

A CODAZZI-LIKE EQUATION AND THE SINGULAR SET FOR C^1 SMOOTH SURFACES IN THE HEISENBERG GROUP

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ABSTRACT. In this paper, we study the structure of the singular set for a C^1 smooth surface in the 3-dimensional Heisenberg group \mathbf{H}_1 . We discover a Codazzi-like equation for the p -area element along the characteristic curves on the surface. Information obtained from this ordinary differential equation helps us to analyze the local configuration of the singular set and the characteristic curves. In particular, we can estimate the size and obtain the regularity of the singular set. We understand the global structure of the singular set through a Hopf-type index theorem. We also justify that Codazzi-like equation by proving a fundamental theorem for local surfaces in \mathbf{H}_1 .

Table of Contents

1. Introduction and statement of the results
2. A Codazzi-like equation and properties:
proofs of Theorem A and Theorem B
3. Local configuration of the singular set
4. Examples
5. Size and regularity of the singular set
6. The local theory of surfaces with
prescribed p -mean curvature
7. Index of the singular set
8. Generalization to pseudohermitian manifolds
9. Appendix: a generalized ODE

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In the recent years, the p -minimal (or \mathbf{H} -minimal) surfaces have been studied extensively in the framework of geometric measure theory (e.g., [8], [7], [15]) and from the viewpoint of partial differential equations and that of differential geometry (e.g., [3], [2], [4], [5]). Motivated by the isoperimetric problem in the Heisenberg group, one also studied nonzero constant p -mean curvature surfaces and the regularity problem (e.g., [14], [1], [11], [12], [18], [13], [16]). In fact, the notion of p -mean curvature ("p" stands for "pseudohermitian") can be defined for (hyper)surfaces in a pseudohermitian manifold. The Heisenberg group as a (flat) pseudohermitian manifold is the simplest model example, and represents a blow-up limit of general pseudohermitian manifolds.

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The equation of prescribed p -mean curvature in the 3-dimensional Heisenberg group \mathbf{H}_1 is one of few known equations having geometric significance in $2D$. For the Plateau or Dirichlet problem with smooth boundary value, we have reasons to believe that the minimizer is at least C^1 (but not C^2 in general). In [5] three of the authors studied the regularity of the nonsingular portion of a C^1 smooth surface in \mathbf{H}_1 . In this paper we study the local structure of the singular set of such a surface through an ordinary differential equation along the characteristic curves. Results on the local structure of the singular set will be used in studying the global structure of the singular set later. Note that the local structure of the singular set for C^2 smooth surfaces has been classified. Namely, on a C^2 smooth surface, a singular point is either isolated or passed through by a C^1 smooth singular curve in a neighborhood under a mild condition on the p -mean curvature (see Theorem 3.3 in [3] and remarks for the general case in Section 7 there). On the other hand, the structure of the singular set of a C^1 smooth surface can be complicated as we will see in this paper. The understanding of the singular set of a C^1 smooth surface would help to solve the isoperimetric problem for C^1 smooth domains in \mathbf{H}_1 (see [18] for more details on the isoperimetric problem).

Let Ω be a domain of R^2 (by a domain we mean an open and connected set) and let $u \in C^1(\Omega)$ be a weak solution to

$$(1.1) \quad \operatorname{div} \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|} = H$$

where \vec{F} (H , resp.) is an L^1_{loc} vector field (function, resp.) in Ω , that is, for any $\varphi \in C_0^\infty(\Omega)$, there holds

$$(1.2) \quad \int_{S_{\vec{F}}(u)} |\nabla \varphi| + \int_{\Omega \setminus S_{\vec{F}}(u)} \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|} \cdot \nabla \varphi + \int_{\Omega} H \varphi \geq 0$$

in which $S_{\vec{F}}(u)$ denotes the singular set of u , consisting of the points (called singular points) where $\nabla u + \vec{F} = 0$ (see (1.2) and (1.3) in [5]). A point $p \in \Omega$ is called nonsingular if $\nabla u + \vec{F} \neq 0$ at p . We call Ω nonsingular if all the points of Ω are nonsingular.

Note that a C^2 smooth solution (i.e. satisfying (1.1) at nonsingular points) may or may not be a C^1 weak solution. The reason is that (1.2) implies some equal-angle condition along $S_{\vec{F}}(u)$ if $S_{\vec{F}}(u)$ is a C^1 smooth curve, which a C^2 smooth solution may not satisfy (see Example 7.3 in [4]). From the variational point of view, that u satisfies the condition (1.2) is more natural than that u satisfies (1.1) pointwise at nonsingular points. Therefore we study the solutions satisfying (1.2).

Let N denote the planar vector $\frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}$ at a nonsingular point. Since N is of unit length, we can write

$$N = (\cos \theta, \sin \theta)$$

locally for some angular function θ ($\in C^0$ if we assume $\vec{F} \in C^0$). Let $D := |\nabla u + \vec{F}|$ ($\in C^0$ a priori) and let $N^\perp := (\sin \theta, -\cos \theta)$. Suppose further that $\vec{F} \in C^1$ and $N(H)$ exists and is continuous. In [5] (see Theorem D therein), we proved that θ is in fact C^1 and $N^\perp D$ exists and is continuous. In this paper we will show that $N^\perp(N^\perp D)$ exists and is continuous. Moreover, D satisfies an ordinary differential equation of second order along any characteristic curve (i.e., integral curve of N^\perp)

(see Theorem A below). For $\vec{F} = (F_1, F_2) \in C^1(\Omega)$ we define

$$\text{curl}\vec{F} := (F_2)_x - (F_1)_y$$

(note that in [5] we used $\text{rot}\vec{F}$ instead of $\text{curl}\vec{F}$). Denote $N^\perp D$ and $N^\perp(N^\perp D)$ by D' and D'' , respectively.

Theorem A. *Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$ such that Ω is nonsingular. Suppose further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous. Then D' and D'' exist and are continuous in Ω . Moreover, D satisfies the following differential equation*

$$(1.3) \quad DD'' = 2\left(D' - \frac{\text{curl}\vec{F}}{2}\right)(D' - \text{curl}\vec{F}) + (N^\perp(\text{curl}\vec{F}))D + (H^2 + N(H))D^2.$$

The proof of (1.3) is based on the construction of so called s, t coordinates using N^\perp and N (see (2.1)). The associated integrability condition

$$(\theta_s)_t = (\theta_t)_s$$

(see (2.5), (2.6)) is exactly (1.3). So (1.3) is a Codazzi-like equation.

When H is the p (or H)-mean curvature, we have $\vec{F} = (-y, x)$ in this case, so $\text{curl}\vec{F} = 2$. The p (or H)-mean curvature is a notion to measure a hypersurface, which respects the ambient pseudohermitian structure (see [3]). Equation (1.3) can then be reduced to

$$(1.4) \quad DD'' = 2(D' - 1)(D' - 2) + (H^2 + N(H))D^2.$$

For a p (or H)-minimal graph, we have $H \equiv 0$, so (1.4) can be further reduced to

$$(1.5) \quad DD'' = 2(D' - 1)(D' - 2).$$

Equation (1.5) is integrable. Namely, we observe that

$$(1.6) \quad \begin{aligned} \frac{2D'}{D} &= \frac{D'D''}{(D' - 1)(D' - 2)} \\ &= \frac{-D''}{D' - 1} + \frac{2D''}{D' - 2} \\ &= -\frac{(D' - 1)'}{D' - 1} + \frac{2(D' - 2)'}{D' - 2} \end{aligned}$$

at the points of a characteristic curve (line), where $D' \neq 1, 2$. Integrating (1.6), we obtain

$$(1.7) \quad |D' - 2|^2 = c|D' - 1|D^2$$

for some constant $0 < c < \infty$. It is not hard to see that if $D' \neq 1$ ($\neq 2$, resp.) at some nonsingular point q , then $D' \neq 1$ ($\neq 2$, resp.) on the whole characteristic curve (line) Γ passing through q . If $D' = 2$ ($= 1$, resp.) at q , then $c = 0$ ($= \infty$, resp.) and $D' \equiv 2$ ($\equiv 1$, resp.) on Γ by the uniqueness of solutions to (1.5), an ordinary differential equation. When a nonsingular point tends to a singular point along a characteristic curve (line), either D' goes to 2 or D' goes to 1 ($D' \equiv 1$ in fact in this case). In general, $(N^\perp(\text{curl}\vec{F}))D + (H^2 + N(H))D^2 \neq 0$ and hence (1.3) is not integrable. But near singular points (where $D = 0$), we consider $(N^\perp(\text{curl}\vec{F}))D + (H^2 + N(H))D^2$ to be a small perturbation term and obtain the following result.

Theorem B. *Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Let $p \in \Omega$ be a singular point. Let $\Gamma : [0, \bar{\rho}) \rightarrow \Omega \in C^1$ be such that $\Gamma(0) = p$ and $\Gamma((0, \bar{\rho}))$ is a characteristic curve with unit-speed parameter $\rho \in (0, \bar{\rho})$. Suppose $\text{curl}\vec{F}(p) \neq 0$. Then the following statements hold:*

(a) *We have either*

$$(1.8) \quad \lim_{\rho \rightarrow 0} D'(\Gamma(\rho)) = \frac{\text{curl}\vec{F}(p)}{2} \text{ or } \lim_{\rho \rightarrow 0} D'(\Gamma(\rho)) = \text{curl}\vec{F}(p).$$

(b) *The sign of $\text{curl}\vec{F}(p)$ determines the direction of N^\perp . That is, if $\text{curl}\vec{F}(p) > 0$, then $N^\perp = \frac{\partial}{\partial \rho}$ while if $\text{curl}\vec{F}(p) < 0$, then $N^\perp = -\frac{\partial}{\partial \rho}$.*

(c) *Let p, q be two distinct singular points in Ω . Suppose $\text{curl}\vec{F} \neq 0$ in Ω . Then there does not exist $\Gamma : [0, \bar{\rho}) \rightarrow \Omega \in C^1$, a characteristic curve on $(0, \bar{\rho})$ with $\Gamma(0) = p$ and $\Gamma(\bar{\rho}) = q$.*

In the Appendix we consider a generalized version of equation (1.3) and prove a result analogous to Theorem B (a) (see Theorem A.1). Applying Theorem A.1 to a general situation for a surface in a pseudohermitian 3-manifold (see (8.23)), we obtain a variant of Theorem B (a) (see Theorem B' in Section 8).

