$\sin(\omega x)$ Can Approximate Almost Every Finite Set of Samples*

Gilbert H. Harman, Sanjeev R. Kulkarni, Hariharan Narayanan†

Abstract

Consider a set of points $(x_1, y_1), \ldots, (x_n, y_n)$ with distinct $0 \leq x_i \leq 1$ and with $-1 < y_i < 1$. The question of whether the function $y = \sin(\omega x)$ can approximate these points arbitrarily closely for a suitable choice of ω is considered. It is shown that such approximation is possible iff the set $\{x_1, \ldots, x_n\}$ is linearly independent over the rationals. Furthermore, a constructive sufficient condition for such approximation is provided. The results provide a sort of counterpoint to the classical sampling theorem for bandlimited signals. They also provide a stronger statement than the well known result that the collection of functions $\{\sin(\omega x) : \omega < \infty\}$ has infinite pseudo dimension.

Index Terms: sinusoids, approximation, Diophantine approximation, VC dimension, pseudo dimension, sampling theorem

1 Introduction

Suppose $x_1, \ldots, x_n$ are distinct with $0 \leq x_i \leq 1$ for $i = 1, \ldots, n$ and that $-1 < y_i < 1$ for $i = 1, \ldots, n$. We are interested in the question of when a function of the form

$$y = \sin(\omega x) \tag{1}$$

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†Gilbert H. Harman is with the Department of Philosophy, Princeton University, Princeton, NJ 08544. His email address is harman@princeton.edu. Sanjeev R. Kulkarni is with the Department of Electrical Engineering, Princeton University, Princeton, NJ 08544. His email address is kulkarni@princeton.edu. Hariharan Narayanan is with the Departments of Statistics and Mathematics, University of Washington, Seattle, WA 98105. His email address is harin@uw.edu.

can approximate the set of points $(x_1, y_1), \ldots, (x_n, , y_n)$ arbitrarily closely for a suitable choice of ω. That is, we address the question of whether for every $\epsilon > 0$ there exists ω such that

$$|\sin(\omega x_i) - y_i| < \epsilon \text{ for } i = 1, \ldots, n$$

In Section 2, we obtain a necessary and sufficient conditions on $x_1, \ldots, x_n$ such that that for *any* $y_1, \ldots, y_n$ with $-1 < y_i < 1$, the points $(x_1, y_1), \ldots (x_n, y_n)$ can be approximated. We call such $x_1, \ldots, x_n$ *approximable*, and we show that $x_1, \ldots, x_n$ are *approximable* if and only if the set $\{x_1, \ldots, x_n\}$ is linearly independent over the rationals. This characterization follows straightforwardly from results in Diophantine analysis.

In Section 3, we provide a different sufficient condition on the x_i (which we call mutually exhaustive) that guarantees approximability for any y_i. Although this condition is not necessary, it gives a constructive way to select a suitable ω for a desired level of approximation in terms of the binary expansions of the x_i. Moreover, we show that if the x_i are chosen randomly (i.i.d. uniform) then the x_i will be mutually exhaustive almost surely.

The classical Shannon sampling theorem, states that a bandlimited signal can be uniquely reconstructed by samples taken at a rate greater than twice the bandwidth of the signal. While rather different than the sampling theorem in a number of ways, the results here provide a sort of rough counterpoint. In the sampling theorem, the reconstructed signal fits the samples exactly, but utilizes a spectrum of sinusoids up to the bandlimit under consideration. In contrast, the results here fit the samples only approximately (though to whatever fidelity desired), but utilize only a single sinusoid, albeit of sufficiently high frequency.

Our results are also related to a result from learning theory and statistical pattern recognition. It's known that the collection of sinusoids of the form in Equation (1) have subgraphs with infinite Vapnik-Chervonenkis (VC) dimension. Alternatively, this class has infinite pseudo-dimension. These results imply the graphs of the collection of sinusoids lie above and/or below *some* collection of points $(x_1, y_1), \ldots (x_n, y_n)$ in all possible combinations for arbitrarily large n. On the other hand, our result implies a similar statement but for *every* choice of y_i as long as the x_i satisfy appropriate conditions.

2 A Necessary and Sufficient Condition from Diophantine Approximation

A collection of real numbers $\alpha_1, \ldots, \alpha_n$ is said to be linearly *dependent* over the rationals if there exist rational numbers $q_1, \ldots, q_n$ such that

$$q_1\alpha_1 + \cdots + q_n\alpha_n = 0$$

If there are no such rational $q_1, \ldots, q_n$ then $\alpha_1, \ldots, \alpha_n$ are said to be linearly *independent* over the rationals.

