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Wilson loop in general representation and RG flow in 1D defect QFT

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Abstract

The generalized Wilson loop operator interpolating between the supersymmetric and the ordinary Wilson loop in $\mathcal{N} = 4$ SYM theory provides an interesting example of renormalization group flow on a line defect: the scalar coupling parameter ζ has a non-trivial beta function and may be viewed as a running coupling constant in a 1D defect QFT. In this paper we continue the study of this operator, generalizing previous results for the beta function and Wilson loop expectation value to the case of an arbitrary representation of the gauge group and beyond the planar limit. Focusing on the scalar ladder limit where the generalized Wilson loop reduces to a purely scalar line operator in a free adjoint theory, and specializing to the case of the rank k symmetric representation of $SU(N)$, we also consider a certain 'semiclassical' limit where k is taken to infinity with the product $k\zeta^2$ fixed. This limit can be conveniently studied using a 1D defect QFT representation in terms of N commuting bosons. Using this representation, we compute the beta function and the circular loop expectation value in the large k limit, and use it to derive constraints on the structure of the beta function for general representation. We discuss the corresponding 1D RG flow and comment on the consistency of the results with the 1D defect version of the F-theorem.

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(Some figures may appear in colour only in the online journal)

Contents

1. Introduction and summary

In this paper we continue our investigation [1–4] of a family of operators that interpolate between the supersymmetric Wilson–Maldacena ($\zeta = 1$) and the standard Wilson ($\zeta = 0$) loop operators [5]

$$W^{(\zeta)}(C) = \text{Tr P}\exp\oint_C \mathrm{d}\tau\ \left[\mathrm{i}\,A_\mu(x)\,\dot{x}^\mu + \zeta\phi_m(x)\,\theta^m\,|\dot{x}|\right],\quad \theta_m^2 = 1. \tag{1.1}$$

Here ϕ_m are the six scalars of the $SU(N)$ $\mathcal{N}=4$ SYM theory. We shall choose the unit vector θ_m to be along the 6th direction, i.e. $\phi_m\theta^m = \phi_6 \equiv \phi$. The study of (1.1) is of interest, in particular, in the context of 1D defect QFT, see e.g. [3, 6–11] for related work, and [12–14] for other examples of RG flows on line defects.

Let us first summarize some previous results. In the simplest case the trace in (1.1) is taken in the fundamental representation; then the expectation value of (1.1) is a function of ζ, N and 't Hooft coupling $\lambda = g^2 N$. For a smooth contour C, $\langle W^{(\zeta)}\rangle$ is logarithmically divergent, requiring a renormalization of the coupling ζ. Its renormalized value obeys the renormalization group (RG) equation

$$\langle W^{(\zeta)}\rangle \equiv \mathrm{W}\,(\lambda;\zeta(\mu),\mu)\,,\qquad \left(\mu\frac{\partial}{\partial\mu} + \beta_\zeta\frac{\partial}{\partial\zeta}\right)\mathrm{W} = 0,\quad \beta_\zeta = \mu\frac{\mathrm{d}\zeta}{\mathrm{d}\mu}. \tag{1.2}$$

At weak coupling in the planar limit the general structure of β_ζ is expected to be[5]

$$\beta_\zeta = b_1\,\lambda\,\zeta(1-\zeta^2) + \lambda^2\,\zeta\,(1-\zeta^2)\,(b_2 + b_3\,\zeta^2) + \lambda^3\,\zeta\,(1-\zeta^2)\,(b_4 + b_5\,\zeta^2 + b_6\,\zeta^4) + \mathcal{O}(\lambda^4). \tag{1.3}$$

The one-loop term in the β_ζ function was found in [5] and the two-loop term in [4]. Explicitly,

$$\beta_\zeta = -\frac{\lambda}{8\pi^2}\,\zeta\,(1-\zeta^2) + \frac{\lambda^2}{64\pi^4}\,\zeta(1-\zeta^4) + \mathcal{O}(\lambda^3). \tag{1.4}$$

The WL ($\zeta = 0$) and WML ($\zeta = 1$) cases are the fixed points to all orders in λ. The running of ζ may be considered as an RG flow in an effective 1D defect theory coupled to the bulk SYM theory. For a circular contour, $\mathrm{F} = \log \mathrm{W}$ has an interpretation of (minus) 1D defect theory free energy on S^1, and $\log\mathrm{W}$ obeys [4] the defect analog of the F-theorem [15, 16] $\mathrm{F}^{(\mathrm{UV})} > \mathrm{F}^{(\mathrm{IR})}$ (cf also [17]). One may also define a defect entropy function that is monotonically decreasing along the flow from UV to IR [11]. On general grounds, consistent with this interpretation, we should have

$$\frac{\partial}{\partial\zeta}\,\log\,\mathrm{W} = \mathcal{C}\beta_\zeta, \tag{1.5}$$

where $\mathcal{C} = \mathcal{C}(\lambda,\zeta)$ admits the weak coupling expansion $\mathcal{C} = \frac{\lambda}{4} + \mathcal{O}(\lambda^2)$ [1].

It is thus positive at least in perturbation theory in small λ.

The expectation value $\mathrm{W} = \langle W^{(\zeta)}\rangle$ on a circle has the following structure [1, 2] (consistent with (1.3) and (1.5))

$$\mathrm{W} = \langle W^{(1)}\rangle\ \left[1 + w_1\,\lambda^2\,(1-\zeta^2)^2 + \lambda^3\,(1-\zeta^2)^2(w_2 + w_3\,\zeta^2) + \cdots\right], \tag{1.6}$$

[5] Note that the one-loop b_1 and two-loop coefficients b_2, b_3 in (1.3) and (1.4), are scheme independent as they are invariant under redefinitions of ζ that do not move the fixed points $\zeta' = \zeta + \zeta\,(1-\zeta^2)\ \left[\lambda\,z_1 + \lambda^2\,(z_2 + z_3\,\zeta^2) + \cdots\right]$.

where $N^{-1}\langle W^{(1)}\rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \mathcal{O}(\lambda^3)$ [18]. The coefficients $w_1 = \frac{1}{128\pi^2}$ [1] and w_2 (which is presently unknown) are scheme-independent. The scheme-dependent coefficient w_3 is finite after the renormalization of ζ [2]

$$w_3 = -\frac{1}{256\,\pi^4}\left(\log\,\mu + \frac{5}{6}\right). \tag{1.7}$$

Here μ is a renormalization scale (in general multiplied by the radius of the circle which is set to 1 here); the coefficient of $\log\mu$ is related (via (1.2)) to the coefficient in the one-loop beta-function (1.4) while the constant $\frac{5}{6}$ is scheme-dependent.

The coefficients of the highest ζ powers at each λ^n order in (1.3), i.e. $b_1, b_3, b_6, \ldots$, may be computed by restricting to diagrams with maximal number of scalar propagators attached to the Wilson line. In particular, these are diagrams that do not have internal vertices, i.e. they are of (scalar) ladder type. Using the vertex renormalization method of [19], we computed them to five-loop order [4]

$$\begin{aligned}\beta_\zeta^{\text{ladder}} = q_1\,\frac{\lambda}{4\pi^2}\,\zeta^3 + q_2\left(\frac{\lambda}{4\pi^2}\right)^2\zeta^5 + q_3\left(\frac{\lambda}{4\pi^2}\right)^3\zeta^7 + q_4\left(\frac{\lambda}{4\pi^2}\right)^4\zeta^9 \\ + q_5\left(\frac{\lambda}{4\pi^2}\right)^5\zeta^{11} + \cdots,\end{aligned} \tag{1.8}$$

$$\begin{aligned}&q_1 = \frac{1}{2}, \qquad q_2 = -\frac{1}{4}, \qquad q_3 = \frac{1}{4} - \frac{\zeta(2)}{8}, \qquad q_4 = -\frac{17}{48} + \frac{\zeta(2)}{3} - \frac{\zeta(3)}{12}, \\ &q_5 = \frac{29}{48} - \frac{37\,\zeta(2)}{48} + \frac{29\,\zeta(3)}{96} + \frac{25\,\zeta(4)}{128}.\end{aligned} \tag{1.9}$$

Here $\zeta(n)$ are the Riemann zeta-function values and q_3 and higher coefficients are scheme dependent. In this ladder approximation, the expectation value of the operator defined on a closed contour parameterized by $\tau \in (0, 2\pi)$ reduces to

$$\langle W^{(\zeta)}\rangle^{\text{ladder}} = \langle \text{Tr P exp}\int_0^{2\pi} \mathrm{d}\tau'\,\zeta\,\phi(\tau')\rangle = \mathrm{W}(\xi), \quad \xi \equiv \lambda\,\zeta^2, \tag{1.10}$$

where we set $\phi(\tau) \equiv \phi(x(\tau))$ and $\langle\ldots\rangle$ is computed in the free adjoint scalar theory

$$\langle\ldots\rangle = \int \mathrm{d}\phi\,\mathrm{e}^{-S}\ldots, \qquad\qquad S = \frac{1}{g^2}\int \mathrm{d}^4\,x\;\mathrm{Tr}(\partial_\alpha\phi\partial^\alpha\phi). \tag{1.11}$$

Redefining the scalar $\phi \to \zeta^{-1}\phi$ we get the one-coupling theory with $\lambda = g^2 N$ in S replaced by ξ defined in (1.10).[6] In the circular or straight line cases the associated 1D propagator $D(\tau - \tau') = \langle\phi(\tau)\phi(\tau')\rangle$ has then the following form[7]

$$\text{circle:}\,D(\tau) = \frac{\xi}{8\pi^2}\frac{1}{4\sin^2\frac{\tau}{2}}, \qquad \text{line:}\,D(\tau) = \frac{\xi}{8\pi^2}\frac{1}{\tau^2}. \tag{1.12}$$

[6] Note also that after factoring out one power of ζ, the expansion in (1.9) may be written in terms of the effective coupling ξ.

[7] We recall that the (bulk) $\mathcal{N} = 4$ SYM action is schematically of the form $S = \frac{1}{g^2}\int \mathrm{d}^4x\,\mathrm{Tr}(F^2 + D\phi D\phi + \phi^4 + \ldots)$, and $\lambda = g^2 N$. Here we also took into account a factor $\frac{1}{2}$ from the relation $T^a T^a = \frac{1}{2}N\mathbf{1}$, for the generators T^a of $SU(N)$ in the fundamental representation.

Let us note that the study of the partition function W of the scalar loop model (1.10) and (1.11) is an interesting problem on its own right, as this is an example of a particularly simple defect QFT. Note that for the case of $SU(2)$, the scalar defect (1.10) may also be thought as describing an impurity in the (free) $O(3)$ vector model (see e.g. [20, 21] and references therein, and also [14] for a related discussion).

The motivation behind the present paper is to try to generalize the expression for the beta function (1.4) and the Wilson loop expectation value (1.6) to the case when the trace in (1.1) is taken in a generic representation R of $SU(N)$ and beyond the planar limit. Let us consider a generic simple group G with coupling g. Then for the circular supersymmetric WML ($\zeta = 1$) in a general representation R of a group G one finds [22, 23] (see also (B.16))

$$\frac{1}{\dim \mathrm{R}}\,\langle W^{(1)}\rangle = 1 + C_\mathrm{R}\,\frac{g^2}{4} + \left(C_\mathrm{R}^2 - \frac{1}{6}C_\mathrm{R}C_\mathrm{A}\right)\frac{g^4}{32} + \left(C_\mathrm{R}^3 - \frac{1}{2}C_\mathrm{R}^2C_\mathrm{A} + \frac{1}{12}C_\mathrm{R}C_\mathrm{A}^2\right)\frac{g^6}{384} + \cdots. \tag{1.13}$$

Here C_A and C_R are the quadratic Casimirs for the adjoint and R representations ($C_\mathrm{A} = N$ for $G = SU(N)$ and $T^aT^a = C_\mathrm{R}\dim\mathrm{R}$, see appendix A for our conventions). For any ζ we then expect to find for the corresponding generalization of the two-loop part of (1.6)

$$\frac{1}{\dim \mathrm{R}}\,\langle W^{(\zeta)}\rangle = 1 + C_\mathrm{R}\,\frac{g^2}{4} + \left[C_\mathrm{R}^2 - \frac{1}{6}C_\mathrm{R}C_\mathrm{A} + (1-\zeta^2)\left(k_1 + k_2\zeta^2\right)\right]\frac{g^4}{32} + \cdots. \tag{1.14}$$

The coefficients k_1 and k_2 may be determined by the methods of [1] and we will find that

$$\frac{1}{\dim \mathrm{R}}\,\langle W^{(\zeta)}\rangle = 1 + C_\mathrm{R}\frac{g^2}{4} + \left[C_\mathrm{R}^2 - \frac{1}{6}C_\mathrm{R}C_\mathrm{A}\left(1 - \frac{3}{\pi^2}(1-\zeta^2)^2\right)\right]\frac{g^4}{32} + \cdots. \tag{1.15}$$

Similarly, the beta function for general representation generalizing the one-loop term in (1.4) is found to be

$$\beta_\zeta = -C_\mathrm{A}\,\zeta\,(1-\zeta^2)\frac{g^2}{8\pi^2} + \cdots. \tag{1.16}$$

Note that (1.15) and (1.16) are related as expected according to (1.5), with $\mathcal{C} = \frac{1}{2}C_\mathrm{R}g^2 + \cdots$. In the formal abelian limit $C_\mathrm{A} = 0$ we recover the expected exponentiation of the one-loop term in (1.15), and the vanishing of the beta function.

Let us note also that the coefficients of the higher powers of C_R in (1.13) and (1.15) are related to the leading one: for the general case of WL in YM theory with matter one expects that powers of C_R exponentiate, i.e. the non-trivial part of $\log\langle W\rangle = \sum_{k=1}^{\infty} g^{2k}\gamma_k$ should start with a term linear in C_R (γ_k are 'maximally non-abelian' color factors; this is a manifestation of the 'non-abelian exponentiation' [24, 25]; see also [26, 27] in the case of light-like WL). Here $\gamma_1 \sim C_\mathrm{R}$, $\gamma_{2,3}$ depend also on C_A, while starting at four loops γ_k contain higher Casimir invariants like Q_R in (1.22) [28, 29]. This then suggests (in view of (1.2) and (1.5) with $\mathcal{C} \sim C_\mathrm{R}\,g^2 + \cdots$) that the one- and two-loop terms in the corresponding β_ζ should depend only on $C_{\rm A}$ while the three-loop term should have dependence on $Q_{\rm R}$.[8] We will confirm this below in the ladder approximation (see (1.21) and (1.25)).

The 'ladder' part of (1.15) (given by highest power of ζ at each order in g) may be written as

$$\frac{1}{\dim \mathrm{R}}\,\langle W^{(\zeta)}\rangle^{\rm ladder} = 1 + C_{\rm R}C_{\rm A}\,\frac{\zeta^4 g^4}{64\pi^2} + \cdots . \tag{1.17}$$

For the fundamental representation R = F of $SU(N)$ using that $C_{\rm F} = \frac{N^2-1}{2N}$ and $C_{\rm A} = N$ we observe that (1.15) reduces to

$$\begin{aligned}\mathrm{R}=\mathrm{F}: \frac{1}{N}\,\langle W^{(\zeta)}\rangle =& 1 + \left[1 - \frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^4}\right)\right]\frac{\lambda}{8}\\ &+ \left[\frac{1}{192} - \frac{5}{384N^2} + \frac{(\zeta^2-1)^2}{128\pi^2}\left(1-\frac{1}{N^2}\right) + \mathcal{O}\left(\frac{1}{N^4}\right)\right]\lambda^2 + \mathcal{O}(\lambda^3).\end{aligned} \tag{1.18}$$

This generalizes the previous planar two-loop result (1.6) to subleading terms in $1/N$.

We may parametrize the three-loop term in (1.15) as

$$\langle W^{(\zeta)}\rangle = \langle W^{(1)}\rangle\left[1 + C_{\rm R}C_{\rm A}(1-\zeta^2)^2\frac{g^4}{64\pi^2} + (1-\zeta^2)^2(\mathrm{w}_2 + \mathrm{w}_3\zeta^2)\,g^6 + \cdots\right], \tag{1.19}$$

where $\langle W^{(1)}\rangle$ is given by (1.13) and $\mathrm{w}_2, \mathrm{w}_3$ are the analogs of w_2, w_3 in (1.6). In particular, we expect that

$$\mathrm{w}_3 = -\frac{1}{128\,\pi^4}\,C_{\rm R}C_{\rm A}^2(\log\,\mu + c_3), \tag{1.20}$$

where in the $SU(N)$ fundamental representation case and at large N (when $C_{\rm R}C_{\rm A}^2 \to \frac{1}{2}N^3$) we should find that $\mathrm{w}_3 \to N^3 w_3$ (so that $c_3 = \frac{5}{6}$ in the same scheme as (1.7)).

For a generic representation R, the structure of the ladder-limit part of three-loop beta function is expected to be the following generalization of (1.9)[9]

$$\begin{aligned}\beta_\zeta^{\rm ladder} =& q_1'\,C_{\rm A}\zeta^3\,\frac{g^2}{4\pi^2} + (q_2'\,C_{\rm A}^2 + q_2''\,C_{\rm A}C_{\rm R})\,\zeta^5\left(\frac{g^2}{4\pi^2}\right)^2\\ &+ \left(q_3'\,C_{\rm A}^3 + q_3''\,C_{\rm A}^2C_{\rm R} + q_3'''\,C_{\rm A}C_{\rm R}^2 + q_3''''\,Q_{\rm R}\right)\zeta^7\left(\frac{g^2}{4\pi^2}\right)^3 + \mathcal{O}(g^8),\end{aligned} \tag{1.21}$$

$$Q_{\rm R} \equiv \frac{d_{\rm A}^{abcd}d_{\rm R}^{abcd}}{C_{\rm R}\ \dim\,\mathrm{R}}. \tag{1.22}$$

Here in $Q_{\rm R}$ the tensor $d_{\rm R}^{abcd}$ is the four-index symmetrized trace $\mathrm{S\,Tr}(T^aT^bT^cT^d)$ (see appendices A and D). The q_n-coefficients are numerical constants independent of representation. We

[8] We are grateful to Korchemsky for this observation and related explanations.

[9] This follows from inspection of the possible color structures. We also impose the condition that the beta-function has to vanish in the abelian limit $C_{\rm A} = 0$.

will show that

$$\beta_\zeta^{\rm ladder} = \frac{1}{2}\,C_{\rm A}\zeta^3\,\frac{g^2}{4\pi^2} - \frac{1}{4}\,C_{\rm A}^2\,\zeta^5\left(\frac{g^2}{4\pi^2}\right)^2 + \left[q_3'\,C_{\rm A}^3 - 3\,\zeta(2)\,Q_{\rm R}\right]\,\zeta^7\left(\frac{g^2}{4\pi^2}\right)^3 + \mathcal{O}(g^8), \tag{1.23}$$

where q_3' is a scheme dependent constant (equal to $\frac{1}{4}$ in the same regularization scheme that led to (1.9)). Note that (1.19), (1.20) and (1.23) are consistent with each other via the RG equation (1.2).

To demonstrate the validity of (1.23) and extract further information about the representation dependence, we will consider the case of R being the *k-symmetric* representation S_k of $SU(N)$. Using perturbation theory in large k at fixed $k\zeta^2 g^2$ and fixed N and comparing with (1.21) we will confirm (1.23).

Our starting point will be the following 1D path integral representation for the Wilson loop in the k-symmetric representation of $SU(N)$ (see, e.g., [30, 31])[10]

$$\mathrm{W}_k = \langle W_k\rangle, \qquad W_k = \int D\chi D\bar\chi\ \delta(\bar\chi\chi - R^2)\,\mathrm{e}^{-S}, \quad R^2 \equiv k + \frac{N}{2}, \tag{1.24}$$

$$S = \int_0^{2\pi}\mathrm{d}\tau\,\left[\,\bar\chi\,\partial_\tau\chi + \zeta\,\phi^a(\tau)\,\bar\chi\,T^a\,\chi\right], \tag{1.25}$$

where we specialized to the purely scalar operator (1.10), and the averaging $\langle\ldots\rangle$ is done over the scalar ϕ as in (1.11). Here $\phi(\tau) = \phi(x(\tau))$, $\tau \in [0, 2\pi]$ and $\chi, \bar\chi$ are periodic bosons transforming in the fundamental representation of $SU(N)$ (T^a are generators in the fundamental representation). After the integration over the free adjoint scalar field ϕ we obtain an effective non-local 1D theory with the action of the following structure

$$S = \int \mathrm{d}\tau\,\bar\chi\,\partial_\tau\chi - \zeta^2\,g^2\int \mathrm{d}\tau\,\mathrm{d}\tau'\,D(\tau-\tau')\,\bar\chi(\tau)T^a\,\chi(\tau)\ \bar\chi(\tau')T^a\chi(\tau'), \tag{1.26}$$

where $D(\tau-\tau') = \langle\phi(\tau)\phi(\tau')\rangle$ (on the line $D \sim \frac{1}{(\tau-\tau')^2}$, cf (1.11) and (1.12)).

The rank k of the symmetric representation enters only through R^2 in the delta-function constraint in (1.24). Rescaling χ by R so that now $\bar\chi\chi = 1$ we get (e.g. on the straight line)

$$S = R^2\left[\int \mathrm{d}\tau\,\bar\chi\,\partial_\tau\chi - \varkappa\int\frac{\mathrm{d}\tau\,\mathrm{d}\tau'}{(\tau-\tau')^2}\,\bar\chi(\tau)T^a\chi(\tau)\ \bar\chi(\tau')T^a\chi(\tau')\right], \tag{1.27}$$

$$\varkappa \equiv \frac{\zeta^2\,g^2\,R^2}{8\pi^2}. \tag{1.28}$$

We may then develop a systematic 'semiclassical' large R^2 or large k perturbation theory at fixed $\varkappa$ and N for W_k and the beta function $\beta_\varkappa$ for the coupling $\varkappa$. Note that since in the ladder approximation the bulk theory is free, the coupling g can take any value (and can actually be absorbed into ζ defining $\bar\xi = \zeta^2 g^2$, cf (1.10)) so the large k limit at fixed $\varkappa$ means also small ζ limit.

[10] This is an example of representing the trace in some representation in terms of an integral over group orbit [32], cf also [33, 34] for a more general discussion.

Explicitly, we will find that for the k-symmetric representation

$$\beta_{\varkappa} = \mu\frac{\mathrm{d}\varkappa}{\mathrm{d}\mu} = \frac{2N}{R^2}\,\frac{\varkappa^2}{1+\pi^2\varkappa^2} - \frac{2N^2}{R^4}\,\frac{\varkappa^3\,(1-\mathrm{b}_1\,\pi^2\varkappa^2)}{(1+\pi^2\varkappa^2)^3} + \mathcal{O}\left(\frac{1}{R^6}\right), \qquad (1.29)$$

where the coefficient b_1 is scheme dependent with $\mathrm{b}_1 = 1$ in a particular momentum cutoff scheme (see also discussion below (6.34)).