Theorem B (b) will be applied to show impossibility of some situations. For instance, in the case of a p (or H)-minimal graph, if a family of characteristic lines converges to another line, then the limit line cannot contain any singular point and must be a characteristic line (see Lemma 3.2).

Theorem B (c) will be used often in the study of the configuration of singular points and characteristic curves in Section 3. Among other things, we have a result about the local structure of singular points. Recall that for a C^2 smooth surface, a singular point is either isolated or passed through by a C^1 smooth singular curve in a neighborhood under a mild condition on H (see Theorem 3.3 in [3]).

Theorem C. *Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Let p be a singular point in Ω . Suppose $\text{curl}\vec{F}(p) \neq 0$. Then we have*

(a) *Either p is an isolated singular point, i.e., there exists a neighborhood $V \subset \Omega$ of p such that V contains no other singular points except p , or there exists at least one C^0 singular curve $\gamma : [0, 1] \rightarrow \Omega$ (i.e., γ is continuous and $\gamma(s)$ is a singular point for each $s \in [0, 1]$) such that $\gamma(0) = p$ and $\gamma(1) \neq p$.*

(b) *Moreover, there is a neighborhood U of p such that for any singular point $q \in U$ there exists a C^0 singular curve $\beta : [0, 1] \rightarrow U$ with $\beta(0) = p$ and $\beta(1) = q$.*

Theorem C (b) implies that the singular set is locally path-connected. It is possible to construct examples having several singular curves meeting at a singular point (see Examples 4.2, 4.3). In case a singular point is isolated, we can describe the local configuration of characteristic curves near such a singular point in a general situation.

Theorem D. *Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Let p be an isolated singular point in Ω . Suppose $\text{curl}\vec{F}(p) \neq 0$. Then there exists $r_0 > 0$ such that for any $q \in \bar{B}_{r_0}(p) \setminus \{p\}$, q is non-singular and the characteristic curve Γ_q passing through q has to meet p . Moreover, the unit tangent vector N^\perp of Γ_q has a limit at p , denoted $v(q)$, and the map $\psi : q \in \partial B_{r_0}(p) \rightarrow v(q) \in T_p\Omega$ is a homeomorphism onto the space of unit tangent vectors at p .*

According to Theorem A in [5], H (plus initial condition) determines θ ($\frac{d\theta}{d\rho} = -H$. Here we use ρ instead of σ as unit-speed parameter) and hence the characteristic curves through (1.8) of [5]. In fact, the characteristic curves $(x(\rho), y(\rho))$ satisfy the following system of ordinary differential equations of second order

$$(1.9) \quad \frac{d^2x}{d\rho^2} = H \frac{dy}{d\rho}, \quad \frac{d^2y}{d\rho^2} = -H \frac{dx}{d\rho}.$$

From Theorem D the value of u near p is completely determined by the values of u at p , \vec{F} , and H by integrating $du + F_1 dx + F_2 dy = 0$ along the characteristic curves.

Corollary E. *Suppose we are in the situation of Theorem D. Then the value of u near p is completely determined by the values of u at p , \vec{F} , and H .*

We remark that in the case of $H = 0$ and $\vec{F} = (-y, x)$, the graph defined by u in Corollary E is a plane. Let \mathcal{H}^2 denote the 2-dimensional Hausdorff measure. For the size of the singular set, we have the following result.

Theorem F. *Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^1(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Suppose also $\text{curl}\vec{F} \neq 0$ in Ω . Then $\mathcal{H}^2(S_{\vec{F}}(u)) = 0$.*

We remark that a C^1 smooth p -minimal graph defined by u is a special case of Theorem F. Because of Theorem F the condition (1.2) for a C^1 smooth p -minimal graph defined by u is reduced to

$$(1.10) \quad \int_{\Omega} N \cdot \nabla \varphi = 0$$

(note that $H = 0$) for any $\varphi \in C_0^\infty(\Omega)$, where $N := \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}$ with $\vec{F} = (-y, x)$. To study the regularity of a singular curve passing through p_0 , we define the notion of the (inverse) expanding rate of characteristic curves along such a singular curve, $\lambda_\pm(p_0)$ (see (5.31)). p_0 is nondegenerate if both $\lambda_+(p_0) \neq 0$ and $\lambda_-(p_0) \neq 0$ (see Definition 5.1 and (5.31)). It is clear from the definition that the set of nondegenerate singular points is relatively open in $S_{\vec{F}}(u)$. The singular curve passing through a nondegenerate point is C^0 a priori for $u \in C^1$. However, we have the regularity result and the equal-angle condition as follows.

Theorem G. *Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^1(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended*

over singular points) in Ω . Suppose $\text{curl } \vec{F}(p) \neq 0$ for any nondegenerate singular point p . Then we have

- (a) the set of nondegenerate singular points consists of C^1 smooth curves.
- (b) two characteristic curves issuing from a nondegenerate singular point p_0 have the same angle with the tangent line of the singular curve through p_0 .

To show Theorem G, we study a more general situation. Consider θ as an independent variable, satisfying the equation

$$(1.11) \quad \text{div}(\cos \theta, \sin \theta) \equiv (\cos \theta)_x + (\sin \theta)_y = H.$$

Suppose $\mathcal{H}^2(K) = 0$ for a subset K in Ω . A solution $\theta \in C^0(\Omega \setminus K)$ is a weak solution to (1.11) (with $H \in L^1_{loc}(\Omega)$, say) if there holds

$$\int_{\Omega} N \cdot \nabla \varphi + \int_{\Omega} H \varphi = 0$$

for any $\varphi \in C_0^\infty(\Omega)$, where we have written N as $(\cos \theta, \sin \theta)$. Suppose $\gamma \subset K$ is a C^0 curve. We can still define what a nondegenerate point of γ is (see Definition 5.1). We also define what a crack point of γ is (see Definition 5.2). Roughly speaking, a crack point is a point at which N has different limits along two characteristic curves.

Theorem G'. *Suppose $\mathcal{H}^2(K) = 0$ for a subset K in Ω . Let $\theta \in C^0(\Omega \setminus K)$ be a weak solution to (1.11) with $H \in C^1(\Omega)$. Then we have*

- (a) the set of nondegenerate crack points consists of C^1 smooth curves.
- (b) two characteristic curves issuing from a nondegenerate crack point p_0 have the same angle with the tangent line of the curve of nondegenerate crack points through p_0 .

Since a singular point is a crack point, we obtain Theorem G from Theorem G' (see Section 5 for more details). On the other hand, in the situation that N ($\equiv (\cos \theta, \sin \theta)$) is defined by u as in Theorem G, we show that a crack point is in fact a singular point (see Theorem 5.4).

In Section 2 we give the proofs of Theorem A and Theorem B. In Section 3 we show those of Theorem C and Theorem D. Some crucial examples are given in Section 4. Theorems F and G are proved in Section 5.

In Riemannian geometry, we have Gauss and Codazzi equations in the submanifold theory. The fundamental theorem for hypersurfaces in a Euclidean space says that the equations of Gauss and Codazzi are exactly the integrability conditions for finding an isometric imbedding with prescribed metric and second fundamental form (see, for instance, page 47 in [10]). In pseudohermitian geometry, we have the analogous fundamental theorem for surfaces in the 3-dimensional Heisenberg group \mathbf{H}_1 . For simplicity we work in the C^∞ category.

Theorem H. *Given a nonzero C^∞ smooth vector field V , a positive C^∞ smooth function D , and a C^∞ smooth function H on an open neighborhood U of a point p in the (ξ, η) plane. Suppose D satisfies the equations*

$$(1.12) \quad DD'' = 2(D' - 1)(D' - 2) + (H^2 + P(H))D^2$$

$$(1.13) \quad L_V P = -\left(\frac{2 - D'}{D}\right)P + HV$$

for a nonzero C^∞ smooth vector field P , transversal to V , such that (V, P) has the same orientation as $(\partial_\xi, \partial_\eta)$, where we denote $V(D)$, $V(V(D))$, and the Lie derivative in the direction V by D' , D'' , and L_V , resp.. Then in a perhaps smaller neighborhood $U' \subset U$ of p

(1) there exist a C^∞ smooth orientation-preserving diffeomorphism: $(\xi, \eta) \rightarrow (x = x(\xi, \eta), y = y(\xi, \eta))$ from U' onto its image in R^2 and a C^∞ smooth function $\theta = \theta(\xi, \eta)$ on U' such that

$$(1.14) \quad \begin{aligned} V(\theta) &= -H \\ V(x) &= \sin \theta, \quad V(y) = -\cos \theta. \end{aligned}$$

In addition, the vector field N satisfying $N(x) = \cos \theta$, $N(y) = \sin \theta$ is equal to P and has the property that

$$(1.15) \quad N(\theta) = \frac{2 - V(D)}{D}.$$

(2) Moreover, there exists a C^∞ smooth function $z = z(\xi, \eta)$ on U' to make a C^∞ smooth embedding: $(\xi, \eta) \in U' \rightarrow (x = x(\xi, \eta), y = y(\xi, \eta), z = z(\xi, \eta)) \in \mathbf{H}_1$ such that the image is a C^∞ smooth graph $z = u(x, y)$, $D = \sqrt{(u_x - y)^2 + (u_y + x)^2}$, $N = \frac{(u_x - y, u_y + x)}{D}$, and

$$(1.16) \quad \operatorname{div} N = H$$

$$(1.17) \quad \operatorname{div} DV = 2$$

where "div" denotes the divergence operator in the x, y coordinates.

