# PANEITZ OPERATOR FOR METRICS NEAR $S^{3}$ 

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#### Abstract

We derive the first and second variation formula for the Green's function pole's value of Paneitz operator on the standard three sphere. In particular it is shown that the first variation vanishes and the second variation is nonpositively definite. Moreover, the second variation vanishes only at the direction of conformal deformation. We also introduce a new invariant of the Paneitz operator and illustrate its close relation with the second eigenvalue and Sobolev inequality of Paneitz operator.


## 1. Introduction

The fourth order $Q$ curvature equation $([\overline{\mathrm{Br}}, \overline{\mathrm{P}})$ has attracted interest due to the successful study in dimension four and its application to conformal geometry in dimension four ( CGY ). We are interested to possibly extend this analysis to dimension three. The effort to understand $Q$ curvature in dimension three motivates many intriguing and challenging problems. In this dimension, the functions in $H^{2}$ are actually $\frac{1}{2}$-Holder continuous, and hence the Green's function has well defined value at its pole. The sign of this value turns out to be an important issue. For the standard sphere, the Green's function is nonpositive everywhere but vanishes exactly at the pole. Our purpose in this article is to study this question for the conformal structures near the standard sphere.

Recall on a three manifold, the $Q$ curvature is given by

$$
\begin{equation*}
Q=-\frac{1}{4} \Delta R-2|R c|^{2}+\frac{23}{32} R^{2} \tag{1.1}
\end{equation*}
$$

and the fourth order Paneitz operator is defined as

$$
\begin{equation*}
P \varphi=\Delta^{2} \varphi+4 \operatorname{div}\left(R c\left(\nabla \varphi, e_{i}\right) e_{i}\right)-\frac{5}{4} \operatorname{div}(R \nabla \varphi)-\frac{1}{2} Q \varphi \tag{1.2}
\end{equation*}
$$

Here $R c$ is the Ricci curvature, $R$ is the scalar curvature and $e_{1}, e_{2}, e_{3}$ is a local orthonormal frame with respect to the metric. For any positive smooth function $\rho$, the operator satisfies

$$
\begin{equation*}
P_{\rho^{-4} g} \varphi=\rho^{7} P_{g}(\rho \varphi) \tag{1.3}
\end{equation*}
$$

As a consequence we know $\operatorname{ker} P_{g}=0 \Leftrightarrow \operatorname{ker} P_{\rho^{-4} g}=0$ and under this assumption, the Green's functions of $P$ satisfy the transformation law

$$
\begin{equation*}
G_{\rho^{-4} g}(p, q)=\rho(p)^{-1} \rho(q)^{-1} G_{g}(p, q) \tag{1.4}
\end{equation*}
$$

Let $(M, g)$ be a smooth compact three dimensional Riemannian manifold, for $u, v \in C^{\infty}(M)$, we write

$$
\begin{align*}
E(u, v) & =\int_{M} P u \cdot v d \mu  \tag{1.5}\\
& =\int_{M}\left(\Delta u \Delta v-4 R c(\nabla u, \nabla v)+\frac{5}{4} R \nabla u \cdot \nabla v-\frac{1}{2} Q u v\right) d \mu
\end{align*}
$$

and $E(u)=E(u, u)$. Here $\mu$ is the measure associated with metric $g$. It is clear that $E(u, v)$ makes sense for $u, v \in H^{2}(M)$.

The scaling invariant quantity $\mu(M)^{\frac{1}{3}} \int_{M} Q d \mu$ satisfies

$$
\mu_{\rho^{-4} g}(M)^{\frac{1}{3}} \int_{M} Q_{\rho^{-4} g} d \mu_{\rho^{-4} g}=-2 E_{g}(\rho)\left\|\rho^{-1}\right\|_{L^{6}(M, g)}^{2}
$$

Hence

$$
\begin{aligned}
\sup _{\widetilde{g} \in[g]} \widetilde{\mu}(M)^{\frac{1}{3}} \int_{M} \widetilde{Q} d \widetilde{\mu} & =-2 \inf _{\rho \in C^{\infty}, \rho>0} E(\rho)\left\|\rho^{-1}\right\|_{L^{6}}^{2} \\
& =-2 \inf _{u \in H^{2}(M), u>0} E(u)\left\|u^{-1}\right\|_{L^{6}}^{2}
\end{aligned}
$$

Here $[g]$ is the conformal class of metrics associated with $g$. As in HY1, we write

$$
\begin{equation*}
I_{4}(u)=E(u)\left\|u^{-1}\right\|_{L^{6}}^{2} \tag{1.6}
\end{equation*}
$$

and

$$
\begin{equation*}
Y_{4}(g)=\inf _{u \in H^{2}(M), u>0} E(u)\left\|u^{-1}\right\|_{L^{6}}^{2} \tag{1.7}
\end{equation*}
$$

From above discussion we see $Y_{4}(\widetilde{g})=Y_{4}(g)$ for $\widetilde{g} \in[g]$.
The question of whether $Y_{4}(g)$ is finite and achieved by some particular metrics was considered in HY1, YZ]. This inequality is analytically different from the one of Yamabe invariant $Y(g)$ (see LP ) due to the negative power involved.

HY1 shows that when $\operatorname{ker} P=0$, the value of the Green's function at pole plays a crucial role. In particular based on explicit calculation of this value on Berger's sphere, we were able to show $Y_{4}(g)$ is achieved on all Berger spheres. In general such an explicit formula is not available. On the other hand, properties of Paneitz operator on the standard three sphere are well understood.

On standard $S^{3}$, we have

$$
\begin{equation*}
P u=\Delta^{2} u+\frac{1}{2} \Delta u-\frac{15}{16} u \tag{1.8}
\end{equation*}
$$

Let $N$ be the north pole, $\pi_{N}: S^{3} \backslash\{N\} \rightarrow \mathbb{R}^{3}$ be the stereographic projection, using $x=\pi_{N}$ as the coordinates, the Green's function of $P$ with pole at $N$ is given by

$$
\begin{equation*}
G_{N}=-\frac{1}{4 \pi} \frac{1}{\sqrt{|x|^{2}+1}} \tag{1.9}
\end{equation*}
$$

In particular $G_{N}(N)=0$.
Proposition 1.1. Let $g$ be the standard metric on $S^{3}$, then for any $p \in S^{3}$ and any smooth symmetric $(0,2)$ tensor $h$,

$$
\left.\partial_{t}\right|_{t=0} G_{g+t h}(p, p)=0
$$

Here $G_{g+t h}$ is the Green's function of the Paneitz operator $P_{g+t h}$.
This calculation leads one to ask about the second variation. We have
Theorem 1.1. Using the stereographic projection $\pi_{N}$ as the coordinate, the standard metric $g$ on $S^{3}$ is written as

$$
\begin{equation*}
g=\frac{4}{\left(|x|^{2}+1\right)^{2}}|d x|^{2}=\tau^{-4}|d x|^{2} \tag{1.10}
\end{equation*}
$$

here

$$
\begin{equation*}
\tau=\sqrt{\frac{|x|^{2}+1}{2}} . \tag{1.11}
\end{equation*}
$$

For any smooth symmetric $(0,2)$ tensor $h$, denote

$$
\begin{equation*}
\theta=\tau^{4} h \tag{1.12}
\end{equation*}
$$

then we have

$$
\begin{align*}
& \left.\partial_{t}^{2}\right|_{t=0} G_{g+t h}(N, N)  \tag{1.13}\\
= & -\frac{1}{64 \pi^{2}} \int_{\mathbb{R}^{3}}\left(\sum_{i j}\left(\theta_{i k j k}+\theta_{j k i k}-(\operatorname{tr} \theta)_{i j}-\Delta \theta_{i j}\right)^{2}-\frac{3}{2}\left(\theta_{i j i j}-\Delta t r \theta\right)^{2}\right) d x
\end{align*}
$$

Here the derivatives $\theta_{i k j k}$ etc are taken with respect to the standard metric on $\mathbb{R}^{3}$.
Using formula (1.13), we will show the second variation is always nonpositive and it vanishes only in the direction generated by conformal diffeomorphism. More precisely,
Proposition 1.2. For any smooth symmetric $(0,2)$ tensor $h$ on $S^{3}$ and $p \in S^{3}$,

$$
\begin{equation*}
\left.\partial_{t}^{2}\right|_{t=0} G_{g+t h}(p, p) \leq 0 \tag{1.14}
\end{equation*}
$$

Moreover, $\left.\partial_{t}^{2}\right|_{t=0} G_{g+t h}(p, p)=0$ if and only if $h=L_{X} g+f \cdot g$ for some smooth vector fields $X$ and smooth function $f$ on $S^{3}$.

It is worth pointing out that in HY2, motivated from recent works GM, HR for $Q$ curvature in dimension five or higher and Proposition 1.1 and 1.2 above, it was shown that for smooth compact three manifold $(M, g)$ with positive scalar and $Q$ curvature, the Paneitz operator must have zero kernel and the Green's function pole's value is strictly negative except when $(M, g)$ is conformal diffeomorphic to the standard $S^{3}$. Further developments can be found in HY3, HY4.

The Sobolev inequality of Paneitz operator on $S^{3}$ was first verified in YZ. Different proofs were given in [H, HY1. The new approach motivates the condition NN and condition P for a Paneitz operator (see [HY1, section 5]). Here we will introduce a quantity for the Paneitz operator whose sign corresponds to condition NN and P. Let $(M, g)$ be a smooth compact three dimensional Riemannian manifold without boundary. For any $p \in M$, denote (recall functions in $H^{2}(M)$ are $\frac{1}{2}$-Holder continuous)

$$
\begin{equation*}
\nu(M, g, p)=\inf \left\{\frac{E(u)}{\int_{M} u^{2} d \mu}: u \in H^{2}(M) \backslash\{0\}, u(p)=0\right\} \tag{1.15}
\end{equation*}
$$

When no confusion could arise we denote it as $\nu(g, p)$ or $\nu_{p}$. We also write

$$
\begin{align*}
\nu(M, g) & =\inf _{p \in M} \nu(M, g, p)  \tag{1.16}\\
& =\inf \left\{\frac{E(u)}{\int_{M} u^{2} d \mu}: u \in H^{2}(M) \backslash\{0\}, u(p)=0 \text { for some } p\right\} .
\end{align*}
$$

The importance of $\nu(M, g)$ lies in that $(M, g)$ satisfies condition P if and only if $\nu(M, g)>0$ and it satisfies condition NN if and only if $\nu(M, g) \geq 0$. For the standard metric $g$ on $S^{3}, \nu\left(S^{3}, g\right)=0$, in fact we have (see Example 4.1) $\nu\left(S^{3}, g, p\right)=0$ for all $p \in S^{3}$.

Theorem 1.2. Let $g$ be the standard metric on $S^{3}$, then for any $p \in S^{3}$ and any smooth symmetric $(0,2)$ tensor $h$,

$$
\begin{equation*}
\left.\partial_{t}\right|_{t=0} \nu(g+t h, p)=0 \tag{1.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\partial_{t}^{2}\right|_{t=0} \nu(g+t h, p)=-\left.16 \partial_{t}^{2}\right|_{t=0} G_{g+t h}(p, p) \tag{1.18}
\end{equation*}
$$

In particular, $\left.\partial_{t}^{2}\right|_{t=0} \nu(g+t h, p) \geq 0$ and it vanishes if and only if $h=L_{X} g+f \cdot g$ for some smooth vector fields $X$ and smooth functions $f$ on $S^{3}$.

Roughly speaking Theorem 1.2 tells us for Riemannian metrics near the standard metric on $S^{3}$, as long as it is not conformal diffeomorphic to the standard sphere, condition P is satisfied and hence $Y_{4}(g)$ is achieved by HY1. More related results can be found in HY2, HY3.