Since g and R^2 are not running, $\beta_{\varkappa}$ is directly related to the ladder beta function for ζ in (1.23). In general, the large k expansion of $\beta_{\varkappa}$ gives an all order prediction for the small ζ expansion of $\beta_\zeta^{\mathrm{ladder}}$: it fixes the coefficient of the highest power of k at each order in ζ. In particular, expanding the 'one-loop' term in (1.29) in powers of ζ yields

$$\beta_\zeta^{\mathrm{ladder}} = \frac{Ng^2}{8\pi^2}\zeta^3 - \frac{Ng^6}{512\pi^2}k^2\zeta^7 + \cdots. \qquad (1.30)$$

Noting that for the k-symmetric representation $Q_{\mathrm{R}} = k^2\frac{N}{4} + O(k)$, this allows to fix the coefficient of the Q_{R} part of the three-loop term in (1.23).

Note also that in the case when k is fixed and N is large the leading $\varkappa^2$ and $\varkappa^3$ terms in the small $\varkappa$ expansion in (1.29) are in agreement with the one-loop and two-loop terms in $\beta_\zeta^{\mathrm{ladder}}$ in (1.8).[11]

For the renormalized value of the scalar ladder Wilson loop expectation value on a circle (of unit radius) in (1.24) defined in the k-symmetric representation we will find that[12]

$$\mathrm{W}_k = \dim\,\mathrm{S}_k\big(1+\pi^2\varkappa^2\big)^{\frac{N-1}{2}}\left[1 + \frac{\mathrm{v}_1}{R^2}\,\frac{N(N-1)\varkappa^3}{(1+\pi^2\varkappa^2)^2} + \mathcal{O}\left(\frac{1}{R^4}\right)\right], \qquad (1.31)$$

$$\mathrm{v}_1 = -2\pi^2(\log\mu + c_3), \qquad (1.32)$$

where $\dim\,\mathrm{S}_k = \frac{(N+k-1)!}{(N-1)!\,k!}$ is the dimension of the k-symmetric representation of $SU(N)$ and c_3 is a scheme-dependent constant as in (1.20). Note that the expression (1.31) effectively resums an infinite set of terms in the ordinary perturbative expansion in powers of ζ.[13] Expanding (1.31) in powers of $\varkappa$, one finds

$$\frac{1}{\dim\,\mathrm{S}_k}\mathrm{W}_k = 1 + \frac{\pi^2}{2}(N-1)\,\varkappa^2 + \frac{\mathrm{v}_1}{R^2}\,N(N-1)\,\varkappa^3 + \cdots. \qquad (1.33)$$

Noting that $C_{\mathrm{S}_k} \sim k^2(N-1)/2N$ at large k, one can see that the term quadratic in $\varkappa$ matches (1.17), while the cubic term matches the $\mathrm{w}_3\zeta^6 g^6$ term in (1.19) and (1.20).[14]

[11] Let us also note that the representation (1.24) applies for any finite k, in particular also to the $k = 1$ case of the fundamental representation. Then naively the large R^2 perturbation could still be applied by taking N large at fixed $\varkappa$ in (1.28) that then becomes $\varkappa \to \frac{1}{16\pi^2}\xi$, where ξ was defined in (1.10). However, since N here is as large as R^2 the $1/R^2$ expansion of the beta-function in (1.29) no longer makes sense, i.e. needs to be resummed. One can still unambiguously extract the lowest order terms in the small ξ expansion and match them with the ζ^3 and ζ^5 terms in $\beta_\zeta^{\mathrm{ladder}}$.

[12] In the case of $SU(2)$ group the prefactor $\big(1+\pi^2\varkappa^2\big)^{1/2}$ was found earlier in [35].

[13] A similar large k limit with $k \gg N$ for the case of the Wilson–Maldacena loop was studied in [36, 37], where it was observed that an exponentiation of the one-loop result occurs in this limit.

[14] Indeed, from (5.38) $\frac{1}{128\pi^4}C_{\mathrm{R}}C_{\mathrm{A}}^2 g^6\zeta^6 = \frac{1}{256\pi^4}N(N-1)k(k+N)g^6\zeta^6$ while from (1.28) we have $\frac{2\pi^2}{R^2}N(N-1)\varkappa^3 = \frac{1}{256\pi^4}N(N-1)(k+\frac{1}{2}N)^2 g^6\zeta^6$ so we get agreement at large k.

The expression (1.31) satisfies the RG equation as in (1.2) and also the analog of the relation (1.5) with $\beta_\varkappa$ given by the one-loop term in (1.29)

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_\varkappa\frac{\partial}{\partial\varkappa}\right)\mathrm{W}_k=0,\qquad \frac{\partial}{\partial\varkappa}\log\,\mathrm{W}_k=\bar{\mathrm{C}}\,\beta_\varkappa,\qquad \bar{\mathrm{C}}=\frac{(N-1)\pi^2R^2}{2\,N\varkappa}>0. \tag{1.34}$$

Let us now discuss properties of the RG flow implied by the $\beta_\varkappa$ function in (1.29). At the leading $1/k$ order we find (using that $\varkappa\geqslant 0$)

$$\frac{\mathrm{d}\varkappa}{\mathrm{d}t}=\frac{2N}{k}\frac{\varkappa^2}{1+\pi^2\varkappa^2},\qquad t\equiv\log\,\mu, \tag{1.35}$$

$$\varkappa(t)=\gamma t+\frac{1}{\pi}\sqrt{1+\pi^2\gamma^2t^2},\qquad \gamma\equiv\frac{N}{\pi^2k}, \tag{1.36}$$

so that the IR ($\mu\to 0$) and UV ($\mu\to\infty$) asymptotics are

$$\mathrm{IR}:\varkappa(t\to-\infty)=\frac{1}{2\pi^2\gamma\,|t|}\to 0,\qquad \mathrm{UV}:\varkappa(t\to+\infty)=2\gamma\,t\to\infty. \tag{1.37}$$

This asymptotic behavior is, in fact, exact, i.e. not changed by higher $1/k$ corrections in $\beta_\varkappa$ since the exact $\beta_\varkappa$ satisfies[15]

$$\beta_\varkappa\Big|_{\varkappa\to 0}\to 0,\qquad \beta_\varkappa\Big|_{\varkappa\to\infty}\to\frac{2N}{k+\frac{1}{2}N}=\mathrm{const}. \tag{1.38}$$

The corresponding asymptotic behavior of the WL expectation value in (1.31)

$$\mathrm{IR}:\mathrm{W}_k\Big|_{\varkappa\to 0}\to\dim\,\mathrm{S}_k,\qquad \mathrm{UV}:\mathrm{W}_k\Big|_{\varkappa\to\infty}\to\dim\,\mathrm{S}_k\varkappa^{\,N-1}, \tag{1.39}$$

$$\log\,\mathrm{W}_k^{(\mathrm{UV})}>\log\,\mathrm{W}_k^{(\mathrm{IR})}. \tag{1.40}$$

This is consistent with 1D version of F-theorem for W_k as partition function on S^1. Furthermore, one may consider the line defect entropy defined in [11] (here a is the radius of S^1)

$$\mathrm{s}\equiv\left(1-\mathrm{a}\frac{\partial}{\partial\mathrm{a}}\right)\log\,\mathrm{W}_k=\left(1-\mu\frac{\partial}{\partial\mu}\right)\log\,\mathrm{W}_k, \tag{1.41}$$

which is equal to $\log\mathrm{W}_k$ at fixed points. Using (1.34) and $\bar{\mathrm{C}}>0$ (which is true at least in perturbation theory) we get

$$\mathrm{s}=\log\,\mathrm{W}_k+\bar{\mathrm{C}}\,\beta_\varkappa^2\geqslant\log\,\mathrm{W}_k. \tag{1.42}$$

To leading order in the $1/k$ perturbation theory, $\mathrm{s}=\log\,\mathrm{W}_k\approx\log\,\dim\,\mathrm{S}_k+\frac{1}{2}(N-1)\log(1+\pi^2\varkappa^2)$, and so both functions monotonically decrease along the RG trajectory. According to the arguments in [11], the defect entropy s should be monotonically decreasing also non-perturbatively.

Let us mention also that if one considers the defect line in a bulk scalar theory in $d=4-\epsilon$ dimensions then the coupling g^2 and thus $\varkappa\sim g^2\zeta^2k$ will get dimension $\epsilon\to 0$. Then the $\beta_\varkappa$

[15] As is clear from (1.29), the $1/R^4$ term vanishes at large $\varkappa$. The same should be true also at higher orders as for large $\varkappa$ the propagator goes as $\varkappa^{-1}$ while vertices in the action (1.27) are proportional to $\varkappa$.

function gets an extra term $-\epsilon\varkappa$, and, in addition to the trivial UV fixed point $\varkappa = 0$, there are two Wilson–Fisher-type UV and IR fixed points

$$\beta_\varkappa = -\epsilon\,\varkappa + \frac{2N}{k}\frac{\varkappa^2}{1+\pi^2\varkappa^2} + \mathcal{O}\left(\frac{1}{k^2}\right), \tag{1.43}$$

$$\beta_\varkappa = 0 : \varkappa_\pm = \frac{N}{\pi^2 k\,\epsilon}\left(1 \pm \sqrt{1 - \frac{\pi^2 k^2\,\epsilon^2}{N^2}}\right) + \mathcal{O}\left(\frac{1}{k^2}\right). \tag{1.44}$$

In order for these fixed points to be real, one should take the small ϵ and large k limits in such a way that the condition $\epsilon k \leqslant \frac{N}{\pi}$ is satisfied (for $\epsilon k = \frac{N}{\pi}$ the two fixed points coincide, and for $\epsilon k > \frac{N}{\pi}$ they become complex). Taking the $\epsilon \to 0$ limit first, the fixed points reduce to

$$\mathrm{UV} : \varkappa_+ = \frac{2N}{\pi^2 k}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \to \infty, \qquad \mathrm{IR} : \varkappa_- = \frac{k}{2N}\epsilon + \mathcal{O}(\epsilon^2) \to 0. \tag{1.45}$$

Like the asymptotics in (1.37) these fixed points are expected to be stable under higher order $1/R^2$ or $1/k$ corrections to $\beta_\varkappa$.

There are several directions that would be interesting to study in the future. One remaining technical problem is the computation of the scheme-independent coefficient w_2 in (1.6). While we studied the case of k-symmetric representation, it is seems straightforward to consider the case of k-antisymmetric representation (in which the auxiliary 1D fields χ_i will be fermions) providing a cross-check on the general representation dependence of the WL and the beta-function for ζ. It would also be important to understand better the dual AdS/CFT counterpart of the RG flow of ζ [7].

The structure of the rest of the paper is as follows. In section 2 we compute the two-loop β_ζ function in ladder approximation (for any N) by applying the vertex renormalization method described in [4]. We also discuss the structure of β_ζ at three-loop level. In section 3 we derive the two-loop expression (1.15) for the expectation value $\langle W^{(\zeta)}\rangle$ in any representation, thus generalizing our previous result in the fundamental representation [1].

In section 4 we introduce the bosonic 1D path integral expression (1.24) and (1.25) for the ladder Wilson loop in the k-symmetric $SU(N)$ representation and discuss some of its general features. It is different from the more standard fermionic 1D path integral (reviewed in appendix B) and convenient for the study of the large k limit considered in section 5. There we first discuss the free $\varkappa = 0$ case (clarifying the role of the constant zero modes of χ) and then compute the Wilson loop at leading order in large $k \sim R^2$ for $\varkappa \neq 0$. Finally, we present the calculation of the $1/R^2$ corrections and, in particular, the logarithmically divergent contributions that determine the leading term in the $\beta_\varkappa$ function.

In section 6 we show that $\beta_\varkappa$ may be computed starting from a two-point correlator of the adjoint scalars inserted on the Wilson line. We first reproduce the $1/R^2$ term in $\beta_\varkappa$ found in section 5 and then study in detail the order $1/R^4$ correction.

In appendix A we recall our group theoretic conventions. Appendix B reviews the 1D fermionic path integral representation [38] for a Wilson loop in any representation. Appendix C presents details of the calculation of the $1/R^4$ contribution to the $\beta_\varkappa$ function in section 6. Appendix D is devoted to a general proof of the universality, in planar limit, of the coefficient of the three-loop ζ^7 term in $\beta_\zeta^{\rm ladder}$ in (1.23). In appendix E we compute the two-loop β_ζ for generic representation using a two-point scalar correlator on the line. In appendix F we apply similar method as in 3 to the closely related case of a multiply wound Wilson loop in the fundamental representation, finding the two-loop term in the weak gauge coupling expansion for generic ζ.

Note added: while completing this paper, we learned about the partially overlapping work [39], which in particular studies the scalar line defect and its large k limit in the $SU(2)$ case (extending some results announced in [35]). We thank the authors for sharing their draft prior to submission. We also learned about another recent related paper [40].

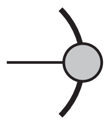
2. β_ζ function in ladder approximation from vertex renormalization

As discussed in detail in [4] the beta function for the ζ coupling in (1.10) can be obtained from the study of the one point function on a long interval $(-L, L)$[16]

$$\frac{\langle \mathrm{Tr}\left(\phi(\tau_0)\,\mathrm{P}\,\mathrm{e}^{\int_{-L}^{L} d\tau'\,\zeta\,\phi(\tau')}\right)\rangle}{\langle \mathrm{Tr}\left(\mathrm{P}\,\mathrm{e}^{\int_{-L}^{L} d\tau'\,\zeta\,\phi(\tau')}\right)\rangle}, \tag{2.1}$$

where the 4D scalar ϕ restricted to the line has a free propagator $D(\tau) = \langle\phi(\tau)\phi(0)\rangle = \frac{g^2}{4\pi^2}\frac{1}{\tau^2}$. Here we shall assume Tr to be in generic representation R of a gauge group. The averaging is done with respect to the free adjoint scalar field as in (1.11).

If we denote by τ the point on the line connected by the propagator to τ_0, then the β_ζ function follows from the renormalization of the vertex V in

$D(\tau_0 - \tau)\,V(\tau,\zeta) =$ τ_0  τ , $V(\tau,\zeta) = \zeta + \text{corrections}.$ (2.2)

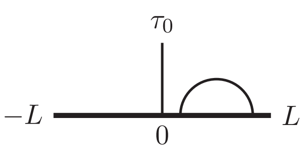
The point τ_0 is at some far part of the Wilson line. We may also choose point τ to be at the origin, $\tau = 0$.

2.1. One-loop order

In dimensional regularization the propagator is

$$D(\tau) = \frac{g^2}{4\pi^2}\frac{1}{|\tau|^{2-\epsilon}}, \quad d = 4 - \epsilon. \tag{2.3}$$

The one-loop planar diagrams in the numerator of (2.1) are



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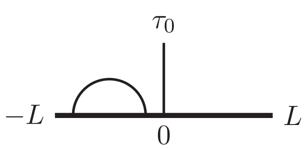
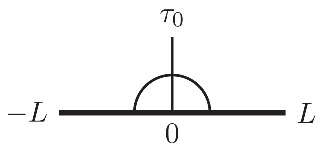


$$= \zeta^3\,\frac{g^2}{4\pi^2}C_\mathrm{R}\,\frac{L^\epsilon}{\epsilon\,(\epsilon-1)}, \tag{2.4}$$

[16] The renormalization of ζ is universal for any contour and thus can be determined by considering the simplest straight-line Wilson loop.

where we used the group generators satisfy $T^aT^a = C_\mathrm{R}\mathbf{1}$. We have also a single non-planar diagram



$$= \zeta^3 \frac{g^2}{4\pi^2}\left(C_\mathrm{R} - \frac{1}{2}C_\mathrm{A}\right) \frac{L^\epsilon(2^\epsilon - 2)}{\epsilon(\epsilon - 1)}, \tag{2.5}$$

where we used (A.9) (C_A corresponds to the adjoint representation, i.e. is equal to N for $SU(N)$). Finally, the denominator of (2.1) contributes

$$= \zeta^2 \frac{g^2}{\pi^2} C_\mathrm{R} \frac{2^{\epsilon-2}L^\epsilon}{\epsilon(\epsilon-1)} . \tag{2.6}$$

The total vertex is then

$$V(\zeta, L) = \zeta + C_\mathrm{A}\,\zeta^3 \frac{g^2}{8\pi^2}\frac{L^\epsilon(2-2^\epsilon)}{\epsilon(\epsilon-1)} + \mathcal{O}(\lambda^2), \tag{2.7}$$

where the dependence on C_R canceled out. V is renormalized by $\zeta_\mathrm{bare} = \zeta \to \zeta_\mathrm{ren} = \zeta(\mu)$

$$\zeta = \mu^{\epsilon/2}\left[\zeta(\mu) + \frac{C_\mathrm{A} g^2}{8\pi^2\epsilon}\zeta^3(\mu) + \mathcal{O}(g^4)\right]. \tag{2.8}$$

The renormalized vertex is then

$$V_\mathrm{ren}(\zeta(\mu), L) = \zeta(\mu) + \frac{C_\mathrm{A} g^2}{8\pi^2}\,\zeta^3(\mu)\left(-1 - \log\frac{L\mu}{2}\right) + \mathcal{O}(g^4), \tag{2.9}$$

and obeys the RG equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\zeta^\mathrm{ladder}\frac{\partial}{\partial\zeta}\right) V^\mathrm{ren}\,(\zeta(\mu), L) = 0, \tag{2.10}$$

with

$$\beta_\zeta^\mathrm{ladder} = C_\mathrm{A}\,\zeta^3\,\frac{g^2}{8\pi^2} + \mathcal{O}(g^4). \tag{2.11}$$

This shows that the one-loop beta-function in (1.6) is universal, i.e. does not depend on a particular representation of the gauge group used to define the WL.[17]

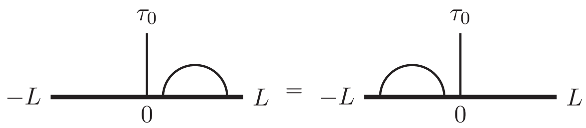
2.2. *Two-loop order*

The two-loop diagrams contributing $V(\zeta, L)$ are much more complicated and we found it convenient to use the propagator with an explicit cutoff as in [4]

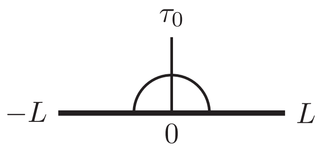
$$D(\tau) = \frac{g^2}{4\pi^2}\frac{1}{(|\tau| + \varepsilon)^2}, \qquad \varepsilon \to 0. \tag{2.12}$$

[17] We have shown that this applies to the ladder part of β_ζ but since it should have $\zeta^2 = 1$ as its zero this should be true also for the full one-loop expression.

We shall focus on logarithmic UV divergences $\log^n \epsilon$. To compare with dimensional regularization, let us first repeat the above one-loop calculation. We find the following analogs of (2.4)–(2.6)



$$= \zeta^3\, C_{\mathrm{R}} \left(\frac{L}{\varepsilon} + \log\frac{\varepsilon}{\varepsilon+L}\right)\frac{g^2}{4\pi^2}, \tag{2.13}$$



$$= -\zeta^3 (C_{\mathrm{R}} - \tfrac{1}{2} C_{\mathrm{A}}) \log\frac{\varepsilon(\varepsilon+2L)}{(\varepsilon+L)^2}\,\frac{g^2}{4\pi^2}. \tag{2.14}$$

$$= \zeta^2 C_{\mathrm{R}} \left(\frac{2L}{\varepsilon} + \log\frac{\varepsilon}{\varepsilon+2L}\right)\frac{g^2}{4\pi^2}, \tag{2.15}$$

so that the total result for the vertex reads

$$V(\zeta, L) = \zeta + \zeta^3\, C_{\mathrm{A}}\, \log\frac{\varepsilon(\varepsilon+2L)}{(\varepsilon+L)^2}\,\frac{g^2}{8\pi^2} + \mathcal{O}(g^4). \tag{2.16}$$

Dependence on C_{R} again drops out and also the linear divergent terms $\frac{L}{\varepsilon}$ cancel. The vertex is renormalized by

$$\zeta \equiv \zeta_{\mathrm{bare}} = \zeta(\mu) - C_{\mathrm{A}}\, \zeta^3(\mu) \log(\mu\varepsilon)\frac{g^2}{8\pi^2} + \mathcal{O}(g^4). \tag{2.17}$$

Then

$$V_{\mathrm{ren}}\,(\zeta(\mu), L) = \zeta(\mu) - C_{\mathrm{A}}\, \zeta^3(\mu) \log\frac{L\mu}{2}\,\frac{g^2}{8\pi^2} + \mathcal{O}(\lambda^2), \tag{2.18}$$

obeys the Callan–Symanzik equation (2.10) with the same beta function as in (2.11).

The same approach can be extended to the two-loop level. We find that the corresponding coupling redefinition and renormalized vertex are

$$\begin{aligned}\zeta \equiv \zeta_{\mathrm{bare}} = {}& \zeta(\mu) - C_{\mathrm{A}}\, \zeta^3(\mu)\, \log(\mu\,\varepsilon)\frac{g^2}{8\pi^2} \\ & + C_{\mathrm{A}}^2\, \zeta^5(\mu) \left[\frac{1}{4}\, \log(\mu\,\varepsilon) + \frac{3}{8}\, \log^2(\mu\,\varepsilon)\right] \frac{g^4}{(4\pi^2)^2} + \mathcal{O}(g^6),\end{aligned} \tag{2.19}$$

$$\begin{aligned}V_{\mathrm{ren}}\,(\zeta(\mu), L) = {}& \zeta(\mu) - C_{\mathrm{A}}\, \zeta^3(\mu) \log\frac{L\mu}{2}\,\frac{g^2}{8\pi^2} - C_{\mathrm{A}}^2\, \zeta^5(\mu) \left[\pi^2 + 12 \log^2 2 \right. \\ & \left. - 3\, \log\frac{\mu L}{2}\left(2 + 3\, \log\frac{\mu L}{2}\right)\right] \frac{g^4}{384\pi^4} + \mathcal{O}(g^6).\end{aligned} \tag{2.20}$$

The corresponding two-loop beta-function is then given by (in agreement with the Callan–Symanzik equation (2.10))

$$\beta_\zeta^{\text{ladder}} = C_{\text{A}}\,\zeta^3\,\frac{g^2}{8\pi^2} - C_{\text{A}}^2\,\zeta^5\,\frac{g^4}{64\pi^4} + \mathcal{O}(g^6). \tag{2.21}$$

Thus the ladder part is again universal, i.e. does not depend on a particular representation. This independence of representation is an accidental two-loop property—we shall see below that it does not hold at three-loop order.

The full two-loop beta-function is then expected to be (cf (1.3) and (1.4))

$$\beta_\zeta = -C_{\text{A}}\,\zeta\,(1-\zeta^2)\frac{g^2}{8\pi^2} + \zeta\,(1-\zeta^2)\,(\text{b}_2 + C_{\text{A}}^2\,\zeta^2)\,\frac{g^4}{64\pi^4} + \mathcal{O}(g^6), \tag{2.22}$$

where b_2 may depend on representation R. Since the beta-function should vanish in the abelian limit b_2 should not contain C_{R}^2 term, i.e. we should have

$$\text{b}_2 = p_1 C_{\text{A}}^2 + p_2 C_{\text{A}} C_{\text{R}}, \tag{2.23}$$

where p_1, p_2 are universal constants. Comparing to the case of the fundamental representation of $SU(N)$ in the planar limit where the two-loop term is given in (1.4) (where $\lambda = g^2 N$, $C_{\text{A}} = N, C_{\text{R}} = \frac{N^2-1}{2N} \to \frac{1}{2}N$) we get the constraint

$$p_1 + \frac{1}{2}p_2 = 1. \tag{2.24}$$

One natural conjecture is that $p_2 = 0$ so that C_{R} does not appear in (2.22), i.e. that like the one-loop beta function, the full two-loop one does not depend on a choice of a particular representation, namely

$$\beta_\zeta = -C_{\text{A}}\,\zeta\,(1-\zeta^2)\frac{g^2}{8\pi^2} + C_{\text{A}}^2\zeta\,(1-\zeta^4)\,\frac{g^4}{64\pi^4} + \mathcal{O}(g^6). \tag{2.25}$$

This C_{R} independence property will be violated at higher-loop orders, as we shall see below.