Note that we can always solve in P for equation (1.13). So the real condition on the given data is (1.12). We remark that in the x, y coordinates, V is identified with N^\perp whose integral curves are the characteristic curves (compare (1.14) with (2.21), (2.23) in [3]). Note that we may consider (1.16) as the (extrinsic) Gauss-like equation in our surface theory. The Codazzi-like equation (1.12) with $P = N$ (only involving the derivatives in the "intrinsic" direction V) can be deduced from (1.17) together with (1.16) through taking the derivative of (1.15) in the direction V and applying (1.13) to θ (also compare with the proof of Theorem A).

Finally we study the global property of the singular set through a Hopf-type index theorem. Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Suppose $\partial\Omega$ consists of finitely many components C_j , $j = 1, 2, \dots, l$, where each C_j is a C^1 smooth, simple closed curve. Assume the singular set $S_{\vec{F}}(u) \subset \Omega$ is compact (which implies that $S_{\vec{F}}(u)$ does not touch the boundary $\partial\Omega$) and the characteristic curves hit each C_j in the following pattern. For $q \in C_j$ except finitely many points, there is only one characteristic curve L_q hitting q transversally (meaning that the vector N^\perp along L_q and the tangent vector of C_j at q are independent). Let p be one of those exceptional points. Consider the line field (1-dimensional distribution) defined by the tangent lines (the lines having the direction $\pm N^\perp(u)$) of the characteristic curves, denoted as \mathcal{D} . Denote the restriction of \mathcal{D} to a small neighborhood $U = B_\varepsilon(p) \cap \Omega$ of p by \mathcal{D}_U . Take another copy of U and the corresponding line field, denoted as U' and \mathcal{D}'_U , resp.. We glue U' with U along the

boundary C_j to get a (two-sided) neighborhood \tilde{U} of p . Denote the line field on \tilde{U} obtained from \mathcal{D}_U and \mathcal{D}'_U by $\tilde{\mathcal{D}}_U$. Define

$$(1.18) \quad \text{index}(p, \mathcal{D}_U) = \frac{1}{2} \text{index}(p, \tilde{\mathcal{D}}_U)$$

where $\text{index}(p, \tilde{\mathcal{D}}_U)$ is the index of p with respect to the line field $\tilde{\mathcal{D}}_U$ (smoothing it near $\tilde{U} \cap C_j$ while keeping the topological type of $\tilde{\mathcal{D}}_U$) (see p.325 in [19]). Note that $\text{index}(p, \mathcal{D}_U)$ is independent of the choice of small neighborhoods U . See Example 7.1 in Section 7.

Let p_1, \dots, p_m denote those exceptional points of C_j . Denote the restriction of \mathcal{D} to a small neighborhood U_k of p_k by \mathcal{D}_{U_k} . We define the index of C_j with respect to u as follows:

$$(1.19) \quad \text{index}(C_j; u) := \sum_{k=1}^m \text{index}(p_k, \mathcal{D}_{U_k}).$$

Let $\chi(\Omega)$ denote the Euler characteristic number of Ω . We can now formulate a Hopf-type index theorem.

Theorem I. *Let Ω be a bounded domain of R^2 . Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^1(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Suppose $\text{curl}\vec{F} \neq 0$ and $\partial\Omega$ consists of finitely many components C_j , $j = 1, 2, \dots, l$, where each C_j is a C^1 smooth, simple closed curve. Assume the singular set $S_{\vec{F}}(u) \subset \Omega$ is compact and the characteristic curves hit each C_j in the pattern mentioned above. Then we have*

$$(1.20) \quad \chi(\Omega) = \# \pi_0(S_{\vec{F}}(u)) + \sum_{j=1}^l \text{index}(C_j; u)$$

where $\# \pi_0(S_{\vec{F}}(u))$ denotes the number of connected components of $S_{\vec{F}}(u)$.

For $u \in C^1(\bar{\Omega})$ we denote the set of singular points in $\bar{\Omega}$ ($\supset \partial\Omega$ in particular) still by $S_{\vec{F}}(u)$. Let $S(u) := S_{\vec{F}}(u)$ for $\vec{F} = (-y, x)$.

Corollary J. *Let Ω be a bounded domain of R^2 with C^1 smooth boundary. Consider a p -minimal graph over $\bar{\Omega}$, defined by $u \in C^1(\bar{\Omega})$. Suppose Ω is convex and $S(u) \subset\subset \Omega$ and nonempty. Then $\# \pi_0(S(u)) = 1$.*

We remark that Corollary J is not the sharpest version for a convex domain to have only one connected component of the singular set. But we won't pursue it in this paper.

For a compact (connected) surface Σ with no boundary, we would like to know the configuration of its singular set S_Σ . When Σ is C^2 smoothly immersed in a 3-dimensional pseudohermitian manifold with bounded p -mean curvature, we learned from [3] that S_Σ consists of isolated (singular) points and closed C^1 curves. By the C^2 theory the characteristic curves meet at any singular curve having the same tangent line, so the singular curves have no index contribution with respect to the line field associated to the characteristic curves. It follows that the Euler characteristic number $\chi(\Sigma)$ equals the number of isolated singular points. Therefore the genus $g(\Sigma)$ of Σ can only be zero or one (see Theorem E in [3]). If we release

the regularity condition, we wonder if $g(\Sigma)$ can be ≥ 2 , say, for Σ being C^1 smooth and of bounded p -mean curvature.

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2. A CODAZZI-LIKE EQUATION AND PROPERTIES: PROOFS OF THEOREM A AND THEOREM B

We first introduce the s, t coordinates. Let N be a C^0 vector field with $|N| \equiv 1$ on a domain $\Omega \subset R^2$. A system of C^1 smooth local coordinates s, t is called a system of characteristic coordinates if s and t have the property that $\nabla s \parallel N^\perp$ and $\nabla t \parallel N$, i.e., ∇s and ∇t are parallel to N^\perp and N , resp.. It follows that $\{t = \text{constant}\}$ are characteristic curves while $\{s = \text{constant}\}$ are seed curves (which are the integral curves of N). In [5] we proved the existence and studied the properties of such a system of special coordinates under some mild conditions.

We now start to prove Theorem A. By Theorem C in [5] (since $u \in C^1(\Omega)$ is a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$ such that Ω is nonsingular), we can find local (near the point concerned) characteristic coordinates s, t and local positive continuous functions f, g with the property that Nf and $N^\perp g$ exist and are continuous, so that

$$(2.1) \quad \frac{\partial}{\partial s} = \frac{1}{f}N^\perp, \quad \frac{\partial}{\partial t} = \frac{1}{gD}N$$

and

$$(2.2) \quad Nf + fH = 0, N^\perp g + \frac{(\text{curl}\vec{F})g}{D} = 0$$

(see (1.9) and (1.10) in [5], resp.). From Theorem D in [5] we learn that D' ($:= N^\perp D$) exists and is continuous. Moreover, $\theta \in C^1$ and (1.13) and (1.12) in [5] read (note that in [5] we used $\text{rot}\vec{F}$ instead of $\text{curl}\vec{F}$)

$$(2.3) \quad \theta_t = \frac{1}{gD^2}(\text{curl}\vec{F} - D')$$

and

$$(2.4) \quad \theta_s = -\frac{H}{f}.$$

Since $N(H)$ and Nf exist and are continuous, $(\theta_s)_t$ exists and is continuous in view of (2.1). From (2.4) and (2.1) we compute

$$\begin{aligned}
(2.5) \quad (\theta_s)_t &= -\frac{1}{gD}N\left(\frac{H}{f}\right) \\
&= -\frac{1}{gD}\left(\frac{N(H)}{f} - \frac{H(Nf)}{f^2}\right) \\
&= -\frac{1}{fgD}(N(H) + H^2) \quad (\text{by (2.2)}).
\end{aligned}$$

The fact that $(\theta_s)_t$ exists and is continuous implies that $(\theta_t)_s$ exists and equals $(\theta_s)_t$ (hence is continuous) by a fundamental result in calculus (see Lemma 5.4 in [5]). It follows that D'' exists and is continuous in view of (2.3) and (2.1) since $N^\perp(\text{curl}\vec{F})$ exists and is continuous by assumption. From (2.1) and (2.3) we compute

$$\begin{aligned}
(2.6) \quad (\theta_t)_s &= \frac{1}{f}N^\perp(\theta_t) \\
&= \frac{1}{fgD^2}(N^\perp(\text{curl}\vec{F}) - D'') \\
&\quad + \frac{1}{f}(\text{curl}\vec{F} - D')\left(-\frac{N^\perp g}{g^2 D^2} - \frac{2D'}{gD^3}\right) \\
&= \frac{1}{fgD}\left[\frac{N^\perp(\text{curl}\vec{F}) - D''}{D} + \frac{(\text{curl}\vec{F} - D')(\text{curl}\vec{F} - 2D')}{D^2}\right].
\end{aligned}$$

Here we have used (2.2) in the last equality of (2.6). Finally by equating (2.5) and (2.6) we obtain (1.3). We have proved Theorem A.