Given a set of real numbers $x_1, \ldots, x_n$ and an integer k, consider the point $\bar{x}_k \in [0,1]^n$ defined by

$$\bar{x}_k = (\{kx_1\}, \ldots, \{kx_n\})$$

where $\{u\}$ denotes the fractional part of u. A well-known result in Diophantine approximation states that the collection of numbers $\bar{x}_1, \bar{x}_2, \ldots$ is dense in $[0,1]^n$ if and only if $\{x_1, \ldots, x_n\}$ are linearly independent over the rationals. This is a form of Kronecker's theorem (see, for example, Chapter 23 of [2]).

This result immediately provides a sufficient condition for $\{x_1, \ldots, x_n\}$ to be approximable as can be seen in Theorem 1 below. Roughly, if $\{x_1, \ldots, x_n\}$ are linearly independent over the rationals then the fractional parts of $(\omega x_1, \ldots, \omega x_n)$ can be made as close to some specified $(\alpha_1, \ldots, \alpha_n)$ as desired so that $(\sin(\omega x_1), \ldots, \sin(\omega x_n))$ can be made as close to $(y_1, \ldots, y_n)$ as desired.

Theorem 1 *A set of distinct points $x_1, \ldots, x_n$ are approximable iff $\{x_1, \ldots, x_n\}$ are linearly independent over the rationals.*

Proof:

Suppose $\{x_1, \ldots, x_n\}$ are linearly independent over the rationals. Consider any $y_1, \ldots, y_n$ with $-1 < y_i < 1$ and consider any $\epsilon > 0$. We need to show that there exists ω such that $|\sin(\omega x_i) - y_i| < \epsilon$ for $i = 1, \ldots, n$.

Let α_i be such that $\sin(\alpha_i) = y_i$, and let δ be such that $|\sin(\beta) - \sin(\alpha)| < \epsilon$ whenever $|\beta - \alpha| < \delta$. Given a real number θ, let $\langle\theta\rangle$ be defined as $2\pi(\frac{\theta}{2\pi} - \lfloor\frac{\theta}{2\pi}\rfloor)$. By Kronecker's theorem, there exists ω such that $|\langle\omega x_i\rangle - \alpha_i| < \delta$ for $i = 1, \ldots, n$. For this ω,

$$|\sin(\omega x_i) - y_i| \quad = \quad |\sin(\omega x_i) - \sin(\alpha_i)|$$

$$
\begin{aligned}
&= |\sin(\langle \omega x_i \rangle) - \sin(\alpha_i)| \\
&< \epsilon
\end{aligned}
$$

For the converse, suppose that $\{x_1, \dots, x_n\}$ are linearly dependent over the rationals. Then there exist integers k_i such that

$$k_1 x_1 + \cdots + k_n x_n = 0$$

Let the linear subspace $\{(x_1, \dots, x_n) | k_1 x_1 + \cdots + k_n x_n = 0\}$ be denoted Λ.

Let T^n denote the n dimensional torus $\{(z_1, \dots, z_n) | z_i \in C, |z_i| = 1\}$. Let

$$\phi : (x_1, \dots, x_n) \mapsto (\exp(2\pi i x_1), \dots, \exp(2\pi i x_n))$$

map $\mathcal{R}^n$ to T^n. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathcal{R}^n$.

Let the vector $(\frac{k_1}{\sqrt{\sum_i k_i^2}}, \dots, \frac{k_n}{\sqrt{\sum_i k_i^2}})$ be denoted $v^\perp$. Let

$$\delta < \left(\prod_{i=1}^{n-1} \sqrt{k_i^2 + k_n^2} \right)^{-1} 2^{-1} (2\pi)^{-n}. \tag{2}$$

We denote by $\Lambda + v^\perp[-\delta, \delta]$, the set of points $x = (x_1, \dots, x_n)$ whose Euclidean distance to the subspace Λ is less or equal to δ.

We see that Λ has a basis consisting of $\{v_1, \dots, v_n\}$, where $v_i := -k_n e_i + k_i e_n$. These basis vectors are integral vectors, therefore

$$\forall x, \exists i \in \{1, \dots, n\}, \phi(x + v_i) = \phi(x). \tag{3}$$

Let P denote the parallelopiped

$$\{\sum_{i=1}^{n} \lambda_i v_i | (\lambda_1, \dots, \lambda_n) \in [0, 1]^n\}.$$

Let $P + v^\perp[-\delta, \delta]$ denote the parallelepiped

$$\{\lambda_0 v^\perp + x | \lambda_0 \in [-\delta, \delta], x \in P\}.$$

Then, by (3), we have

$$\phi(P + v^\perp[-\delta, \delta]) = \phi\left(\Lambda + v^\perp[-\delta, \delta]\right) \tag{4}$$

as sets.

