\Delta \left(Rc_{ij}h_{ij}^2\right) \\ & + \frac{t^2}{2} \left(2h_{ikk} - (trh)_i\right) h_{ij}R_j + \frac{t^2}{2} \left(2h_{ikj} - h_{ijk}\right) h_{ij}R_k + t^2h_{ij}^2R_{ij} + O\left(t^3\right). \end{split}$$

Recall

$$Q = -\frac{1}{4}\Delta R - 2\left|Rc\right|^2 + \frac{23}{32}R^2.$$
 (5.12)

Plug in the formulas above we get

$$\begin{split} \bar{Q} & (5.13) \\ = & Q - \frac{t}{4} \Delta \left(h_{ijij} - \Delta trh - Rc_{ij}h_{ij} \right) - 2tRc_{ij} \left(2h_{ikjk} - (trh)_{ij} - \Delta h_{ij} \right) \\ & + \frac{23t}{16} R \left(h_{ijij} - \Delta trh - Rc_{ij}h_{ij} \right) + \frac{t}{8} \left(2h_{ijj} - (trh)_i \right) R_i + \frac{t}{4} h_{ij} R_{ij} + 4tRc_{ij} Rc_{ik} h_{jk} \\ & - \frac{t^2}{4} \Delta \left[\left(\Delta h_{ij} + (trh)_{ij} - h_{ikjk} - h_{ikkj} \right) h_{ij} \right] + \frac{t^2}{4} h_{ij} \left(h_{klkl} - \Delta trh - Rc_{kl} h_{kl} \right)_{ij} \\ & - \frac{t^2}{16} \Delta \sum_{ijk} \left(h_{ikj} + h_{jki} - h_{ijk} \right)^2 + \frac{t^2}{16} \Delta \sum_i \left(2h_{ijj} - (trh)_i \right)^2 \\ & + \frac{t^2}{8} \left(2h_{ijj} - (trh)_i \right) \left(h_{klkl} - \Delta trh - Rc_{kl} h_{kl} \right)_i - \frac{t^2}{4} \Delta \left(Rc_{ij} h_{ij}^2 \right) \\ & - \frac{t^2}{2} \sum_{ij} \left(h_{ikjk} + h_{jkik} - (trh)_{ij} - \Delta h_{ij} \right)^2 + 2t^2 Rc_{ij} h_{kl} \left(2h_{ikjl} - h_{klij} - h_{ijkl} \right) \\ & + 4t^2 Rc_{ij} h_{ik} \left(h_{jlkl} + h_{kljl} - (trh)_{jk} - \Delta h_{jk} \right) + \frac{23t^2}{32} \left(h_{ijij} - \Delta trh - Rc_{ij} h_{ij} \right)^2 \\ & + \frac{23t^2}{16} R \left(\Delta h_{ij} + (trh)_{ij} - h_{ikkj} - h_{ikkj} \right) h_{ij} - t^2 Rc_{ij} \left(h_{ikl} + h_{kli} - h_{ilk} \right) \left(h_{jkl} + h_{klj} - h_{jlk} \right) \\ & + t^2 Rc_{ij} \left(2h_{ikj} - h_{ijk} \right) \left(2h_{kll} - (trh)_k \right) + \frac{23t^2}{64} R \sum_{ijk} \left(h_{ikj} + h_{jki} - h_{ijk} \right) h_{ij} R_k \\ & - \frac{23t^2}{64} R \sum_i \left(2h_{ijj} - (trh)_i \right)^2 - \frac{t^2}{8} \left(2h_{ikk} - (trh)_i \right) h_{ij} R_j - \frac{t^2}{8} \left(2h_{ikj} - h_{ijk} \right) h_{ij} R_k \\ & - \frac{t^2}{4} h_{ij}^2 R_{ij} - 4t^2 Rc_{ij} Rc_{ik} h_{jk}^2 - 2t^2 Rc_{ij} Rc_{kl} h_{ikh} h_{jl} + \frac{23t^2}{16} R \cdot Rc_{ij} h_{ij}^2 + O \left(t^3 \right). \end{split}$$

Since the Paneitz operator can be written as

$$P\varphi = \Delta^2 \varphi + 4Rc_{ij}\varphi_{ij} - \frac{5}{4}R\Delta\varphi + \frac{3}{4}\nabla R \cdot \nabla\varphi - \frac{1}{2}Q\varphi, \qquad (5.14)$$

calculation shows

$$P_{g,h}^{(1)}\varphi$$

$$= -h_{ij} (\Delta\varphi)_{ij} - \Delta (h_{ij}\varphi_{ij}) - \frac{1}{2} (2h_{ijj} - (trh)_i) (\Delta\varphi)_i - \frac{1}{2}\Delta [(2h_{ijj} - (trh)_i)\varphi_i]$$

$$+ 2 (2h_{ikjk} - \Delta h_{ij} - (trh)_{ij})\varphi_{ij} - 8Rc_{ij}h_{ik}\varphi_{jk} + \frac{5}{4}Rh_{ij}\varphi_{ij}$$

$$- \frac{5}{4} (h_{ijij} - \Delta trh - Rc_{ij}h_{ij}) \Delta\varphi - 2Rc_{ij} (2h_{ikj} - h_{ijk})\varphi_k + \frac{5}{8}R (2h_{ijj} - (trh)_i)\varphi_i$$

$$+ \frac{3}{4} (h_{jkjk} - \Delta trh - Rc_{jk}h_{jk})_i\varphi_i - \frac{3}{4}h_{ij}R_i\varphi_j + \frac{1}{8}\Delta (h_{ijij} - \Delta trh - Rc_{ij}h_{ij}) \cdot \varphi$$

$$- \frac{1}{8}h_{ij}R_{ij}\varphi - \frac{1}{16} (2h_{ijj} - (trh)_i)R_i\varphi + Rc_{ij} (2h_{ikjk} - \Delta h_{ij} - (trh)_{ij})\varphi$$

$$- 2Rc_{ij}Rc_{ik}h_{jk}\varphi - \frac{23}{32}R (h_{ijij} - \Delta trh - Rc_{ij}h_{ij})\varphi.$$
(5.15)

In particular on standard S^3 we have

$$P_{g,h}^{(1)}\varphi$$

$$= -h_{ij} \left(\Delta\varphi\right)_{ij} - \Delta\left(h_{ij}\varphi_{ij}\right) - \frac{1}{2} \left(2h_{ijj} - (trh)_i\right) \left(\Delta\varphi\right)_i - \frac{1}{2}\Delta\left[\left(2h_{ijj} - (trh)_i\right)\varphi_i\right]$$

$$+ 2 \left(2h_{ikjk} - \Delta h_{ij} - (trh)_{ij}\right)\varphi_{ij} - \frac{17}{2}h_{ij}\varphi_{ij} - \frac{5}{4} \left(h_{ijij} - \Delta trh - 2trh\right)\Delta\varphi$$

$$- \frac{1}{4} \left(2h_{ijj} - (trh)_i\right)\varphi_i + \frac{3}{4} \left(h_{jkjk} - \Delta trh - 2trh\right)_i\varphi_i + \frac{1}{8}\Delta\left(h_{ijij} - \Delta trh - 2trh\right)\varphi$$

$$- \frac{5}{16} \left(h_{ijij} - \Delta trh - 2trh\right)\varphi.$$

$$(5.16)$$

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