We are going to prove Theorem B. Let $l = \left|\frac{\text{curl}\vec{F}(p)}{2}\right| > 0$. Let $m = \frac{\text{curl}\vec{F}(p)}{2}$ ($\text{curl}\vec{F}(p)$, resp.) if $\text{curl}\vec{F}(p) > 0$ ($\text{curl}\vec{F}(p) < 0$, resp.). Let $M = \text{curl}\vec{F}(p)$ ($\frac{\text{curl}\vec{F}(p)}{2}$, resp.) if $\text{curl}\vec{F}(p) > 0$ ($\text{curl}\vec{F}(p) < 0$, resp.). So $m < M$. From equation (1.3) we establish the following statement

$$\begin{aligned}
(2.7) \quad &\text{Given } 0 < \delta < \frac{l}{3}, \text{ there exists } 0 < \varepsilon = \varepsilon(\delta) < \bar{\rho} \text{ such that} \\
&\quad \text{for any } 0 < \rho < \varepsilon(\delta) \\
&\text{if } D'(\Gamma(\rho)) \in (-\infty, m - \delta) \cup (M + \delta, \infty), \text{ then } D''(\Gamma(\rho)) > 0 \text{ while} \\
&\text{if } D'(\Gamma(\rho)) \in (m + \delta, M - \delta), \text{ then } D''(\Gamma(\rho)) < 0.
\end{aligned}$$

Next we claim that

$$\begin{aligned}
(2.8) \quad &\text{For any } a \in (-\infty, m - \delta) \cup (m + \delta, M - \delta) \cup (M + \delta, \infty), \text{ there exists} \\
&\text{at most one } \rho \in (0, \varepsilon(\delta)) \text{ such that } D'(\Gamma(\rho)) = a.
\end{aligned}$$

Suppose there are $0 < \rho_1 < \rho_2 < \varepsilon(\delta)$ such that $D'(\rho_1) = D'(\rho_2) = a$. Then there exist $\rho_3, \rho_4, \rho_1 \leq \rho_3 < \rho_4 \leq \rho_2$, such that $D'(\Gamma(\rho_3)) = D'(\Gamma(\rho_1)) = D'(\Gamma(\rho_4)) = D'(\Gamma(\rho_2)) = a$, and either $D'(\Gamma(\rho)) \geq a$ for all $\rho \in [\rho_3, \rho_4]$ or $D'(\Gamma(\rho)) \leq a$ for all $\rho \in [\rho_3, \rho_4]$. In both cases, we have either $D''(\Gamma(\rho_3)) \geq 0$ while $D''(\Gamma(\rho_4)) \leq 0$ or $D''(\Gamma(\rho_3)) \leq 0$ while $D''(\Gamma(\rho_4)) \geq 0$. This contradicts (2.7). We have shown (2.8). (2.8) will be used to show that $\lim_{\rho \rightarrow 0} D'(\Gamma(\rho))$ exists.

Let $a_1 = \liminf_{\rho \rightarrow 0^+} D'(\Gamma(\rho))$ and $a_2 = \limsup_{\rho \rightarrow 0^+} D'(\Gamma(\rho))$. Suppose $a_1 < a_2$. Then there exists $a \in (a_1, a_2)$ and $a \neq m, M$. We can then choose small $\delta \in (0, \frac{l}{3})$

such that $a \in (-\infty, m - \delta) \cup (m + \delta, M - \delta) \cup (M + \delta, \infty)$. By (2.8) we have at most one $\rho \in (0, \varepsilon(\delta))$ such that $D'(\Gamma(\rho)) = a$. On the other hand, there are infinitely many $\rho_j \rightarrow 0$ satisfying $D'(\Gamma(\rho_j)) = a$ by the continuity of $D' \circ \Gamma$ and the definition of a_1 and a_2 . The contradiction implies $a_1 = a_2$, and hence $\lim_{\rho \rightarrow 0} D'(\Gamma(\rho))$ exists.

Let $\bar{a} = \lim_{\rho \rightarrow 0} D'(\Gamma(\rho))$. Note that \bar{a} may be $\pm\infty$. Suppose $\bar{a} \neq m, M$. Then we can find small $\delta \in (0, \frac{1}{3})$ and associated ε such that $D'(\Gamma(\rho))$ is monotonically increasing or decreasing for $\rho \in (0, \varepsilon)$ according to $N^\perp = \pm \frac{\partial}{\partial \rho}$ ($N^\perp = \mp \frac{\partial}{\partial \rho}$, resp.) by (2.7) in case $\bar{a} \in [-\infty, m - \delta) \cup (M + \delta, \infty]$ ($\bar{a} \in (m + \delta, M - \delta)$, resp.). We then make estimate from (1.3) and integrate the resulting inequality to reach a contradiction.

In case $\bar{a} \in [-\infty, m - \delta) \cup (M + \delta, \infty]$, there exists a constant $C_1 > 0$ such that $DD'' \geq C_1$ for $\rho \in (0, \varepsilon)$. Multiplying this inequality by $\frac{D'}{D}$, we obtain

$$(2.9) \quad \frac{1}{2}[(D')^2]' = D'D'' \geq C_1 \frac{D'}{D} \quad (\leq C_1 \frac{D'}{D}, \text{ resp.})$$

if $N^\perp = \frac{\partial}{\partial \rho}$ ($N^\perp = -\frac{\partial}{\partial \rho}$, resp.). Note that if $N^\perp = \frac{\partial}{\partial \rho}$, we have

$$\begin{aligned} D' & : \quad = N^\perp D = \frac{\partial}{\partial \rho} D \\ & = \lim_{\rho \rightarrow 0^+} \frac{D(\Gamma(\rho)) - D(\Gamma(0))}{\rho} = \lim_{\rho \rightarrow 0^+} \frac{D(\Gamma(\rho))}{\rho} \geq 0 \end{aligned}$$

at $p = \Gamma(0)$, where $D \geq 0$ and $D(p) = 0$ since p is a singular point. Integrating (2.9) (for both cases) over $[\rho, \rho_0] \subset (0, \varepsilon)$ gives

$$(2.10) \quad \begin{aligned} & \frac{1}{2}(D')^2(\Gamma(\rho_0)) - \frac{1}{2}(D')^2(\Gamma(\rho)) \\ & \geq C_1 [\log D(\Gamma(\rho_0)) - \log D(\Gamma(\rho))]. \end{aligned}$$

Letting $\rho \rightarrow 0$ in (2.10), we reach a contradiction since the left hand side of (2.10) is bounded from above while the right hand side goes to $+\infty$ in view of $\log D(\Gamma(0)) = \log D(p) = \log 0 = -\infty$.

In case $\bar{a} \in (m + \delta, M - \delta)$, there exists a constant $C_2 > 0$ such that $DD'' \leq -C_2$ for $\rho \in (0, \varepsilon)$. Multiplying this inequality by $\frac{D'}{D}$ and integrating as above give

$$(2.11) \quad \begin{aligned} & \frac{1}{2}(D')^2(\Gamma(\rho_0)) - \frac{1}{2}(D')^2(\Gamma(\rho)) \\ & \leq -C_2 [\log D(\Gamma(\rho_0)) - \log D(\Gamma(\rho))]. \end{aligned}$$

Letting $\rho \rightarrow 0$ in (2.11), we observe that the left hand side is a finite number while the right hand side goes to $-\infty$, a contradiction. Altogether we can conclude that $\bar{a} = m$ or M . We have proved (1.8) and the statement (b) in Theorem B follows.

To prove (c) in Theorem B, we observe that $\text{curl} \vec{F}$ is continuous, nonzero, and hence $\text{curl} \vec{F}(p)$ and $\text{curl} \vec{F}(q)$ have the same sign if p and q are connected by a characteristic curve Γ . Now N^\perp points in an inward (outward, resp.) direction of Γ at both p and q if $\text{curl} \vec{F} > 0$ ($\text{curl} \vec{F} < 0$, resp.) at p and q . This contradicts the continuity of N^\perp (on Γ). We have shown the nonexistence of Γ connecting p and q , and hence (c).

In the Appendix we generalize equation (1.3) and prove a result analogous to Theorem B (a).

3. LOCAL CONFIGURATION OF THE SINGULAR SET

Let Ω be a domain of R^2 . Let $u \in C^1(\Omega)$ and $\vec{F} = (F_1, F_2) \in C^1(\Omega)$. Recall that a point $p \in \Omega$ is called singular if $\nabla u + \vec{F} = 0$ at p . Let $S_{\vec{F}}(u)$ denote the set of all singular points. Define $\vec{G}^\perp := (G_2, -G_1)$ for $\vec{G} = (G_1, G_2)$. Recall that $\text{curl}\vec{F} := (F_2)_x - (F_1)_y$.

Lemma 3.1. *Suppose $\text{curl}\vec{F} \neq 0$ in Ω . Then $S_{\vec{F}}(u)$ is nowhere dense in Ω .*

Proof. First note that $S_{\vec{F}}(u)$ is closed. So if $S_{\vec{F}}(u)$ is not nowhere dense in Ω , then there is a point $p_1 \in S_{\vec{F}}(u)$ such that $S_{\vec{F}}(u)$ contains $B_{r_1}(p_1)$, a ball of center p_1 with radius $r_1 > 0$. Take a sequence of C^∞ smooth functions u_n such that u_n converges to u in C^1 norm on the closure of $B_{r_2}(p_1)$ for $0 < r_2 < r_1$. Since $\nabla u + \vec{F} = 0$ in $B_{r_1}(p_1)$, we have (ν denotes the unit outer normal)

$$\begin{aligned}
(3.1) \quad 0 &= \oint_{\partial B_{r_2}(p_1)} (\nabla u + \vec{F})^\perp \cdot \nu \\
&= \lim_{n \rightarrow \infty} \oint_{\partial B_{r_2}(p_1)} (\nabla u_n + \vec{F})^\perp \cdot \nu \\
&= \int_{B_{r_2}(p_1)} \text{div}(\nabla u_n + \vec{F})^\perp \quad (\text{by the divergence theorem}) \\
&= \int_{B_{r_2}(p_1)} \text{div}\vec{F}^\perp \quad (\text{since } \text{div}(\nabla u_n)^\perp = 0) \\
&= \int_{B_{r_2}(p_1)} \text{curl}\vec{F}.
\end{aligned}$$

Since $\text{curl}\vec{F}$ is continuous on $B_{r_2}(p_1)$, a connected set, we must have either $\text{curl}\vec{F} > 0$ in $B_{r_2}(p_1)$ or $\text{curl}\vec{F} < 0$ in $B_{r_2}(p_1)$. This contradicts (3.1). \square

When H is the p (or H)-mean curvature (see (1.1)), $\vec{F} = (-y, x)$, so $\text{curl}\vec{F} = 2$ and hence Lemma 3.1 applies. For simplicity we will only consider the case of p (or H)-minimal graphs in the following discussion. Since $H = 0$, the characteristic curves are straight lines. We often call them characteristic lines (here line may just mean line segment). Recall that by a domain we mean an open and connected set. For a subset $A \subset R^n$ we define an ε -neighborhood $N_\varepsilon(A)$ by

$$N_\varepsilon(A) := \{p \in R^n \mid d(p, A) < \varepsilon\}$$

where $d(p, A) := \inf\{d(p, q) \mid q \in A\}$ and $d(\cdot, \cdot)$ is the Euclidean distance. For $A, B \subset R^n$, we define the Hausdorff distance between A and B to be the infimum of $\varepsilon > 0$ such that $B \subset N_\varepsilon(A)$, $A \subset N_\varepsilon(B)$.