In section 2 below we will introduce technique simplifying various calculations. In section 3 we will derive the first and second variation formulas and justify its nonpositivity. In section4, we will study the quantity $\nu(g)$ and its relations to $Y_{4}(g)$ and the second eigenvalue of Paneitz operator. Some of the lengthy calculations are collected in the appendix to streamline the discussions.

## 2. Some preparations

Because the formula of $Q$ curvature and Paneitz operator are relatively complicated, it is crucial to take advantage of the conformal covariant property (1.3) to simplify the calculation of first and second variation of the Green's function pole's value. To achieve this we observe that the Paneitz operator gives us a sequence of fourth order conformal covariant operators. Indeed for smooth metric $g$ and symmetric $(0,2)$ tensor $h$, we define the operator $P_{g, h}^{(k)}$ by the Taylor expansion

$$
\begin{equation*}
P_{g+t h} \varphi \sim \sum_{k=0}^{\infty} t^{k} P_{g, h}^{(k)} \varphi \tag{2.1}
\end{equation*}
$$

Here $\sim$ means for any $m \geq 0$,

$$
P_{g+t h} \varphi=\sum_{k=0}^{m} t^{k} P_{g, h}^{(k)} \varphi+O\left(t^{m+1}\right)
$$

as $t \rightarrow 0$.
Lemma 2.1. For any smooth function $\varphi$ and positive smooth function $\rho$,

$$
\begin{equation*}
P_{\rho^{-4} g, \rho^{-4} h}^{(k)} \varphi=\rho^{7} P_{g, h}^{(k)}(\rho \varphi) \tag{2.2}
\end{equation*}
$$

This is the conformal covariant property of $P_{g, h}^{(k)}$. Indeed for $t$ near 0 ,

$$
P_{\rho^{-4}(g+t h)} \varphi=\rho^{7} P_{g+t h}(\rho \varphi)=P_{\rho^{-4} g+t \rho^{-4} h} \varphi
$$

hence

$$
\sum_{k=0}^{\infty} t^{k} P_{\rho^{-4} g, \rho^{-4} h}^{(k)} \varphi \sim \rho^{7} \sum_{k=0}^{\infty} t^{k} P_{g, h}^{(k)}(\rho \varphi)
$$

Equation (2.2) follows.

Careful calculation shows (see appendix)

$$
\begin{align*}
& P_{g, h}^{(1)} \varphi  \tag{2.3}\\
= & -h_{i j}(\Delta \varphi)_{i j}-\Delta\left(h_{i j} \varphi_{i j}\right)-\frac{1}{2}\left(2 h_{i j j}-(t r h)_{i}\right)(\Delta \varphi)_{i}-\frac{1}{2} \Delta\left(\left(2 h_{i j j}-(t r h)_{i}\right) \varphi_{i}\right) \\
& +2\left(2 h_{i k j k}-\Delta h_{i j}-(t r h)_{i j}\right) \varphi_{i j}-8 R c_{i j} h_{i k} \varphi_{j k}+\frac{5}{4} R h_{i j} \varphi_{i j} \\
& -\frac{5}{4}\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right) \Delta \varphi-2 R c_{i j}\left(2 h_{i k j}-h_{i j k}\right) \varphi_{k}+\frac{5}{8} R\left(2 h_{i j j}-(t r h)_{i}\right) \varphi_{i} \\
& +\frac{3}{4}\left(h_{k l k l}-\Delta t r h-R c_{k l} h_{k l}\right)_{i} \varphi_{i}-\frac{3}{4} h_{i j} R_{i} \varphi_{j}+\frac{1}{8} \Delta\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right) \varphi \\
& -\frac{1}{8} h_{i j} R_{i j} \varphi-\frac{1}{16}\left(2 h_{i j j}-(t r h)_{i}\right) R_{i} \varphi+R c_{i j}\left(2 h_{i k j k}-\Delta h_{i j}-(t r h)_{i j}\right) \varphi \\
& -2 R c_{i j} R c_{i k} h_{j k} \varphi-\frac{23}{32} R\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right) \varphi .
\end{align*}
$$

On the other hand the formula of $P_{g, h}^{(2)} \varphi$ is much more complicated, and we will not write it down here. Instead we observe that $P_{g, h}^{(2)} \varphi$ is a fourth order operator in $\varphi$ and $P_{g, h}^{(2)} 1$ can be written down in a reasonable way. Indeed (see appendix)

$$
\begin{align*}
& Q_{g+t h}  \tag{2.4}\\
= & Q-\frac{t}{4} \Delta\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right)-2 t R c_{i j}\left(2 h_{i k j k}-(t r h)_{i j}-\Delta h_{i j}\right) \\
& +\frac{23 t}{16} R\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right)+\frac{t}{8}\left(2 h_{i j j}-(t r h)_{i}\right) R_{i}+\frac{t}{4} h_{i j} R_{i j}+4 t R c_{i j} R c_{i k} h_{j k} \\
& -\frac{t^{2}}{4} \Delta\left[\left(\Delta h_{i j}+(t r h)_{i j}-h_{i k j k}-h_{i k k j}\right) h_{i j}\right]+\frac{t^{2}}{4} h_{i j}\left(h_{k l k l}-\Delta t r h-R c_{k l} h_{k l}\right)_{i j} \\
& -\frac{t^{2}}{16} \Delta \sum_{i j k}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)^{2}+\frac{t^{2}}{16} \Delta \sum_{i}\left(2 h_{i j j}-(t r h)_{i}\right)^{2} \\
& +\frac{t^{2}}{8}\left(2 h_{i j j}-(t r h)_{i}\right)\left(h_{k l k l}-\Delta t r h-R c_{k l} h_{k l}\right)_{i}-\frac{t^{2}}{4} \Delta\left(R c_{i j} h_{i j}^{2}\right) \\
& -\frac{t^{2}}{2} \sum_{i j}\left(h_{i k j k}+h_{j k i k}-(t r h)_{i j}-\Delta h_{i j}\right)^{2}+2 t^{2} R c_{i j} h_{k l}\left(2 h_{i k j l}-h_{k l i j}-h_{i j k l}\right) \\
& +4 t^{2} R c_{i j} h_{i k}\left(h_{j l k l}+h_{k l j l}-(t r h)_{j k}-\Delta h_{j k}\right)+\frac{23 t^{2}}{32}\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right)^{2} \\
& +\frac{23 t^{2}}{16} R\left(\Delta h_{i j}+(t r h)_{i j}-h_{i k j k}-h_{i k k j}\right) h_{i j}-t^{2} R c_{i j}\left(h_{i k l}+h_{k l i}-h_{i l k}\right)\left(h_{j k l}+h_{k l j}-h_{j l k}\right) \\
& +t^{2} R c_{i j}\left(2 h_{i k j}-h_{i j k}\right)\left(2 h_{k l l}-(t r h)_{k}\right)+\frac{23 t^{2}}{64} R \sum_{i j k}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)^{2} \\
& -\frac{23 t^{2}}{64} R \sum_{i}\left(2 h_{i j j}-(t r h)_{i}\right)^{2}-\frac{t^{2}}{8}\left(2 h_{i k k}-(t r h)_{i}\right) h_{i j} R_{j}-\frac{t^{2}}{8}\left(2 h_{i k j}-h_{i j k}\right) h_{i j} R_{k} \\
& -\frac{t^{2}}{4} h_{i j}^{2} R_{i j}-4 t^{2} R c_{i j} R c_{i k} h_{j k}^{2}-2 t^{2} R c_{i j} R c_{k l} h_{i k} h_{j l}+\frac{23 t^{2}}{16} R \cdot R c_{i j} h_{i j}^{2}+O\left(t^{3}\right) .
\end{align*}
$$

Because

$$
P_{g+t h} 1=-\frac{1}{2} Q_{g+t h}
$$

we deduce that

$$
\begin{align*}
& P_{g, h}^{(2)} 1  \tag{2.5}\\
= & -\frac{1}{8} h_{i j}\left(h_{k l k l}-\Delta t r h-R c_{k l} h_{k l}\right)_{i j}+\frac{1}{8} \Delta\left[\left(\Delta h_{i j}+(t r h)_{i j}-h_{i k j k}-h_{i k k j}\right) h_{i j}\right] \\
& -\frac{1}{16}\left(2 h_{i j j}-(t r h)_{i}\right)\left(h_{k l k l}-\Delta t r h-R c_{k l} h_{k l}\right)_{i}+\frac{1}{32} \Delta \sum_{i j k}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)^{2} \\
& -\frac{1}{32} \Delta \sum_{i}\left(2 h_{i j j}-(t r h)_{i}\right)^{2}+\frac{1}{8} \Delta\left(R c_{i j} h_{i j}^{2}\right)+\frac{1}{4} \sum_{i j}\left(h_{i k j k}+h_{j k i k}-(t r h)_{i j}-\Delta h_{i j}\right)^{2} \\
& -R c_{i j} h_{k l}\left(2 h_{i k j l}-h_{k l i j}-h_{i j k l}\right)-2 R c_{i j} h_{i k}\left(h_{j l k l}+h_{k l j l}-(t r h)_{j k}-\Delta h_{j k}\right) \\
& -\frac{23}{64}\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right)^{2}-\frac{23}{32} R\left(\Delta h_{i j}+(t r h)_{i j}-h_{i k j k}-h_{i k k j}\right) h_{i j} \\
& +\frac{1}{2} R c_{i j}\left(h_{i k l}+h_{k l i}-h_{i l k}\right)\left(h_{j k l}+h_{k l j}-h_{j l k}\right)-\frac{1}{2} R c_{i j}\left(2 h_{i k j}-h_{i j k}\right)\left(2 h_{k l l}-(t r h)_{k}\right) \\
& -\frac{23}{128} R \sum_{i j k}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)^{2}+\frac{23}{128} R \sum_{i}\left(2 h_{i j j}-(t r h)_{i}\right)^{2} \\
& +\frac{1}{8}\left[h_{i j}^{2} R_{i j}+\frac{1}{2}\left(2 h_{i k k}-(t r h)_{i}\right) h_{i j} R_{j}+\frac{1}{2}\left(2 h_{i k j}-h_{i j k}\right) h_{i j} R_{k}\right]+2 R c_{i j} R c_{i k} h_{j k}^{2} \\
& +R c_{i j} R c_{k l} h_{i k} h_{j l}-\frac{23}{32} R \cdot R c_{i j} h_{i j}^{2} .
\end{align*}
$$

In general, $P_{g, h}^{(1)}$ is not self adjoint, instead we have
Lemma 2.2. For every $\varphi, \psi \in C^{\infty}$,

$$
\begin{equation*}
\int_{M} P_{g, h}^{(1)} \varphi \cdot \psi d \mu=\int_{M} \varphi P_{g, h}^{(1)} \psi d \mu-\frac{1}{2} \int_{M}(P \varphi \cdot \psi-\varphi P \psi) \operatorname{trh} d \mu \tag{2.6}
\end{equation*}
$$

Indeed this follows from the Taylor expansion in $t$ for

$$
\int_{M} P_{g+t h} \varphi \cdot \psi d \mu_{g+t h}=\int_{M} \varphi P_{g+t h} \psi d \mu_{g+t h}
$$

To derive a variational formula for the Green's function pole's value we write

$$
\begin{equation*}
G_{g+t h}(p, q)=G(p, q)+t I(p, q, h)+t^{2} I I(p, q, h)+O\left(t^{3}\right) \tag{2.7}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\left.\partial_{t}\right|_{t=0} G_{g+t h}(p, p)=I(p, p, h) \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\partial_{t}^{2}\right|_{t=0} G_{g+t h}(p, p)=2 I I(p, p, h) \tag{2.9}
\end{equation*}
$$

We can write $I$ and $I I$ in terms of $P_{g, h}^{(1)}$ and $P_{g, h}^{(2)}$.
Lemma 2.3. For any smooth symmetric $(0,2)$ tensor $h$, let $I$ and $I I$ be defined in (2.7), then

$$
\begin{equation*}
I(p, q, h)=-\int_{M} P_{g, h}^{(1)} G_{q} \cdot G_{p} d \mu-\frac{1}{2} G(p, q) \operatorname{trh}(q) \tag{2.10}
\end{equation*}
$$

and

$$
\begin{align*}
& I I(p, p, h)  \tag{2.11}\\
= & -\int_{M}\left(P_{g, h}^{(2)} G_{p} \cdot G_{p}+P_{g, h}^{(1)} G_{p} \cdot I_{p}+\frac{1}{2} P_{g, h}^{(1)} G_{p} \cdot G_{p} \operatorname{trh}\right) d \mu \\
& -\frac{1}{2} I(p, p, h) \operatorname{trh}(p)-\frac{1}{8} G(p, p)(\operatorname{trh}(p))^{2}+\frac{1}{4} G(p, p)|h(p)|^{2} .
\end{align*}
$$

Here $G_{p}(q)=G(p, q), I_{p}(q, h)=I(p, q, h)$. The integration should be understood in distribution sense.