2.3. Three-loop order

As already mentioned in the introduction (cf (1.21)), from the analysis of possible contributions to the four-loop WL expectation value the general structure of the three-loop beta function in ladder approximation is expected to be

$$(\beta_\zeta^{\text{ladder}})^{(3)} = \left(q_3'\,C_{\text{A}}^3 + q_3''\,C_{\text{A}}^2 C_{\text{R}} + q_3'''\,Q_{\text{A}} C_{\text{R}}^2 + q_3''''\,Q_{\text{R}}\right)\,\zeta^7\left(\frac{g^2}{4\pi^2}\right)^3, \tag{2.26}$$

where Q_{R} was defined in (1.22). This satisfies the condition of vanishing in the abelian limit when $C_{\text{A}} = 0$. Here the tensor d_R^{abcd} is the symmetrized trace of the product of four generators

$$d_{\text{R}}^{a_1\ldots a_n} = \text{Str}(T^a T^b T^c T^d) = \frac{1}{n!}\,\text{Tr}\sum_{\sigma\in\mathcal{S}_n} T^{a_{\sigma(1)}}\ldots T^{a_{\sigma(n)}}. \tag{2.27}$$

To constrain the numerical coefficients $q_3', q_3'', \ldots$ we shall consider the case of R being k-symmetric representation of $SU(N)$ in the limit of $k \gg 1$. Then (see, e.g., [22])

$$C_{\mathrm{R}} = \frac{k(N-1)(N+k)}{2N} \to k^2 \frac{N-1}{2N}, \tag{2.28}$$

$$Q_{\mathrm{R}} = \frac{d_{\mathrm{A}}^{abcd} d_{\mathrm{R}}^{abcd}}{C_{\mathrm{R}} \dim \mathrm{R}} = \frac{N}{24}\left[N^2 - 6N + 6k(k+N)\right] \to k^2 \frac{N}{4}. \tag{2.29}$$

As we shall demonstrate below, the ladder beta function is expected to vanish (for generic N) in the 'classical' limit [35]

$$k \to \infty, \qquad \zeta \to 0, \qquad k\,\zeta^2 = \text{fixed}. \tag{2.30}$$

This implies that $q_3''' = 0$. Then the remaining terms give the following large k limit (at fixed ζ)

$$(\beta_\zeta^{\text{ladder}})^{(3)} \stackrel{k \gg 1}{=} \left(\frac{1}{2} q_3'' N(N-1) + \frac{1}{4} q_3 q_3'''' N\right) k^2\, \zeta^7 \left(\frac{g^2}{4\pi^2}\right)^3. \tag{2.31}$$

Below we will compute (2.31) explicitly determining the two constants q_3'' and q_3'''. Plugging them into (2.26) will lead to the three-loop expression quoted in (1.23).

3. Two-loop term in $\langle W^{(\zeta)} \rangle$ for generic representation

Here we sketch the computation of the non-trivial two-loop term in $\langle W^{(\zeta)} \rangle$ quoted in (1.15). We generalize the calculation in [1] in order to determine the coefficients k_1 and k_2 in (1.14) which applies to a generic representation R of a simple gauge group G.

We can decompose the two-loop contributions to $\langle W^{(\zeta)} \rangle$ into planar ladder diagrams, self-energy diagrams, spider diagrams involving three-vertices, and non-planar ladders, cf figure 1. Using the results in [1] and introducing explicit color factors, we obtain

$$\begin{aligned}\frac{1}{\dim \mathrm{R}} \langle W^{(\zeta)} \rangle = 1 + 2C_{\mathrm{R}}\, \mathrm{W}^{(\zeta)}_{\text{tree}}\, g^2 + \Big[& 4C_{\mathrm{R}}^2\, \mathrm{W}^{(\zeta)}_{\text{planar ladder}} + 2C_{\mathrm{R}}\, C_{\mathrm{A}}\, \mathrm{W}^{(\zeta)}_{\text{self}} + 2C_{\mathrm{R}}\, C_{\mathrm{A}}\, \mathrm{W}^{(\zeta)}_{3-\text{vertex}} \\ & + 4C_{\mathrm{R}}\left(C_{\mathrm{R}} - \frac{1}{2}C_{\mathrm{A}}\right) \mathrm{W}^{(\zeta)}_{\text{non-planar ladder}} \Big]\, g^4 + \cdots .\end{aligned} \tag{3.1}$$

The expressions of all planar pieces in dimensional regularization are [1] (in this appendix we follow the notation of [1] where $d = 2\omega = 4 - 2\epsilon$)

$$\begin{aligned}\mathrm{W}^{(\zeta)}_{\text{tree}} &= \frac{1}{8} - \frac{1}{8}\zeta^2 \epsilon, \qquad \mathrm{W}^{(\zeta)}_{\text{planar ladder}} = \frac{1}{192} + (1-\zeta^2) \\ &\quad \times \left(\frac{1}{64\pi^2 \epsilon} + \frac{1}{128\pi^2}(7 - 3\zeta^2) + \frac{\log(\pi e^{\gamma_E})}{32\pi^2}\right), \\ \mathrm{W}^{(\zeta)}_{\text{self}} &= \zeta^2 \left(-\frac{1}{64\pi^2\epsilon} - \frac{1}{32\pi^2} - \frac{\log(\pi e^{\gamma_E})}{32\pi^2}\right) + (1-\zeta^2) \\ &\quad \times \left(-\frac{1}{64\pi^2\epsilon} - \frac{1}{16\pi^2} - \frac{\log(\pi e^{\gamma_E})}{32\pi^2}\right), \\ \mathrm{W}^{(\zeta)}_{3-\text{vertex}} &= -\mathrm{W}^{(\zeta=1)}_{\text{self}} + (1-\zeta^2)\left(-\frac{1}{64\pi^2\epsilon} - \frac{1}{64\pi^2} - \frac{\log(\pi e^{\gamma_E})}{32\pi^2}\right).\end{aligned} \tag{3.2}$$

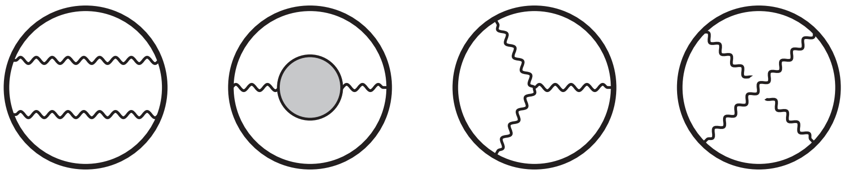


Figure 1. Order λ^2 contributions to the standard Wilson loop. The first three diagrams are planar: (a) ladder diagram; (b) self-energy one-loop correction in SYM theory (with vector, ghost, scalar and fermion fields in the loop); (c) spider type diagram involving the trilinear gauge field self-coupling; (d) non-planar ladder diagram. For the ζ-deformed loop there are additional diagrams with scalar propagators attached to the loop replacing some of the vector ones.

The non-planar ladder contribution is

$$W^{(\zeta)}_{\text{non-planar ladder}} = \frac{[\Gamma(1-\epsilon)]^2}{64\pi^{4-2\epsilon}} \int_{\tau_1>\tau_2>\tau_3>\tau_4} d^4\tau \, \frac{(\zeta^2-\cos\tau_{13})\,(\zeta^2-\cos\tau_{24})}{\left(4\sin^2\frac{\tau_{13}}{2}\ 4\sin^2\frac{\tau_{24}}{2}\right)^{1-\epsilon}}. \tag{3.3}$$

Computing it by the method described in [1], we find

$$W^{(\zeta)}_{\text{non-planar ladder}} = \frac{\zeta^2-1}{64\pi^2\,\epsilon} + \frac{1}{384} + \frac{(\zeta^2-1)(3\zeta^2-7)}{128\pi^2} + (\zeta^2-1)\frac{\log(\pi e^{\gamma_E})}{32\pi^2}. \tag{3.4}$$

Substituting the expressions in (3.2) and (3.4) into (3.1) and also expressing the bare coupling ζ by its renormalized value using the one-loop beta function $\beta_\zeta = -C_A\zeta(1-\zeta^2)\frac{g^2}{8\pi^2} + \cdots$,[18] we finally find the expression in (1.15).

4. 1D path integral for ladder Wilson loop in k-symmetric $SU(N)$ representation

As was mentioned in [4], one may also study the (fundamental) WL renormalization and compute β_ζ using more conventional approach in which path ordering is replaced by a functional integral over the auxiliary 1D fermions ψ_i ($i = 1, \ldots, N$) as in [38, 41, 42]. We will review this representation in appendix B. Then in the ladder approximation when the bulk theory reduces to just a free 4D adjoint scalar field integrating the scalar out leads to an effective theory of $\psi_i(\tau)$ with a non-local 1D action of the form (cf (1.26))

$$S = \int d\tau\, \bar{\psi}_i \partial_\tau \psi^i - \frac{\lambda\zeta^2}{8\pi^2} \int d\tau\, d\tau'\, \bar{\psi}_j(\tau)\,\psi^i(\tau)\,\frac{1}{(\tau-\tau')^2}\,\bar{\psi}_i(\tau')\,\psi^j(\tau'). \tag{4.1}$$

Below we will be interested in the case of k-symmetric $SU(N)$ representation in which a different 1D effective representation in terms of 1D bosons [30, 31] is more convenient (cf also [35]).[19]

[18] That the one-loop term in (1.4) does not depend on representation R follows from a direct inspection of the possible color factors, and using also the condition of the vanishing of beta-function in the abelian limit.

[19] In [30, 31], this representation was discussed in the context of the half-BPS Wilson loop, but it applies the same way to the generalized Wilson loop (1.1) or the purely scalar loop (1.10).

Let us start with the following partition function of periodic bosons χ^i in the fundamental representation of $SU(N)$

$$Z = \int D\chi D\bar{\chi}\ \mathrm{e}^{\mathrm{i}\int_0^{2\pi}\mathrm{d}\tau\mathscr{L}}, \qquad \mathscr{L} = \mathrm{i}\,\bar{\chi}\partial_\tau\chi + \mathrm{i}\,\bar{\chi}\phi(\tau)\chi, \quad \phi = \phi^a T^a. \tag{4.2}$$

In the case of our interest $\phi(\tau)$ will be the adjoint free scalar ϕ of the ladder model restricted to the τ-line (1.10) and (1.11) (up to rescaling by ζ).

In the operator quantization (with $[\chi^j, \bar{\chi}_i] = \delta_i^j$) we have $Z = \mathrm{tr}_\chi[\mathrm{T}\ \exp\mathrm{i}\int \mathrm{d}\tau\mathcal{H}(\tau)]$ where the time dependent local Hamiltonian is $\mathcal{H}(\tau) \equiv \hat{\phi} = -\mathrm{i}\,\bar{\chi}\phi\chi$. Here, time-ordering is interpreted as path-ordering and we have

$$Z = \mathrm{tr}_\chi\left[\mathrm{P}\ \exp\left(\mathrm{i}\int_0^{2\pi}\mathrm{d}\tau\,\hat{\phi}(\tau)\right)\right], \tag{4.3}$$

where the trace is over the Hilbert space of $\chi^i, \bar{\chi}_i$. The state space is built starting from $\chi^i|0\rangle = 0$ and acting with $\bar{\chi}_i$. Z may be written as a sum of partition functions restricted to the subspace where the particle number operator $\nu = \bar{\chi}_i\chi^i$ has fixed value. On the many-particle states with $\nu = k$ the action of $\bar{\chi}T^a\chi$ is the same as that of the generator T^a in the k-symmetric representation (that we will denote as S_k)[20]. Hence, Z computes the sum of all 'Wilson loops' in the k-symmetric representations

$$Z = \sum_{k=0}^{\infty} W_k, \qquad W_k = \mathrm{Tr}_k\,\mathrm{P}\ \exp\left(\int_0^{2\pi}\mathrm{d}\tau\,\phi(\tau)\right). \tag{4.4}$$

To select a particular W_k contribution we may add the constraint on $\bar{\chi}\chi$ with a Lagrange multiplier $A = A(\tau)$ as

$$\mathscr{L} = \mathrm{i}\,\bar{\chi}\partial_\tau\chi + \mathrm{i}\bar{\chi}\phi(\tau)\chi + A(\bar{\chi}\chi - k - \frac{N}{2}). \tag{4.5}$$

The extra constant shift by $\frac{N}{2}$ is due to the choice of Weyl ordering[21]. In what follows we shall use the notation

$$R^2 = k + \frac{N}{2}. \tag{4.6}$$

Note that R^2 appears in the action as the coefficient of the 1D Chern–Simons term $\int \mathrm{d}\tau\,A$, and one may argue as usual that it should not be renormalized since it is quantized.

We shall see shortly that this shift by $\frac{N}{2}$ leads indeed to the correct result for W_k in the simplest case of $\phi = 0$, namely, that it is equal to the dimension $\mathrm{Tr}_k\,1 = \dim\,\mathrm{S}_k = \frac{(N+k-1)!}{k!\,(N-1)!}$ of the k-symmetric representation

$$W_{k,0} = \int D\chi D\bar{\chi}\,\mathrm{e}^{-\int\mathrm{d}\tau\,\bar{\chi}\partial_\tau\chi}\ \delta(\bar{\chi}\chi - R^2) = \dim\,\mathrm{S}_k = \binom{N+k-1}{k}. \tag{4.7}$$

[20] For example, on one-particle state (corresponding to fundamental representation) we have $(\bar{\chi}T^a\chi)\bar{\chi}_i|0\rangle = \bar{\chi}_k(T^a)^k_j\chi^j\bar{\chi}_i|0\rangle = (T^a)^j_i\bar{\chi}_j|0\rangle$.

[21] In the path integral formulation the number operator ν corresponds to $\bar{\chi}\chi - \frac{N}{2}$: if $\chi, \bar{\chi}$ are operators, using the symmetric (Weyl) ordering prescription we have $\nu = \frac{1}{2}(\bar{\chi}_i\chi^i + \chi^i\bar{\chi}_i) - \frac{N}{2}$.

This requires careful definition of the path integral over the Lagrange multiplier A, which can be interpreted as a 1D $U(1)$ gauge field. Indeed, the path integral

$$W_{k,0}=\int D\chi D\bar{\chi}DA\ \exp\left(\mathrm{i}\int_0^{2\pi}\mathrm{d}\tau\ \left[\mathrm{i}\bar{\chi}\partial_\tau\chi+A(\bar{\chi}\chi-R^2)\right]\right) \tag{4.8}$$

is invariant under

$$\chi^i\to\mathrm{e}^{\mathrm{i}\alpha}\chi^i,\qquad \bar{\chi}_i\to\mathrm{e}^{-\mathrm{i}\alpha}\bar{\chi}_i,\qquad A\to A+\partial_\tau\alpha,\quad \alpha=\alpha(\tau). \tag{4.9}$$

The function α compatible with periodic boundary conditions on χ should satisfy $\alpha(2\pi)-\alpha(0)=2\pi n$ where n is an integer, i.e.

$$\alpha(\tau)=\alpha_0(\tau)+n\tau,\quad \alpha_0(2\pi)=\alpha_0(0), \tag{4.10}$$

α_0 corresponds to the 'small' gauge transformation. It allows to gauge fix A to be a constant

$$A=\mu,\quad \mu=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\tau\, A. \tag{4.11}$$

Under the 'large' gauge transformation $\alpha(\tau)=n\tau$, A changes by an integer n. Naively one would expect this to be a symmetry of the path integral (4.8) only if $R^2=k+\frac{N}{2}$ is an integer, which would require N to be even. However, as we shall see below, $\mu\to\mu+n$ is in fact a symmetry for any N, due to an 'anomalous' contribution of the functional determinant coming from integration over χ and $\bar{\chi}$. The redundancy under $\mu\to\mu+n$ can be fixed by restricting the integration over μ to the interval $[0,1]$

$$W_{k,0}=\int_0^1\mathrm{d}\mu\int D\chi D\bar{\chi}\ \exp\left(\mathrm{i}\int_0^{2\pi}\mathrm{d}\tau\ \left[\mathrm{i}\bar{\chi}\partial_\tau\chi+\mu(\bar{\chi}\chi-R^2)\right]\right). \tag{4.12}$$

The functional integral over χ and $\bar{\chi}$ gives $[\det(\mathrm{i}\partial_\tau+\mu)]^{-N}$ where the determinant can be defined as usual with the ζ-function prescription (recall that $\chi(2\pi)=\chi(0)$)

$$\begin{aligned}\det(\mathrm{i}\partial_\tau+\mu)&=\prod_{n=-\infty}^{\infty}(n+\mu)=\mu\prod_{n=1}^{\infty}(\mu^2-n^2)=\mu\prod_{n=1}^{\infty}\frac{n^2-\mu^2}{n^2}\prod_{n=1}^{\infty}(-n^2)\\ &=\frac{\sin(\pi\mu)}{\pi}\prod_{n=1}^{\infty}(-n^2)=\frac{\sin(\pi\mu)}{\pi}\,\mathrm{e}^{\log(-1)\zeta(0)-2\zeta'(0)}=-2\mathrm{i}\ \sin(\pi\mu).\end{aligned} \tag{4.13}$$

This leads to the expected result in (4.7)[22]

$$\begin{aligned}W_{k,0}&=\int_0^1\mathrm{d}\mu\,\mathrm{e}^{-2\pi\mathrm{i}\mu R^2}\,[-2\mathrm{i}\ \sin(\pi\mu)]^{-N}=\int_0^1\mathrm{d}\mu\ \frac{\mathrm{e}^{-2\pi\mathrm{i}k\mu}}{(1-\mathrm{e}^{2\pi\mathrm{i}\mu})^N}\\ &=\begin{pmatrix}R^2+\frac{N}{2}-1\\ R^2-\frac{N}{2}\end{pmatrix}=\begin{pmatrix}N+k-1\\ k\end{pmatrix}.\end{aligned} \tag{4.14}$$

Note that as was claimed above, the integrand here is indeed invariant under $\mu\to\mu+n$.

Before proceeding, let us point out as a side remark that a similar 1D action (4.5), with χ taken to be N anticommuting fermions with antiperiodic boundary conditions, describes instead the Wilson loop in the rank k antisymmetric representation [30] (note that this is different from the fermionic representation of [38, 41, 42] reviewed in the appendix B). Further

[22] Here in computing the integral we use analytic continuation in N.

generalizations with (bosonic or fermionic) χ fields carrying an additional $U(M)$ index and a 1D $U(M)$ gauge field on the defect can also be used to describe Wilson loops in representations corresponding to a general Young tableau.

5. Large *k* perturbative expansion in scalar ladder model

In this section we will work out the large k expansion of the Wilson loop in symmetric representation S_k in the scalar ladder approximation. We will begin with the free theory ($\zeta = 0$) case to explain the strategy of perturbative $1/k$ expansion and then move on to the general $\zeta \neq 0$ case.

5.1. Free theory

Since the parameter k appears only in the combination (4.6), it will be convenient to work out the large k expansion as an expansion in inverse powers $1/R^2$. Thus, our aim will be to reproduce the large R^2 expansion of

$$\begin{aligned} W_{k,0} &= \dim\, S_k = \binom{N+k-1}{k} = \binom{R^2+\frac{N}{2}-1}{R^2-\frac{N}{2}} \\ &= \frac{R^{2(N-1)}}{(N-1)!}\left[1 - \frac{N(N-1)(N-2)}{24\,R^4} + \cdots\right]. \end{aligned} \tag{5.1}$$

Starting with the exact integral representation (4.14) for $W_{k,0}$ we may write it in the form amenable to $1/R^2$ expansion[23]

$$\begin{aligned} W_{k,0} &= \int_{-1/2}^{1/2} \mathrm{d}\mu\, \mathrm{e}^{-2\pi\mathrm{i}\mu R^2}\,[-2\mathrm{i}\,\sin(\pi\mu)]^{-N} \\ &= \left(-\frac{2\pi\mathrm{i}}{R^2}\right)^{-N} \frac{1}{R^2}\int_{-\frac{R^2}{2}}^{\frac{R^2}{2}} \mathrm{d}\mu'\,\mathrm{e}^{-2\pi\mathrm{i}\mu'}\,\mu'^{-N}\left(1+\frac{\pi^2\mu'^2 N}{6R^4}+\cdots\right). \end{aligned} \tag{5.2}$$

Taking R large and thus setting the integration limits to $\pm\infty$,[24] and using the analytic continuation in the integral

$$\int_{-\infty}^{\infty} \mathrm{d}\mu\,\mathrm{e}^{-2\pi\mathrm{i}\mu}\mu^{\alpha} = -\frac{1}{(2\pi)^{\alpha}}\,\mathrm{e}^{\frac{\mathrm{i}\pi\alpha}{2}}\,\frac{\alpha}{\Gamma(1-\alpha)}, \qquad -1<\mathrm{Re}(\alpha)<0, \tag{5.3}$$

we find

$$W_{k,0} = \frac{R^{2N-2}}{\Gamma(N)}\left(1 - \frac{N(N-1)(N-2)}{24R^4} + \cdots\right), \tag{5.4}$$

in agreement with (5.1).

This perturbative procedure has a direct counterpart at the level of the path integral (4.12), i.e. before integrating out $\chi, \bar{\chi}$ in terms of a functional determinant. Once again, we expect to

[23] We shifted μ by -1 which is a symmetry of the integral in (4.14).