Lemma 3.2. *Consider a C^1 smooth p -minimal graph defined by u over a plane convex domain Ω . Let $\tilde{\Gamma}_\infty$ be a straight line such that $\Gamma_\infty := \tilde{\Gamma}_\infty \cap \Omega$ divides Ω into two disjoint nonempty domains Ω^+ , Ω^- . Let $\tilde{\Gamma}_j$ be a family of straight lines such that all $\Gamma_j := \tilde{\Gamma}_j \cap \Omega$ are characteristic. Suppose $\{\Gamma_j\}$ converges to Γ_∞ in the sense that the Hausdorff distance between Γ_j and Γ_∞ tends to zero as $j \rightarrow \infty$.*

Then Γ_∞ is a characteristic line (segment). In particular, Γ_∞ contains no singular point.

Proof. We first observe that since the singular set $S(u)$ is closed, $\Gamma_\infty \setminus S(u)$ is open in Γ_∞ . So $\Gamma_\infty \setminus S(u)$ is empty or the union of open line segments Γ_∞^k , $k = 1, 2, \dots$. Each Γ_∞^k is characteristic since it is the (Hausdorff) limit of characteristic lines. If one of the Γ_∞^k 's has two singular end points in Ω , we reach a contradiction since we cannot have a characteristic line connecting two singular points by Theorem B (c). Therefore we have only four possibilities:

Case 1– $\Gamma_\infty \setminus S(u)$ is empty, i.e., $\Gamma_\infty \subset S(u)$;

Case 2– $\Gamma_\infty \setminus S(u) = \Gamma_\infty^1$;

Case 3– $\Gamma_\infty \setminus S(u) = \Gamma_\infty^1 \cup \Gamma_\infty^2$;

Case 4– $\Gamma_\infty \setminus S(u) = \Gamma_\infty$, i.e., Γ_∞ contains no singular points,

where each of Γ_∞^1 and Γ_∞^2 has only one singular end point in Ω . In Case 3, if there is only one singular point $p \in \Gamma_\infty$, then we can decompose Γ_∞ as a disjoint union $\Gamma_\infty^+ \cup \{p\} \cup \Gamma_\infty^-$ where $\Gamma_\infty^+ = \Gamma_\infty^1$ and $\Gamma_\infty^- = \Gamma_\infty^2$ are characteristic rays emitted by p in opposite directions respectively (see Figure 3.1 below). We claim that this is impossible.

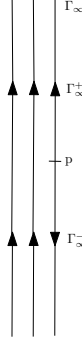


Figure 3.1

Let $B_r(p) \subset \Omega$ denote a ball of center p with radius r . Take $p_j^+ \in \Gamma_\infty^+ \cap B_r(p)$ and $p_j^- \in \Gamma_\infty^- \cap B_r(p)$ approaching p . There exist a large integer $n(j)$ and $q_j^+, q_j^- \in \Gamma_{n(j)}$ such that

$$(3.2) \quad |N^\perp(q_j^\pm) - N^\perp(p_j^\pm)| < \frac{1}{j}$$

by the continuity of N^\perp . On the other hand, we have

$$(3.3) \quad \lim_{p_j^+ \rightarrow p} N^\perp(p_j^+) = - \lim_{p_j^- \rightarrow p} N^\perp(p_j^-)$$

according to Theorem B (b) in Section 1 (for the case of Γ_∞ being a straight line, $N^\perp(p_j^+) = -N^\perp(p_j^-)$ considered as free vectors). From (3.2), we get

$$\begin{aligned} \lim_{p_j^+ \rightarrow p} N^\perp(p_j^+) &= \lim_{j \rightarrow \infty} N^\perp(q_j^+) \\ &= \lim_{j \rightarrow \infty} N^\perp(q_j^-) = \lim_{p_j^- \rightarrow p} N^\perp(p_j^-) \end{aligned}$$

which contradicts (3.3) (for the second equality, we have used $N^\perp(q_j^+) = N^\perp(q_j^-)$ since $\Gamma_{n(j)}$ is a straight line). We have proved our claim. So in Case 3, the remaining situation is that $\Gamma_\infty = \Gamma_\infty^1 \cup I \cup \Gamma_\infty^2$ (disjoint union) where $I \subset S(u)$ is a closed line segment. Case 1, Case 2, and this situation of Case 3 have the common feature that Γ_∞ contains an open (and hence a shorter closed) line segment which consists of singular points. Let $\Upsilon \subset \Gamma_\infty$ denote such a closed singular line segment. We are going to show that this is impossible.

Near Υ , we can parametrize Γ_∞ (Γ_j , resp.) by the map γ (γ_j , resp.): $(-a, a) \rightarrow \Gamma_\infty$ (Γ_j , resp.), where $s \in (-a, a)$ is the unit-speed parameter with $\frac{\partial}{\partial s} = N^\perp$ on Γ_j , such that $\gamma([- \varepsilon, \varepsilon]) \subset \Upsilon$ for some $0 < \varepsilon < a$ and $\gamma_j(s) \rightarrow \gamma(s)$ as $j \rightarrow \infty$ for $s \in [- \varepsilon, \varepsilon]$. We claim that if there is a point $p_0 \in \Gamma_j$ with $D'(p_0) > 2$ ($D'(p_0) = 2$, $1 < D'(p_0) < 2$, $D'(p_0) = 1$, $D'(p_0) < 1$, resp.), then $D'(p) > 2$ ($D'(p) = 2$, $1 < D'(p) < 2$, $D'(p) = 1$, $D'(p) < 1$, resp.) for all $p \in \Gamma_j$. First observe that $D' \equiv 2$ ($D' \equiv 1$, resp.) on Γ_j is a solution to (1.5). Our claim follows from the uniqueness of solutions to (1.5) with initial data $(D(p_0), D'(p_0))$. By the pigeonhole principle, at least one of the five cases: $D' > 2$, $D' = 2$, $1 < D' < 2$, $D' = 1$, $D' < 1$ holds on Γ_j for infinitely many j 's. Suppose that $D' > 2$ on Γ_j for infinitely many j 's. Then we have

$$(3.4) \quad D(\gamma_j(\varepsilon)) - D(\gamma_j(0)) \geq 2\varepsilon$$

for infinitely many j 's. Letting $j \rightarrow \infty$ in (3.4) and noting that $D = 0$ on Υ give

$$0 = D(\gamma(\varepsilon)) - D(\gamma(0)) \geq 2\varepsilon,$$

a contradiction. Similarly we can reach a contradiction for the cases $D' = 2$, $1 < D' < 2$, $D' = 1$, resp.. Now we are left to deal with the remaining case: $D' < 1$ on infinitely many Γ_j 's (still denoted as Γ_j). From (1.5) we learn that

$$D'' = \frac{2(D' - 1)(D' - 2)}{D} > 0.$$

So D' is strictly increasing on each Γ_j . Hence one of the three cases: $D'(\gamma_j(-\frac{\varepsilon}{3})) \leq -\frac{1}{2}$, $D'(\gamma_j(\frac{\varepsilon}{3})) \geq \frac{1}{2}$, $-\frac{1}{2} \leq D' \leq \frac{1}{2}$ on $\gamma_j([-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}])$ holds for infinitely many j 's by the pigeonhole principle. Suppose that $D'(\gamma_j(-\frac{\varepsilon}{3})) \leq -\frac{1}{2}$ for infinitely many j 's. Then $D' \leq -\frac{1}{2}$ on $\gamma_j([-\frac{2\varepsilon}{3}, -\frac{\varepsilon}{3}])$ since D' is strictly increasing on each Γ_j . It follows that

$$D(\gamma_j(-\frac{2\varepsilon}{3})) - D(\gamma_j(-\frac{\varepsilon}{3})) \geq \frac{1}{2} \cdot \frac{\varepsilon}{3}.$$

Taking $j \rightarrow \infty$, we obtain that $0 = D(\gamma(-\frac{2\varepsilon}{3})) - D(\gamma(-\frac{\varepsilon}{3})) \geq \frac{\varepsilon}{6}$, a contradiction. Next for the case $D'(\gamma_j(\frac{\varepsilon}{3})) \geq \frac{1}{2}$, we have $D' \geq \frac{1}{2}$ on $\gamma_j([\frac{\varepsilon}{3}, \frac{2\varepsilon}{3}])$ since D' is strictly increasing on each Γ_j . A similar argument as above will lead to a contradiction. Now let us assume that $-\frac{1}{2} \leq D' \leq \frac{1}{2}$ on $\gamma_j([-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}])$ for infinitely many j 's. From (1.5) we have

$$(3.5) \quad \begin{aligned} D'' &= \frac{2(D' - 1)(D' - 2)}{D} \\ &\geq \frac{3}{2} \cdot \frac{1}{D} \end{aligned}$$

on $\gamma_j([-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}])$. On the other hand, we have the Taylor expansion to the second order for D :

$$(3.6) \quad \begin{aligned} & D(\gamma_j(\frac{\varepsilon}{3})) - D(\gamma_j(-\frac{\varepsilon}{3})) \\ &= D'(\gamma_j(-\frac{\varepsilon}{3}))\frac{2\varepsilon}{3} + \frac{D''(\gamma_j(\xi_j))}{2}(\frac{2\varepsilon}{3})^2 \end{aligned}$$

for some ξ_j between $-\frac{\varepsilon}{3}$ and $\frac{\varepsilon}{3}$. From (3.6), condition on D' , and (3.5), we have

$$(3.7) \quad \begin{aligned} & D(\gamma_j(\frac{\varepsilon}{3})) - D(\gamma_j(-\frac{\varepsilon}{3})) \\ &\geq -\frac{1}{2} \cdot \frac{2\varepsilon}{3} + \frac{3}{4} \cdot \frac{1}{D(\gamma_j(\xi_j))} \cdot (\frac{2\varepsilon}{3})^2. \end{aligned}$$

Letting $j \rightarrow \infty$ in (3.7) and noting that $D(\gamma_j(\xi_j)) \rightarrow 0$, we obtain that $0 = D(\gamma(\frac{\varepsilon}{3})) - D(\gamma(-\frac{\varepsilon}{3})) \geq \infty$, a contradiction. Altogether we have excluded Cases 1, 2, and 3. But Case 4 just means that Γ_∞ contains no singular point and hence it is a characteristic line. \square

We can extend Lemma 3.2 as follows. Although Lemma 3.2' is more general than Lemma 3.2, we include the above proof of Lemma 3.2 for the reader to understand the situation better.