Proof. For any smooth function $\varphi$ we have

$$
\varphi(p)=\int_{M} P_{g+t h} \varphi \cdot G_{g+t h, p} d \mu_{g+t h}
$$

expand everything into power series of $t$, using

$$
d \mu_{g+t h}=\left[1+\frac{t r h}{2} \cdot t+\left(\frac{(t r h)^{2}}{8}-\frac{|h|^{2}}{4}\right) t^{2}+O\left(t^{3}\right)\right] d \mu
$$

we see

$$
\int_{M}\left(I_{p} \cdot P \varphi+G_{p} P_{g, h}^{(1)} \varphi+\frac{1}{2} G_{p} t r h \cdot P \varphi\right) d \mu=0
$$

and

$$
\begin{aligned}
0= & \int_{M}\left[P \varphi \cdot I I_{p}+P_{g, h}^{(1)} \varphi \cdot I_{p}+P_{g, h}^{(2)} \varphi \cdot G_{p}+\frac{1}{2} P \varphi \cdot I_{p} t r h+\frac{1}{2} P_{g, h}^{(1)} \varphi \cdot G_{p} t r h\right. \\
& \left.+\frac{1}{8} P \varphi \cdot G_{p}(t r h)^{2}-\frac{1}{4} P \varphi \cdot G_{p}|h|^{2}\right] d \mu
\end{aligned}
$$

By approximation we know the same formula remains true for $\varphi \in H^{2}(M)$. Let $\varphi=G_{q}$ or $G_{p}$, we get the lemma.

## 3. First and second variation of Green's function pole's value

Let $N$ be the north pole on $S^{3}$ and $\pi_{N}: S^{3} \backslash\{N\} \rightarrow \mathbb{R}^{3}$ be the stereographic projection. Using $x=\pi_{N}$ as the coordinate, we have the standard metric $g$ on $S^{3}$ can be written as

$$
\begin{equation*}
g=\frac{4}{\left(|x|^{2}+1\right)^{2}}|d x|^{2}=\tau^{-4}|d x|^{2} \tag{3.1}
\end{equation*}
$$

here

$$
\begin{equation*}
\tau=\sqrt{\frac{|x|^{2}+1}{2}} \tag{3.2}
\end{equation*}
$$

By conformal invariance property (1.3), the Green's function of $P$ with pole at $N$ is given by

$$
\begin{equation*}
G_{N}=-\frac{1}{4 \pi} \frac{1}{\sqrt{|x|^{2}+1}} \tag{3.3}
\end{equation*}
$$

More generally

$$
\begin{equation*}
G(x, y)=-\frac{1}{4 \pi} \frac{|x-y|}{\sqrt{|x|^{2}+1} \sqrt{|y|^{2}+1}} \tag{3.4}
\end{equation*}
$$

We are ready to compute the first variation of Green's function pole value.

Proposition 3.1. For any $p \in S^{3}$ and smooth symmetric ( 0,2 ) tensor $h, I(p, p, h)=$ 0 .

Proposition 1.1 follows from Proposition 3.1 and (2.8).
Proof. By symmetry we can assume $p=N$. For convenience we write

$$
I(h)=I(N, N, h) .
$$

Because we need to discuss various function's behavior near $N$, we denote $S$ as the south pole of $S^{3}, \pi_{S}: S^{3} \backslash\{S\} \rightarrow \mathbb{R}^{3}$ as the stereographic projection with respect to $S$. We can use $y=\pi_{S}$ as the coordinate. By Lemma 2.3 and the fact $G_{N}(N)=0$,

$$
I(h)=-\int_{S^{3}} P_{g, h}^{(1)} G_{N} \cdot G_{N} d \mu \text { (in distribution sense). }
$$

Let $\eta \in C^{\infty}\left(\mathbb{R}^{3}\right)$ such that $\left.\eta\right|_{B_{1}}=1,\left.\eta\right|_{\mathbb{R}^{3} \backslash B_{2}}=0$ and $0 \leq \eta \leq 1$. For $\varepsilon>0$, we write $\eta_{\varepsilon}=\eta\left(\frac{y}{\varepsilon}\right)$. By [HY1, Lemma 2.2] $\eta_{\varepsilon} G_{N} \rightarrow 0$ in $H^{2}\left(S^{3}\right)$, hence

$$
\begin{aligned}
I(h) & =-\lim _{\varepsilon \rightarrow 0^{+}} \int_{S^{3}} P_{g, h}^{(1)}\left(\left(1-\eta_{\varepsilon}\right) G_{N}\right) \cdot G_{N} d \mu \\
& =-\int_{S^{3}} P_{g, h}^{(1)} G_{N} \cdot G_{N} d \mu \text { (in pointwise product sense). }
\end{aligned}
$$

Note here we have used the dominated convergence theorem and the fact near $N$,

$$
\left|P_{g, h}^{(1)} G_{N} \cdot G_{N}\right| \leq c|y|^{-2}, \quad\left|P_{g, h}^{(1)}\left(\left(1-\eta_{\varepsilon}\right) G_{N}\right) \cdot G_{N}\right| \leq c|y|^{-2}
$$

here $c$ is independent of $\varepsilon$. For convenience we denote $\theta=\tau^{4} h$. By Lemma 2.1 we have

$$
\begin{aligned}
& I(h) \\
= & -\lim _{\varepsilon \rightarrow 0^{+}} \int_{S^{3} \backslash B_{\varepsilon}(N)} P_{g, h}^{(1)} G_{N} \cdot G_{N} d \mu \\
= & -\frac{1}{32 \pi^{2}} \lim _{R \rightarrow \infty} \int_{|x| \leq R} P_{|d x|^{2}, \theta}^{(1)} 1 d x \\
= & -\frac{1}{256 \pi^{2}} \lim _{R \rightarrow \infty} \int_{|x| \leq R} \Delta\left(\theta_{i j i j}-\Delta t r \theta\right) d x \\
= & -\frac{1}{256 \pi^{2}} \lim _{R \rightarrow \infty} \int_{|x|=R}\left(\theta_{i j i j}-\Delta t r \theta\right)_{k} \frac{x_{k}}{R} d S
\end{aligned}
$$

To understand the boundary term we use the following notation: let $f$ be a smooth function defined outside a ball, we say $f=O^{(\infty)}\left(|x|^{a}\right)$ as $|x| \rightarrow \infty$ if for any $m, \partial_{i_{1} \cdots i_{m}} f(x)=O\left(|x|^{a-m}\right)$. In particular $h_{i j}=O^{(\infty)}\left(|x|^{-4}\right), \tau=O^{(\infty)}(|x|)$, hence $\theta_{i j}=O^{(\infty)}(1)$ and

$$
\left(\theta_{i j i j}-\Delta t r \theta\right)_{k}=O^{(\infty)}\left(|x|^{-3}\right)
$$

this implies $I(h)=0$.
It is worth pointing out that there are other ways to calculate $I(N, N, h)$. For example one may do this by using the formula of $P_{g, h}^{(1)}$ on $S^{3}$ (see (5.16)). However the method in the above proof will be crucial for the calculation of second variation formula.

To continue we need the expression of $I(N, q, h)$.

Lemma 3.1. Let $\theta=\tau^{4} h$, under the stereographic projection with respect to $N$, we denote the coordinate of $q$ as $y$, then

$$
\begin{align*}
& I(N, q, h)  \tag{3.5}\\
= & -\frac{1}{256 \pi^{2}} \int_{\mathbb{R}^{3}}\left(\theta_{i j i j}-\Delta \operatorname{tr} \theta\right) \cdot \frac{1}{\sqrt{|y|^{2}+1}}\left(\frac{2}{|x-y|}-\frac{2}{\sqrt{|x|^{2}+1}}-\frac{1}{\left(\sqrt{|x|^{2}+1}\right)^{3}}\right) d x \\
& -\frac{G(N, q)}{8} \int_{S^{3}}\left(\Delta G_{N}-\frac{5}{2} G_{N}\right) h_{i j i j} d \mu-\frac{G(N, q)}{8} \int_{S^{3}}\left(\Delta G_{N}+\frac{5}{16} G_{N}\right) \operatorname{trh} d \mu \\
& +\frac{1}{8} G(N, q) \operatorname{trh}(N) .
\end{align*}
$$

Proof. Indeed it follows from Lemma 2.3 that

$$
\begin{aligned}
& I(N, q, h) \\
= & I(q, N, h) \\
= & -\int_{S^{3}} P_{g, h}^{(1)} G_{N} \cdot G_{q} d \mu-\frac{1}{2} G(N, q) \operatorname{trh}(N) \\
= & -\int_{S^{3}} P_{g, h}^{(1)} G_{N} \cdot\left(G_{q}-G_{q}(N)\right) d \mu-G(N, q) \int_{S^{3}} P_{g, h}^{(1)} G_{N} d \mu-\frac{1}{2} G(N, q) \operatorname{trh}(N) .
\end{aligned}
$$

By Lemma 2.2 we have

$$
\begin{aligned}
& \int_{S^{3}} P_{g, h}^{(1)} G_{N} d \mu \\
= & \int_{S^{3}} G_{N} P_{g, h}^{(1)} 1 d \mu-\frac{1}{2} \int_{S^{3}}\left(P G_{N}-G_{N} \cdot P 1\right) \operatorname{trh} d \mu \\
= & \int_{S^{3}} G_{N}\left(\frac{1}{8} \Delta\left(h_{i j i j}\right)-\frac{5}{16} h_{i j i j}\right) d \mu+\frac{1}{8} \int_{S^{3}}\left(\Delta G_{N}+\frac{5}{16} G_{N}\right) \operatorname{trh} d \mu-\frac{5}{8} \operatorname{trh}(N) .
\end{aligned}
$$

Using the fact $G_{q}-G_{q}(N)$ vanishes at $N$, by the same method in the proof of Proposition 3.1

$$
\begin{aligned}
& \int_{S^{3}} P_{g, h}^{(1)} G_{N} \cdot\left(G_{q}-G_{q}(N)\right) d \mu \\
= & \lim _{\varepsilon \rightarrow 0^{+}} \int_{S^{3} \backslash B_{\varepsilon}(N)} P_{g, h}^{(1)} G_{N} \cdot\left(G_{q}-G_{q}(N)\right) d \mu \\
= & \frac{1}{32 \pi^{2}} \lim _{R \rightarrow \infty} \int_{|x| \leq R} P_{|d x|^{2}, \theta} 1 \cdot \frac{|x-y|-\sqrt{|x|^{2}+1}}{\sqrt{|y|^{2}+1}} d x \\
= & \frac{1}{256 \pi^{2}} \int_{\mathbb{R}^{3}} \Delta\left(\theta_{i j i j}-\Delta t r \theta\right) \cdot \frac{|x-y|-\sqrt{|x|^{2}+1}}{\sqrt{|y|^{2}+1}} d x \\
= & \frac{1}{256 \pi^{2}} \int_{\mathbb{R}^{3}}\left(\theta_{i j i j}-\Delta t r \theta\right) \cdot \frac{1}{\sqrt{|y|^{2}+1}}\left(\frac{2}{|x-y|}-\frac{2}{\sqrt{|x|^{2}+1}}-\frac{1}{\left(\sqrt{|x|^{2}+1}\right)^{3}}\right) d x .
\end{aligned}
$$