[24] The effect of large gauge transformations is not visible in large R^2 perturbation theory so the restriction on the range of μ-integration can be relaxed.

find

$$W_{k,0} = \int_{-1/2}^{1/2} \mathrm{d}\mu \int D\chi D\bar{\chi}\ \mathrm{e}^{\mathrm{i}\int \mathrm{d}\tau\ [\mathrm{i}\bar{\chi}\partial_\tau\chi+\mu(\bar{\chi}\chi-R^2)]} = \frac{1}{R^2}\int_{-\frac{R^2}{2}}^{\frac{R^2}{2}} \mathrm{d}\mu'\ \mathrm{e}^{-2\pi\mathrm{i}\mu'}\left[J(\frac{\mu'}{R^2})\right]^N, \tag{5.5}$$

$$J(\mu) \equiv \int D\chi D\bar{\chi}\ \mathrm{e}^{\mathrm{i}\int \mathrm{d}\tau\ (\mathrm{i}\bar{\chi}\partial_\tau\chi+\mu\,\bar{\chi}\chi)} = [-2\,\mathrm{i}\ \sin(\pi\mu)]^{-1} = \frac{\mathrm{i}}{2\pi\mu} + \frac{\mathrm{i}\pi\mu}{12} + \frac{7\mathrm{i}\pi^3\mu^3}{720} + \cdots, \tag{5.6}$$

where in (5.6) χ is now a singlet field. Let us show how to reproduce (5.6) in small mass expansion. This requires isolating the contribution of the constant zero mode of the ∂_τ kinetic operator, i.e.

$$\chi = n + \chi', \qquad \bar{\chi} = \bar{n} + \bar{\chi}', \qquad \int \mathrm{d}\tau\,\chi' = \int \mathrm{d}\tau\,\bar{\chi}' = 0, \tag{5.7}$$

$$S = \int \mathrm{d}\tau(\mathrm{i}\bar{\chi}\partial_\tau\chi + \mu\bar{\chi}\chi) = \int \mathrm{d}\tau(\mathrm{i}\bar{\chi}'\partial_\tau\chi' + \mu\bar{\chi}'\bar{\chi}' + \mu\,\bar{n}\,n). \tag{5.8}$$

The Gaussian integration over the constants n and $\bar{n}$ gives the $\frac{1}{\mu}$ factor and the rest of the small μ expansion is then regular[25]

$$\begin{aligned} J(\mu) &= \frac{1}{\mu}\int D\chi' D\bar{\chi}'\ \mathrm{e}^{-\int \mathrm{d}\tau\ \bar{\chi}'\partial_\tau\chi'} \\ &\quad\times\left(1 + \mathrm{i}\mu\int \mathrm{d}\tau\bar{\chi}'\chi' - \frac{\mu^2}{2}\int \mathrm{d}\tau\bar{\chi}'\chi'\int \mathrm{d}\tau'\bar{\chi}'\chi' + \cdots\right) \\ &= \frac{\mathrm{i}}{2\pi\mu}\left(1 + \mathrm{i}\mu\,\langle\int \mathrm{d}\tau\bar{\chi}'\chi'\rangle - \frac{\mu^2}{2}\langle\int \mathrm{d}\tau\bar{\chi}'\chi'\int \mathrm{d}\tau'\bar{\chi}'\chi'\rangle + \cdots\right). \end{aligned} \tag{5.9}$$

The expectation values in (5.9) are computed using the propagator for the non-constant mode, i.e.

$$\begin{aligned} \mathscr{D}(\tau) &= \mathscr{D}(\tau + 2\pi) = \langle\chi'(\tau)\bar{\chi}'(0)\rangle = \sum_{\ell\neq 0}\frac{1}{2\pi\mathrm{i}\ell}\,\mathrm{e}^{\mathrm{i}\ell\tau} \\ &= \frac{1}{\mathrm{i}\pi}\sum_{\ell=1}^{\infty}\frac{\sin(\ell\tau)}{\ell}, \qquad \mathscr{D}(\tau) = -\mathscr{D}(-\tau), \end{aligned} \tag{5.10}$$

so that $\mathscr{D}(0) = 0$. The explicit form of $\mathscr{D}$ restricted to the interval $\tau \in (0, 2\pi)$ is

$$\mathscr{D}(\tau) = \mathrm{i}\,\frac{\tau - \pi}{2\pi}, \quad 0 < \tau < 2\pi. \tag{5.11}$$

Thus $\langle\int \mathrm{d}\tau\bar{\chi}'\chi'\rangle = 0$ and

[25] The factor $\frac{\mathrm{i}}{2\pi}$ comes from $\det'(\mathrm{i}\partial_t)^{-1}$: starting from (4.13), removing the zero mode and then sending $\mu \to 0$ one finds $(-2\mathrm{i}\,\frac{\sin(\pi\mu)}{\mu})^{-1} \to \frac{\mathrm{i}}{2\pi}$.

$$\langle\int \mathrm{d}\tau \bar{\chi}'\chi' \int \mathrm{d}\tau' \bar{\chi}'\chi'\rangle = \int_0^{2\pi}\mathrm{d}\tau\int_0^{2\pi}\mathrm{d}\tau' \left[\mathscr{D}(\tau-\tau')\right]^2$$
$$= 2\pi\int_0^{2\pi}\mathrm{d}\tau \, [\mathscr{D}(\tau)]^2 = -\frac{\pi^2}{3}, \tag{5.12}$$

where we used (5.11). As a result, we reproduce (5.2).

5.2. Interacting case

Starting with the ladder scalar model on the circle (4.2), let us make the dependence on the coupling ζ explicit by $\phi \to \zeta\phi$ and integrate out the scalar field. This gives the effective 1D action

$$S = \int \mathrm{d}\tau \left[\mathrm{i}\,\bar{\chi}\partial_\tau\chi + \mu(\bar{\chi}\chi - R^2)\right]$$
$$-\frac{\mathrm{i}\zeta^2 g^2}{8\pi^2}\int \frac{\mathrm{d}\tau\,\mathrm{d}\tau'}{4\sin^2\frac{\tau-\tau'}{2}}\,\bar{\chi}(\tau)T^a\,\chi(\tau)\,\bar{\chi}(\tau')T^a\chi(\tau'), \tag{5.13}$$

where we used the explicit form (1.12) of the scalar propagator restricted to the circle. Let us introduce a compact notation for the integration measure

$$\widehat{\mathrm{d}^2\tau} = \frac{\mathrm{d}\tau\,\mathrm{d}\tau'}{4\sin^2\frac{\tau-\tau'}{2}}. \tag{5.14}$$

The effective coupling that will play a central role below is

$$\varkappa \equiv \frac{\zeta^2 g^2 R^2}{8\pi^2}, \quad R^2 = k + \frac{N}{2}. \tag{5.15}$$

Redefining χ and $\bar{\chi}$ by a factor of R we may then write (5.13) as

$$S = S_2 + S_4 = R^2\int \mathrm{d}\tau\,[\mathrm{i}\,\bar{\chi}\partial_\tau\chi + \mu(\bar{\chi}\chi - 1)]$$
$$-\,\mathrm{i}\,\varkappa R^2\int \widehat{\mathrm{d}^2\tau}\,\bar{\chi}(\tau)T^a\chi(\tau)\,\bar{\chi}(\tau')T^a\chi(\tau'), \tag{5.16}$$

where S_4 stands for the quartic term. As in (5.7) let us separate the constant part of χ as

$$\chi = n + \frac{1}{R}\chi', \qquad \bar{\chi} = \bar{n} + \frac{1}{R}\bar{\chi}', \qquad \int \mathrm{d}\tau\,\chi' = \int \mathrm{d}\tau\,\bar{\chi}' = 0. \tag{5.17}$$

Then

$$S_2 = \int \mathrm{d}\tau \left[\mathrm{i}\,\bar{\chi}'\partial_\tau\chi' + \mu'\left(\bar{n}n - 1 + \frac{1}{R^2}\bar{\chi}'\chi'\right)\right], \quad \mu' = R^2\mu, \tag{5.18}$$

$$S_4 = -\mathrm{i}\,\varkappa R^2\int \widehat{\mathrm{d}^2\tau}\left(\bar{n} + \frac{1}{R}\bar{\chi}'(\tau)\right)T^a\left(n + \frac{1}{R}\chi'(\tau)\right)\left(\bar{n} + \frac{1}{R}\bar{\chi}'(\tau')\right)T^a$$
$$\times\left(n + \frac{1}{R}\chi'(\tau')\right). \tag{5.19}$$

Note that in addition to $1/R^2$ term in (5.18) the action (5.19) contains $1/R$ cubic and $1/R^2$ quartic interaction vertices. Integrating over μ' we get for the resulting path integral measure

$$\int D\chi\, D\bar{\chi} \;\to\; \int D\chi'\, D\bar{\chi}' \int \mathrm{d}n\, \mathrm{d}\bar{n}\; \delta\left(\bar{n}n - 1 + \frac{1}{2\pi R^2}\int \mathrm{d}\tau\, \bar{\chi}'\chi'\right). \tag{5.20}$$

5.2.1. Leading (one-loop) order at large R. Expanding (5.18) and (5.19) at large R^2 for fixed $\varkappa$ we note that at leading order the delta-function in (5.20) imposes that

$$\bar{n}_i n_i = 1. \tag{5.21}$$

Then

$$\begin{aligned} S_2 &= \mathrm{i}\int \mathrm{d}\tau\, \bar{\chi}'\partial_\tau\chi', \\ S_4 &= -\mathrm{i}\,\varkappa \int \widehat{\mathrm{d}^2\tau}\, \big[\bar{n}T^a\chi'(\tau)\; \bar{n}T^a\chi'(\tau') + \bar{n}T^a\chi'(\tau)\; \bar{\chi}'(\tau')T^a n \\ &\quad + \bar{\chi}'(\tau)T^a n\; \bar{n}T^a\chi'(\tau') + \bar{\chi}'(\tau)T^a n\; \bar{\chi}'(\tau')T^a n\big] + \mathcal{O}(R^{-1}). \end{aligned} \tag{5.22}$$

We used the following remarkable property of the measure $\widehat{\mathrm{d}^2\tau}$ in (5.14) (valid in dimensional regularization, or up to power divergences that we will neglect): for a generic function $f(\tau)$

$$\int \widehat{\mathrm{d}^2\tau}\, f(\tau) = 0. \tag{5.23}$$

Using that for the T^a in the fundamental representation

$$(\bar{\alpha}\, T^a\beta)\; (\bar{\gamma}\, T^a\delta) = \frac{1}{2}(\bar{\alpha}\delta)(\bar{\gamma}\delta) - \frac{1}{2N}(\bar{\alpha}\beta)(\bar{\gamma}\delta), \tag{5.24}$$

we then have from (5.19)

$$S = S^{(2)} + \frac{1}{R}S^{(3)} + \frac{1}{R^2}S^{(4)}, \tag{5.25}$$

$$\begin{aligned} S^{(2)} &= \mathrm{i}\int \mathrm{d}\tau\, \bar{\chi}'\partial_\tau\chi' - \frac{\mathrm{i}}{2}\varkappa \int \widehat{\mathrm{d}^2\tau} \\ &\quad \times \left[\left(1 - \frac{1}{N}\right)\big(\chi_i'(\tau')\bar{n}_i\bar{n}_j\chi_j'(\tau) + \bar{\chi}_i'(\tau')n_i n_j\bar{\chi}_j'(\tau)\big)\right. \\ &\quad \left. + 2\bar{\chi}_i'(\tau')\left(\delta_{ij} - \frac{1}{N}n_i\bar{n}_j\right)\chi_j'(\tau)\right], \end{aligned} \tag{5.26}$$

where we used (5.21). The explicit form of the cubic $S^{(3)}$ and quartic $S^{(4)}$ terms in the action will be discussed later. In momentum space representation

$$\chi'(\tau) = \sum_{\ell\in\mathbb{Z}\setminus\{0\}} a(\ell)\, \mathrm{e}^{\mathrm{i}\ell\tau}, \qquad \bar{\chi}'(\tau) = \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \bar{a}(\ell)\, \mathrm{e}^{\mathrm{i}\ell\tau}, \tag{5.27}$$

S_2 in (5.23) becomes

$$\mathrm{i}\int \mathrm{d}\tau\, \bar{\chi}'\partial_\tau\chi' = 2\pi \sum_{\ell\in\mathbb{Z}\backslash\{0\}} \ell\, \bar{a}(\ell)\, a(-\ell). \tag{5.28}$$

Using that[26]

$$\sum_{\ell=1}^{\infty}(-\ell)\cos(\ell\tau) = \frac{1}{4\sin^2\frac{\tau}{2}}, \tag{5.29}$$

we have

$$\begin{aligned}\int \widehat{\mathrm{d}^2\tau}\ \bar{\chi}'_i(\tau')\chi'_j(\tau) &= -2\pi^2\sum_{\ell=1}^{\infty}\ell\ \left[\bar{a}_i(\ell)\,a_j(-\ell)+\bar{a}_i(-\ell)\,a_j(\ell)\right]\\ &= -2\pi^2\sum_{\ell\in\mathbb{Z}\backslash\{0\}}|\ell|\,\bar{a}_i(\ell)a_j(-\ell),\end{aligned} \tag{5.30}$$

and a similar expression for the integral of two χ''s or two $\bar{\chi}'$'s. The resulting quadratic part (5.26) of the total action that determines the leading contribution at large R is

$$\begin{aligned}S^{(2)} = 2\pi\sum_{\ell\in\mathbb{Z}\backslash\{0\}}&\left\{\ell\,\bar{a}_i(\ell)a_j(-\ell)+\frac{\mathrm{i}\pi\varkappa}{2}\,|\ell|\left[\left(1-\frac{1}{N}\right)\right.\right.\\ &\times\left(\bar{n}_i\bar{n}_j a_i(\ell)a_j(-\ell)+n_i n_j\bar{a}_i(\ell)\bar{a}_j(-\ell)\right)+2\left(\delta_{ij}-\frac{1}{N}n_i\bar{n}_j\right)\,\bar{a}_i(\ell)\\ &\left.\left.\times\, a_j(-\ell)\right]\right\} = 2\pi\sum_{\ell\in\mathbb{Z}\backslash\{0\}}\mathrm{A}_u\,(\ell)\,Q_{uv}(\ell)\,\mathrm{A}_v(-\ell),\end{aligned} \tag{5.31}$$

where $\mathrm{A}_u = (a_1,\dots,a_N,\bar{a}_1,\dots,\bar{a}_N)$ and Q_{uv} is the $2N\times 2N$ matrix

$$Q(\ell) = \frac{1}{2}\ell\,\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+\frac{\mathrm{i}\pi\varkappa}{2}\,|\ell|\,\begin{pmatrix}\left(1-\frac{1}{N}\right)n\otimes n & 1-\frac{1}{N}n\otimes\bar{n}\\ 1-\frac{1}{N}\bar{n}\otimes n & \left(1-\frac{1}{N}\right)\bar{n}\otimes\bar{n}\end{pmatrix}. \tag{5.32}$$

Using that $\bar{n}n = 1$ its determinant evaluates to

$$\det\,Q(\ell) = \frac{1}{4^N}\,\ell^2\,\left(\ell^2+\pi^2\varkappa^2|\ell|^2\right)^{N-1} = \frac{1}{4^N}\ell^{2N}(1+\pi^2\varkappa^2)^{N-1}. \tag{5.33}$$

A short-cut way to this result is to use the rotational symmetry of the problem implying that determinant can only depend on length on n_i which is 1 and then to choose this constant vector $n_i = (1,0,\dots,0)$.

[26] This follows, for instance, from $\frac{1}{2}\,\log(1+b^2-2b\,\cos\,\theta) = -\sum_{n=1}^{\infty}\frac{b^n}{n}\,\cos(n\theta)$, after applying $(b\partial_b)^2$ and setting $b = 1$.

Thus the integral over $\mathrm{A}_u = (a_i, \bar{a}_i)$ gives

$$\prod_{\ell\neq 0}[\det\, Q(\ell)]^{-1/2} \propto\ (1+\pi^2\varkappa^2)^{\frac{N-1}{2}}, \tag{5.34}$$

where we used that in the ζ function regularization[27]

$$\prod_{\ell\neq 0} c = \prod_{\ell=1}^{\infty} c^2 = \exp\big(\zeta(0)\log\, c^2\big) = c^{-1}. \tag{5.35}$$

The $\varkappa$-independent proportionality constant in (5.34) and the normalization of the path integral measure can be accounted for at the end by observing that for $\varkappa = 0$ the action (5.16) becomes free and thus the partition function should be given by (4.14) (or its large R expansion in (5.1)) as discussed above.

We thus find for the ladder Wilson loop expectation value

$$\mathrm{W}_k = \dim\, \mathrm{S}_k\big(1+\pi^2\varkappa^2\big)^{\frac{N-1}{2}}\ (1+\Gamma), \quad \Gamma = \Gamma_2+\Gamma_4+\cdots, \quad \Gamma_{2n} = \mathcal{O}(R^{-2n}), \tag{5.36}$$

or, equivalently[28],

$$\log\, \mathrm{W}_k = \log\, \dim\, \mathrm{S}_k + \frac{N-1}{2}\, \log\left(1+\frac{\zeta^4\, g^4\, R^4}{64\pi^2}\right) + \Gamma_2 + \mathcal{O}(R^{-4}), \tag{5.37}$$

where Γ stands for subleading corrections at large R and fixed $\varkappa = \frac{\zeta^2\, g^2\, R^2}{8\pi^2}$.

Using that $R^2 = k + \frac{N}{2}$ and expanding in powers of $\zeta^2 g^2$ we may compare (5.37) with the ladder part of the two-loop expression for the WL expectation value in (1.17). Since for k-symmetric representation of $SU(N)$ one has

$$C_{\mathrm{A}} = N, \qquad C_{\mathrm{R}} = \frac{k(k+N)(N-1)}{2N} \tag{5.38}$$

so that $C_{\mathrm{A}}\, C_{\mathrm{R}} = \frac{1}{2}(N-1)(k^2+Nk)$ and thus we find the agreement with the leading $\zeta^4 g^4$ term in the expansion of (5.37) in both leading and subleading orders in large k expansion.

5.2.2. Propagators for the $\chi', \bar{\chi}'$ fluctuation. To develop perturbation theory in $1/R^2$ starting with (5.18) and (5.19), i.e. to compute the effect of interaction terms that complement the quadratic part of the action (5.31) we need to find the propagators for the corresponding fluctuation fields. Using the covariance with respect to the rotation of the constant vector n_i in (5.17) and (5.21) and of the fluctuation fields we may write the quadratic action (5.26) in the special frame where

$$n = \bar{n} = (0, \ldots, 0, 1). \tag{5.39}$$

Let us label the components of non-constant fluctuation χ'_i as

$$\chi'_i = (\eta_1, \ldots, \eta_{N-1}, \varphi), \qquad \bar{\chi}'_i = (\bar{\eta}_1, \ldots, \bar{\eta}_{N-1}, \bar{\varphi}). \tag{5.40}$$

[27] This derivation of the $(1+\pi^2\varkappa^2)^{\frac{N-1}{2}}$ prefactor in W_k is formally very similar to the one of the Born–Infeld factor in the disc partition function of an open string in external abelian gauge field [43].

[28] In the case of $SU(2)$ equivalent result was announced earlier in [35].

Then the quadratic action (5.26) reads ($r = 1, \ldots, N-1$)

$$S^{(2)} = \mathrm{i}\int \mathrm{d}\tau\ (\bar{\varphi}\partial_\tau\varphi + \bar{\eta}_r\partial_\tau\eta_r) - \mathrm{i}\,\varkappa \int \widehat{\mathrm{d}^2\tau}\left[\left(1-\frac{1}{N}\right)\right.$$
$$\left.\times\left(\frac{1}{2}\varphi(\tau')\,\varphi(\tau) + \frac{1}{2}\bar{\varphi}(\tau')\,\bar{\varphi}(\tau) + \bar{\varphi}(\tau')\,\varphi(\tau)\right) + \bar{\eta}_r(\tau')\ \eta_r(\tau)\right]. \tag{5.41}$$

Going to momentum space, inverting the 2×2 matrix in the $\varphi, \bar{\varphi}$ sector and using (5.28), we find for the corresponding propagators

$$\begin{aligned}
&\mathscr{D}_{\varphi\varphi}(\tau-\tau') = \langle\varphi(\tau)\varphi(\tau')\rangle = \langle\bar{\varphi}(\tau)\bar{\varphi}(\tau')\rangle = -\frac{N-1}{2N}\varkappa\sum_{\ell\neq 0}\frac{1}{|\ell|}\,\mathrm{e}^{\mathrm{i}\ell(\tau-\tau')},\\
&\mathscr{D}_{\bar{\varphi}\varphi}(\tau-\tau') = \langle\bar{\varphi}(\tau)\varphi(\tau')\rangle = \sum_{\ell\neq 0}\left(\frac{\mathrm{i}}{2\pi\ell} + \frac{N-1}{2N}\,\varkappa\,\frac{1}{|\ell|}\right)\,\mathrm{e}^{\mathrm{i}\ell(\tau-\tau')},\\
&\mathscr{D}_{\eta\eta}(\tau-\tau') = \frac{1}{2\pi}\sum_{\ell\neq 0}\frac{\mathrm{i}}{\ell + \mathrm{i}\pi\varkappa\,|\ell|}\,\mathrm{e}^{\mathrm{i}\ell(\tau-\tau')},\\
&\langle\bar{\eta}_r(\tau)\eta_s(\tau')\rangle = \delta_{rs}\mathscr{D}_{\eta\eta}(\tau-\tau').
\end{aligned} \tag{5.42}$$

Computing the sums and restricting to the interval $0 < \tau < 2\pi$ the propagators may be written explicitly as

$$\begin{aligned}
&\mathscr{D}_{\varphi\varphi}(\tau) = \langle\varphi(\tau)\varphi(0)\rangle = \langle\bar{\varphi}(\tau)\bar{\varphi}(0)\rangle = \varkappa\,\frac{N-1}{2N}\ \log\left(4\sin^2\frac{\tau}{2}\right),\\
&\mathscr{D}_{\bar{\varphi}\varphi}(\tau) = \langle\bar{\varphi}(\tau)\varphi(0)\rangle = \frac{1}{2\pi}(\tau-\pi) - \varkappa\,\frac{N-1}{2N}\ \log\left(4\sin^2\frac{\tau}{2}\right),\\
&\mathscr{D}_{\eta\eta}(\tau) = \frac{1}{2\pi}\,\frac{1}{1+\pi^2\varkappa^2}\left[\tau - \pi - \pi\varkappa\,\log\left(4\sin^2\frac{\tau}{2}\right)\right].
\end{aligned} \tag{5.43}$$

Then they can be extended to all τ by periodicity. Note that the linear in τ part is not continuous at $\tau = 0$ where it has a jump. For the corresponding propagators in momentum space we then have[29]

$$\begin{aligned}
&\langle\varphi_p\varphi_q\rangle = \langle\bar{\varphi}_p\bar{\varphi}_q\rangle = -\varkappa\frac{N-1}{2N}\,\frac{1}{|p|}\,\delta_{p+q,0},\\
&\langle\bar{\varphi}_p\varphi_q\rangle = \left(\frac{\mathrm{i}}{2\pi\,p} + \varkappa\,\frac{N-1}{2N}\,\frac{1}{|p|}\right)\,\delta_{p+q,0},\\
&\langle\bar{\eta}_{r,p}\eta_{s,q}\rangle = \delta_{rs}\,\frac{1}{2\pi}\,\frac{\mathrm{i}}{p+\mathrm{i}\pi\,\varkappa\,|p|}\,\delta_{p+q,0}, \qquad r,s = 1,\ldots,N-1.
\end{aligned} \tag{5.44}$$

In the following, it will be convenient to use these propagators in the more general case when $\bar{n}n = u$ (where u is a positive constant) and thus when $n = \bar{n} = \sqrt{u}\,(0,\ldots,0,1)$. Then (5.44)–(5.53) generalize simply by the replacement $\varkappa \to u\varkappa$, cf (5.23) and (5.25).

[29] In all cases $\langle A(\tau)B(\tau')\rangle = \sum_{\ell\neq 0} K_\ell\,\mathrm{e}^{\mathrm{i}\ell(\tau-\tau')}$. Hence $\langle A_pB_q\rangle = \frac{1}{(2\pi)^2}\int \mathrm{d}\tau\,\mathrm{d}\tau'\sum_{\ell\neq 0} K_\ell\,\mathrm{e}^{\mathrm{i}\ell(\tau-\tau')}\,\mathrm{e}^{-\mathrm{i}p\tau-\mathrm{i}q\tau'} = K_p\delta_{p+q,0}$.