Lemma 3.2'. *Let Ω be a bounded domain of R^2 . Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Suppose $\text{curl}\vec{F} \neq 0$. Take $p_0 \in \Omega$, $B_{r_1}(p_0) \subset\subset \Omega$ such that $0 < r_1 \leq (\sup_{B_{r_1}(p_0)} |H|)^{-1}$. Take a sequence of nonsingular points $p_j \in B_{r_1}(p_0)$ converging to p_0 . Suppose for each j , the characteristic curve passing through p_j does not hit p_0 or any singular points in $B_{r_1}(p_0)$ before it meets $\partial B_{r_1}(p_0)$ at two points q_j^1, q_j^2 . Let $\tilde{\Gamma}_j$ denote the characteristic curve passing through p_j with q_j^1, q_j^2 as two end points. Then*

(a) *There exists $0 < r_2 < r_1$ such that a subsequence of closed arcs $\Gamma_j \subset \tilde{\Gamma}_j \cap B_{r_2}(p_0)$, containing p_j and an open arc, converges to a closed arc $\Gamma_\infty \subset B_{r_2}(p_0)$ in C^2 with respect to a certain parametrization.*

(b) *Γ_∞ contains p_0 and an open arc, but contains no singular points. Moreover, Γ_∞ is a characteristic curve passing through p_0 .*

Proof. Take $0 < r_2 \ll r_1$ ($\leq (\sup_{B_{r_1}(p_0)} |H|)^{-1}$ by assumption) such that $\Gamma_j^0 := \tilde{\Gamma}_j \cap B_{r_2}(p_0)$ is a connected arc passing through p_j (note that $-H$ is the curvature of the curve $\tilde{\Gamma}_j$ [5]) (see Figure 3.1' below).

For j large, there exists s_0 (independent of j) > 0 such that near p_j we can parametrize Γ_j^0 by the (C^2 smooth according to [5]) map $\gamma_j : [-s_0, s_0] \rightarrow \Gamma_j^0 \subset B_{r_2}(p_0)$, $\gamma_j(0) = p_j$, where $s \in [-s_0, s_0]$ is the unit-speed parameter with $\frac{\partial}{\partial s} = N^\perp$ on Γ_j . We choose an angular function θ ranging in $[0, 2\pi)$, which works for all Γ_j^0 . Let $\theta_j(s) := \theta(\gamma_j(s))$. It follows that for j large and $s \in [-s_0, s_0]$, we have

$$(3.8) \quad \begin{aligned} |\theta_j(s)| &\leq C_1 \\ |\theta'_j(s)| &= |-H(\gamma_j(s))| \leq C_2 \end{aligned}$$

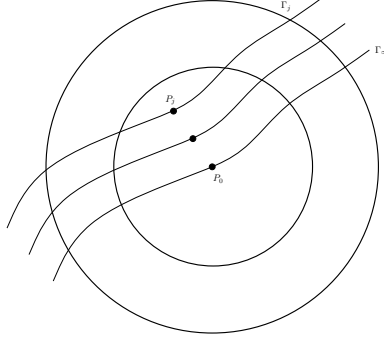


Figure 3.1'

(Theorem A in [5]) where the constants C_1 and C_2 are independent of j and s . By the Arzela-Ascoli theorem in view of (3.8), we can find a subsequence of θ_j , still denoted θ_j , such that θ_j converges to θ_∞ in $C^0([-s_0, s_0])$. So $\{\theta_j\}$ is Cauchy. We claim that $\{\gamma_j\}$ is Cauchy (with respect to C^0 -topology). Write

$$(3.9) \quad \begin{aligned} \gamma_j(s) - \gamma_k(s) &= (\gamma_j(s) - \gamma_j(0)) - (\gamma_k(s) - \gamma_k(0)) \\ &\quad + (\gamma_j(0) - \gamma_k(0)). \end{aligned}$$

From the definition of characteristic curves, we have

$$\gamma_j(s) - \gamma_j(0) = \int_0^s (\sin \theta_j(\tau), -\cos \theta_j(\tau)) d\tau$$

and hence

$$(3.10) \quad \begin{aligned} &(\gamma_j(s) - \gamma_j(0)) - (\gamma_k(s) - \gamma_k(0)) \\ &= \int_0^s [(\sin \theta_j(\tau) - \sin \theta_k(\tau), -\cos \theta_j(\tau) + \cos \theta_k(\tau))] d\tau. \end{aligned}$$

Since $\{\theta_j\}$ is Cauchy, we have $(\gamma_j(s) - \gamma_j(0)) - (\gamma_k(s) - \gamma_k(0))$ is small uniformly in $s \in [-s_0, s_0]$ for j, k large enough by (3.10). On the other hand, $\gamma_j(0) = p_j$ ($\in R^2$) converges to p_0 , so $\{p_j\}$ is Cauchy and hence $\gamma_j(0) - \gamma_k(0)$ is small for j, k large enough. Altogether we have shown that $\{\gamma_j\}$ is Cauchy in view of (3.9). Thus γ_j converges in $C^0([-s_0, s_0], R^2)$ and we denote the limit by γ_∞ . So if we take Γ_j in (a) to be $\gamma_j([-s_0, s_0])$, then $\Gamma_\infty = \gamma_\infty([-s_0, s_0])$. Since θ_j converges to θ_∞ in $C^0([-s_0, s_0])$, γ_j converges to γ_∞ in $C^1([-s_0, s_0], R^2)$ with $\frac{d\gamma_\infty(s)}{ds} = (\sin \theta_\infty(s), -\cos \theta_\infty(s))$ as the following argument shows. Write $\gamma_j(s) = (x_j(s), y_j(s))$ and $\gamma_\infty(s) = (x_\infty(s), y_\infty(s))$. From the mean value theorem we have

$$\frac{x_j(s_2) - x_j(s_1)}{s_2 - s_1} = \sin \theta_j(\hat{s}), \quad \frac{y_j(s_2) - y_j(s_1)}{s_2 - s_1} = -\cos \theta_j(\bar{s})$$

for $s_1 < \hat{s}, \bar{s} < s_2$. Taking $j \rightarrow \infty$ in the above formulas, we obtain

$$\frac{x_\infty(s_2) - x_\infty(s_1)}{s_2 - s_1} = \sin \theta_\infty(\hat{s}), \quad \frac{y_\infty(s_2) - y_\infty(s_1)}{s_2 - s_1} = -\cos \theta_\infty(\bar{s}).$$

Then letting $s_2 \rightarrow s_1$ and hence $\hat{s}, \bar{s} \rightarrow s_1$, we get

$$(3.11) \quad \frac{dx_\infty(s_1)}{ds} = \sin \theta_\infty(s_1), \quad \frac{dy_\infty(s_1)}{ds} = -\cos \theta_\infty(s_1).$$

Therefore $\gamma'_j = (\sin \theta_j, -\cos \theta_j)$ converges in $C^0([-s_0, s_0], R^2)$ to $(\sin \theta_\infty, -\cos \theta_\infty)$ which equals γ'_∞ by (3.11). Let $H_j(s) := H(\gamma_j(s))$ and $H_\infty(s) := H(\gamma_\infty(s))$. A similar argument with $x_j, \sin \theta_j$ replaced by $\theta_j, -H_j$, respectively in the above argument shows that

$$(3.12) \quad \frac{d\theta_\infty(s_1)}{ds} = -H_\infty(s_1)$$

(noting that θ_j is C^1 smooth and $\theta'_j(s) = -H_j(s)$ from Theorem A in [5]). Now we compute

$$\begin{aligned} \gamma''_j &= (\cos \theta_j, \sin \theta_j)\theta'_j \\ &= (\cos \theta_j, \sin \theta_j)(-H_j) \rightarrow (\cos \theta_\infty, \sin \theta_\infty)(-H_\infty) \end{aligned}$$

in $C^0([-s_0, s_0])$, which equals γ''_∞ according to (3.11), (3.12). We have shown that Γ_j converges to Γ_∞ in C^2 with respect to the parametrization given by γ_j, γ_∞ , respectively. We have completed the proof of (a).