Let

$$\psi : T^n \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^n$$

map the point $(\exp(i\theta_1), \dots, \exp(i\theta_n))$ to $(\alpha_1, \dots, \alpha_n)$ where the

$$\alpha_i \in [-\pi/2, \pi/2]$$

are given by $\sin^{-1}(\sin(\theta_i)) = \alpha_i$.

By (4),

$$\psi\left(\phi\left(P + v^{\perp}[-\delta, \delta]\right)\right) = \psi\left(\phi\left(\Lambda + v^{\perp}[-\delta, \delta]\right)\right). \tag{5}$$

Denoting the path-metric by d, we see that $d(\psi(x), \psi(y)) \leq d(x, y)$ for any $x, y \in T$. So the volume of $\psi\left(\phi\left(P + v^{\perp}[-\delta, \delta]\right)\right)$ is less or equal to the volume of $\phi\left(P + v^{\perp}[-\delta, \delta]\right)$, which (by our choice of δ) is less than or equal to 1, and hence less or equal to $(\pi/3)^n$. The volume of $[-\pi/6, \pi/6]^n$ is $(\pi/3)^n$, therefore there must be some point $\alpha = (\alpha_1, \dots, \alpha_n) \in [-\frac{\pi}{6}, \frac{\pi}{6}]^n$ that does not belong to $\psi\left(\phi\left(P + v^{\perp}[-\delta, \delta]\right)\right)$. By (5) there must be some point $\alpha = (\alpha_1, \dots, \alpha_n) \in [-\frac{\pi}{6}, \frac{\pi}{6}]^n$ that does not belong to $\psi\left(\phi\left(\Lambda + v^{\perp}[-\delta, \delta]\right)\right)$.

We claim that for this α,

$$y = (\sin(\alpha_1), \dots, \sin(\alpha_n))$$

cannot be approximated arbitrarily well by a point of the form $(\sin(\omega x_1), \dots, \sin(\omega x_n))$. Indeed, suppose that for each i,

$$|\sin(\omega x_i) - \sin(\alpha_i)| < \frac{\delta}{2\sqrt{n}}.$$

Since $\psi \circ \psi : T \to [-\pi/2, \pi/2]^n$ is a contraction of metric spaces and since $\alpha \in [-\frac{\pi}{6}, \frac{\pi}{6}]^n$, we have $\psi(\phi(\alpha)) = \alpha$, and have, for each i,

$$|\sin(\psi(\phi(\omega x_i))) - \sin(\alpha_i)| < \frac{\delta}{2\sqrt{n}}.$$

Since n is greater or equal to 1, by (2)

$$\delta \leq 2^{-1} < \pi/6.$$

Also, the derivative of $\sin(x)$ (which of course is $\cos(x)$) is greater than $1/2$ in the interval $(-\pi/3, \pi/3)$. So we must have

$$|\psi(\phi(\omega x_i)) - \alpha_i| < \left(\frac{\delta}{2\sqrt{n}}\right)\left(\sup_{\alpha \in (-\pi/3, \pi/3)} \cos(x)^{-1}\right) = \left(\frac{\delta}{\sqrt{n}}\right).$$

This implies that the distance between $(\alpha_1, \ldots, \alpha_n)$ and the set $\psi(\phi(\Lambda))$ is less than δ. However, our assumption that $\alpha = (\alpha_1, \ldots, \alpha_n)$ does not belong to $\psi\left(\phi\left(\Lambda + v^{\perp}[-\delta, \delta]\right)\right)$, implies that the distance of α to any point in $\psi\left(\phi\left(\Lambda\right)\right)$ must be greater or equal to δ. This is a contradiction.

$\square$

3 A Constructive Sufficient Condition

Let $x_i = .b_{i,1}b_{i,2}\cdots$ represent the binary expansion of x_i. That is, $b_{i,j} \in \{0, 1\}$ and

$$x_i = \sum_{j=1}^{\infty} b_{i,j} 2^{-j}$$

Define x_i to be *exhaustive* iff every finite binary string appears somewhere in the binary expansion of x_i. That is, x_i is exhaustive iff for any k and any binary string $s_1 s_2 \cdots s_k$ of length k, there exists m such that

$$b_{i,m-1+j} = s_j \text{ for } j = 1, \ldots, k$$

Define the set $x_1, \ldots, x_n$ to be *mutually exhaustive* iff for every set of n finite binary strings of the same length, there is a common index at which these n strings appear respectively in the binary expansions of $x_1, \ldots, x_n$ starting at this index. That is, the set $x_1, \ldots, x_n$ is *mutually exhaustive* iff for any k and any set of length k binary strings $s_{i,1}s_{i,2}\cdots s_{i,k}$ for $i = 1, \ldots, n$, there exists $m < \infty$ such that

$$b_{i,m-1+j} = s_{i,j} \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, k$$