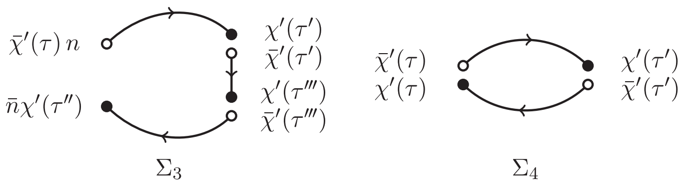


Figure 2. Diagrammatic representation of sample contractions contributing to Σ_3 and Σ_4 in (5.45). All positions are integrated with the non-local measure $\widehat{\mathrm{d}^2\tau}$ defined in (5.14).

5.2.3. $1/R^2$ *order: logarithmic divergence and one-loop beta function.* The next step is to compute the leading $1/R^2$ term Γ_2 in Γ in (5.37). It is given by the sum of the three contributions (which are effectively two-loop ones from the path integral point of view)

$$\Gamma_2 = \mathsf{D} + \Sigma_4 + \Sigma_3. \tag{5.45}$$

Here D is the contribution of the $1/R^2$ term in S_2 in (5.18) or in the delta-function in (5.20). Σ_4 is the contribution of the quartic interaction terms in (5.19) or $S^{(4)}$ in (5.25) and Σ_3 comes from the contraction of two cubic $1/R$ vertices in $S^{(3)}$ in (5.25), see figure 2 for a schematic illustration of the relevant diagrams. We will focus on the logarithmic UV divergent part of (5.45). Its renormalization will determine the leading one-loop $1/R^2$ term in the beta function for $\varkappa$.

D-term. The D-contribution in (5.45) comes from the delta-function constraint in (5.20). Expanding this delta-function in $1/R^2$ gives

$$\begin{aligned}\delta\,(\bar n n - 1 + M) &= \delta(\bar n n - 1) + \delta'(\bar n n - 1)M + \cdots = \delta(\bar n n - 1)\\ &\quad - \frac{\partial}{\partial u}\delta(\bar n n - u)\,M\bigg|_{u=1} + \cdots,\end{aligned} \tag{5.46}$$

$$M \equiv \frac{1}{2\pi R^2}\int \mathrm{d}\tau\, \bar\chi'\chi' = \frac{1}{R^2}\sum_{i=1}^{N}\sum_{\ell\neq 0}\bar\chi_i'(\ell)\chi_i'(-\ell), \tag{5.47}$$

where we introduced an auxiliary parameter u. Then in the subleading term the integration in (5.20) is done with the constraint $\bar n n = u$ with u set to 1 at the end. We get using (5.40)–(5.42)

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{\ell\neq 0}\langle\bar\chi_i'(\ell)\chi_i'(-\ell)\rangle_{\bar n n = u}\\ &\quad= \sum_{\ell\neq 0}\left[\langle\bar\varphi_\ell\varphi_{-\ell}\rangle_{\bar n n=u} + \sum_{r=1}^{N-1}\langle\bar\eta_{r,\ell}\eta_{r,\ell}\rangle_{\bar n n=u}\right]\\ &\quad= \sum_{\ell\neq 0}\left[\frac{\mathrm{i}}{2\pi\,\ell} + u\,\varkappa\,\frac{N-1}{2N\,|\ell|} + \frac{\mathrm{i}(N-1)}{2\pi(\ell + \mathrm{i}\pi\,u\,\varkappa\,|\ell|)}\right]\\ &\quad= \frac{u\,\varkappa(N-1)(N+1+\pi^2u^2\varkappa^2)}{N(1+\pi^2u^2\varkappa^2)}\,I_0,\end{aligned} \tag{5.48}$$

$$I_0 \equiv \sum_{\ell=1}^{\infty} \frac{e^{-\varepsilon\,\ell}}{\ell} = -\log\varepsilon + \mathcal{O}(\varepsilon), \tag{5.49}$$

where we introduced an exponential cut-off in the sum over ℓ. Then the contribution of the correction term in (5.46) is found to be

$$\begin{aligned}\mathsf{D} = &-\frac{1}{R^2} I_0 \frac{1}{(1+\pi^2\varkappa^2)^{\frac{N-1}{2}}} \frac{\partial}{\partial u} \\ &\times \left[u^{N-1}(1+u^2\pi^2\varkappa^2)^{\frac{N-1}{2}} \frac{u\,\varkappa\,(N-1)(N+1+\pi^2u^2\varkappa^2)}{N(1+\pi^2u^2\varkappa^2)} \right]\Bigg|_{u=1}. \end{aligned} \tag{5.50}$$

Here in the square bracket we included the factor of u^{N-1} from the integral over n, i.e. $\int \mathrm{d}n\,\mathrm{d}\bar{n}\,\delta(\bar{n}n-u)$, and the leading $\varkappa$ dependent factor in (5.34) and (5.36) generalized to the present case of $u \neq 1$. Computing the derivative over u at $u=1$ we finish with

$$\mathsf{D} = -\frac{1}{R^2} I_0 \frac{(N-1)\,\varkappa}{N(1+\pi^2\varkappa^2)^2} \left[N(N+1) + \pi^2\varkappa^2(2N^2-1) + \pi^4\varkappa^4(2N-1) \right]. \tag{5.51}$$

Quartic terms. The contribution Σ_4 in (5.45) comes from the quartic terms in (5.19) after expanding e^{iS}

$$\Sigma_4 = i\langle S^{(4)}\rangle, \quad S^{(4)} = -i\frac{\varkappa}{R^2} \int \widehat{d^2\tau}\, [\bar{\chi}'(\tau)T^a\chi'(\tau)]\; [\bar{\chi}'(\tau')T^a\chi'(\tau')]. \tag{5.52}$$

Using again the $SU(N)$ fusion relation (5.24), i.e.

$$\begin{aligned}\bar{\chi}'(\tau)T^a\chi'(\tau)\,\bar{\chi}'(\tau')T^a\chi'(\tau') = &\frac{1}{2}[\bar{\chi}'(\tau')\chi'(\tau)][\bar{\chi}'(\tau)\chi'(\tau')] \\ &- \frac{1}{2N}[\bar{\chi}'(\tau)\chi'(\tau)][\bar{\chi}'(\tau')\chi'(\tau')], \end{aligned} \tag{5.53}$$

we obtain

$$\begin{aligned}\Sigma_4 = \frac{1}{R^2}\frac{N-1}{2N}\,\varkappa \int \widehat{d^2\tau}\, \Big[&[\mathcal{D}_{\bar{\varphi}\varphi}(0)]^2 + \mathcal{D}_{\bar{\varphi}\varphi}(\tau-\tau')\mathcal{D}_{\bar{\varphi}\varphi}(-\tau+\tau') \\ &- 2\mathcal{D}_{\bar{\varphi}\varphi}(0)\mathcal{D}_{\eta\eta}(0) + [\mathcal{D}_{\eta\eta}(0)]^2 + N\mathcal{D}_{\bar{\varphi}\varphi}(-\tau+\tau')\mathcal{D}_{\eta\eta}(\tau-\tau') \\ &+ N\mathcal{D}_{\bar{\varphi}\varphi}(\tau-\tau')\mathcal{D}_{\eta\eta}(-\tau+\tau') - \mathcal{D}_{\eta\eta}(\tau-\tau')\mathcal{D}_{\eta\eta}(-\tau+\tau') \\ &- N\mathcal{D}_{\eta\eta}(\tau-\tau')\mathcal{D}_{\eta\eta}(-\tau+\tau') + N^2\mathcal{D}_{\eta\eta}(\tau-\tau')\mathcal{D}_{\eta\eta}(-\tau+\tau') \\ &+ [\mathcal{D}_{\varphi\varphi}(\tau-\tau')]^2 \Big]. \end{aligned} \tag{5.54}$$

Ignoring constant divergent terms (that drop out after integrating with measure $\widehat{d^2\tau}$, cf (5.23)) and using the symmetry under $\tau \leftrightarrow \tau'$, we get

$$\begin{aligned}\Sigma_4 = \frac{1}{R^2}\frac{N-1}{2N}\,\varkappa \int \widehat{\mathrm{d}^2\tau}\, \big[&\mathscr{D}_{\bar\varphi\varphi}(\tau-\tau')\mathscr{D}_{\bar\varphi\varphi}(-\tau+\tau') \\ &+ 2N\mathscr{D}_{\bar\varphi\varphi}(-\tau+\tau')\mathscr{D}_{\eta\eta}(\tau-\tau') \\ &+ (N^2-N-1)\mathscr{D}_{\eta\eta}(\tau-\tau')\mathscr{D}_{\eta\eta}(-\tau+\tau') + [\mathscr{D}_{\varphi\varphi}(\tau-\tau')]^2\big]\,. \end{aligned} \tag{5.55}$$

Using the translation invariance gives

$$\begin{aligned}\Sigma_4 = \frac{2\pi}{R^2}\frac{N-1}{2N}\,\varkappa \int_0^{2\pi} \frac{\mathrm{d}\tau}{4\sin^2\frac{\tau}{2}}\, \big[&\mathscr{D}_{\bar\varphi\varphi}(\tau)\mathscr{D}_{\bar\varphi\varphi}(2\pi-\tau) \\ &+ 2N\mathscr{D}_{\bar\varphi\varphi}(\tau)\mathscr{D}_{\eta\eta}(2\pi-\tau) \\ &+ (N^2-N-1)\mathscr{D}_{\eta\eta}(\tau)\mathscr{D}_{\eta\eta}(2\pi-\tau) + [\mathscr{D}_{\varphi\varphi}(\tau)]^2\big]\,. \end{aligned} \tag{5.56}$$

The propagators $\mathscr{D}$ in (5.43) have a linear part $\sim \tau-\pi$ and a log part $\sim \log(4\sin^2\frac{\tau}{2})$. Due to parity around $\tau=\pi$ there cannot be crossed contributions. The logarithmic divergences may come only the linear in τ terms[30]. The linear in τ parts are

$$\mathscr{D}^{\mathrm{lin}}_{\eta\eta}(\tau) = \frac{1}{2\pi}\frac{1}{1+\pi^2\varkappa^2}(\tau-\pi), \qquad \mathscr{D}^{\mathrm{lin}}_{\varphi\varphi}(\tau) = 0, \qquad \mathscr{D}^{\mathrm{lin}}_{\bar\varphi\varphi}(\tau) = \frac{1}{2\pi}(\tau-\pi). \tag{5.57}$$

Using (5.29) with mode regularization, i.e. $\frac{1}{4\sin^2\frac{\tau}{2}} \to \sum_{\ell=1}^{\infty} \mathrm{e}^{-\varepsilon\,\ell}(-\ell)\cos(\ell\tau)$ we have (cf (5.49))

$$\begin{aligned} &\int_0^{2\pi} \frac{\mathrm{d}\tau}{4\sin^2\frac{\tau}{2}} = 0, \\ &\int_0^{2\pi} \frac{\mathrm{d}\tau}{4\sin^2\frac{\tau}{2}}\,\tau = 0, \\ &\int_0^{2\pi} \frac{\mathrm{d}\tau}{4\sin^2\frac{\tau}{2}}\,\tau^2 = -4\pi\sum_{\ell=1}^{\infty}\frac{\mathrm{e}^{-\varepsilon\,\ell}}{\ell} = -4\pi I_0, \end{aligned} \tag{5.58}$$

[30] The contributions of purely logarithmic terms in $\mathscr{D}$ are finite. This may be easily shown in dimensional regularization. In mode regularization, where we add a factor $\exp(-\varepsilon\ell)$ to the ℓth Fourier mode [1], this is also true up to a power-like divergence $\frac{\log\varepsilon}{\varepsilon}$. Indeed, using $\log(4\sin^2\frac{\tau}{2}) \to -2\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{e}^{-\varepsilon\,n}\cos(n\tau)$, we have

$$\begin{aligned} \int_0^{2\pi}\frac{\mathrm{d}\tau}{4\sin^2\frac{\tau}{2}}\left[\log 4\left(\sin^2\frac{\tau}{2}\right)\right]^2 &= 4\sum_{n,p,q=1}^{\infty}\mathrm{e}^{-(n+p+q)\varepsilon}\frac{-n}{pq}\int_0^{2\pi}\mathrm{d}\tau\,\cos(n\tau)\cos(p\tau) \\ &\quad\times\cos(q\tau) = 4\left[-\frac{\pi}{2}\sum_{p,q=1}^{\infty}\frac{p+q}{pq}\mathrm{e}^{-2(p+q)\varepsilon} - 2\times\frac{\pi}{2}\sum_{1\leqslant q<p<\infty}\mathrm{e}^{-2p\varepsilon}\frac{p-q}{pq}\right] \\ &= 4\pi\frac{1+(1+\mathrm{e}^{2\varepsilon})\log(1-\mathrm{e}^{-2\varepsilon})}{-1+\mathrm{e}^{2\varepsilon}} \\ &= \frac{2\pi}{\varepsilon}(1+2\,\log\,2+2\,\log\,\varepsilon) - 6\pi + \mathcal{O}(\varepsilon). \end{aligned}$$

and then finally the logarithmically divergent part of Σ_4 is found to be

$$\Sigma_4^{\mathrm{UV}} = \frac{1}{R^2} I_0 \, \frac{(N-1)\,\varkappa}{N\,(1+\pi^2\varkappa^2)^2} \, \left[N(N+1) + 2\pi^2\varkappa^2(N+1) + \pi^4\varkappa^4\right]. \tag{5.59}$$

Combined with (5.51) this gives

$$\mathsf{D} + \Sigma_4^{\mathrm{UV}} = -\frac{\pi^2}{R^2} I_0 \, \frac{(N-1)\,\varkappa^3}{N\,(1+\pi^2\varkappa^2)^2} \left[-3 - 2N + 2N^2 + 2\pi^2\varkappa^2(N-1)\right]. \tag{5.60}$$

Cubic terms. The term Σ_3 in (5.45) is coming from contraction of two cubic vertices $S^{(3)}$ in the action (5.19) and (5.25) (i.e. from the quadratic term in the expansion of $\mathrm{e}^{\mathrm{i}S_4}$).

Explicitly, expanding S_4 in (5.19) near $n = \bar{n} = (1, 0, \ldots, 0)$ using (5.17) and (5.40) gives

$$\begin{aligned} S_4 = &-\mathrm{i}\varkappa R^2 \int \widehat{\mathrm{d}^2\tau} \left(\frac{1}{2}[\bar{\chi}(\tau')\chi(\tau)][\bar{\chi}(\tau)\chi(\tau')] \right.\\ &\left. - \frac{1}{2N}[\bar{\chi}(\tau)\chi(\tau)][\bar{\chi}(\tau')\chi(\tau')]\right) \to -\frac{\mathrm{i}\varkappa}{R}\int \widehat{\mathrm{d}^2\tau} \\ &\times \left[\frac{1}{2}(\bar{\varphi}(\tau') + \varphi(\tau))[\bar{\chi}'(\tau)\chi'(\tau')] + \frac{1}{2}[\bar{\chi}'(\tau')\chi'(\tau)](\bar{\varphi}(\tau) + \varphi(\tau'))\right. \\ &- \frac{1}{2N}(\bar{\varphi}(\tau) + \varphi(\tau))[\bar{\chi}'(\tau')\chi'(\tau')] \\ &\left.- \frac{1}{2N}[\bar{\chi}'(\tau)\chi'(\tau)](\bar{\varphi}(\tau') + \varphi(\tau'))\right] = -\frac{\mathrm{i}\varkappa}{R}\int \widehat{\mathrm{d}^2\tau} \\ &\times \left[(\bar{\varphi}(\tau') + \varphi(\tau))[\bar{\chi}'(\tau)\chi'(\tau')] - \frac{1}{N}(\varphi(\tau) + \bar{\varphi}(\tau))[\bar{\chi}'(\tau')\chi'(\tau')]\right]. \end{aligned} \tag{5.61}$$

Thus (using that $\bar{\chi}' n = \bar{\varphi}$, etc)

$$\begin{aligned} S^{(3)} &= -\frac{\mathrm{i}\varkappa}{R}\int \widehat{\mathrm{d}^2\tau}\left[(\bar{\varphi}(\tau') + \varphi(\tau))[\bar{\varphi}(\tau)\varphi(\tau') + \bar{\eta}(\tau)\eta(\tau')]\right. \\ &\quad \left. - \frac{1}{N}(\varphi(\tau) + \bar{\varphi}(\tau))[\bar{\varphi}(\tau')\varphi(\tau') + \bar{\eta}(\tau')\eta(\tau')]\right] \\ &= -\frac{\mathrm{i}\varkappa}{R}\int \widehat{\mathrm{d}^2\tau}\left[\left(1 - \frac{1}{N}\right)\varphi(\tau)\bar{\varphi}(\tau)[\varphi(\tau') + \bar{\varphi}(\tau')] + \left[\bar{\varphi}(\tau')\right.\right. \\ &\quad \left.\left. + \varphi(\tau)\right]\,\bar{\eta}(\tau)\eta(\tau') - \frac{1}{N}[\varphi(\tau) + \bar{\varphi}(\tau)]\,\bar{\eta}(\tau')\eta(\tau')\right]. \end{aligned} \tag{5.62}$$

For three generic non-constant functions of τ we have the following expression in terms of their Fourier modes

$$\int \widehat{\mathrm{d}^2\tau} A(\tau)B(\tau)C(\tau') = -2\pi^2 \sum_{\substack{\ell,p\in\mathbb{Z}\setminus\{0\} \\ \ell\neq p}} |n|\, A_p B_{\ell-p} C_{-\ell}. \tag{5.63}$$

Hence, introducing the mode regularization factor we get

$$S^{(3)} = -\frac{2\pi^2 \mathrm{i}\varkappa}{R} \sum_{\substack{\ell,p\in\mathbb{Z}\setminus\{0\}\\ \ell\neq p}} |\ell|\, \mathrm{e}^{-\varepsilon\,|\ell|}$$
$$\times \left[\left(1-\frac{1}{N}\right)\varphi_p\bar{\varphi}_{\ell-p}(\varphi_{-\ell}+\bar{\varphi}_{-\ell}) + \bar{\varphi}_p\eta_{\ell-p}\bar{\eta}_{-\ell}\right.$$
$$\left. + \varphi_p\bar{\eta}_{\ell-p}\eta_{-\ell} - \frac{1}{N}\bar{\eta}_p\eta_{\ell-p}(\varphi_{-\ell}+\bar{\varphi}_{-\ell})\right]. \tag{5.64}$$

Taking the expectation value of $[S^{(3)}]^2$ using the momentum space propagators in cf (5.44) we obtain a sum of triple products of propagators, that after the integration can be reduced to a set of double sums. Regulating all infinite sums with an exponential mode cutoff and dropping power-like singular terms of the form $\frac{1}{\varepsilon}$ or $\frac{\log\,\varepsilon}{\varepsilon}$, we find for the UV logarithmically divergent part

$$\Sigma_3^{\mathrm{UV}} = \frac{1}{2}\langle\left[i\,S^{(3)}\right]^2\rangle = \frac{\pi^2}{R^2}\,\log\,\varepsilon\,\frac{(N-1)\,\varkappa^3}{N\,(1+\pi^2\varkappa^2)^2}\left[2N+3-2\pi^2\varkappa^2(N-1)\right]. \tag{5.65}$$

Summing this up with (5.60) gives the total log divergence at order $1/R^2$

$$\Gamma_2 = \mathsf{D} + \Sigma_4^{\mathrm{UV}} + \Sigma_3^{\mathrm{UV}} = \frac{2\pi^2}{R^2}N(N-1)\,\frac{\varkappa^3}{(1+\pi^2\varkappa^2)^2}\,\log\,\varepsilon + \cdots. \tag{5.66}$$

Log divergence and beta-function. Using (5.37) the ladder Wilson loop expectation value is thus

$$\log\,\mathrm{W}_k = \log\,\dim\,\mathrm{S}_k + \frac{N-1}{2}\,\log(1+\pi^2\varkappa^2) + \Gamma_2 + \mathcal{O}(R^{-4}) = \log\,\dim\,\mathrm{S}_k$$
$$+\frac{N-1}{2}\,\log(1+\pi^2\varkappa^2) + \frac{2\pi^2}{R^2}N(N-1)\,\frac{\varkappa^3}{(1+\pi^2\varkappa^2)^2}\,\log\,\varepsilon + \cdots, \tag{5.67}$$

where dots stand for finite parts and higher R^{-4} corrections.

The divergence in (5.67) can be absorbed into renormalization of $\varkappa$ (which is equivalent to renormalization of ζ as this is the only running coupling, cf (2.19))

$$\varkappa \equiv \varkappa_{\mathrm{bare}} \to \varkappa(\mu) - \frac{2N}{R^2}\frac{\varkappa^2(\mu)}{1+\pi^2\varkappa^2(\mu)}\,\log(\mu\varepsilon) + \mathcal{O}(R^{-4}), \tag{5.68}$$

so that the renormalized W_k expressed in terms of renormalized $\varkappa(\mu)$ (cf (1.31)) satisfies

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\varkappa\frac{\partial}{\partial\varkappa}\right)\,\mathrm{W}_k = 0, \tag{5.69}$$

$$\beta_\varkappa = \mu\frac{\mathrm{d}\varkappa}{\mathrm{d}\mu} = \frac{2N}{R^2}\frac{\varkappa^2}{1+\pi^2\varkappa^2} + \mathcal{O}(R^{-4}). \tag{5.70}$$

The corresponding $\varkappa^3 \log \mu \sim \zeta^6 g^6 \log \mu$ term in renormalized $\log \mathrm{W}_k$ is in agreement with the $\zeta^6 g^6 \log \mu$ term in (1.19) and (1.20):

$$-\frac{1}{128\pi^4}\, C_{\mathrm{R}} C_{\mathrm{A}}^2\, \zeta^6 g^6 = -\frac{2\pi^2}{k} N(N-1)\, \varkappa^3 + \cdots . \tag{5.71}$$

Here we used (5.38), (2.28) and expanded at large k with fixed $\varkappa = \frac{1}{8\pi^2}\zeta^2 g^2 R^2$, $R^2 = k + \frac{1}{2}N$. The beta-function (5.70) written in terms of ζ gives

$$\beta_\varkappa \to \beta_\zeta^{\text{ladder}} = \mu \frac{\mathrm{d}\zeta}{\mathrm{d}\mu} = \frac{\zeta^3 N g^2}{8\pi^2 \left(1 + \frac{1}{64\pi^2}\zeta^4 g^4 R^4\right)} + \mathcal{O}(R^{-4}). \tag{5.72}$$

Expanding in small ζ the leading ζ^3 term here is in agreement with the one-loop beta-function in (1.4) (in the large N limit $\lambda = N g^2$). The first correction from the denominator in (5.72) comes only at order $\zeta^7 \lambda^3$.