Equation (3.5) follows.
Theorem 3.1. For any smooth symmetric $(0,2)$ tensor $h$, denote $\theta=\tau^{4} h$, then

$$
\begin{align*}
& I I(N, N, h)  \tag{3.6}\\
= & -\frac{1}{128 \pi^{2}} \int_{\mathbb{R}^{3}}\left(\sum_{i j}\left(\theta_{i k j k}+\theta_{j k i k}-(\operatorname{tr} \theta)_{i j}-\Delta \theta_{i j}\right)^{2}-\frac{3}{2}\left(\theta_{i j i j}-\Delta t r \theta\right)^{2}\right) d x .
\end{align*}
$$

Theorem 1.1 follows from Theorem 3.1 and (2.9).
Proof. By Lemma 2.3,

$$
\begin{align*}
& I I(N, N, h)  \tag{3.7}\\
= & -\int_{S^{3}}\left(P_{g, h}^{(2)} G_{N} \cdot G_{N}+P_{g, h}^{(1)} G_{N} \cdot I_{N}+\frac{1}{2} P_{g, h}^{(1)} G_{N} \cdot G_{N} t r h\right) d \mu .
\end{align*}
$$

First we note that because $G_{N}(N)=0$, the same argument as in the proof of Proposition 3.1 shows

$$
\begin{aligned}
& \int_{S^{3}} P_{g, h}^{(2)} G_{N} \cdot G_{N} d \mu \\
= & \lim _{\varepsilon \rightarrow 0^{+}} \int_{S^{3} \backslash B_{\varepsilon}(N)} P_{g, h}^{(2)} G_{N} \cdot G_{N} d \mu \\
= & \frac{1}{32 \pi^{2}} \lim _{R \rightarrow \infty} \int_{|x| \leq R} P_{|d x|^{2}, \theta}^{(2)} 1 d x \\
= & \frac{1}{128 \pi^{2}} \int_{\mathbb{R}^{3}}\left[\sum_{i j}\left(\theta_{i k j k}+\theta_{j k i k}-(t r \theta)_{i j}-\Delta \theta_{i j}\right)^{2}\right. \\
& \left.-\frac{1}{16}\left(23 \theta_{i j i j}-19 \Delta t r \theta\right)\left(\theta_{k l k l}-\Delta t r \theta\right)\right] d x
\end{aligned}
$$

Here we have used (2.5). Next using $I_{N}(N)=0$ we have

$$
\begin{aligned}
\int_{S^{3}} P_{g, h}^{(1)} G_{N} \cdot I_{N} d \mu & =\lim _{\varepsilon \rightarrow 0^{+}} \int_{S^{3} \backslash B_{\varepsilon}(N)} P_{g, h}^{(1)} G_{N} \cdot I_{N} d \mu \\
& =-\frac{1}{4 \pi \sqrt{2}} \lim _{R \rightarrow \infty} \int_{|x| \leq R} P_{|d x|^{2}, \theta}^{(1)} 1 \cdot \tau I_{N} d x
\end{aligned}
$$

Since

$$
\lim _{R \rightarrow \infty} \int_{|x| \leq R} P_{|d x|^{2}, \theta}^{(1)} 1 \cdot \tau G_{N} d x=0
$$

(see the proof of Proposition 3.1), by Lemma 3.1 we have

$$
\begin{aligned}
& \int_{S^{3}} P_{g, h}^{(1)} G_{N} \cdot I_{N} d \mu \\
= & \frac{1}{2048 \pi^{3}} \int_{\mathbb{R}^{3}} P_{|d x|^{2}, \theta}^{(1)} 1\left(\int_{\mathbb{R}^{3}}\left(\theta_{i j i j}-\Delta t r \theta\right)(y)\left(\frac{2}{|x-y|}-\frac{2}{\sqrt{|y|^{2}+1}}-\frac{1}{\left(\sqrt{|y|^{2}+1}\right)^{3}}\right) d y\right) d x \\
= & \frac{1}{16384 \pi^{3}} \int_{\mathbb{R}^{3}} d y\left(\theta_{i j i j}-\Delta t r \theta\right)(y) \int_{\mathbb{R}^{3}} \Delta\left(\theta_{k l k l}-\Delta \operatorname{tr} \theta\right)(x)\left(\frac{2}{|x-y|}-\frac{2}{\sqrt{|y|^{2}+1}}-\frac{1}{\left(\sqrt{|y|^{2}+1}\right)^{3}}\right) d x \\
= & -\frac{1}{2048 \pi^{2}} \int_{\mathbb{R}^{3}}\left(\theta_{i j i j}-\Delta t r \theta\right)^{2} d x .
\end{aligned}
$$

Similarly for the third term in (3.7) we have

$$
\begin{aligned}
\int_{S^{3}} \frac{1}{2} P_{g, h}^{(1)} G_{N} \cdot G_{N} t r h d \mu & =\frac{1}{2} \lim _{\varepsilon \rightarrow 0^{+}} \int_{S^{3} \backslash B_{\varepsilon}(N)} P_{g, h}^{(1)} G_{N} \cdot G_{N} t r h d \mu \\
& =\frac{1}{64 \pi^{2}} \lim _{R \rightarrow \infty} \int_{|x| \leq R} P_{|d x|^{2}, \theta}^{(1)} 1 \cdot \operatorname{tr} \theta d x \\
& =\frac{1}{512 \pi^{2}} \int_{\mathbb{R}^{3}} \Delta\left(\theta_{i j i j}-\Delta \operatorname{tr} \theta\right) \cdot \operatorname{tr} \theta d x \\
& =\frac{1}{512 \pi^{2}} \int_{\mathbb{R}^{3}} \Delta \operatorname{tr} \theta \cdot\left(\theta_{i j i j}-\Delta \operatorname{tr} \theta\right) d x
\end{aligned}
$$

Sum up we get (3.6).
Next we will study sign of the second variation. For convenience we write

$$
\begin{equation*}
I I(h)=I I(N, N, h) . \tag{3.8}
\end{equation*}
$$

First we observe that by conformal covariant property, for any smooth vector field $X$ and function $f$,

$$
\begin{equation*}
I I\left(L_{X} g+f g\right)=0 \tag{3.9}
\end{equation*}
$$

Indeed let $\phi_{t}$ be the flow generated by $X$, then for $t$ near 0 ,

$$
\begin{aligned}
G_{(1+t f) \phi_{t}^{*} g}(N, N) & =(1+t f(N))^{\frac{1}{2}} G_{\phi_{t}^{*} g}(N, N) \\
& =(1+t f(N))^{\frac{1}{2}} G_{g}\left(\phi_{t}(N), \phi_{t}(N)\right)=0 .
\end{aligned}
$$

Since $\left.\partial_{t}\right|_{t=0}\left((1+t f) \phi_{t}^{*} g\right)=L_{X} g+f g$, (3.9) follows. In fact we can say a little more: let $I I(h, k)$ be the symmetric form associated with $I I(h)$, if

$$
\begin{equation*}
\theta=\tau^{4} h, \quad \kappa=\tau^{4} k, \tag{3.10}
\end{equation*}
$$

then it follows from Theorem 3.1 that

$$
\begin{align*}
& I I(h, k)  \tag{3.11}\\
= & -\frac{1}{128 \pi^{2}} \int_{\mathbb{R}^{3}}\left[\sum_{i j}\left(\theta_{i k j k}+\theta_{j k i k}-(t r \theta)_{i j}-\Delta \theta_{i j}\right)\left(\kappa_{i l j l}+\kappa_{j l i l}-(t r \kappa)_{i j}-\Delta \kappa_{i j}\right)\right. \\
& \left.-\frac{3}{2}\left(\theta_{i j i j}-\Delta t r \theta\right)\left(\kappa_{k l k l}-\Delta t r \kappa\right)\right] d x .
\end{align*}
$$

Lemma 3.2. Given smooth symmetric $(0,2)$ tensor $h$, vector field $X$ and function $f$, we have

$$
\begin{equation*}
I I\left(h, L_{X} g+f g\right)=0 \tag{3.12}
\end{equation*}
$$

To achieve this we need the following technical fact:
Lemma 3.3. If $h$ is a smooth symmetric $(0,2)$ tensor on $S^{3}$, then there exists a smooth vector field $X$ such that

$$
\left(h-L_{X} g\right)(N)=0, \quad D\left(h-L_{X} g\right)(N)=0
$$

To derive this lemma, we start with the following linear algebra fact.
Lemma 3.4. Denote

$$
\mathcal{P}_{m}=\left\{\text { homogeneous degree } m \text { polynomials on } \mathbb{R}^{3}\right\} .
$$

If for $1 \leq i, j \leq 3, H_{i j} \in \mathcal{P}_{1}, H_{i j}=H_{j i}$, then there exists unique $A_{i} \in \mathcal{P}_{2}$ such that

$$
\partial_{i} A_{j}+\partial_{j} A_{i}=H_{i j}
$$

Proof. Let

$$
\mathcal{X}=\left\{\left[\begin{array}{l}
A_{1} \\
A_{2} \\
A_{3}
\end{array}\right]: A_{i} \in \mathcal{P}_{2}\right\}
$$

and

$$
\mathcal{Y}=\left\{\left[H_{i j}\right]_{1 \leq i, j \leq 3}: H_{i j} \in \mathcal{P}_{1}, H_{i j}=H_{j i}\right\}
$$

Note $\operatorname{dim} \mathcal{X}=\operatorname{dim} \mathcal{Y}=18$. Let

$$
\phi: \mathcal{X} \rightarrow \mathcal{Y}:\left[\begin{array}{l}
A_{1} \\
A_{2} \\
A_{3}
\end{array}\right] \mapsto\left[H_{i j}\right]_{1 \leq i, j \leq 3}
$$

be given by $H_{i j}=\partial_{i} A_{j}+\partial_{j} A_{i}$. We need to show $\phi$ is a linear isomorphism. We only need to prove $\operatorname{ker} \phi=0$. Indeed if $\left[\begin{array}{c}A_{1} \\ A_{2} \\ A_{3}\end{array}\right] \in \operatorname{ker} \phi$, then

$$
\partial_{i} A_{j}+\partial_{j} A_{i}=0
$$

This implies

$$
x_{i} x_{j} \partial_{i} A_{j}+x_{i} x_{j} \partial_{j} A_{i}=0 .
$$

Since $x_{i} \partial_{i} A_{j}=2 A_{j}$, we see $x_{i} A_{i}=0$. Hence

$$
0=\partial_{j}\left(x_{i} A_{i}\right)=A_{j}+x_{i} \partial_{j} A_{i}=A_{j}-x_{i} \partial_{i} A_{j}=-A_{j}
$$

The lemma follows.
Now we will use Taylor expansion to prove Lemma 3.3.
Proof of Lemma 3.3. By standard cut-off argument we see the conclusion is in fact a local statement. We choose a local coordinate near $N$, say $y_{1}, y_{2}, y_{3}$ such that $y_{i}(N)=0$. Assume $X=X^{i} \frac{\partial}{\partial y_{i}}$, let $\alpha$ be the associated 1-form i.e. $\alpha_{i}=g_{i j} X^{j}$, then we only need to find $\alpha$ with

$$
\alpha_{i j}+\alpha_{j i}=h_{i j}+O\left(|y|^{2}\right)
$$

as $y \rightarrow 0$. In another way the equation is

$$
\partial_{i} \alpha_{j}+\partial_{j} \alpha_{i}-2 \Gamma_{i j}^{k} \alpha_{k}=h_{i j}+O\left(|y|^{2}\right) .
$$

We will look for $\alpha_{i}=\alpha_{i}^{(1)}+\alpha_{i}^{(2)}$, here $\alpha_{i}^{(l)} \in \mathcal{P}_{l}$. We have the Taylor expansion of $h_{i j}$ as $h_{i j}=h_{i j}(0)+h_{i j}^{(1)}+O\left(|y|^{2}\right)$. So the equation becomes

$$
\begin{equation*}
\partial_{i} \alpha_{j}^{(1)}+\partial_{j} \alpha_{i}^{(1)}=h_{i j}(0) \tag{3.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial_{i} \alpha_{j}^{(2)}+\partial_{j} \alpha_{i}^{(2)}-2 \Gamma_{i j}^{k}(0) \alpha_{k}^{(1)}=h_{i j}^{(1)} . \tag{3.14}
\end{equation*}
$$

For (3.13), we can simply choose $\alpha_{i}^{(1)}=\frac{1}{2} h_{i k}(0) y_{k}$. Using Lemma 3.4 we see (3.14) also has a solution. Lemma 3.3 follows.