For the proof of (b), it is clear that $p_0 \in \Gamma_\infty$ and Γ_∞ contains the open arc $\gamma_\infty((-s_0, s_0))$. To show that Γ_∞ contains no singular points, we mimic the reasoning as in the proof of Lemma 3.2. In view of Theorem B (b), (c) we only have to deal with and exclude (by contradiction) the situation that Γ_∞ contains an open (and hence a shorter closed) arc which consists of singular points. Without loss of generality we may just assume that Γ_∞ is such a closed singular arc. That is to say, $D \equiv 0$ on Γ_∞ . Let

$$\lambda := \inf_{p \in B_{r_2}(p_0)} \text{curl } \vec{F}(p).$$

From the assumption $\text{curl } \vec{F} \neq 0$, we may assume $\lambda > 0$ without loss of generality. Let $\bar{\Gamma}_j := \gamma_j([-s_0, s_0])$ for $0 < \bar{s}_0 < s_0$, independent of j . Let

$$m_j := \inf_{p \in \bar{\Gamma}_j} D'(p), \quad M_j := \sup_{p \in \bar{\Gamma}_j} D'(p).$$

We claim that for j large enough, there holds $M_j < \frac{\lambda}{4}$. Suppose the converse holds. Then there exists a subsequence j_k such that $M_{j_k} \geq \frac{\lambda}{4}$. Hence we can find a sequence of points $q_k \in \bar{\Gamma}_{j_k}$ such that $D'(q_k) \geq \frac{\lambda}{8}$. We then extract a convergent subsequence of q_k , still denoted as q_k . Let $q_\infty = \lim q_k$. It follows that $q_\infty \in \bar{\Gamma}_\infty := \gamma_\infty([-s_0, s_0])$ since γ_j converges to γ_∞ (in C^2). Let $s_k \leq \bar{s}_0$ such that $\gamma_{j_k}(s_k) = q_k$. We are going to show that for k large, there holds

$$(3.13) \quad D'(\gamma_{j_k}(s)) \geq \frac{\lambda}{8}$$

for all $s \in [s_k, s_0] \supset [s_0, s_0]$. If not, there are a subsequence of k , still denoted as k , and a sequence $t_k \in [s_k, s_0]$ such that $D'(\gamma_{j_k}(t_k)) < \frac{\lambda}{8}$. May assume that $D'(\gamma_{j_k}(t_k))$ achieves its minimum over $[s_k, s_0]$ at t_k . From (1.3) we evaluate

$$(3.14) \quad D'' = \frac{2(D' - \frac{\text{curl } \vec{F}}{2})(D' - \text{curl } \vec{F})}{D} + (N^\perp(\text{curl } \vec{F})) + (H^2 + N(H))D$$

at $\gamma_{j_k}(t_k)$. It is easy to see that $D''(\gamma_{j_k}(t_k)) > 0$ for k large enough since $D(\gamma_{j_k}(t_k)) \rightarrow 0$ by the assumption: $D \equiv 0$ on Γ_∞ . So D' is strictly increasing at $\gamma_{j_k}(t_k)$. This contradicts that it achieves a minimum at $\gamma_{j_k}(t_k)$ unless $t_k = s_k$. But at s_k

$D'(\gamma_{j_k}(s_k)) = D'(q_k) \geq \frac{\lambda}{8}$, a contradiction. Now by the mean-value theorem and (3.13), we have

$$(3.15) \quad \begin{aligned} D(\gamma_{j_k}(s_0)) - D(\gamma_{j_k}(\bar{s}_0)) &= D'(\gamma_{j_k}(\tilde{s}_k))(s_0 - \bar{s}_0) \\ &\geq \frac{\lambda}{8}(s_0 - \bar{s}_0). \end{aligned}$$

where $\tilde{s}_k \in [\bar{s}_0, s_0]$. Letting $k \rightarrow \infty$ in the left-hand side of (3.15), we get

$$\begin{aligned} 0 &= 0 - 0 = D(\gamma_\infty(s_0)) - D(\gamma_\infty(\bar{s}_0)) \\ &\geq \frac{\lambda}{8}(s_0 - \bar{s}_0) > 0, \end{aligned}$$

a contradiction. We have proved that for j large enough, there holds

$$(3.16) \quad M_j < \frac{\lambda}{4}.$$

Similarly, we can show that for j large enough, there holds

$$(3.17) \quad m_j > -\frac{\lambda}{4}.$$

We then write

$$(3.18) \quad \begin{aligned} D(\gamma_{j_k}(s_0)) - D(\gamma_{j_k}(\bar{s}_0)) &= D'(\gamma_{j_k}(\bar{s}_0))(s_0 - \bar{s}_0) \\ &\quad + D''(\gamma_{j_k}(\hat{s}_k)) \frac{(s_0 - \bar{s}_0)^2}{2} \end{aligned}$$

for $\hat{s}_k \in [\bar{s}_0, s_0]$. Evaluate (3.14) at $\gamma_{j_k}(\hat{s}_k)$. Observe that in the right-hand side of (3.14), the numerator of the first term is bounded away from zero by (3.16), (3.17) while the denominator D goes to zero, the second term is bounded, and the third term goes to zero. It follows that $D''(\gamma_{j_k}(\hat{s}_k)) \rightarrow \infty$ as $k \rightarrow \infty$. On the other hand, the two terms in the left-hand side of (3.18) tends to zero as $k \rightarrow \infty$ while the first term in the right-hand side of (3.18) is bounded due to (3.16), (3.17). Altogether we get $0 = \infty$ as $k \rightarrow \infty$ in (3.18), a contradiction. We can then conclude that Γ_∞ contains no singular points. Since Γ_∞ is a C^2 limit of Γ_j , it must be characteristic. We have proved (b). \square

Proposition 3.3. *Consider a C^1 smooth p -minimal graph over a planar domain Ω . Let p be a singular point of Ω . Then p emits at least one characteristic ray. That is to say, there exists at least one characteristic line with p as an end point in Ω .*

Proof. By Lemma 3.1 we can find a sequence of nonsingular points q_j converging to p . Let Γ_j denote the characteristic line passing through q_j . Observe that Γ_j can only hit at most one singular point in Ω by Theorem B (c) in Section 1. Suppose there exists a subsequence of Γ_j which do not hit singular points in a ball $B_\delta(p) \subset \Omega$, $\delta > 0$. Then Γ_∞ , the limit of Γ_j , must pass through p . On the other hand, Γ_∞ is a characteristic line by Lemma 3.2. This contradicts the fact that $p \in \Gamma_\infty$ is a singular point. So we may assume that each Γ_j hits a singular point $s_j \in B_\varepsilon(p)$ for some $\varepsilon > 0$ (s_j may coincide with p). We still let Γ_∞ be the limit of Γ_j . Let s_∞ be a limit of (any subsequence) of s_j . Since $q_j \rightarrow p$ and a characteristic line can only

hit at most one singular point (see Theorem B (c)), we must have $s_\infty = p$ and Γ_∞ hits p . So Γ_∞ is the characteristic ray that we want. \square

Next we will discuss the situation in which p emits two different characteristic rays Γ_1, Γ_2 . Denote the fan-shaped region surrounded by Γ_1, p , and Γ_2 by $\{\Gamma_1 p \Gamma_2\}$ (see Figure 3.2 below).

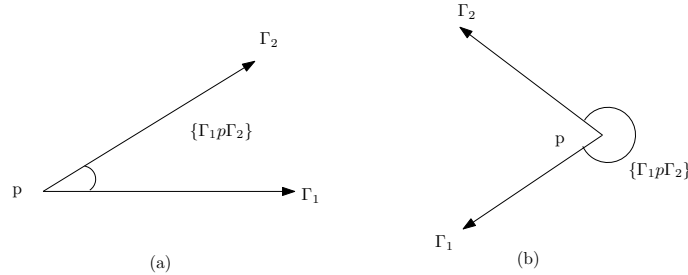


Figure 3.2

Proposition 3.4. *Consider a C^1 smooth p -minimal graph over a planar domain Ω . Let p be a singular point of Ω . Suppose that p emits two different characteristic rays Γ_1, Γ_2 . Suppose $\{\Gamma_1 p \Gamma_2\} \cap \Omega$ contains no singular points. Then p emits characteristic rays (which stop when hitting $\partial\Omega$) at all the directions pointing inside $\{\Gamma_1 p \Gamma_2\}$ (see Figure 3.3).*

Proof. Case 1. Suppose that the angle between Γ_1 and Γ_2 in the region $\{\Gamma_1 p \Gamma_2\}$ is less than π (see Figure 3.3 (a)). Then there exists $\varepsilon > 0$ such that the tangent line L_q at each point $q \in \partial B_\varepsilon(p) \cap \{\Gamma_1 p \Gamma_2\}$ hits either Γ_1 or Γ_2 and lies in Ω before hitting one of them. Now the characteristic line Γ_q passing through q cannot be L_q since two characteristic lines do not intersect at a nonsingular points (see Theorem B' in [5]). So Γ_q must hit $(\Gamma_1 \cup \{p\} \cup \Gamma_2) \cap B_\varepsilon(p)$. Since Γ_q cannot intersect with either Γ_1 or Γ_2 , Γ_q has to hit p . We are done.

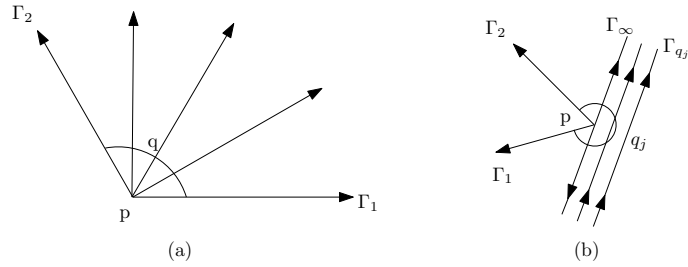


Figure 3.3

Case 2. Suppose that the angle between Γ_1 and Γ_2 in the region $\{\Gamma_1 p \Gamma_2\}$ is larger than π . Assume that the conclusion fails. Then there exists a sequence of characteristic lines Γ_{q_j} through q_j as $q_j \rightarrow p$, which do not hit p (see Figure 3.3 (b)).