We also get the following analog of the relation (1.5)

$$\frac{\partial}{\partial \varkappa} \log \, \mathrm{W}_k = \bar{\mathcal{C}}\, \beta_\varkappa, \qquad \bar{\mathcal{C}} = \frac{(N-1)\pi^2 R^2}{2N\varkappa} + \cdots , \tag{5.73}$$

where the leading one-loop term (5.70) in $\beta_\varkappa$ comes directly from the leading finite one-loop term $\frac{N-1}{2}\, \log(1 + \pi^2 \varkappa^2)$ in (5.67).[31]

6. $\beta_\varkappa$-function from two-point correlator on Wilson line

The Wilson line may be viewed as defining a defect 1D CFT with basic correlation functions of local operators inserted on the line defined by (here $\mathcal{W} = \exp[\zeta \int \mathrm{d}\tau \phi]$ is the scalar Wilson factor)

$$\langle\!\langle \mathcal{O}_1(\tau_1) \ldots \mathcal{O}_n(\tau_n) \rangle\!\rangle \equiv \frac{\langle \mathrm{Tr}\, [\mathrm{P}\, \mathcal{O}_1(\tau_1) \ldots \mathcal{O}_n(\tau_n)\, \mathcal{W}] \rangle}{\langle \mathrm{Tr}\, [\mathrm{P}\, \mathcal{W}] \rangle}. \tag{6.1}$$

The corresponding diagrams are shown in figure 3. If we consider the scalar ladder model as a subsector of $\mathcal{N} = 4$ SYM then for the two-point function of a 'transverse' scalar $\phi_\perp$ not coupled to the loop (i.e. not appearing in the Wilson factor $\mathcal{W}$) there is no genuine anomalous dimension, i.e. all divergences in

$$G_\perp(\tau_{12}) = \varkappa\, \langle\!\langle \phi_\perp(\tau_1)\ \phi_\perp(\tau_2) \rangle\!\rangle, \tag{6.2}$$

can be absorbed into ζ or $\varkappa$ only (i.e. no extra Z factor is needed). Thus, the renormalized two-point function should satisfy

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\varkappa \frac{\partial}{\partial \varkappa} \right) G_\perp^{\text{ren}} = 0. \tag{6.3}$$

In this section we discuss how we can use this relation to extract the beta-function $\beta_\varkappa$ from the two-point function $G_\perp^{\text{ren}}$.

This way of deriving $\beta_\varkappa$ has several advantages. First, the propagator on the line is simpler than on the circle, cf (1.12). Second, there are no constant zero modes on the line and thus

[31] An apparent singularity of $\bar{\mathcal{C}}$ in $\varkappa$ is just an artifact of the expression of (5.73) in terms of $\varkappa \sim \zeta^2$ rather than $\sqrt{\varkappa} \sim \zeta$.

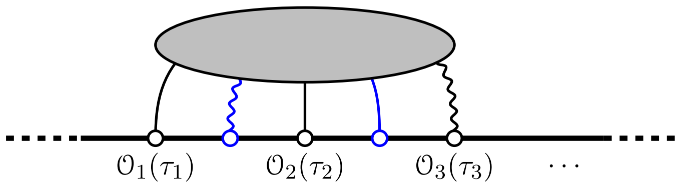


Figure 3. Graphical representation of the defect correlator in (6.1). The horizontal thick line is the Wilson loop. The local operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ are built out of scalars, vectors or fermions and are inserted at positions τ_1, τ_2, τ_3. They are connected to the gray ellipse containing SYM bulk vertices that may be attached to some of the fields coupled to the loop (blue scalar and vector lines ending on the loop in the figure).

it will be possible to treat the delta-function constraint $\bar{\chi}\chi = R^2$ (cf (4.7) in the free case) by solving it directly.

Our starting point will be the bosonic 1D action on the line (cf (5.13); we rescaled χ by R)

$$S = \mathrm{i}R^2 \int \mathrm{d}\tau\, \bar{\chi}\partial_\tau\chi - \mathrm{i}\varkappa R^2 \int \frac{\mathrm{d}\tau\,\mathrm{d}\tau'}{(\tau-\tau')^2}\, \bar{\chi}(\tau)T^a\chi(\tau)\,\bar{\chi}(\tau')T^a\chi(\tau'), \quad \bar{\chi}\chi = 1. \tag{6.4}$$

This action has the same local $U(1)$ invariance as in (4.9)[32]

$$\chi_i \to \mathrm{e}^{\mathrm{i}\alpha(\tau)}\chi_i, \qquad \bar{\chi}_i \to \mathrm{e}^{-\mathrm{i}\alpha(\tau)}\bar{\chi}_i. \tag{6.5}$$

We can use this symmetry to gauge fix χ_N to be real; then solving the constraint we get

$$\chi_N = \bar{\chi}_N = (1 - \bar{\eta}_r\eta_r)^{1/2}, \qquad \eta_r \equiv (\chi_1, \ldots, \chi_{N-1}). \tag{6.6}$$

In the following, we shall use the notation η for the $N-1$ independent components χ_r. The kinetic term in (6.4) becomes simply $\bar{\eta}_r\partial_\tau\eta_r$ (since χ_N is real, it contributes only a total derivative).

The two-point function (6.2) may be written as

$$G_\perp(\tau_{12}) = \varkappa\, \langle\!\langle [\bar{\chi}\phi_\perp\chi(\tau_1)]\; [\bar{\chi}\phi_\perp\chi(\tau_2)] \rangle\!\rangle, \tag{6.7}$$

where the indices of the adjoint scalar $\phi_\perp = \phi_\perp^a T^a$ are contracted with the 1D bosons. The average is done with the effective action (6.4) (already incorporating the effect of the integral over free coupled scalar) and with the free scalar bulk action (1.11) for $\phi_\perp$.[33] Computing first the expectation value with respect to the bulk field $\phi_\perp$ one gets ($\tau_{12} = \tau_1 - \tau_2$)

$$G_\perp(\tau_{12}) = \frac{\varkappa\, g^2}{8\pi^2\tau_{12}^2}\langle \left([\bar{\chi}(\tau_2)\chi(\tau_1)][\bar{\chi}(\tau_1)\chi(\tau_2)] - \frac{1}{N} \right)\rangle, \tag{6.8}$$

where $\langle \ldots \rangle$ is the remaining averaging over 1D bosons χ.

[32] Due to the constraint, the kinetic term is invariant up to an irrelevant total derivative.

[33] One may view (6.7) as originating from the generating functional with the coupling $\zeta\phi + h(\tau)\phi_\perp$ and then differentiating twice over the source function $h(\tau)$.

6.1. One-loop $1/R^2$ contribution

Then writing this in terms of independent $N-1$ components $\eta_r = (\chi_1, \ldots, \chi_{N-1})$ in (6.6) we get

$$G_\perp(\tau_{12}) = \frac{\varkappa\, g^2}{8\pi^2\tau_{12}^2}\, \Big\langle \Big[1 - \frac{1}{N} + \bar\eta(\tau_2)\eta(\tau_1) + \bar\eta(\tau_1)\eta(\tau_2) - \bar\eta(\tau_1)\eta(\tau_1) - \bar\eta(\tau_2)\eta(\tau_2) + \mathcal{O}(\eta^4) \Big] \Big\rangle = \left(1 - \frac{1}{N}\right) \frac{\varkappa\, g^2}{8\pi^2\tau_{12}^2} \times \left[1 + \frac{N}{R^2}\, [\mathscr{D}(\tau_{12}) + \mathscr{D}(-\tau_{12}) - 2\mathscr{D}(0)] + \mathcal{O}(\langle\eta^4\rangle) \right]. \tag{6.9}$$

Here $\mathscr{D}$ is the infinite line analog of the exact propagator (5.42) on the circle that is found from the action (6.4) after using (6.6)[34]

$$\mathscr{D}(\tau) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{i}}{p + \mathrm{i}\,\pi\,\varkappa\,|p|}\, \mathrm{e}^{\mathrm{i}p\tau} = \frac{1}{\pi(1+\pi^2\varkappa^2)} \int_0^\infty \frac{\mathrm{d}p}{p}\, [\pi\,\varkappa\,\cos(p\tau) - \sin(p\tau)]\,. \tag{6.10}$$

While $\mathscr{D}(\tau)$ is singular in the IR (at $p = 0$) the combination appearing in (6.9)

$$\mathscr{D}(\tau) + \mathscr{D}(-\tau) - 2\mathscr{D}(0) = \frac{2\varkappa}{1+\pi^2\varkappa^2} \int_0^\infty \mathrm{d}p\, \frac{\cos(p\tau) - 1}{p} \tag{6.11}$$

is regular at $p = 0$. Its UV divergence at $p \to \infty$ can be regularized with a hard cutoff $|p| < \Lambda$:

$$\int_0^\Lambda \mathrm{d}p\, \frac{\cos(p\tau) - 1}{p} = -\log(\bar\Lambda\tau) + \mathcal{O}(\bar\Lambda^{-1}), \quad \bar\Lambda = \Lambda\, \mathrm{e}^{\gamma_\mathrm{E}}. \tag{6.12}$$

This is equivalent to mode regularization $\mathrm{e}^{-\varepsilon\ell}$ if used in (5.42) after identifying $\varepsilon = \bar\Lambda^{-1}$ (cf also (5.49)), i.e.

$$\int_0^\infty \mathrm{d}p \frac{\cos(p\tau) - 1}{p}\, \mathrm{e}^{-p/\bar\Lambda} = -\frac{1}{2}\, \log\left(1 + \bar\Lambda^2\tau^2\right) = -\log(\bar\Lambda\,\tau) + \mathcal{O}(\bar\Lambda^{-1}). \tag{6.13}$$

Thus we find for the log divergent part of (6.9) (we assume $\tau > 0$)[35]

$$G_\perp(\tau) = \left(1 - \frac{1}{N}\right) \frac{\varkappa\, g^2}{8\pi^2\tau_{12}^2} \times \left[1 - \frac{1}{R^2} f_1^{(1)}(\varkappa) \log(\bar\Lambda\,\tau) + \cdots \right], \quad f_1^{(1)}(\varkappa) = \frac{2N\varkappa}{1+\pi^2\varkappa^2}. \tag{6.14}$$

[34] We explicitly extracted the $\frac{1}{R^2}$ prefactor which is due to the normalization of $\chi_r \equiv \eta_r$ in (6.4); $\langle\bar\eta_r(\tau)\eta_s(0)\rangle = \delta_{rs}\mathscr{D}(\tau)$.

[35] Note that here the UV scale $\bar\Lambda$ enters only together with τ so that there are no IR divergences. Thus we can safely take the limit of the infinite length of the line as in the similar computations in [4].

Then renormalizing $\varkappa$ as in (5.68) we find that the renormalized $G_\perp(\tau)$ satisfies the CS equation (6.3) with

$$\beta_\varkappa = \frac{1}{R^2}\,\varkappa\, f_1^{(1)}(\varkappa) + \mathcal{O}\left(\frac{1}{R^4}\right) = \frac{2N}{R^2}\frac{\varkappa^2}{1+\pi^2\varkappa^2} + \mathcal{O}\left(\frac{1}{R^4}\right), \tag{6.15}$$

which is the same as in (5.70).

6.2. Subleading $1/R^4$ contribution

At the next order we expect to find the following $1/R^4$ corrections in (6.14)

$$\begin{aligned} G_\perp(\tau) = \left(1-\frac{1}{N}\right)\frac{\varkappa g^2}{8\pi^2\tau_{12}^2}\Bigg[1 - \frac{1}{R^2}\, f_1^{(1)}(\varkappa)\,\log(\bar\Lambda\tau) \\ -\frac{1}{R^4}\left(f_0^{(2)}(\varkappa) + f_1^{(2)}(\varkappa)\,\log(\bar\Lambda\tau) + f_2^{(2)}(\varkappa)\log^2(\bar\Lambda\tau)\right) + \cdots\Bigg]. \end{aligned} \tag{6.16}$$

Assuming renormalizability or using the CS equation (6.3) we have (prime is derivative over $\varkappa$)

$$f_2^{(2)} = -\frac{1}{2} f_1^{(1)}\left[f_1^{(1)} + \varkappa (f_1^{(1)})'\right] = -4N^2\,\frac{\varkappa^2}{(1+\pi^2\varkappa^2)^3}, \tag{6.17}$$

$$\beta_\varkappa = \frac{1}{R^2}\,\varkappa\, f_1^{(1)} + \frac{1}{R^4}\,\varkappa\, f_1^{(2)} + \mathcal{O}\left(\frac{1}{R^6}\right), \tag{6.18}$$

where in (6.17) we used the one-loop expression in (6.14).

In (6.16) we assumed that all IR divergences cancel, i.e. the UV cutoff enters together with τ. Thus to check (6.17) and to find the two-loop coefficient $f_1^{(2)}$ we may concentrate on extracting the $\frac{1}{R^4}\,\log\tau$ terms.

To compute corrections to (6.9) we note that in general they come from the following expectation value computed with the effective propagator $\mathscr{D}$ in (6.10) (we again use $\eta \equiv (\chi_r)$, $r = 1, \ldots, N-1$)

$$X = \left\langle \left[(1-\bar\eta(\tau_2)\eta(\tau_2))^{1/2}(1-\bar\eta(\tau_1)\eta(\tau_1))^{1/2} + \bar\eta(\tau_2)\eta(\tau_1)\right][\tau_1 \leftrightarrow \tau_2]\ \mathrm{e}^{\mathrm{i}S_{\mathrm{int}}}\right\rangle, \tag{6.19}$$

where S_{int} contains interacting (higher than quadratic in η) parts of the quartic part of the action in (6.4) after one eliminates χ_N using (6.6). The relevant quartic interaction term in S_{int} is given by

$$\begin{aligned} S_{\mathrm{int}}^{(4)} = -\frac{1}{2}\mathrm{i}\varkappa R^2 \int \frac{\mathrm{d}\tau\,\mathrm{d}\tau'}{(\tau-\tau')^2}\,\{[\bar\eta(\tau)\eta(\tau')][\bar\eta(\tau')\eta(\tau)] - [\bar\eta(\tau)\chi(\tau')] \\ \times [\bar\eta(\tau)\eta(\tau)] - [\bar\eta(\tau')\chi(\tau)][\bar\eta(\tau)\eta(\tau)] + [\bar\eta(\tau')\chi(\tau')][\bar\eta(\tau)\eta(\tau)]\}\,. \end{aligned} \tag{6.20}$$

In general

$$X = X_1 + X_2 + X_3 + \cdots, \tag{6.21}$$

where X_n is given by sums of products of n propagators $\mathscr{D}$. The $1/R^4$ correction will come from either doing contractions of four η in the prefactor in (6.19) between themselves (X_2 term) or from contractions of two η with one power of $S^{(4)}_{\rm int}$ from the expansion of $e^{iS_{\rm int}}$ (X_3 term). We do not need to include disconnected contractions as they cancel against the contributions of the normalization factor in (6.1).

We thus find

$$G_\perp(\tau_{12}) = \left(1-\frac{1}{N}\right)\frac{\varkappa g^2}{8\pi^2\tau_{12}^2}\left[1+\frac{N}{N-1}(X_1+X_2+X_3)+\mathcal{O}\left(\frac{1}{R^6}\right)\right], \tag{6.22}$$

$$X_1 = \frac{1}{R^2}(N-1)\left[\mathcal{D}(\tau_{12})+\mathcal{D}(-\tau_{12})\right], \quad \mathcal{D}(\tau)\equiv\mathscr{D}(\tau)-\mathscr{D}(0), \tag{6.23}$$

$$X_2 = \frac{1}{R^4}N(N-1)\,\mathcal{D}(\tau_{12})\ \mathcal{D}(-\tau_{12}), \tag{6.24}$$

$$X_4 = \varkappa\frac{N(N-1)}{4R^4}\int\frac{d\tau\, d\tau'}{(\tau-\tau')^2}Y_3(\tau,\tau',\tau_{12}), \tag{6.25}$$

where Y_3 is the relevant connected part given by the sum of products of three propagators

$$\begin{aligned} Y_3 = &-2\mathcal{D}(\tau-\tau_{12})\left[\mathcal{D}(-\tau)\,\mathcal{D}(\tau-\tau')+\mathcal{D}(-\tau)\,\mathcal{D}(-\tau+\tau')-2\mathcal{D}(-\tau')\right.\\ &\left.\times\,\mathcal{D}(-\tau+\tau')\right]-2\mathcal{D}(-\tau+\tau_{12})\left[\mathcal{D}(\tau)\,\mathcal{D}(\tau-\tau')-2\mathcal{D}(\tau-\tau')\,\mathcal{D}(\tau')\right.\\ &\left.+\,\mathcal{D}(\tau)\,\mathcal{D}(-\tau+\tau')\right]+2\mathcal{D}(\tau-\tau_{12})\,\mathcal{D}(-\tau+\tau_{12})\\ &\times\left[\mathcal{D}(\tau-\tau')+\mathcal{D}(-\tau+\tau')\right]-4\mathcal{D}(-\tau+\tau_{12})\,\mathcal{D}(-\tau_{12}+\tau')\,\mathcal{D}(\tau-\tau'). \end{aligned} \tag{6.26}$$

All terms in (6.22) are expressed in terms of the shifted propagator $\mathcal{D}(\tau)$ that is regular in the IR (cf (6.10) and (6.11))

$$\begin{aligned} \mathcal{D}(\tau) = R^2\left[\mathscr{D}(\tau)-\mathscr{D}(0)\right] &= \int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{i}{p+i\pi\varkappa|p|}(e^{ip\tau}-1)\\ &= \frac{1}{\pi(1+\pi^2\varkappa^2)}\int_0^\infty\frac{dp}{p}\left(\pi\,\varkappa\left[\cos(p\tau)-1\right]-\sin(p\tau)\right). \end{aligned} \tag{6.27}$$

The terms in the third line of (6.26) become independent of τ_{12} after shifting of τ and τ' by τ_{12} under the integral in (6.25). The remaining terms (in the first and the second line) can be written, using also the symmetry $(\tau,\tau')\leftrightarrow(-\tau,-\tau')$ of (6.25) as

$$\begin{aligned} Y_3 &= -4\left[\mathcal{D}(-\tau)\,\mathcal{D}(\tau-\tau')-\mathcal{D}(-\tau')\,\mathcal{D}(-\tau+\tau')\right]\left[\mathcal{D}(\tau-\tau_{12})+\mathcal{D}(\tau+\tau_{12})\right]\\ &= -4\,\mathcal{D}(-\tau)\mathcal{D}(\tau-\tau')\left[\mathcal{D}(\tau-\tau_{12})+\mathcal{D}(\tau+\tau_{12})-\mathcal{D}(\tau'-\tau_{12})\right.\\ &\quad\left.-\,\mathcal{D}(\tau'+\tau_{12})\right]. \end{aligned} \tag{6.28}$$

Thus (6.22) is given by

$$G_\perp(\tau) = \left(1-\frac{1}{N}\right)\frac{\varkappa\, g^2}{8\pi^2\tau_{12}^2}\left[1-\frac{2N}{R^2}\,\frac{\varkappa}{1+\pi^2\varkappa^2}\,\log(\bar{\Lambda}\tau)+\frac{N^2}{R^4}\,\mathcal{D}(\tau_{12})\right.$$
$$\left.\times\,\mathcal{D}(-\tau_{12})+\frac{N^2}{4R^4}\,\varkappa\int\frac{\mathrm{d}\tau\,\mathrm{d}\tau'}{(\tau-\tau')^2}\,Y_3(\tau,\tau',\tau_{12})+\mathcal{O}\left(\frac{1}{R^6}\right)\right]. \tag{6.29}$$

The computation of the logarithmically divergent part of the $1/R^4$ correction in the second line of (6.29) is quite non-trivial and is presented in appendix C. Here we just quote the result

$$G_\perp(\tau) = \left(1-\frac{1}{N}\right)\frac{\varkappa\, g^2}{8\pi^2\tau_{12}^2}\left[1-\frac{2N}{R^2}\,\frac{\varkappa}{1+\pi^2\varkappa^2}\,\log(\bar{\Lambda}\tau)+\frac{N^2}{R^4}\right.$$
$$\left.\times\,\frac{4\varkappa^2}{(1+\pi^2\varkappa^2)^3}\log^2(\bar{\Lambda}\tau)+\frac{N^2}{R^4}\,\frac{2\,\varkappa^2\,(1-\mathrm{b}_1\,\pi^2\varkappa^2)}{(1+\pi^2\varkappa^2)^3}\,\log(\bar{\Lambda}\tau)+\cdots\right]. \tag{6.30}$$

The coefficient b_1 in general is scheme dependent; in the momentum cutoff scheme we found that (see appendix C)

$$\mathrm{b}_1 = 1. \tag{6.31}$$

The $\log^2$ term obeys the RG condition (6.17) while the log term leads to the two-loop term in the beta-function (6.18)

$$\beta_\varkappa = \frac{2N}{R^2}\,\frac{\varkappa^2}{1+\pi^2\varkappa^2}-\frac{2N^2}{R^4}\,\frac{\varkappa^3\,(1-\mathrm{b}_1\,\pi^2\varkappa^2)}{(1+\pi^2\varkappa^2)^3}+\mathcal{O}\left(\frac{1}{R^6}\right). \tag{6.32}$$

As already discussed below (1.29) the lowest $\varkappa^3$ term in the $1/R^4$ correction corresponds precisely to the ζ^5 term in the two-loop ladder beta function for ζ in (1.4) and (1.23).

6.3. Comments on scheme dependence and three-loop $\beta_\zeta^{\text{ladder}}$ in general representation

Let us comment on the scheme dependence of the beta-function (6.32). In general, in this one-coupling theory (with only ζ or $\varkappa$ running and expansion going in powers of $\hbar=\frac{1}{R^2}$) the scheme freedom should correspond to coupling $\varkappa$ redefinitions

$$\varkappa\to\varkappa+\frac{1}{R^2}q_1(\varkappa)+\cdots, \tag{6.33}$$

$$\beta_\varkappa=\mu\frac{\mathrm{d}\varkappa}{\mathrm{d}\mu}=\frac{1}{R^2}b_1(\varkappa)+\frac{1}{R^4}b_2(\varkappa)+\cdots\to\beta_\varkappa$$
$$+\frac{1}{R^4}\left[q_1(\varkappa)\,b_1'(\varkappa)-b_1(\varkappa)\,q_1'(\varkappa)\right]+\cdots. \tag{6.34}$$

Thus unless $q(\xi)$ is exactly proportional to the one-loop beta function term $b_1(\xi)$ (as it happens in simplest cases of one-coupling theories) the two-loop $1/R^4$ term is not, in general, invariant. For example, considering small $\varkappa$ expansion, with $q_1=c_1\varkappa^2+c_2\varkappa^4+\cdots$ and using that the one-loop term in (6.32) is $b_1=2N(\varkappa^2-\pi^2\varkappa^4+\cdots)$ we find that $q_1b_1'-b_1q_1'=-4N(c_2+\pi^2c_1)\varkappa^5+\cdots$.

Thus while the coefficient of the leading $\varkappa^3$ term in the two-loop correction in (6.32) is invariant, the coefficient b_1 of the first subleading $\varkappa^5$ term is, in general, scheme dependent. At the same time the denominator $(1+\pi^2\varkappa)^{-3}$ structure originating from $(1+\pi^2\varkappa^2)^{-1}$ factors in the exact propagator (6.27) appears to be universal (at least in a natural class of regularization schemes that do not substantially modify the structure of (6.11) and (6.27)).