With Lemma 3.3 at hand, we can proceed to prove Lemma 3.2 ,
Proof of Lemma 3.2. Note that

$$
f=O^{(\infty)}(1), \quad \theta_{i j}=O^{(\infty)}(1)
$$

as $|x| \rightarrow \infty$. By (3.10),

$$
\begin{aligned}
& I I(h, f g) \\
= & -\frac{1}{128 \pi^{2}} \int_{\mathbb{R}^{3}}\left[-\left(\theta_{i k j k}+\theta_{j k i k}-(\operatorname{tr} \theta)_{i j}-\Delta \theta_{i j}\right)\left(f_{i j}+\Delta f \cdot \delta_{i j}\right)\right. \\
& \left.+3\left(\theta_{i j i j}-\Delta t r \theta\right) \Delta f\right] d x
\end{aligned}
$$

By integration by parts, we have

$$
\int_{\mathbb{R}^{3}}-\left(\theta_{i k j k}+\theta_{j k i k}-(t r \theta)_{i j}-\Delta \theta_{i j}\right) f_{i j} d x=\int_{\mathbb{R}^{3}}\left(-\theta_{i j i j}+\Delta t r \theta\right) \Delta f d x
$$

and

$$
\int_{\mathbb{R}^{3}}-\left(\theta_{i k j k}+\theta_{j k i k}-(\operatorname{tr} \theta)_{i j}-\Delta \theta_{i j}\right) \Delta f \cdot \delta_{i j} d x=\int_{\mathbb{R}^{3}}\left(-2 \theta_{i j i j}+2 \Delta t r \theta\right) \Delta f d x
$$

Sum up we get $I I(h, f g)=0$.
Next we will show $I I\left(h, L_{X} g\right)=0$. First we note that it follows from (3.9) that for any smooth vector fields $X^{\prime}, X$ and smooth functions $f^{\prime}, f$, we have

$$
I I\left(L_{X^{\prime}} g+f^{\prime} g, L_{X} g+f g\right)=0
$$

To continue we can assume $h$ satisfies

$$
h(N)=0, \quad D h(N)=0
$$

Indeed given any smooth $h$, by Lemma 3.3, we can find a smooth vector field $X^{\prime}$ such that

$$
\left(h-L_{X^{\prime}} g\right)(N)=0, \quad D\left(h-L_{X^{\prime}} g\right)(N)=0
$$

It follows that

$$
I I\left(h, L_{X} g\right)=I I\left(h-L_{X^{\prime}} g, L_{X} g\right)
$$

Under the additional assumption on $h$, we have

$$
\theta_{i j}=O^{(\infty)}\left(|x|^{-2}\right), \quad X_{i}=O^{(\infty)}\left(|x|^{2}\right)
$$

as $x \rightarrow \infty$. Here $X=X_{i} \frac{\partial}{\partial x_{i}}$. Let $\kappa=\tau^{4} L_{X} g$, then

$$
\kappa_{i j}=-4 \tau^{-1} X \tau \cdot \delta_{i j}+X_{i j}+X_{j i}
$$

Hence

$$
\kappa_{i k j k}+\kappa_{j k i k}-(t r \kappa)_{i j}-\Delta \kappa_{i j}=-4\left(\tau^{-1} X \tau\right)_{i j}-4 \Delta\left(\tau^{-1} X \tau\right) \cdot \delta_{i j}
$$

and

$$
\kappa_{i j i j}-\Delta t r \kappa=8 \Delta\left(\tau^{-1} X \tau\right)
$$

It implies

$$
\begin{aligned}
& I I\left(h, L_{X} g\right) \\
& =\frac{1}{32 \pi^{2}} \int_{\mathbb{R}^{3}}\left[-\left(\theta_{i k j k}+\theta_{j k i k}-(\operatorname{tr} \theta)_{i j}-\Delta \theta_{i j}\right)\left(\left(\tau^{-1} X \tau\right)_{i j}+\Delta\left(\tau^{-1} X \tau\right) \cdot \delta_{i j}\right)\right. \\
& \left.\quad+3\left(\theta_{i j i j}-\Delta \operatorname{tr} \theta\right) \Delta\left(\tau^{-1} X \tau\right)\right] d x
\end{aligned}
$$

Note that $\tau^{-1} X \tau=O^{(\infty)}(|x|)$ and $\theta_{i j}=O^{(\infty)}\left(|x|^{-2}\right)$, the same integration by parts argument as the beginning shows $I I\left(h, L_{X} g\right)=0$.

Proposition 3.2. For any smooth symmetric $(0,2)$ tensor $h, I I(h) \leq 0$. Moreover, $I I(h)=0$ if and only if $h=L_{X} g+f \cdot g$ for some smooth vector fields $X$ and smooth functions $f$ on $S^{3}$.

Proof. In view of Lemma 3.2 and 3.3 we can assume $h(N)=0$ and $D h(N)=0$. Under such assumption we have

$$
\theta_{i j}=O^{(\infty)}\left(|x|^{-2}\right)
$$

as $x \rightarrow \infty$. In particular $\theta_{i j} \in L^{2}\left(\mathbb{R}^{3}\right)$. Let

$$
a_{i j}(\xi)=\widehat{\theta_{i j}}(\xi)
$$

then $a_{i j} \in L^{2}\left(\mathbb{R}^{3}\right)$. By Parseval relation we have

$$
\begin{aligned}
& \int_{\mathbb{R}^{3}}\left(\sum_{i j}\left(\theta_{i k j k}+\theta_{j k i k}-(\operatorname{tr} \theta)_{i j}-\Delta \theta_{i j}\right)^{2}-\frac{3}{2}\left(\theta_{i j i j}-\Delta t r \theta\right)^{2}\right) d x \\
= & \int_{\mathbb{R}^{3}}\left(-2 a_{i j} \bar{a}_{i k} \xi_{j} \xi_{k}|\xi|^{2}+\frac{1}{2} a_{i j} \bar{a}_{k l} \xi_{i} \xi_{j} \xi_{k} \xi_{l}+\frac{1}{2} a_{i j} \xi_{i} \xi_{j} \bar{a}_{k k}|\xi|^{2}+\frac{1}{2} \bar{a}_{i j} \xi_{i} \xi_{j} a_{k k}|\xi|^{2}\right. \\
& \left.+\sum_{i j}\left|a_{i j}\right|^{2}|\xi|^{4}-\frac{1}{2}\left|\sum_{i} a_{i i}\right|^{2}|\xi|^{4}\right) d \xi .
\end{aligned}
$$

We will show the integrand in nonnegative. Indeed, if we write

$$
A=\left[a_{i j}\right]_{1 \leq i, j \leq 3},
$$

then $A$ is symmetric and the integrand is equal to

$$
\begin{align*}
& -2|A \xi|^{2}|\xi|^{2}+\frac{1}{2}\left|\xi^{T} A \xi\right|^{2}+\frac{1}{2} \xi^{T} A \xi \cdot \overline{\operatorname{tr} A}|\xi|^{2}+\frac{1}{2} \xi^{T} \bar{A} \xi \cdot \operatorname{tr} A|\xi|^{2}  \tag{3.15}\\
& -\frac{1}{2}|\operatorname{tr} A|^{2}|\xi|^{4}+|A|^{2}|\xi|^{4}
\end{align*}
$$

Assume $\xi \neq 0$, then we may find an orthogonal matrix $O$ such that

$$
\xi=O\left[\begin{array}{c}
|\xi| \\
0 \\
0
\end{array}\right]
$$

Denote

$$
B=O^{T} A O
$$

then $B$ is symmetric and the integrand (3.15) is equal to

$$
\begin{equation*}
\left(\frac{1}{2}\left|b_{22}-b_{33}\right|^{2}+2\left|b_{23}\right|^{2}\right)|\xi|^{4} \tag{3.16}
\end{equation*}
$$

and hence nonnegative. This implies $I I(h) \leq 0$.
If $I I(h)=0$, then we have $b_{22}=b_{33}$ and $b_{23}=0$. this implies

$$
\begin{equation*}
a_{i j}=\alpha \delta_{i j}+\beta_{i} \xi_{j}+\beta_{j} \xi_{i} \tag{3.17}
\end{equation*}
$$

here $\alpha$ and $\beta_{i}$ depend on $\xi$. To continue we recall the orthogonal decomposition [B, p130, lemma 4.57],

$$
\begin{equation*}
\mathcal{S}^{2} S^{3}=\mathcal{A} \oplus \mathcal{B} \tag{3.18}
\end{equation*}
$$

Here

$$
\begin{aligned}
\mathcal{S}^{2} S^{3} & =\left\{C^{\infty} \text { symmetric }(0,2) \text { tensors on } S^{3}\right\} \\
\mathcal{A} & =\left\{L_{X} g+f g: X \text { is a } C^{\infty} \text { vector field, } f \text { is a } C^{\infty} \text { function }\right\} \\
\mathcal{B} & =\left\{k \in \mathcal{S}^{2} S^{3}: \operatorname{tr} k=0, k_{i j j}=0\right\}
\end{aligned}
$$

To show $h \in \mathcal{A}$, we only need to prove $h \perp \mathcal{B}$. Indeed if $k \in \mathcal{B}$, let $\kappa=\tau^{4} k=$ $\kappa_{i j} d x_{i} d x_{j}$, then

$$
\begin{equation*}
\kappa_{i i}=0, \quad\left(\tau^{-6} \kappa_{i j}\right)_{j}=0 \tag{3.19}
\end{equation*}
$$

On the other hand we have

$$
\begin{equation*}
\theta_{i j}=O^{(\infty)}\left(|x|^{-2}\right), \quad \tau^{-6} \kappa_{i j}=O^{(\infty)}\left(|x|^{-6}\right) \tag{3.20}
\end{equation*}
$$

By Fourier transform we have

$$
\begin{equation*}
\widehat{\tau^{-6} \kappa_{i i}}=0, \quad \widehat{\tau^{-6} \kappa_{i j}} \xi_{j}=0 \tag{3.21}
\end{equation*}
$$

Hence

$$
\begin{aligned}
\int_{S^{3}}\langle h, k\rangle d \mu & =\int_{\mathbb{R}^{3}} \theta_{i j} \kappa_{i j} \tau^{-6} d x \\
& =\int_{\mathbb{R}^{3}} \widehat{\theta_{i j}} \overline{\tau^{-6} \kappa_{i j}} d \xi \\
& =\int_{\mathbb{R}^{3}}\left(\alpha \delta_{i j}+\beta_{i} \xi_{j}+\beta_{j} \xi_{i}\right) \widehat{\widehat{\tau^{-6} \kappa_{i j}}} d \xi \\
& =0
\end{aligned}
$$

here we have used (3.21) in the last step. Hence $h=L_{X} g+f g$ for some smooth vector field $X$ and smooth function $f$.