Theorem 2 *If a distinct set of points $x_1, \ldots, x_n$ with $0 \le x_i \le 1$ are mutually exhaustive then $x_1, \ldots, x_n$ are approximable. That is, for any $y_1, \ldots, y_n$ with $-1 < y_i < 1$, a function of the form*

$$y = \sin(\omega x)$$

can approximate the set of points $(x_1, y_1), \ldots, (x_n, , y_n)$ arbitrarily closely in the sense that for every $\epsilon > 0$ there exists ω such that

$$|\sin(\omega x_i) - y_i| < \epsilon \text{ for } i = 1, \ldots, n$$

Proof:

Choose $\alpha_i \in (0,1]$ such that $\sin(2\pi\alpha_i) = y_i$, and let δ be such that $|\sin(2\pi\alpha) - \sin(2\pi\beta)| < \epsilon/2$ whenever $|\alpha - \beta| < \delta$.

Choose k large enough so that $2^{-k} < \delta$. Let β_i be the first k bits in the binary expansion of α_i. Since $\alpha_i, \beta_i \in [0,1]$ and they agree on the first k bits, we have $|\beta_i - \alpha_i| \leq 2^{-k} < \delta$, so that

$$|\sin(2\pi\beta_i) - \sin(2\pi\alpha_i)| \leq \epsilon/2$$

Since $x_1, \ldots, x_n$ are mutually exhaustive, there exists $m < \infty$ so that for each $i = 1, \ldots, m$, the string β_i appears in the binary expansion of x_i starting at position m.

Let $r(x) = x - \lfloor x \rfloor$ denote the non-integer part of x. Then since $r(2^m x_i), \beta_i \in [0,1]$ and they agree on the first k bits, we have

$$|r(2^m x_i) - \beta_i| \leq 2^{-k} < \delta$$

so that

$$|\sin(2\pi\, r(2^m x_i)) - \sin(2\pi\beta_i)| \leq \epsilon/2$$

Thus, if we let $\omega = 2\pi 2^m$ then

$$\begin{aligned} |\sin(\omega x_i) - y_i| &= |\sin(2\pi 2^m x_i) - \sin(2\pi\alpha_i)| \\ &= |\sin(2\pi\, r(2^m x_i) - \sin(2\pi\alpha_i)| \\ &\leq |\sin(2\pi\, r(2^m x_i)) - \sin(2\pi\beta_i)| + |\sin(2\pi\beta_i) - \sin(2\pi\alpha_i)| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

Think of x as the result of an series of tosses of a fair coin with 1 representing heads and 0 representing tails. With respect to the resulting probability distribution, we can say that *almost every* x is ϕ iff the probability that x is ϕ is 1.

Theorem 3
(a) Almost every x such that $0 \leq x \leq 1$ is exhaustive.
(b) Almost every set of values $x_1, \ldots, x_n$ such that $0 \leq x_i \leq 1$ is mutually exhaustive.

Proof of (a): For any string s of length k, the probability that s occurs starting at the first position in x is $p = 1 - 1/2^k$. Similarly for the probability that s occurs starting at any other position in x. The probability that s does not occur at any given one of these positions is $(1 - p)$. Since the probabilities that s does not occur starting at positions $1 + kq$ for all nonnegative integers q are independent, the probability that s does not occur starting at the first j of those positions is $(1 - p)^j$, which approaches 0 as n approaches infinity. So the probability that s does not occur anywhere in x is 0. Since there are only finitely many strings of length k, the probability that not all strings of length k occur in x is also 0. Furthermore, the probability that not all strings occur in x is no greater than the sum for k from 0 to infinity of the probabilities that not all strings of length k occur in x, which is 0. □

Proof of (b): The proof is similar to the proof of (a). Consider any n strings $s_1, s_2, \ldots, s_n$ each of length k. There is a small but finite probability $p = 1/2^{nk}$ that these strings occur starting at the first position in $x_1, x_2, \ldots, x_n$ respectively. The probabilities that these strings do not occur starting at the same position $1 + kq$ in each of these strings for any nonnegative integer q are independent of each other and are each equal to $(1-p)$. Since $(1-p)^n$ approaches 0 as n goes to infinity, the probability that there is no position at which each of the strings appears in $x_1, x_2, \ldots, x_n$ is therefore 0. This is true no matter what length k of strings is considered. So, the probability that $x_1, x_2, \ldots, x_n$ are mutually exhaustive is 1. □

Note that Theorems 2 and 3 together immediately imply that almost every set of points $x_1, \ldots, x_n$ with $0 \leq x_i \leq 1$ is approximable. This also follows from Theorem 1 and the fact that almost every set of points $x_1, \ldots, x_n$ with $0 \leq x_i \leq 1$ are linearly independent over the rationals.

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