Next, let us elaborate on the implications of the structure of $\beta_\varkappa$ in (6.32) (see comments below (5.70)). Using the definition of $\varkappa$ we may turn (6.32) into a perturbative large $R^2 \sim k$ expansion of $\beta_\zeta^{\text{ladder}}$

$$\beta_\zeta^{\text{ladder}} = \frac{N}{2}\zeta^3\frac{g^2}{4\pi^2} - \frac{N^2}{4}\zeta^5\left(\frac{g^2}{4\pi^2}\right)^2 - \frac{\pi^2 N k^2}{8}\,\zeta^7\left(\frac{g^2}{4\pi^2}\right)^3 + \cdots. \tag{6.35}$$

Note that three loop $g^6\zeta^7$ term in (6.35) comes entirely from the expansion of the denominator in the first term in (6.32) or from (5.72). Indeed, the $1/R^4$ term in (6.32) produces only $\zeta^5 g^4 + \zeta^9 g^8 + \cdots$ terms. Comparing with (2.31) for general N fixes the coefficients there as

$$q_3'' = 0, \qquad q_3 q_3''' = -3\,\zeta(2) = -\frac{1}{2}\pi^2. \tag{6.36}$$

We thus obtain the following three-loop ladder beta function for general representation (cf (1.21) and (1.22))

$$\begin{aligned}\beta_\zeta^{\text{ladder}} &= \frac{1}{2}\,C_{\mathrm{A}}\zeta^3\,\frac{g^2}{4\pi^2} - \frac{1}{4}\,C_{\mathrm{A}}^2\,\zeta^5\left(\frac{g^2}{4\pi^2}\right)^2\\ &\quad + \left(q_3'\,C_{\mathrm{A}}^3 - 3\,\zeta(2)\,Q_{\mathrm{R}}\right)\,\zeta^7\left(\frac{g^2}{4\pi^2}\right)^3 + \mathcal{O}(g^8).\end{aligned} \tag{6.37}$$

We will prove in appendix D that in the planar limit, for any irreducible representation R of $SU(N)$ the coefficient Q_{R} of the $\zeta(2)\zeta^7$ term in (6.37) is universal, i.e. one has ($\lambda = g^2 N$)

$$\beta_\zeta^{\text{ladder}} = \frac{1}{2}\,\zeta^3\,\frac{\lambda}{4\pi^2} - \frac{1}{4}\,\zeta^5\left(\frac{\lambda}{4\pi^2}\right)^2 + \left(q_3' - \frac{\zeta(2)}{8}\right)\,\zeta^7\left(\frac{\lambda}{4\pi^2}\right)^3 + \mathcal{O}(\lambda^4). \tag{6.38}$$

Comparing with (1.8), we see that the Q_{R} term in (6.37) corresponds to the $\zeta(2)$ transcendental part of the coefficient $q_3 = \frac{1}{4} - \frac{\zeta(2)}{8}$ in (1.8).

This agreement is remarkable given that the three loop beta function is, in general, expected to be scheme dependent. Indeed, the expansion (1.8) has been derived in dimensional regularization while (6.32) and (6.35) have been obtained in a mode regularization. This suggests that only q_3' term in (6.37) is actually scheme dependent while q_3'''' in (6.36) is scheme independent. An explanation of this scheme independence is that this coefficient comes from the κ^4 term in the expansion of the one-loop term in $\beta_\varkappa$ in (1.29), i.e. from the first scheme-independent term in the perturbative $1/k$ expansion.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. *SU*(*N*) conventions

For the $SU(N)$ generators in the fundamental representation we have ($a = 1, \ldots, N^2 - 1$; $i = 1, \ldots, N$)

$$\begin{aligned}
[T^a, T^b] &= i\, f^{abc}\, T^c, \\
\mathrm{Tr}\, T^a &= 0, \\
\mathrm{Tr}\,(T^a T^b) &= \frac{1}{2}\delta^{ab}, \\
(T^a T^a)_{ij} &= \frac{N^2-1}{2N}\,\delta_{ij},
\end{aligned} \tag{A.1}$$

$$T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right), \qquad f^{acd} f^{bcd} = N\delta^{ab}. \tag{A.2}$$

Then also

$$\begin{aligned}
\mathrm{Tr}(T^aT^aT^bT^b) &= \frac{1}{2}\mathrm{Tr}(1)\,\mathrm{Tr}(T^bT^b) - \frac{1}{2N}\,\mathrm{Tr}(T^bT^b) \\
&= \left(\frac{N}{2} - \frac{1}{2N}\right)\frac{N^2-1}{2} = \frac{(N^2-1)^2}{4N},
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\mathrm{Tr}(T^aT^bT^aT^b) &= \frac{1}{2}\,\mathrm{Tr}(T^b)\,\mathrm{Tr}(T^b) - \frac{1}{2N}\,\mathrm{Tr}(T^bT^b) \\
&= -\frac{1}{2N}\frac{N^2-1}{2} = -\frac{N^2-1}{4N},
\end{aligned} \tag{A.4}$$

$$\mathrm{Tr}(T^aT^bT^bT^a) = \mathrm{Tr}(T^aT^aT^bT^b) = \frac{(N^2-1)^2}{4N}. \tag{A.5}$$

For a generic representation R we define the index C_{R} by

$$T^aT^a = C_{\mathrm{R}}\,\mathbf{1}, \quad \mathrm{Tr}(T^aT^a) = C_{\mathrm{R}}\,\dim \mathrm{R}. \tag{A.6}$$

In the special case of the fundamental representation

$$\mathrm{Tr}(T^aT^a) = \frac{1}{2}(N^2-1) \to C_{\mathrm{F}} = \frac{N^2-1}{2N}. \tag{A.7}$$

For the adjoint representation $(T^a_{\mathrm{adj}})_{bc} = -if_{abc}$ so from (A.2) we have $C_{\mathrm{A}} = N$.

For the k-symmetric representation S_k

$$\dim \mathrm{S}_k = \binom{N+k-1}{k}, \qquad C_{\mathrm{S}_k} = \frac{k(N-1)(N+k)}{2N}. \tag{A.8}$$

Let us note also the following relation[36]

$$T^a T^b T^a = \left(C_\mathrm{R} - \frac{1}{2}C_\mathrm{A}\right) T^b. \tag{A.9}$$

Also, if X is some matrix (e.g. a product of some generators) then

$$T^a T^b XT^a T^b = T^a T^b XT^b T^a - \frac{1}{2}C_\mathrm{A}\, T^a XT^a. \tag{A.10}$$

Useful examples are

$$\mathrm{Tr}(T^a T^a T^b T^b) = C_\mathrm{R}^2 \dim \mathrm{R}, \tag{A.11}$$

$$\mathrm{Tr}(T^a T^b T^a T^b) = \left(C_\mathrm{R} - \frac{1}{2}C_\mathrm{A}\right) \mathrm{Tr}(T^b T^b) = \left(C_\mathrm{R} - \frac{1}{2}C_\mathrm{A}\right) C_\mathrm{R} \dim \mathrm{R}. \tag{A.12}$$

Appendix B.1 D fermionic representation for the Wilson loop

The Wilson loop admits a 1D fermionic representation [38, 41, 42] that we will review here for the case of a general representation of gauge group. We start with the path-ordered exponential

$$U_\mathrm{ab} = \left[\mathrm{P}\,\exp \int_{\tau_1}^{\tau_2} F(\tau)\right]_\mathrm{ab}, \tag{B.1}$$

where $F(\tau) = F^a(\tau) T^a$ is a Lie algebra valued function in the representation R (with the corresponding indices being a, b). We can write

$$\begin{aligned} U_\mathrm{ab} &= \mathrm{e}^{\Omega} \left[\frac{\delta^2}{\delta \bar{u}_\mathrm{a}(\tau_2)\, \delta u_\mathrm{b}(\tau_1)} \right. \\ &\quad \left. \times \exp\left(\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \frac{\delta}{\delta u_\mathrm{c}(\tau)} F_{\mathrm{cc}'}(\tau) \frac{\delta}{\delta \bar{u}_{\mathrm{c}'}(\tau)}\right)\right]_{u=\bar{u}=0} \mathrm{e}^{-\Omega}, \\ \Omega &= \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \mathrm{d}\tau'\ \bar{u}_\mathrm{c}(\tau) u_\mathrm{c}(\tau')\, \theta(\tau - \tau'), \end{aligned} \tag{B.2}$$

where u and $\bar{u}$ are (Grassmann) vectors of R and θ is the step function. Ω admits the following representation in terms of path integral over anticommuting fields ψ_a and $\bar{\psi}_\mathrm{a}$ (which are vectors in the representation R) with the antiperiodic boundary condition $\psi(\tau_2) = -\psi(\tau_1)$[37]

$$\mathrm{e}^{-\Omega} = \int D\psi D\bar{\psi}\, \exp \int \mathrm{d}\tau\ \left[\bar{\psi}\partial_\tau \psi + \mathrm{i}(\bar{u}\psi + \bar{\psi} u)\right]. \tag{B.3}$$

[36] For other similar relations see, for instance, section 3.1 of [44].
[37] This follows from $\theta(\tau)$ being the propagator associated with the first order kinetic term $\partial_\tau \theta(\tau) = \delta(\tau)$.

Then for a closed loop the trace of U in (B.2) may be written as

$$\mathrm{Tr}\,U = \frac{\delta^2}{\delta\bar{u}_a(0)\delta u_a(0)}\left[\log\int D\psi D\bar{\psi}\right.$$
$$\left.\times\ \exp\int \mathrm{d}\tau\,\left(\bar{\psi}\partial_\tau\psi - \bar{\psi}F\psi + \mathrm{i}(\bar{u}\psi + \bar{\psi}u)\right)\right]_{u=\bar{u}=0}. \tag{B.4}$$

Equation (B.4) represents the two-point function $\langle\bar{\psi}(2\pi)\,\psi(0)\rangle$ whose perturbative expansion is expressed in terms of factors of $F(\tau)$ connected by z-propagators, i.e. by theta functions that implement path-ordering.

As an example, let us consider the scalar ladder model. Integrating the free scalar field we get the corresponding 1D effective action

$$S = \int \mathrm{d}\tau\,\bar{\psi}\partial_\tau\psi + \frac{1}{2}\zeta^2\int \mathrm{d}\tau\,\mathrm{d}\tau'\ D(\tau-\tau')\bar{\psi}(\tau)T^a\psi(\tau)\,\bar{\psi}(\tau')T^a\psi(\tau'). \tag{B.5}$$

Here $D(\tau-\tau') = \langle\phi(\tau)\phi(\tau')\rangle$ is the scalar propagator restricted to the line. Introducing an auxiliary 1D field $\sigma_a(\tau)$ we may write the corresponding ladder WL expectation value as

$$\mathrm{W} = \left\langle \mathrm{Tr}\left[\frac{1}{\partial_\tau - \sigma_a(\tau)\,T^a}\right]\right\rangle, \tag{B.6}$$

where $\langle\ldots\rangle$ amounts to Wick contractions of the free fields $\sigma_a(\tau)$ with the propagator $\langle\sigma_a(\tau)\sigma_b(\tau')\rangle = \delta_{ab}D(\tau-\tau')$. This reconstructs the standard perturbative evaluation of the Wilson loop like (1.1) or (1.10).

The same steps may be repeated in the case of the circular $\frac{1}{2}$-BPS loop in $\mathcal{N}=4$ SYM where the function F can be read off from (1.1) with $\zeta=1$. Assuming interaction terms in the SYM action do not contribute (as turns out to be true) and integrating out the free scalar and vector fields we obtain

$$\langle W^{(1)}\rangle = \frac{\int D\psi D\bar{\psi}\ \bar{\psi}(2\pi)\,\psi(0)\,\exp\left[\int \mathrm{d}\tau\,\bar{\psi}\partial_\tau\psi + \frac{g^2}{16\pi^2}\left(\int \mathrm{d}\tau\,\bar{\psi}T^a\psi\right)^2\right]}{\int D\psi D\bar{\psi}\ \exp\left[\int \mathrm{d}\tau\,\bar{\psi}\partial_\tau\psi + \frac{g^2}{16\pi^2}\left(\int \mathrm{d}\tau\,\bar{\psi}T^a\psi\right)^2\right]}. \tag{B.7}$$

We used that here the effective propagator corresponding to the combination $(AA+\phi\phi)^{ab}$ is constant [18], i.e. $D(\tau) = D_0 = \frac{g^2\delta^{ab}}{8\pi^2}$. Introducing an auxiliary constant field σ_a, we may write the quartic action in a local form

$$\int \mathrm{d}\tau\,\bar{\psi}\partial_\tau\psi + \frac{g^2}{16\pi^2}\left(\int \mathrm{d}\tau\,\bar{\psi}(\tau)T^a\psi(\tau)\right)^2$$
$$\to \int \mathrm{d}\tau\,\bar{\psi}\partial_\tau\psi - \frac{g}{2\pi}\sigma_a\int \mathrm{d}\tau\bar{\psi}(\tau)T^a\psi(\tau) - \sigma_a^2. \tag{B.8}$$

Integrating out the fermions $\psi,\bar{\psi}$, we then obtain another equivalent representation

$$\langle W^{(1)}\rangle = \left\langle \mathrm{Tr}\left[\frac{1}{\partial_\tau - \frac{g}{2\pi}\sigma_a\,T^a}\right]\right\rangle, \qquad \langle\ldots\rangle = \int\prod_a \mathrm{d}\sigma_a\,\mathrm{e}^{-\sigma_a^2}\ldots \tag{B.9}$$

Let us show how (B.9) can be used to reproduce the perturbative expansion in (1.13). We expand the trace using $\langle\tau_2|(\partial_\tau)^{-1}|\tau_1\rangle = \theta(\tau_2-\tau_1)$. For instance,

$$\langle 2\pi|(\partial_\tau)^{-3}|0\rangle = \int_0^{2\pi} \mathrm{d}\tau\,\mathrm{d}\tau'\,\theta(\tau-0)\theta(\tau'-\tau)\theta(2\pi-\tau') = \int_{\tau<\tau'} \mathrm{d}^2\tau. \quad \text{(B.10)}$$

We then obtain

$$\mathrm{Tr}\left[\frac{1}{\partial_\tau - \frac{g}{2\pi}\sigma_a T^a}\right] = \dim \mathrm{R} + \mathrm{Tr}(T^aT^b)\sigma_a\sigma_b\frac{g^2}{4\pi^2}\int_{\tau_1<\tau_2}\mathrm{d}^2\tau + \mathrm{Tr}\left(T^aT^bT^c \right.$$
$$\left. \times T^d\right)\sigma_a\sigma_b\sigma_c\sigma_d\left(\frac{g^2}{4\pi^2}\right)^2\int_{\tau_1<\ldots<\tau_4}\mathrm{d}^4\tau + \cdots \quad \text{(B.11)}$$

Taking the average using that

$$\langle\sigma_a\sigma_b\rangle = \frac{1}{2}\delta_{ab}, \qquad \langle\sigma_a\sigma_b\sigma_c\sigma_d\rangle = \frac{1}{4}(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}), \quad \text{(B.12)}$$
$$\mathrm{Tr}(T^aT^b)\delta_{ab} = \mathrm{Tr}(T^aT^a) = \dim \mathrm{R}\, C_\mathrm{R}, \quad \text{(B.13)}$$

$$\mathrm{Tr}(T^aT^bT^cT^d)(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}) = 2\,\mathrm{Tr}(T^aT^aT^bT^b) + \mathrm{Tr}(T^aT^bT^aT^b)$$
$$= \dim\left[2C_\mathrm{R}^2 + C_\mathrm{R}\left(C_\mathrm{R} - \frac{1}{2}C_\mathrm{A}\right)\right], \quad \text{(B.14)}$$

$$\int_{\tau_1<\tau_2}\mathrm{d}^2\tau = 2\pi^2, \qquad \int_{\tau_1<\ldots<\tau_4}\mathrm{d}^4\tau = \frac{2}{3}\pi^4, \quad \text{(B.15)}$$

we find that

$$\frac{1}{\dim \mathrm{R}}\langle W^{(1)}\rangle = 1 + \frac{1}{4}C_\mathrm{R}\,g^2 + \frac{1}{192}C_\mathrm{R}(6C_\mathrm{R}-C_\mathrm{A})\,g^4 + \mathcal{O}(g^6), \quad \text{(B.16)}$$

which is in agreement with (1.13).

Appendix C. Computation of divergent part of $1/R^4$ term in scalar two-point function

To find the divergent part of the $\int \mathrm{d}\tau\,\mathrm{d}\tau'(\tau-\tau')^{-2}Y_3$ term in (6.29) we shall use somewhat eclectic direct cutoff method. First, let us introduce a UV cutoff $a\to 0$ in the $(\tau-\tau')^{-2}$ kernel (which originated from the 4D scalar propagator restricted to the line) as

$$\frac{1}{(\tau-\tau')^2} \to \frac{1}{(\tau-\tau')^2+a^2} = \int_{-\infty}^{\infty}\mathrm{d}p\,\frac{\mathrm{e}^{-a\,|p|}}{2a}\,\mathrm{e}^{\mathrm{i}p(\tau-\tau')}, \quad a\to 0. \quad \text{(C.1)}$$

Using the expression for the propagator $\mathcal{D}$ in (6.27) in Y_3 in (6.28) and integrating over τ, τ' we then obtain[38]

$$\bar{Y}_3 \equiv \int \frac{\mathrm{d}\tau\,\mathrm{d}\tau'}{(\tau-\tau')^2+a^2}\, Y_3 = \int_0^\infty \mathrm{d}p_1 \int_0^\infty \mathrm{d}p_2\, f(p_1,p_2), \tag{C.2}$$

$$f(p_1,p_2)\Big|_{p_1<p_2} = \frac{4\mathrm{e}^{-a(p_1+p_2)}(1-\mathrm{e}^{ap_1})\varkappa\cos(p_1\tau_{12})}{a(1+\pi^2\varkappa^2)^3\, p_1^2 p_2} \times \left[1+\pi^2\varkappa^2+\mathrm{e}^{ap_1}(3-\pi^2\varkappa^2)+2\,\mathrm{e}^{ap_2}(1-\pi^2\varkappa^2)\right].$$

Introducing an extra hard momentum cutoff $p_1, p_2 < \Lambda$ (which we will later relate to $1/a$) and integrating over p_2 we get

$$\begin{aligned}\bar{Y}_3 = \int_0^\Lambda \mathrm{d}p_1\, \frac{4\varkappa\cos(p_1\tau_{12})}{a(1+\pi^2\varkappa^2)^3 p_1^2}\, &\left[\mathrm{e}^{-ap_1}(1-\mathrm{e}^{ap_1})\left[1+\pi^2\varkappa^2+\mathrm{e}^{ap_1}(3-\pi^2\varkappa^2)\right]\right. \\ &\times \mathrm{Ei}(-a\Lambda)+\mathrm{e}^{ap_1}(3-\pi^2\varkappa^2)\mathrm{Ei}(-ap_1)+\mathrm{e}^{-ap_1}(-3+\pi^2\varkappa^2)\mathrm{Ei}(ap_1) \\ &\left.+2\,\mathrm{e}^{-ap_1}(-1+\mathrm{e}^{ap_1})(-1+\pi^2\varkappa^2)\log(a\bar{\Lambda})\right],\end{aligned} \tag{C.3}$$

where $\mathrm{Ei}(z) = -\int_{-z}^\infty \mathrm{d}t\, \frac{\mathrm{e}^{-t}}{t}$, and $\bar{\Lambda} = \Lambda\,\mathrm{e}^{\gamma_\mathrm{E}}$. To perform the last integration over p_1 we consider the integrand in the limit $\Lambda \to \infty$, $a \to 0$. Dropping power divergent terms $\sim 1/a$ and integrating over $p_1 < \Lambda$ we find that the terms depending on τ_{12} are

$$\begin{aligned}\frac{\varkappa}{4}\bar{Y}_2 = &-\frac{\varkappa^2(-3+\pi^2\varkappa^2)}{(1+\pi^2\varkappa^2)^3}\log^2(\bar{\Lambda}\tau_{12}) \\ &-\frac{2\varkappa^2\,(-5+2\,\log(a\bar{\Lambda})+\pi^2\varkappa^2)}{(1+\pi^2\varkappa^2)^3}\log(\bar{\Lambda}\tau_{12})+\cdots.\end{aligned} \tag{C.4}$$

To this we need to add the contribution of the X_2 term in (6.24) or $\mathcal{D}(\tau_{12})\mathcal{D}(-\tau_{12})$ in (6.29). Introducing the same momentum cutoff Λ in the propagators $\mathcal{D}$ (6.27) ($\int_0^\infty \mathrm{d}p \to \int_0^\Lambda \mathrm{d}p$) and integrating over p we get for $\tau > 0$

$$\mathcal{D}(\pm\tau) = -\frac{\varkappa}{1+\pi^2\varkappa^2}\log(\bar{\Lambda}|\tau|) \mp \frac{1}{2(1+\pi^2\varkappa^2)} + \mathcal{O}(\Lambda^{-1}), \tag{C.5}$$

$$\mathcal{D}(\tau_{12})\mathcal{D}(-\tau_{12}) = \frac{\varkappa^2}{(1+\pi^2\varkappa^2)^2}\log^2(\bar{\Lambda}\tau_{12}) + \text{finite}. \tag{C.6}$$

Combining the contributions of (C.4) and (C.6) we get for the relevant divergent terms in (6.29)

$$\begin{aligned}G_\perp(\tau) = \left(1-\frac{1}{N}\right)\frac{\varkappa\, g^2}{8\pi^2\tau_{12}^2}&\left[1-\frac{2N}{R^2}\,\frac{\varkappa}{1+\pi^2\varkappa^2}\log(\bar{\Lambda}\tau)+\frac{N^2}{R^4}\frac{4\varkappa^2}{(1+\pi^2\varkappa^2)^3}\right. \\ &\left.\times\log^2(\bar{\Lambda}\tau)+\frac{N^2}{R^4}\,\frac{2\,\varkappa^2\left[5-2\,\log(a\bar{\Lambda})-\pi^2\varkappa^2\right]}{(1+\pi^2\varkappa^2)^3}\log(\bar{\Lambda}\tau)+\cdots\right].\end{aligned} \tag{C.7}$$

[38] Note that singular terms like $\frac{1}{p+\mathrm{i}\pi\varkappa|p|}$ at $p \to 0$ that appear at intermediate steps cancel.

The coefficient of the $\log^2$ term here agrees with the one following from the RG constraint (6.17).

The resulting coefficient of the leading $\frac{1}{R^4}\varkappa^3$ term in the beta function will then be $5 - 2\log(a\bar{\Lambda})$ (cf (6.30) and (6.32)). To match the known two-loop coefficient of ζ^5 term in $\beta_\zeta^{\text{ladder}}$ in (1.3) and (1.8) we need to require that a and $\bar{\Lambda}$ are related so that $\log(a\bar{\Lambda}) = 1$. It is clearly desirable to find a more systematic regularization approach in which this value will appear automatically. In principle, it should be sufficient to introduce a UV cutoff only in the bulk propagator kernel $\frac{1}{(\tau-\tau')^2}$ appearing in the 1D effective action (6.4). Then this cutoff will appear also in the exact $\varkappa$-dependent propagator (6.10). However, our attempts to use some natural choices like dimensional regularization led to complicated integrals that we did not manage to evaluate.

Appendix D. Universal form of planar limit of three-loop term in $\beta_\zeta^{\text{ladder}}$

The three-loop ζ^7 term in $\beta_\zeta^{\text{ladder}}$ in (1.23) contains the group-theoretic coefficient

$$Q_{\text{R}} = \frac{d_{\text{A}}^{abcd} d_{\text{R}}^{abcd}}{C_{\text{R}}\ \dim \text{R}}. \tag{D.1}$$

Here we shall prove that for any irreducible representation R of $SU(N)$ one has

$$\lim_{N\to\infty} \frac{Q_{\text{R}}}{N^3} = \frac{1}{24}, \tag{D.2}$$

independently on R, leading to the universal coefficient of the $\zeta(2)\zeta^7$ term in (6.38).