## 4. A new invariant for Paneitz operator

Let $(M, g)$ be a smooth compact three dimensional Riemannian manifold. For any $p \in M$, we set

$$
\begin{equation*}
\nu_{p}=\inf \left\{\frac{E(u)}{\int_{M} u^{2} d \mu}: u \in H^{2}(M) \backslash\{0\}, u(p)=0\right\} . \tag{4.1}
\end{equation*}
$$

$\nu_{p}$ is always finite and achieved. Indeed we let $u_{i} \in H^{2}(M)$ such that $u_{i}(p)=0$, $\left\|u_{i}\right\|_{L^{2}}=1$ and $E\left(u_{i}\right) \rightarrow \nu_{p}$. In view of the fact

$$
E\left(u_{i}\right) \geq c_{1}\left\|u_{i}\right\|_{H^{2}(M)}^{2}-c_{2}\left\|u_{i}\right\|_{L^{2}}^{2}
$$

for some positive constants $c_{1}$ and $c_{2}$, we see $\left\|u_{i}\right\|_{H^{2}(M)}^{2} \leq c$, independent of $i$. After passing to a subsequence we can assume $u_{i} \rightharpoonup u$ weakly in $H^{2}(M)$. It follows that $u_{i} \rightarrow u$ uniformly and hence $u(p)=0$ and $\|u\|_{L^{2}}=1$. By lower semicontinuity we have

$$
E(u) \leq \lim \inf _{i \rightarrow \infty} E\left(u_{i}\right)=\nu_{p}
$$

Hence $E(u)=\nu_{p}$ and $u$ is a minimizer.
Note $u$ satisfies

$$
E(u, \varphi)=\nu_{p} \int_{M} u \varphi d \mu
$$

for any $\varphi \in H^{2}(M)$ with $\varphi(p)=0$. For any $\psi \in H^{2}(M)$, let $\varphi=\psi-\psi(p)$ we see

$$
E(u, \psi)=\nu_{p} \int_{M} u \psi d \mu+\alpha \psi(p)
$$

Here $\alpha$ is a constant. In another word, we have

$$
\begin{equation*}
P u=\nu_{p} u+\alpha \delta_{p} \tag{4.2}
\end{equation*}
$$

in distribution sense and

$$
\begin{equation*}
u \in H^{2}(M),\|u\|_{L^{2}}=1, u(p)=0 \tag{4.3}
\end{equation*}
$$

Sometime to avoid confusion we write $u=u_{p}$ and $\alpha=\alpha_{p}$.
Example 4.1. Using [HY1, Lemma 7.1 and Corollary 7.1], we see on standard $S^{3}, \nu_{N}=0$ and it is achieved on constant multiple of the Green's function $G_{N}$. Calculation shows $\left\|G_{N}\right\|_{L^{2}}=\frac{1}{4}$, hence $u_{N}=4 G_{N}, \alpha_{N}=4$.

Let

$$
\begin{align*}
\nu(M, g) & =\inf _{p \in M} \nu(M, g, p)  \tag{4.4}\\
& =\inf \left\{\frac{E(u)}{\int_{M} u^{2} d \mu}: u \in H^{2}(M) \backslash\{0\}, u \text { vanishes somewhere }\right\}
\end{align*}
$$

We will write $\nu(g)$ when no confusion could happen. Same argument as before shows $\nu(g)$ is finite and achieved. It is clear that condition P is satisfied if and only if $\nu(g)>0$, condition NN is satisfied if and only if $\nu(g) \geq 0$. By Example 4.1 and symmetry, we see $\nu\left(S^{3}, g_{S^{3}}\right)=0$.

Here we make some general discussion about $\nu_{p}$ and $\nu(g)$. For convenience we write $\nu=\nu(g)$. Let

$$
\lambda_{1} \leq \lambda_{2} \leq \lambda_{3} \leq \cdots
$$

be eigenvalues and $\varphi_{i}$ be the associated orthonormal eigenfunctions of Paneitz operator $P$, then

$$
\begin{equation*}
\lambda_{1} \leq \nu_{p} \leq \lambda_{2}, \quad \lambda_{1} \leq \nu \leq \lambda_{2} \tag{4.5}
\end{equation*}
$$

Indeed, for given $p \in M$, there exists $c_{1}, c_{2}$ not all zeroes such that $c_{1} \varphi_{1}(p)+$ $c_{2} \varphi_{2}(p)=0$, then

$$
\nu_{p} \leq \frac{E\left(c_{1} \varphi_{1}+c_{2} \varphi_{2}\right)}{\left\|c_{1} \varphi_{1}+c_{2} \varphi_{2}\right\|_{L^{2}}^{2}}=\frac{\lambda_{1} c_{1}^{2}+\lambda_{2} c_{2}^{2}}{c_{1}^{2}+c_{2}^{2}} \leq \lambda_{2}
$$

The inequality of $\nu$ follows.
Assume $u \in H^{2}(M),\|u\|_{L^{2}}=1$ and $u(p)=0$ for some $p$ with $E(u)=\nu$ i.e. $u$ is a minimizer for the $\nu$ problem, then

$$
\begin{equation*}
P u=\nu u+\alpha \delta_{p} . \tag{4.6}
\end{equation*}
$$

If $\sharp u^{-1}(0)>1$ i.e. $u$ vanishes at two or more points, then

$$
\begin{equation*}
P u=\nu u, \quad \lambda_{1}=\nu . \tag{4.7}
\end{equation*}
$$

Indeed assume $u\left(p_{1}\right)=0$ and $u\left(p_{2}\right)=0$ for $p_{1} \neq p_{2}$, then for $\varphi \in H^{2}$ with either $\varphi\left(p_{1}\right)=0$ or $\varphi\left(p_{2}\right)=0$,

$$
\begin{equation*}
E(u, \varphi)=\nu \int_{M} u \varphi d \mu \tag{4.8}
\end{equation*}
$$

Hence (4.8) is valid for any $\varphi \in H^{2}$. In another word $P u=\nu u$. If $\lambda_{1}<\nu$, then $\lambda_{1}<\lambda_{2}$ and $\varphi_{1}$ does not vanish anywhere. Using

$$
\int_{M} \varphi_{1} \varphi_{2} d \mu=0
$$

we see $\varphi_{2}$ must change sign. Hence for $\varepsilon>0$ small

$$
\lambda_{1}<\nu \leq \frac{E\left(\varepsilon \varphi_{1}+\varphi_{2}\right)}{\left\|\varepsilon \varphi_{1}+\varphi_{2}\right\|_{L^{2}}^{2}}=\frac{\varepsilon^{2} \lambda_{1}+\lambda_{2}}{\varepsilon^{2}+1}<\lambda_{2}
$$

A contradiction with the fact $\nu$ is an eigenvalue. Hence $\nu$ must be the first eigenvalue.

Now we can state the following interesting relation between condition NN and the sign of $\lambda_{2}$.
Proposition 4.1. Assume the Yamabe invariant $Y(g)>0$ and there exists $a$ $\widetilde{g} \in[g]$ such that $\widetilde{Q} \geq 0$ and not identically zero, then the following statements are equivalent
(1) $Y_{4}(g)>-\infty$.
(2) $\lambda_{2}(P)>0$.
(3) $\nu(g) \geq 0$ i.e. $P$ satisfies condition $N N$.

Proof. It follows from the assumption that $\lambda_{1}<0$. By [HY3, Proposition 1.2] and (1.4) we have ker $P=0$ and $G_{P}(p, q)<0$ for $p \neq q$. Here $G_{P}$ is the Green's function of the Paneitz operator. Let $m \geq 1$ be the natural number such that $\lambda_{m}<0$ and $\lambda_{m+1}>0$ i.e. $\lambda_{m}$ is the largest negative eigenvalue. By applying the classical Krein-Rutman theorem to the operator

$$
T f(p)=-\int_{M} G_{P}(p, q) f(q) d \mu(q)
$$

we know $\lambda_{m}$ must be simple and $\varphi_{m}$ can not touch zero (see [HY3, section 4]). Without losing of generality, we assume $\varphi_{m}>0$.
$(1) \Rightarrow(2)$ : If $\lambda_{2}<0$, then $m \geq 2$ and the first eigenfunction $\varphi_{1}$ must change sign. Let

$$
\kappa=-\min _{p \in M} \frac{\varphi_{1}(p)}{\varphi_{m}(p)}>0
$$

then $\varphi_{1}+\kappa \varphi_{m} \geq 0$ and it touches zero somewhere. On the other hand $E\left(\varphi_{1}+\kappa \varphi_{m}\right)=\lambda_{1}+\kappa^{2} \lambda_{m}<0$, hence

$$
Y_{4}(g) \leq\left\|\left(\varphi_{1}+\kappa \varphi_{m}+\varepsilon\right)^{-1}\right\|_{L^{6}}^{2} E\left(\varphi_{1}+\kappa \varphi_{m}+\varepsilon\right) \rightarrow-\infty
$$

as $\varepsilon \downarrow 0$, a contradiction.
$(2) \Rightarrow(3)$ : Since $\lambda_{2}>0$, we get $m=1$. Let $u \in H^{2}(M)$ such that $u$ touches zero somewhere, $\|u\|_{L^{2}}=1$ and $E(u)=\nu$. We claim $\sharp u^{-1}(0)=1$. Indeed if $\sharp u^{-1}(0)>1$, then by the discussion before Proposition4.1 we know $P(u)=$ $\nu u$ and $\nu=\lambda_{1}$. Its eigenfunction $u$ can not touch zero, a contradiction. The claim follows i.e. $u$ touches 0 exactly once. Assume $u(p)=0$ and $u>0$ on $M \backslash\{p\}$, then

$$
P(u)=\nu u+\alpha \delta_{p}
$$

Hence

$$
\int_{M} P(u) G_{L, p}^{-1} d \mu=\nu \int_{M} u G_{L, p}^{-1} d \mu
$$

Here $G_{L}$ is the Green's function of the conformal Laplacian operator $L=$ $-8 \Delta+R$. On the other hand it follows from [HY3, Proposition 2.1] that

$$
\int_{M} P(u) G_{L, p}^{-1} d \mu=\int_{M} u G_{L, p}^{-1}\left|R c_{G_{L, p}^{4} g}\right|_{g}^{2} d \mu
$$

Combine the two equalities above we get

$$
\nu \int_{M} u G_{L, p}^{-1} d \mu=\int_{M} u G_{L, p}^{-1}\left|R c_{G_{L, p}^{4} g}\right|_{g}^{2} d \mu
$$

Hence $\nu \geq 0$.
$(3) \Rightarrow(1)$ : If $E(u)=0, u$ is not identically zero but $u(p)=0$, then $u=c G_{p}$ (see HY1, section 5]). Hence $G_{p}(p)=0$. It follows from HY3, Proposition 1.2] that $(M, g)$ is conformal diffeomorphic to standard $S^{3}$. In this case we know $Y_{4}(g)>-\infty$ (see [HY1, YZ]). On the other hand if $E(u)>0$ for any $u \in H^{2} \backslash\{0\}$ and $u$ touches zero somewhere, then the Paneitz operator satisfies condition P and $Y_{4}(g)>-\infty$ (see [HY1]).

Indeed the above proof gives us the following
Corollary 4.1. Assume the Yamabe invariant $Y(g)>0,(M, g)$ is not conformal diffeomorphic to the standard $S^{3}$ and there exists a $\widetilde{g} \in[g]$ such that $\widetilde{Q} \geq 0$ and not identically zero, then the following statements are equivalent
(1) $Y_{4}(g)>-\infty$.
(2) $\lambda_{2}(P)>0$.
(3) $\nu(g)>0$ i.e. $P$ satisfies condition $P$.

We remark that Proposition 4.1 provides another argument for the third conclusion of HY2, Theorem 1.1]. This approach does not need the connecting path to Berger's sphere and [HY1, Theorem 1.3].

Let $\widetilde{g}=g+t h$, for quantities in (4.2) and (4.3) we write

$$
\begin{align*}
\nu(g+t h, p) & =\nu(p)+\nu^{(1)}(p, h) t+\nu^{(2)}(p, h) t^{2}+O\left(t^{3}\right)  \tag{4.9}\\
\alpha(g+t h, p) & =\alpha(p)+\alpha^{(1)}(p, h) t+\alpha^{(2)}(p, h) t^{2}+O\left(t^{3}\right)  \tag{4.10}\\
u_{p}(g+t h, q) & =u_{p}(q)+u_{p}^{(1)}(q, h) t+u_{p}^{(2)}(q, h) t^{2}+O\left(t^{3}\right) \tag{4.11}
\end{align*}
$$

Hence $u_{p}^{(i)}(p, h)=0$ for $i=1,2$. Note because

$$
\int_{M} \widetilde{u}_{p}^{2} d \widetilde{\mu}=1
$$

we have

$$
\begin{equation*}
\int_{M}\left(2 u_{p} \cdot u_{p}^{(1)}+\frac{1}{2} u_{p}^{2} t r h\right) d \mu=0 \tag{4.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{M}\left[2 u_{p} \cdot u_{p}^{(2)}+\left(u_{p}^{(1)}\right)^{2}+u_{p} \cdot u_{p}^{(1)} t r h+\frac{1}{8} u_{p}^{2}(t r h)^{2}-\frac{1}{4} u_{p}^{2}|h|^{2}\right] d \mu=0 \tag{4.13}
\end{equation*}
$$

For any smooth function $\varphi$, it follows from (4.2) that

$$
\int_{M} \widetilde{u}_{p}\left(\widetilde{P}-\widetilde{\nu}_{p}\right) \varphi d \widetilde{\mu}=\widetilde{\alpha}_{p} \varphi(p)
$$

Hence

$$
\begin{align*}
& \alpha_{p}^{(1)} \varphi(p)  \tag{4.14}\\
= & \int_{M}\left(u_{p}\left(P_{g, h}^{(1)}-\nu_{p}^{(1)}\right) \varphi+u_{p}^{(1)}\left(P-\nu_{p}\right) \varphi+\frac{1}{2} u_{p}\left(P-\nu_{p}\right) \varphi \cdot \operatorname{trh}\right) d \mu
\end{align*}
$$

and

$$
\begin{align*}
& \int_{M}\left[u_{p}^{(2)}\left(P-\nu_{p}\right) \varphi+u_{p}^{(1)}\left(P_{g, h}^{(1)}-\nu_{p}^{(1)}\right) \varphi+u_{p}\left(P_{g, h}^{(2)}-\nu_{p}^{(2)}\right) \varphi\right.  \tag{4.15}\\
& +\frac{1}{2} u_{p}^{(1)}\left(P-\nu_{p}\right) \varphi \cdot \operatorname{trh}+\frac{1}{2} u_{p}\left(P_{g, h}^{(1)}-\nu_{p}^{(1)}\right) \varphi \cdot \operatorname{trh} \\
& \left.+\frac{1}{8} u_{p}\left(P-\nu_{p}\right) \varphi \cdot(t r h)^{2}-\frac{1}{4} u_{p}\left(P-\nu_{p}\right) \varphi \cdot|h|^{2}\right] d \mu \\
= & \alpha_{p}^{(2)} \varphi(p)
\end{align*}
$$

By approximation it is also true for $\varphi \in H^{2}(M)$ too. Hence taking $\varphi=u_{p}$, we get

$$
\begin{equation*}
\nu_{p}^{(1)}=\int_{M} u_{p} P_{g, h}^{(1)} u_{p} d \mu \tag{4.16}
\end{equation*}
$$

Similar arguments show

$$
\begin{equation*}
=\int_{M}^{\nu_{p}^{(2)}} u_{p} P_{g, h}^{(2)} u_{p} d \mu+\int_{M} u_{p}^{(1)} P_{g, h}^{(1)} u_{p} d \mu+\frac{1}{2} \int_{M} u_{p} P_{g, h}^{(1)} u_{p} \cdot \operatorname{trh} d \mu-\frac{1}{4} \nu_{p}^{(1)} \int_{M} u_{p}^{2} \operatorname{trh} d \mu \tag{4.17}
\end{equation*}
$$

Proposition 4.2. Let $\left(S^{3}, g\right)$ be the standard sphere, then for any $p \in S^{3}$ and smooth symmetric $(0,2)$ tensor $h$, we have

$$
\nu^{(1)}(p, h)=0
$$

and

$$
\nu^{(2)}(p, h)=-16 I I(p, p, h)
$$

Theorem 1.2 follows from Proposition 3.2 and 4.2. By symmetry we can assume $p=N$, then it follows from (4.16) that

$$
\begin{aligned}
\nu^{(1)}(N, h) & =\int_{S^{3}} u_{N} P_{g, h}^{(1)} u_{N} d \mu \\
& =16 \int_{S^{3}} G_{N} P_{g, h}^{(1)} G_{N} d \mu \\
& =0
\end{aligned}
$$

by the proof of Proposition 3.1 .
Next we note that (4.17) implies

$$
\begin{equation*}
=\int_{S^{3}} u_{N} P_{g, h}^{(2)} u_{N} d \mu+\int_{S^{3}} u_{N}^{(1)} P_{g, h}^{(1)} u_{N} d \mu+\frac{1}{2} \int_{S^{3}} u_{N} P_{g, h}^{(1)} u_{N} \cdot \operatorname{tr} h d \mu . \tag{4.18}
\end{equation*}
$$

To compute $u_{N}^{(1)}$, we observe that (4.14) implies

$$
\int_{S^{3}}\left(u_{N} P_{g, h}^{(1)} \varphi+u_{N}^{(1)} P \varphi+\frac{1}{2} u_{N} P \varphi \cdot \operatorname{trh}\right) d \mu=\alpha_{N}^{(1)} \varphi(N)
$$

for any $\varphi \in H^{2}\left(S^{3}\right)$. Take $\varphi=u_{p}=4 G_{p}$, we see

$$
\begin{equation*}
\int_{S^{3}} u_{N} P_{g, h}^{(1)} u_{p} d \mu+4 u_{N}^{(1)}(p)+2 u_{N}(p) \operatorname{trh}(p)=\alpha_{N}^{(1)} u_{p}(N) \tag{4.19}
\end{equation*}
$$

Since

$$
u_{p}(N)=4 G_{p}(N)=4 G_{N}(p)=u_{N}(p)
$$

it follows from Lemma 2.3 and (4.19) that

$$
\begin{equation*}
u_{N}^{(1)}(p)=4 I_{N}(p)+\frac{1}{4} \alpha_{N}^{(1)} u_{N}(p) . \tag{4.20}
\end{equation*}
$$

Hence

$$
\begin{aligned}
& \nu^{(2)}(N, h) \\
= & 16 \int_{S^{3}} G_{N} P_{g, h}^{(2)} G_{N} d \mu+16 \int_{S^{3}} P_{g, h}^{(1)} G_{N} \cdot I_{N} d \mu+8 \int_{S^{3}} P_{g, h}^{(1)} G_{N} \cdot G_{N} t r h d \mu \\
= & -16 I I(N, N, h)
\end{aligned}
$$

Here we have used $\int_{S^{3}} G_{N} P_{g, h}^{(1)} G_{N} d \mu=0$ (which follows from the proof of Proposition (3.1) and (3.7). Proposition 4.2 follows.
5. Appendix: TAylor expansion formula for $Q$ curvature of metrics depending on a parameter

Assume on a smooth three dimensional Riemannian manifold we have $\widetilde{g}=g+t h$, then under a local orthonormal frame with respect to $g$, we have

$$
\begin{equation*}
\widetilde{g}_{i j}=g_{i j}+t h_{i j} \tag{5.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\widetilde{g}^{i j}=\delta_{i j}-t h_{i j}+t^{2} h_{i j}^{2}+O\left(t^{3}\right) \tag{5.2}
\end{equation*}
$$

Here $h^{2}$ is the tensor given by

$$
h_{i j}^{2}=h_{i k} h_{j k}
$$

The measure associated with $\widetilde{g}$ is given by

$$
\begin{equation*}
d \widetilde{\mu}=\left[1+\frac{t}{2} t r h+t^{2}\left(\frac{(t r h)^{2}}{8}-\frac{|h|^{2}}{4}\right)+O\left(t^{3}\right)\right] d \mu . \tag{5.3}
\end{equation*}
$$

Here

$$
|h|^{2}=\sum_{i j}\left(h_{i j}\right)^{2} .
$$

The Christofel symbol satisfies

$$
\begin{align*}
& \widetilde{\Gamma}_{i j}^{k}-\Gamma_{i j}^{k}  \tag{5.4}\\
= & \frac{t}{2}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)-\frac{t^{2}}{2}\left(h_{i \alpha j}+h_{j \alpha i}-h_{i j \alpha}\right) h_{k \alpha}+O\left(t^{3}\right) .
\end{align*}
$$

Note that the left hand side is a tensor. The Riemann curvature tensor satisfies

$$
\begin{align*}
\widetilde{R}_{i j k}^{l}= & R_{i j k}^{l}+\frac{t}{2}\left(h_{i l k j}+h_{j k l i}+h_{k l i j}-h_{j l k i}-h_{i k l j}-h_{k l j i}\right)  \tag{5.5}\\
& -\frac{t^{2}}{2}\left(h_{i \alpha k j}+h_{j k \alpha i}+h_{k \alpha i j}-h_{j \alpha k i}-h_{i k \alpha j}-h_{k \alpha j i}\right) h_{l \alpha} \\
& +\frac{t^{2}}{4}\left(h_{i \alpha l}+h_{l \alpha i}-h_{i l \alpha}\right)\left(h_{j \alpha k}+h_{k \alpha j}-h_{j k \alpha}\right) \\
& -\frac{t^{2}}{4}\left(h_{i \alpha k}+h_{k \alpha i}-h_{i k \alpha}\right)\left(h_{j \alpha l}+h_{l \alpha j}-h_{j l \alpha}\right)+O\left(t^{3}\right) .
\end{align*}
$$

In particular, the Ricci tensor is given by

$$
\begin{align*}
& \widetilde{R} c_{i j}  \tag{5.6}\\
= & R c_{i j}+\frac{t}{2}\left(h_{i k j k}+h_{j k i k}-(t r h)_{i j}-\Delta h_{i j}\right) \\
& -\frac{t^{2}}{2}\left(h_{i k j l}+h_{j k i l}-h_{k l j i}-h_{i j k l}\right) h_{k l} \\
& +\frac{t^{2}}{4}\left(h_{i k l}+h_{k l i}-h_{i l k}\right)\left(h_{j k l}+h_{k l j}-h_{j l k}\right) \\
& -\frac{t^{2}}{4}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)\left(2 h_{k l l}-(t r h)_{k}\right)+O\left(t^{3}\right)
\end{align*}
$$

here the tensor $\Delta h$ is given by

$$
\Delta h_{i j}=h_{i j k k}
$$

The scalar curvature is given by

$$
\begin{align*}
& \widetilde{R}  \tag{5.7}\\
= & R+t\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right) \\
& +t^{2}\left(\Delta h_{i j}+(t r h)_{i j}-h_{i k j k}-h_{i k k j}\right) h_{i j}+\frac{t^{2}}{4} \sum_{i j k}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)^{2} \\
& -\frac{t^{2}}{4} \sum_{i}\left(2 h_{i j j}-(t r h)_{i}\right)^{2}+t^{2} R c_{i j} h_{i j}^{2}+O\left(t^{3}\right) .
\end{align*}
$$