To prove (D.2) we need first to recall some definitions. The Chern character in representation R with generators T_{R}^a is a function of X^a ($a = 1, \ldots, N^2 - 1$) defined by

$$\text{Ch(R)} = \text{Tr}\left[e^{X^a T_{\text{R}}^a}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} d_{\text{R}}^{a_1\ldots a_n} X^{a_1}\ldots X^{a_n}, \tag{D.3}$$

$$\text{Ch}(\text{R}_1 \times \text{R}_2) = \text{Ch}(\text{R}_1)\,\text{Ch}(\text{R}_2), \qquad \text{Ch}(\text{R}_1 + \text{R}_2) = \text{Ch}(\text{R}_1) + \text{Ch}(\text{R}_2). \tag{D.4}$$

For symmetric S_k and antisymmetric A_k representations it is known that

$$\begin{aligned} \text{Ch}(\text{S}_k) &= \sum_{k=\sum_i n_i m_i} \prod_i \frac{1}{m_i!}\left[\frac{\text{Ch}(n_i\text{F})}{n_i}\right]^{m_i}, \\ \text{Ch}(\text{A}_k) &= (-1)^k \sum_{k=\sum_i n_i m_i} \prod_i \frac{1}{m_i!}\left[-\frac{\text{Ch}(n_i\text{F})}{n_i}\right]^{m_i}, \end{aligned} \tag{D.5}$$

where the sums are over all integer partitions of k (n_i appears in the partition with multiplicity m_i). Tensoring representations one can obtain expressions for characters in terms of fundamental characters, see examples below. For a generic irreducible representation with n_{R} blocks in the Young tableau, the leading large N power comes from the term with a maximal power of Ch(F)

$$\text{Ch(R)} = \frac{1}{h_{\text{R}}}[\text{Ch(F)}]^{n_{\text{R}}} + \cdots. \tag{D.6}$$

In (D.6) h_R is obtained as the product over all blocks B in the Young tableau of their hook length, defined as one plus the number of blocks below and to the right to B. The relevant terms in (D.3) are then

$$\begin{aligned}\mathrm{Ch(R)} = \frac{1}{h_\mathrm{R}}\Bigg[&N^{n_\mathrm{R}} + n_\mathrm{R}\,N^{n_\mathrm{R}-1}\left(\frac{1}{2}d_\mathrm{F}^{ab}X^aX^b + \frac{1}{4!}d_\mathrm{F}^{abcd}X^aX^bX^cX^d + \cdots\right)\\ &+ \frac{n_\mathrm{R}(n_\mathrm{R}-1)}{2!}N^{n_\mathrm{R}-2}\\ &\times\left(\frac{1}{2}d_\mathrm{F}^{ab}X^aX^b + \frac{1}{4!}d_\mathrm{F}^{abcd}X^aX^bX^cX^d + \cdots\right)^2\Bigg],\end{aligned} \tag{D.7}$$

where subscript F refers to fundamental representation. Picking the terms with 0, 2, 4 factors of X^a gives the leading power of N in

$$\begin{aligned}\dim\,\mathrm{R} &= \frac{N^{n_\mathrm{R}}}{h_\mathrm{R}} + \cdots,\\ C_\mathrm{R} &= \frac{N^2-1}{\dim\,\mathrm{R}}\frac{n_\mathrm{R}N^{n_\mathrm{R}-1}}{2h_\mathrm{R}} + \cdots = \frac{n_\mathrm{R}N}{2} + \cdots,\\ d_\mathrm{R}^{abcd} &= \frac{n_\mathrm{R}N^{n_\mathrm{R}-1}}{h_\mathrm{R}}\,d_\mathrm{F}^{abcd} + \cdots.\end{aligned} \tag{D.8}$$

Using now that

$$d_\mathrm{F}^{abcd}d_\mathrm{A}^{abcd} = \frac{N(N^2-1)(N^2+6)}{48} = \frac{N^5}{48} + \cdots, \tag{D.9}$$

we obtain for (D.1)

$$Q_\mathrm{R} = \frac{N^3}{24} + \cdots, \tag{D.10}$$

implying (D.2).

Let us now present some explicit examples of particular representations that check the expansions (D.8). Let us begin with the case of the representation $\yng(2,1)$ which is the minimal one not included in S_k and A_k series. We start with

$$\yng(1)\times\yng(1)\times\yng(1) = \yng(3) + 2\,\yng(2,1) + \yng(1,1,1)\,. \tag{D.11}$$

Using (D.4) and (D.5) one obtains [22]

$$\mathrm{Ch}(\yng(2,1)) = \frac{1}{3}\left[\mathrm{Ch(F)}\right]^3 - \frac{1}{3}\mathrm{Ch(3F)}. \tag{D.12}$$

Using (D.3) and expanding to 4th order gives then

$$\dim_{\yng(2,1)} = \frac{N(N^2-1)}{3}, \qquad C_{\yng(2,1)} = \frac{3(N^2-3)}{2N}, \tag{D.13}$$

$$d_{\yng(2,1)}^{abcd} = (N^2-27)\,d_\mathrm{F}^{abcd} + \frac{N}{2}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}). \tag{D.14}$$

Contracting with d_{A}^{abcd} for adjoint representation and using $d_{\mathrm{A}}^{aabc} = \frac{5}{6}N^2\,\delta^{bc}$

$$d_{\ydiagram{2,1}}^{abcd}\, d_{\mathrm{A}}^{abcd} = \frac{1}{48}N(N^2-1)(N^4+39N^2-162)\ , \qquad \lim_{N\to\infty}\frac{Q_{\mathrm{R}}}{N^3}\Big|_{\mathrm{R}=\ydiagram{2,1}} = \frac{1}{24}. \tag{D.15}$$

As a next example we consider is $\ydiagram{3,1}$.. From

$$\ydiagram{1} \times \ydiagram{3} = \ydiagram{4} + \ydiagram{3,1}, \tag{D.16}$$

we obtain

$$\mathrm{Ch}(\ydiagram{3,1}) = \frac{1}{8}\big[\mathrm{Ch}(\mathrm{F})\big]^4 + \frac{1}{4}\big[\mathrm{Ch}(\mathrm{F})\big]^2\mathrm{Ch}(2\mathrm{F}) - \frac{1}{8}\big[\mathrm{Ch}(2\mathrm{F})\big]^2 - \frac{1}{4}\mathrm{Ch}(4\mathrm{F}). \tag{D.17}$$

Expanding as in (D.3) gives

$$\dim_{\ydiagram{3,1}} = \frac{N(N^2-1)(N+2)}{8}, \qquad C_{\ydiagram{3,1}} = \frac{2(N^2+N-4)}{N}, \tag{D.18}$$

$$d_{\ydiagram{3,1}}^{abcd}\, d_{\mathrm{A}}^{abcd} = \frac{1}{96}N(N^2-1)(N+2)(N^4+7N^3+74N^2+48N-384), \tag{D.19}$$

and thus again

$$\lim_{N\to\infty}\frac{Q_{\mathrm{R}}}{N^3}\Big|_{\mathrm{R}=\ydiagram{3,1}} = \frac{1}{24}. \tag{D.20}$$

Our final example is $\ydiagram{2,2}$.. Using

$$\ydiagram{1} \times \ydiagram{2,1} = \ydiagram{3,1} + \ydiagram{2,2} + \ydiagram{2,1,1}, \qquad \ydiagram{1} \times \ydiagram{1,1,1} = \ydiagram{2,1,1} + \ydiagram{1,1,1,1} \tag{D.21}$$

and (D.12) and (D.17) we get

$$\begin{gathered}
\mathrm{Ch}(\ydiagram{2,2}) = \frac{1}{12}\big[\mathrm{Ch}(\mathrm{F})\big]^4 + \frac{1}{4}\big[\mathrm{Ch}(2\mathrm{F})\big]^2 - \frac{1}{3}\mathrm{Ch}(\mathrm{F})\,\mathrm{Ch}(3\mathrm{F}),\\
\dim_{\ydiagram{2,2}} = \frac{N^2(N^2-1)}{12}, \qquad C_{\ydiagram{2,2}} = \frac{2(N^2-4)}{N}, \qquad d_{\ydiagram{2,2}}^{abcd}\, d_{\mathrm{A}}^{abcd} = \frac{N^2(N^2-1)(N^2-4)(N^2+42)}{144},\\
\lim_{N\to\infty}\frac{Q_{\mathrm{R}}}{N^3}\Big|_{\mathrm{R}=\ydiagram{2,2}} = \frac{1}{24}.
\end{gathered} \tag{D.22}$$

Appendix E. Two-loop ladder beta function from two-point correlators on the line

In [4] we showed how to compute the two-loop ladder beta function $\beta_\zeta^{\text{ladder}}$ in the planar limit by considering the defect two-point function of the scalar fields (either coupled to the loop or 'transverse' to it) in the case when the scalar Wilson loop in the fundamental representation is defined on a straight line of length $2L$. Here we will extend that calculation to the case of a generic representation of $SU(N)$ at finite N.

E.1. Transverse scalar

For one 'transverse' scalar denoted by $\phi_\perp$ which does not appear in the Wilson line exponent we want to compute

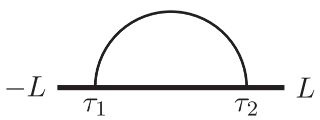
$$G_\perp(\tau) = \langle\!\langle \phi_\perp(0)\,\phi_\perp(\tau)\rangle\!\rangle = \frac{\left\langle \mathrm{Tr}\left[\mathrm{P}\,\phi_\perp(0)\,\phi_\perp(\tau)\,\exp\int_{-L}^{L}\mathrm{d}\tau'\,\phi(\tau')\right]\right\rangle}{\left\langle \mathrm{Tr}\left[\mathrm{P}\,\exp\int_{-L}^{L}\mathrm{d}\tau'\,\phi(\tau')\right]\right\rangle}. \tag{E.1}$$

Here ϕ is rescaled by ζ so that the relevant coupling that appears in the propagator is $\bar{\xi} = \zeta^2 g^2 = N^{-1}\,\xi$ (where ξ was defined in (1.10)). The propagator on the line (cf (1.11) and (1.12)) in dimensional regularization is given by (cf (2.3))

$$D(\tau) = \frac{N\bar{\xi}}{8\pi^2}\frac{1}{|\tau|^{2-\epsilon}}, \quad d = 4-\epsilon, \quad \bar{\xi} = \zeta^2\,g^2. \tag{E.2}$$

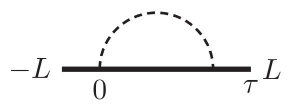
One loop. At the tree and one loop level we have from (E.1)

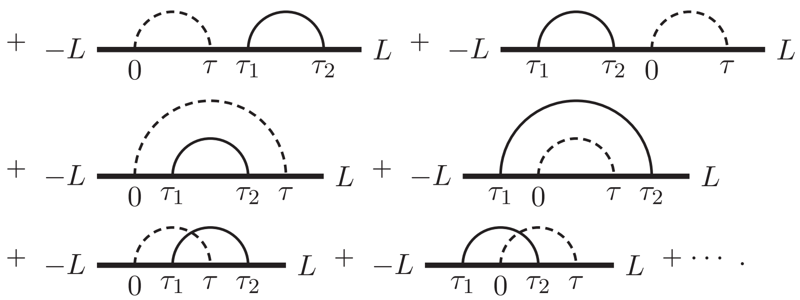
$$\langle \mathrm{Tr}\big[\mathrm{P}\,\exp\int_{-L}^{L} d\tau'\,\phi(\tau')\big]\rangle = 1 + $$



$$+\cdots\,, \tag{E.3}$$

$$\langle \mathrm{Tr}\big[\mathrm{P}\,\phi_\perp(0)\,\phi_\perp(\tau)\,\exp\int_{-L}^{L} d\tau'\,\phi(\tau')\big]\rangle = $$





$$+\cdots\,. \tag{E.4}$$

A given diagram contributes with factor $[\frac{\bar{\xi}}{4\pi^2}]^\nu, \nu =$ number of loops. The planar diagrams have color factor $[C_\text{R}]^\nu$, while the last two two-loop non-planar diagrams have a factor of $C_\text{R}(C_\text{R} - \frac{1}{2}C_\text{A})$. As a result, we find for (E.1)

$$G_\perp(\tau) = \frac{\tau^{-2+\epsilon}C_\text{R}\bar{\xi}}{4\pi^2} + \frac{\tau^{-2+\epsilon}\left[(L-\tau)^\epsilon + 2\tau^\epsilon - (L+\tau)^\epsilon\right]C_\text{A}C_\text{R}\bar{\xi}^2}{32\pi^4(-1+\epsilon)\epsilon} + \mathcal{O}(\bar{\xi}^3), \tag{E.5}$$

where the dependence on C_R is just by an overall factor. This is renormalized by setting (cf (2.8))

$$\bar{\xi} = \mu^\epsilon\left[\bar{\xi}(\mu) + \frac{p_1}{\epsilon}\,\bar{\xi}^{\,2}(\mu) + \cdots\right], \quad p_1 = \frac{C_\text{A}}{4\pi^2}. \tag{E.6}$$

One can then take $L \to \infty$ and finally

$$G_\perp^\text{ren}(\tau,\mu) = C_\text{R}\,\frac{\bar{\xi}}{4\pi^2}\,\frac{1}{\tau^2}\left[1 - C_\text{A}\frac{\bar{\xi}}{4\pi^2}\,(\log(\mu\tau)+1) + \mathcal{O}(\bar{\xi}^2)\right]. \tag{E.7}$$

Note that here there is no need for an additional Z-factor so that we have

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_{\bar{\xi}}\frac{\partial}{\partial\bar{\xi}}\right)G_\perp^\text{ren}(\tau;\mu) = 0. \tag{E.8}$$

Two loops. At two loops, the most convenient scheme is the regularization discussed in [4] where the propagator is as in (2.12)

$$D(\tau) = \frac{N\bar{\xi}}{8\pi^2}\frac{1}{(|\tau|+\varepsilon)^2}, \quad \varepsilon \to 0. \tag{E.9}$$

We find that the renormalized two-point function is then[39]

$$G_\perp^\text{ren}(\tau;\mu) = C_\text{R}\,\frac{\bar{\xi}}{4\pi^2}\frac{1}{\tau^2}\left[1 - C_\text{A}\frac{\bar{\xi}}{4\pi^2}\,\log(\mu\tau) + C_\text{A}^2\left(\frac{\bar{\xi}}{4\pi^2}\right)^2\right.$$
$$\left.\times\left(\frac{\pi^2}{24} + \frac{1}{2}\,\log(\mu\tau) + \log^2(\mu\tau)\right) + \mathcal{O}(\bar{\xi}^3)\right], \tag{E.10}$$

so that (E.8) is satisfied with

$$\beta_{\bar{\xi}} = C_\text{A}\,\frac{\bar{\xi}^2}{4\pi^2} - \frac{1}{2}C_\text{A}^2\,\frac{\bar{\xi}^3}{(4\pi^2)^2} + \mathcal{O}(\bar{\xi}^4). \tag{E.11}$$

[39] Compared to (E.7) found in dimensional regularization in this regularization scheme the one-loop correction contains just $\log(\mu\tau)$ term, i.e. the two μ parameters in (E.7) and in (E.10) are related by a factor of e.

E.2. Coupled scalar

The same analysis for the scalar field ϕ coupled to the Wilson line requires the evaluation of around 200 different diagrams. All of them can be treated with the regularization (E.9) with the final result

$$G^{\mathrm{ren}}(\tau;\mu) = C_{\mathrm{R}}\,\frac{\bar{\xi}}{4\pi^2}\frac{1}{\tau^2}\left[1 + C_{\mathrm{A}}\frac{\bar{\xi}}{4\pi^2}\,(1-3\,\log(\mu\tau)) + C_{\mathrm{A}}^2\left(\frac{\bar{\xi}}{4\pi^2}\right)^2 \right.$$
$$\left. \times\left(-2+\frac{5\pi^2}{24}-\frac{3}{2}\,\log(\mu\tau)+6\log^2(\mu\tau)\right)+\mathcal{O}(\bar{\xi}^3)\right], \qquad \text{(E.12)}$$

where in addition to renormalization of $\bar{\xi}$ one needs to introduce a Z-factor, i.e. $G^{\mathrm{ren}} = ZG$ with

$$Z = 1 - \frac{C_{\mathrm{A}}\,\bar{\xi}}{2\pi^2}\log(\varepsilon\,L) + \frac{C_{\mathrm{A}}^2\,\bar{\xi}^2}{16\pi^4}\left[2\,\log(\varepsilon\,L)+\log^2(\varepsilon\,L)\right] + \mathcal{O}(\bar{\xi}^3). \qquad \text{(E.13)}$$

As a result, G^{ren} satisfies the Callan–Symanzik equation with an anomalous dimension Δ (see a discussion in [4])

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_{\bar{\xi}}\frac{\partial}{\partial\bar{\xi}}+2(\Delta-1)\right]\left(\bar{\xi}^{-1}\,G^{\mathrm{ren}}(\tau;\mu)\right)=0, \qquad \text{(E.14)}$$

where $\beta_{\bar{\xi}}$ is as in (E.11) and

$$\Delta = 1 + \frac{3\,C_{\mathrm{A}}\,\bar{\xi}}{8\pi^2} - \frac{5\,C_{\mathrm{A}}^2\,\bar{\xi}^2}{64\pi^4} + \mathcal{O}(\bar{\xi}^3). \qquad \text{(E.15)}$$

Appendix F. Multiply wound Wilson loop

Our results have a simple application to the case of k-wound Wilson loop in the fundamental representation F. This generalization amounts to the replacement $\mathrm{Tr}_{\mathrm{F}}\,U \to \mathrm{Tr}_{\mathrm{F}}(U^k)$ in the definition of the Wilson loop (1.1). To start, let us write

$$\mathrm{Tr}_{\mathrm{F}}\,U^k = \sum_i {c^k}_i\,\mathrm{Tr}_{\mathrm{R}_i}\,U, \qquad \text{(F.1)}$$

where the sum is over all irreducible representations appearing in $\mathrm{F}^{\otimes k}$. Then, (F.1) implies the following relation for the associated Wilson loops

$$W_{k-\mathrm{wound}} = \sum_i {c^k}_i W_{\mathrm{R}_i}. \qquad \text{(F.2)}$$

The coefficients $\{{c^k}_i\}$ in (F.1) appear in the inversion of the Frobenius formula [45]

$$\mathrm{Tr}_{\mathrm{R}}\,U = \frac{1}{|\mathrm{R}|!}\sum_{\sigma\in S_{|\mathrm{R}|}}\chi_{\mathrm{R}}(\sigma)\prod_{i=1,2,\ldots}\mathrm{Tr}_{\mathrm{F}}\,U^{k_i(\sigma)}. \qquad \text{(F.3)}$$

Here $K = |R|$ is the number of blocks in the Young tableau of R, $k_i(\sigma)$ is the length of the ith cycle of the permutation σ. The symmetric group characters $\chi_{\mathrm{R}}(\sigma)$ are obtained as

$$\chi_{\mathrm{R}}(\sigma) = \text{coeff. of } x_1^{\ell_1}\ldots x_K^{\ell_K} \text{ in } \Delta(x)\prod_{j\geqslant 1}^{n} P_j(x)^{\nu_j(\sigma)}, \qquad \text{(F.4)}$$

where λ_i are the rows of the Young tableau of R, padded with zero to have K entries, $\nu_j(\sigma)$ is the number of cycles of length j in σ, and $\Delta(x) = \prod_{1\leqslant i<j\leqslant K}(x_i - x_j)$. For $K = 2$ this gives the well known relations[40]

$$\mathrm{Tr}_{(1,1)}U = \frac{1}{2}(\mathrm{Tr}_\mathrm{F}\, U)^2 - \frac{1}{2}\mathrm{Tr}_\mathrm{F}\, U^2, \qquad \mathrm{Tr}_{(2)}\, U = \frac{1}{2}(\mathrm{Tr}_\mathrm{F}\, U)^2 + \frac{1}{2}\mathrm{Tr}_\mathrm{F}\, U^2, \tag{F.5}$$

so that the inversion of (F.3) reads

$$\mathrm{Tr}_\mathrm{F}\, U^2 = \mathrm{Tr}_{(2)}\, U - \mathrm{Tr}_{(1,1)}\, U. \tag{F.6}$$

Repeating the same procedure for higher values of k, (F.6) generalizes to

$$\begin{aligned}
&\mathrm{Tr}_\mathrm{F}\, U^3 = \mathrm{Tr}_{(3)}\, U - \mathrm{Tr}_{(2,1)}\, U + \mathrm{Tr}_{(1,1,1)}\, U,\\
&\mathrm{Tr}_\mathrm{F}\, U^4 = \mathrm{Tr}_{(4)}\, U - \mathrm{Tr}_{(3,1)}\, U + \mathrm{Tr}_{(2,1,1)}\, U - \mathrm{Tr}_{(1,1,1,1)}\, U,\\
&\mathrm{Tr}_\mathrm{F}\, U^5 = \mathrm{Tr}_{(5)}\, U - \mathrm{Tr}_{(4,1)}\, U + \mathrm{Tr}_{(3,1,1)}\, U - \mathrm{Tr}_{(3,1,1,1)}\, U + \mathrm{Tr}_{(1,1,1,1,1)}\, U,
\end{aligned} \tag{F.7}$$

and so on. To evaluate (1.15), we need the sum of C_R and C_R^2 based on the decomposition (F.1), i.e. the effective k-dependent coefficients

$$\begin{aligned}
&C_\mathrm{R} \to \sum C_\mathrm{R} = k^2\,\frac{N^2-1}{2N},\\
&C_\mathrm{R}^2 \to \sum C_\mathrm{R}^2 = k^2\,\frac{(N^2-1)[N^2+k^2(-3+2N^2)]}{12N^2},
\end{aligned} \tag{F.8}$$

leading to

$$\begin{aligned}
\frac{1}{N}\,\langle W\rangle_{k-\mathrm{wound}} = 1 + k^2\,\frac{N^2-1}{8N}\,g^2 + k^2\,(N^2-1)&\\
\times\left[\frac{k^2}{192}\left(1-\frac{3}{2N^2}\right) + \frac{1}{128\pi^2}(1-\zeta^2)^2\right] g^4 + \mathcal{O}(g^6).&
\end{aligned} \tag{F.9}$$

We remark that, for $\zeta = 1$, the winding is implemented by the simple substitution rule $g^2 \to k^2 g^2$, which is clear in the matrix model representation of the BPS WML (at any finite N). For generic ζ, we notice that the coefficient of the $(1-\zeta^2)$ term is instead $\sim k^2 g^4$, i.e. has a different scaling with k.

The same analysis applies to the two point functions or more general correlators. Once we write $\langle \mathrm{Tr}_\mathrm{F}[\mathcal{O}(\tau_1)\dots\mathcal{O}(\tau_n)\, U^k]\rangle$ as derivatives of $\langle\langle \mathrm{Tr}_\mathrm{F} U(\eta)^k\rangle\rangle$ where $\eta(s)$ is a local coupling to $\mathcal{O}$, and we can treat $\mathrm{Tr}_\mathrm{F}[U(\eta)]^k$ as above.

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[40] We use $(s_1, s_2, \dots)$ to label traces in representations R by the corresponding Young tableau.

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