The Laplacian satisfies

$$
\begin{align*}
= & \widetilde{\Delta} \varphi  \tag{5.8}\\
& \Delta \varphi-\frac{t}{2}\left(2 h_{i j j}-(t r h)_{i}\right) \varphi_{i}-t h_{i j} \varphi_{i j} \\
& +\frac{t^{2}}{2}\left(2 h_{i k k}-(t r h)_{i}\right) h_{i j} \varphi_{j}+\frac{t^{2}}{2}\left(2 h_{i k j}-h_{i j k}\right) h_{i j} \varphi_{k}+t^{2} h_{i j}^{2} \varphi_{i j}+O\left(t^{3}\right) .
\end{align*}
$$

As a consequence we have

$$
\begin{align*}
& \widetilde{R}^{2}  \tag{5.9}\\
= & R^{2}+2 t R\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right) \\
& +t^{2}\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right)^{2}+2 t^{2} R\left(\Delta h_{i j}+(t r h)_{i j}-h_{i k j k}-h_{i k k j}\right) h_{i j} \\
& +\frac{t^{2} R}{2} \sum_{i j k}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)^{2}-\frac{t^{2} R}{2} \sum_{i}\left(2 h_{i j j}-(t r h)_{i}\right)^{2}+2 t^{2} R \cdot R c_{i j} h_{i j}^{2} \\
& +O\left(t^{3}\right)
\end{align*}
$$

and

$$
\begin{align*}
& |\widetilde{R c}|^{2}  \tag{5.10}\\
= & |R c|^{2}+t R c_{i j}\left(2 h_{i k j k}-(t r h)_{i j}-\Delta h_{i j}\right)-2 t R c_{i j} R c_{i k} h_{j k} \\
& +\frac{t^{2}}{4} \sum_{i j}\left(h_{i k j k}+h_{j k i k}-(t r h)_{i j}-\Delta h_{i j}\right)^{2}-t^{2} R c_{i j} h_{k l}\left(2 h_{i k j l}-h_{k l i j}-h_{i j k l}\right) \\
& -2 t^{2} R c_{i j} h_{i k}\left(h_{j l k l}+h_{k l j l}-(t r h)_{j k}-\Delta h_{j k}\right) \\
& +\frac{t^{2}}{2} R c_{i j}\left(h_{i k l}+h_{k l i}-h_{i l k}\right)\left(h_{j k l}+h_{k l j}-h_{j l k}\right) \\
& -\frac{t^{2}}{2} R c_{i j}\left(2 h_{i k j}-h_{i j k}\right)\left(2 h_{k l l}-(t r h)_{k}\right)+2 t^{2} R c_{i j} R c_{i k} h_{j k}^{2}+t^{2} R c_{i j} R c_{k l} h_{i k} h_{j l} \\
& +O\left(t^{3}\right)
\end{align*}
$$

and

$$
\begin{align*}
& \widetilde{\Delta} \widetilde{R}  \tag{5.11}\\
= & \Delta R+t \Delta\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right)-\frac{t}{2}\left(2 h_{i j j}-(t r h)_{i}\right) R_{i}-t h_{i j} R_{i j} \\
& +t^{2} \Delta\left[\left(\Delta h_{i j}+(t r h)_{i j}-h_{i k j k}-h_{i k k j}\right) h_{i j}\right]-t^{2} h_{i j}\left(h_{k l k l}-\Delta t r h-R c_{k l} h_{k l}\right)_{i j} \\
& +\frac{t^{2}}{4} \Delta \sum_{i j k}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)^{2}-\frac{t^{2}}{4} \Delta \sum_{i}\left(2 h_{i j j}-(t r h)_{i}\right)^{2} \\
& -\frac{t^{2}}{2}\left(2 h_{i j j}-(t r h)_{i}\right)\left(h_{k l k l}-\Delta t r h-R c_{k l} h_{k l}\right)_{i}+t^{2} \Delta\left(R c_{i j} h_{i j}^{2}\right) \\
& +\frac{t^{2}}{2}\left(2 h_{i k k}-(t r h)_{i}\right) h_{i j} R_{j}+\frac{t^{2}}{2}\left(2 h_{i k j}-h_{i j k}\right) h_{i j} R_{k}+t^{2} h_{i j}^{2} R_{i j}+O\left(t^{3}\right) .
\end{align*}
$$

Recall

$$
\begin{equation*}
Q=-\frac{1}{4} \Delta R-2|R c|^{2}+\frac{23}{32} R^{2} \tag{5.12}
\end{equation*}
$$

Plug in the formulas above we get

$$
\begin{align*}
& \widetilde{Q}  \tag{5.13}\\
= & Q-\frac{t}{4} \Delta\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right)-2 t R c_{i j}\left(2 h_{i k j k}-(t r h)_{i j}-\Delta h_{i j}\right) \\
& +\frac{23 t}{16} R\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right)+\frac{t}{8}\left(2 h_{i j j}-(t r h)_{i}\right) R_{i}+\frac{t}{4} h_{i j} R_{i j}+4 t R c_{i j} R c_{i k} h_{j k} \\
& -\frac{t^{2}}{4} \Delta\left[\left(\Delta h_{i j}+(t r h)_{i j}-h_{i k j k}-h_{i k k j}\right) h_{i j}\right]+\frac{t^{2}}{4} h_{i j}\left(h_{k l k l}-\Delta t r h-R c_{k l} h_{k l}\right)_{i j} \\
& -\frac{t^{2}}{16} \Delta \sum_{i j k}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)^{2}+\frac{t^{2}}{16} \Delta \sum_{i}\left(2 h_{i j j}-(t r h)_{i}\right)^{2} \\
& +\frac{t^{2}}{8}\left(2 h_{i j j}-(t r h)_{i}\right)\left(h_{k l k l}-\Delta t r h-R c_{k l} h_{k l}\right)_{i}-\frac{t^{2}}{4} \Delta\left(R c_{i j} h_{i j}^{2}\right) \\
& -\frac{t^{2}}{2} \sum_{i j}\left(h_{i k j k}+h_{j k i k}-(t r h)_{i j}-\Delta h_{i j}\right)^{2}+2 t^{2} R c_{i j} h_{k l}\left(2 h_{i k j l}-h_{k l i j}-h_{i j k l}\right) \\
& +4 t^{2} R c_{i j} h_{i k}\left(h_{j l k l}+h_{k l j l}-(t r h)_{j k}-\Delta h_{j k}\right)+\frac{23 t^{2}}{32}\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right)^{2} \\
& +\frac{23 t^{2}}{16} R\left(\Delta h_{i j}+(t r h)_{i j}-h_{i k j k}-h_{i k k j}\right) h_{i j}-t^{2} R c_{i j}\left(h_{i k l}+h_{k l i}-h_{i l k}\right)\left(h_{j k l}+h_{k l j}-h_{j l k}\right) \\
& +t^{2} R c_{i j}\left(2 h_{i k j}-h_{i j k}\right)\left(2 h_{k l l}-(t r h)_{k}\right)+\frac{23 t^{2}}{64} R \sum_{i j k}\left(h_{i k j}+h_{j k i}-h_{i j k}\right)^{2} \\
& -\frac{23 t^{2}}{64} R \sum_{i}\left(2 h_{i j j}-(t r h)_{i}\right)^{2}-\frac{t^{2}}{8}\left(2 h_{i k k}-(t r h)_{i}\right) h_{i j} R_{j}-\frac{t^{2}}{8}\left(2 h_{i k j}-h_{i j k}\right) h_{i j} R_{k} \\
& -\frac{t^{2}}{4} h_{i j}^{2} R_{i j}-4 t^{2} R c_{i j} R c_{i k} h_{j k}^{2}-2 t^{2} R c_{i j} R c_{k l} h_{i k} h_{j l}+\frac{23 t^{2}}{16} R \cdot R c_{i j} h_{i j}^{2}+O\left(t^{3}\right) .
\end{align*}
$$

Since the Paneitz operator can be written as

$$
\begin{equation*}
P \varphi=\Delta^{2} \varphi+4 R c_{i j} \varphi_{i j}-\frac{5}{4} R \Delta \varphi+\frac{3}{4} \nabla R \cdot \nabla \varphi-\frac{1}{2} Q \varphi \tag{5.14}
\end{equation*}
$$

calculation shows

$$
\begin{align*}
& P_{g, h}^{(1)} \varphi  \tag{5.15}\\
= & -h_{i j}(\Delta \varphi)_{i j}-\Delta\left(h_{i j} \varphi_{i j}\right)-\frac{1}{2}\left(2 h_{i j j}-(t r h)_{i}\right)(\Delta \varphi)_{i}-\frac{1}{2} \Delta\left[\left(2 h_{i j j}-(t r h)_{i}\right) \varphi_{i}\right] \\
& +2\left(2 h_{i k j k}-\Delta h_{i j}-(t r h)_{i j}\right) \varphi_{i j}-8 R c_{i j} h_{i k} \varphi_{j k}+\frac{5}{4} R h_{i j} \varphi_{i j} \\
& -\frac{5}{4}\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right) \Delta \varphi-2 R c_{i j}\left(2 h_{i k j}-h_{i j k}\right) \varphi_{k}+\frac{5}{8} R\left(2 h_{i j j}-(t r h)_{i}\right) \varphi_{i} \\
& +\frac{3}{4}\left(h_{j k j k}-\Delta t r h-R c_{j k} h_{j k}\right)_{i} \varphi_{i}-\frac{3}{4} h_{i j} R_{i} \varphi_{j}+\frac{1}{8} \Delta\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right) \cdot \varphi \\
& -\frac{1}{8} h_{i j} R_{i j} \varphi-\frac{1}{16}\left(2 h_{i j j}-(t r h)_{i}\right) R_{i} \varphi+R c_{i j}\left(2 h_{i k j k}-\Delta h_{i j}-(t r h)_{i j}\right) \varphi \\
& -2 R c_{i j} R c_{i k} h_{j k} \varphi-\frac{23}{32} R\left(h_{i j i j}-\Delta t r h-R c_{i j} h_{i j}\right) \varphi .
\end{align*}
$$

In particular on standard $S^{3}$ we have

$$
\begin{align*}
& P_{g, h}^{(1)} \varphi  \tag{5.16}\\
= & -h_{i j}(\Delta \varphi)_{i j}-\Delta\left(h_{i j} \varphi_{i j}\right)-\frac{1}{2}\left(2 h_{i j j}-(t r h)_{i}\right)(\Delta \varphi)_{i}-\frac{1}{2} \Delta\left[\left(2 h_{i j j}-(t r h)_{i}\right) \varphi_{i}\right] \\
& +2\left(2 h_{i k j k}-\Delta h_{i j}-(t r h)_{i j}\right) \varphi_{i j}-\frac{17}{2} h_{i j} \varphi_{i j}-\frac{5}{4}\left(h_{i j i j}-\Delta t r h-2 t r h\right) \Delta \varphi \\
& -\frac{1}{4}\left(2 h_{i j j}-(t r h)_{i}\right) \varphi_{i}+\frac{3}{4}\left(h_{j k j k}-\Delta t r h-2 t r h\right)_{i} \varphi_{i}+\frac{1}{8} \Delta\left(h_{i j i j}-\Delta t r h-2 t r h\right) \varphi \\
& -\frac{5}{16}\left(h_{i j i j}-\Delta t r h-2 t r h\right) \varphi .
\end{align*}
$$

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