Observe that $\{\Gamma_1 p \Gamma_2\} \cap \Omega$ contains no singular points and Γ_{q_j} 's do not intersect, say, in a ball $B_r(p) \subset \Omega$. It follows that a subsequence of Γ_{q_j} converges to a straight line Γ_∞ passing through p . According to Lemma 3.2, Γ_∞ is a characteristic line (so containing no singular point). We have reached a contradiction. \square

Lemma 3.5. *Consider a C^1 smooth p -minimal graph over a planar domain Ω . Let p and q be two singular points in Ω such that $\bar{B}_{d(p,q)}(p) \subset \Omega$ where $d(p, q)$ denotes the Euclidean distance between p and q . Then there exists a C^0 singular curve $\gamma : [0, 1] \rightarrow \bar{B}_{d(p,q)}(p)$ such that $\gamma(0) = p$ and $\gamma(1) \in \partial B_{d(p,q)}(p)$ (note that $\gamma(1)$ may not be q).*

Proof. First by Proposition 3.3 we have a characteristic ray Γ_p emitted from p . We orient the circle $\mathfrak{S} := \partial B_{d(p,q)}(p)$ counterclockwise and view the point $q_0 = \Gamma_p \cap \mathfrak{S}$ as the starting point. Then there exists a point $\bar{q} \in \mathfrak{S}$ (may be q_0) between q_0 and q such that for any point q' on the closed arc $q_0 \bar{q}$, the characteristic line passing through q' meets p and for $\zeta \in \mathfrak{S}$ beyond \bar{q} , but near \bar{q} , the characteristic line Γ_ζ passing through ζ does not meet p (see Figure 3.4 below). Note that since \bar{q} is nonsingular (otherwise, we contradict Lemma 3.2), we may assume that all nearby ζ are nonsingular.

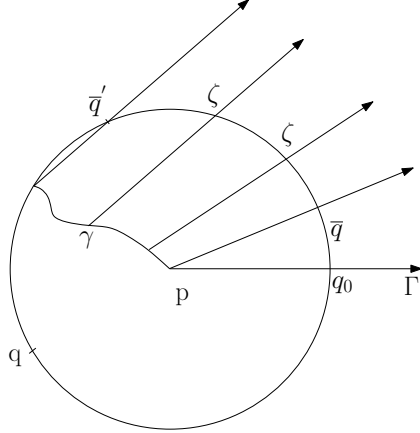


Figure 3.4

We claim that Γ_ζ has to meet a singular point other than p in $B_{d(p,q)}(p)$ for any ζ in the open arc $\bar{q}q'$ where $q' \in \mathfrak{S} \setminus \{\text{closed arc } q_0 \bar{q}\}$ is close enough to \bar{q} . Suppose the converse holds. Then there exists a sequence $\{\zeta_j\}$ such that $\zeta_j \rightarrow \bar{q}$ and Γ_{ζ_j} does not meet any singular point in $B_{d(p,q)}(p)$ for any ζ_j . On the other hand, the sequence of lines $\Gamma_{\zeta_j} \cap B_{d(p,q)}(p)$ converges to a straight line containing $\Gamma_{\bar{q}}$ as $\zeta_j \rightarrow \bar{q}$. Since $\Gamma_{\bar{q}}$ contains the singular point p , we have reached a contradiction to Lemma 3.2. Now let $\mathfrak{s}(\zeta)$ denote the first singular point that Γ_ζ meets in $B_{d(p,q)}(p)$. We claim that \mathfrak{s} is continuous on the open arc $\bar{q}q'$ and can be continuously extended to \bar{q} and q' so that $\mathfrak{s}(\bar{q}) = p$. Suppose the converse holds. Then there exists a sequence $\{\zeta_j\} \rightarrow \bar{\zeta}$ in the closed arc $\bar{q}q'$ such that $\lim \mathfrak{s}(\zeta_j) \in \bar{B}_{d(p,q)}(p)$ exists and

is different from $\mathfrak{s}(\bar{\zeta})$. Let $\bar{\mathfrak{s}} = \lim \mathfrak{s}(\zeta_j)$. Since the set of singular points is closed, $\bar{\mathfrak{s}}$ is a singular point. Observe that $\{\Gamma_{\zeta_j}\}$ converges to a line segment $\bar{\Gamma}$ ending at $\bar{\mathfrak{s}}$. By the continuity of N^\perp at $\bar{\zeta}$, $\bar{\Gamma}$ contains $\Gamma_{\bar{\zeta}}$ since $\mathfrak{s}(\bar{\zeta})$ is the first singular point that $\Gamma_{\bar{\zeta}}$ meets. We are in a situation in which Lemma 3.2 applies. But $\bar{\mathfrak{s}} \neq \mathfrak{s}(\bar{\zeta})$ implies that the limit $\bar{\Gamma}$ contains $\mathfrak{s}(\bar{\zeta})$, a singular point. This contradicts the conclusion of Lemma 3.2. Thus the map \mathfrak{s} with $\mathfrak{s}(\bar{q}) = p$ is continuously defined and the domain closed arc $\bar{q}\bar{q}'$ can be extended so that either $\mathfrak{s}(\bar{q}') \in \mathfrak{S}$ or \bar{q}' becomes a singular point (before hitting q or $= q$). In the latter case, we claim that $\lim_{\zeta \rightarrow \bar{q}'} \mathfrak{s}(\zeta) = \bar{q}' \in \mathfrak{S}$. Otherwise, there is a sequence $\{\zeta'_j\} \rightarrow \bar{q}'$ such that $\lim_{\zeta'_j \rightarrow \bar{q}'} \mathfrak{s}(\zeta'_j)$ exists in $\bar{B}_{d(p,q)}(p)$, but is different from \bar{q}' . Now the limit line segment Γ_∞ of $\Gamma_{\zeta'_j}$ is characteristic by Lemma 3.2 while Γ_∞ meets two distinct singular points, which contradicts Theorem B (c). We have shown that $\mathfrak{s} : \text{closed arc } \bar{q}\bar{q}' \rightarrow \bar{B}_{d(p,q)}(p)$ is continuous with $\mathfrak{s}(\bar{q}) = p$ and $\mathfrak{s}(\bar{q}') \in \mathfrak{S} (:= \partial B_{d(p,q)}(p))$. Parametrize the closed arc $\bar{q}\bar{q}'$ continuously by the map $l : [0, 1] \rightarrow \text{closed arc } \bar{q}\bar{q}'$ so that $l(0) = \bar{q}$, $l(1) = \bar{q}'$. Let $\gamma = \mathfrak{s} \circ l$. It is now clear that γ satisfies the required property. \square

By a similar argument replacing characteristic lines by characteristic curves, we can extend Proposition 3.3 and Lemma 3.5 to a general situation, based on Lemma 3.2'.

Proposition 3.3'. *Let Ω be a bounded domain of R^2 . Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Suppose $\text{curl}\vec{F} \neq 0$. Let p be a singular point of Ω . Then there exists at least one characteristic curve with p as an end point in Ω .*

Lemma 3.5'. *Let Ω be a bounded domain of R^2 . Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Suppose $\text{curl}\vec{F} \neq 0$. Let p and q be two singular points in Ω such that $\bar{B}_{d(p,q)}(p) \subset \Omega$ where $d(p, q)$ denotes the Euclidean distance between p and q . Assume $|H| \leq \frac{1}{2d(p,q)}$ in $B_{d(p,q)}(p)$. Then there exists a C^0 singular curve $\gamma : [0, 1] \rightarrow \bar{B}_{d(p,q)}(p)$ such that $\gamma(0) = p$ and $\gamma(1) \in \partial B_{d(p,q)}(p)$ (note that $\gamma(1)$ may not be q).*

Note that the condition on the bound of H is to guarantee that characteristic curves behave like straight lines in a ball.

Proof. (of Theorem C) Suppose that p is not an isolated singular point. Then near p we can find another singular point q such that $\bar{B}_{d(p,q)}(p) \subset \Omega$ and $|H| \leq \frac{1}{2d(p,q)}$ in $B_{d(p,q)}(p)$. Now it is clear that the C^0 singular curve obtained in Lemma 3.5' satisfies the desired property. We have proved (a).

Let $A \subset S(u)$ denote the path-connected component containing p . To prove (b), suppose the converse holds. Then there exist a sequence $\varepsilon_j \rightarrow 0$ and a sequence $p_j \in [S(u) \cap B_{\varepsilon_j}(p)] \setminus A$. By Proposition 3.3' we can find a characteristic curve Γ_{p_j} emitted from p_j . Take a ball $B_\delta(p)$ of radius δ such that $|H| \leq \frac{1}{2\delta}$ in $B_\delta(p)$. For j large, Γ_{p_j} must go out of the ball $B_\delta(p)$ and hit a nonsingular point \tilde{p}_j on $\partial B_\delta(p)$

by Theorem B (c). There is a subsequence, still denoted \tilde{p}_j , converging to $\tilde{p}_\infty \in \partial B_\delta(p)$. By a similar argument as in the proof of Lemma 3.2', we can show that Γ_{p_j} converges to a characteristic curve $\Gamma_\infty \subset B_\delta(p)$, emitted from \tilde{p}_∞ , in C^1 on a compact parameter interval. Since $p_j \rightarrow p$, Γ_∞ cannot hit another singular point before hitting p by Lemma 3.2' and the uniqueness of characteristic curves through a point (see Theorem B' of [5]). So Γ_∞ connects p with \tilde{p}_∞ . From Theorem B (c) we conclude that \tilde{p}_∞ is a nonsingular point since p is singular. Let $\mathcal{V} \subset \partial B_\delta(p) \setminus S(u)$ be a neighborhood of \tilde{p}_∞ . For $\tilde{p} \in \mathcal{V}$, we define $\mathfrak{s}(\tilde{p})$ to be the (first) singular point which the characteristic curve through \tilde{p} hits in $B_\delta(p)$. Now in view of Lemma 3.2' the map $\mathfrak{s} : \tilde{p} \in \mathcal{V}' \rightarrow \mathfrak{s}(\tilde{p}) \in S(u) \cap B_\delta(p)$ is defined and continuous for a perhaps smaller neighborhood $\mathcal{V}' \subset \mathcal{V}$. Thus $p = \mathfrak{s}(\tilde{p}_\infty)$ and $p_j = \mathfrak{s}(\tilde{p}_j)$ are path-connected in $\mathfrak{s}(\mathcal{V}')$. So $p_j \in A$, a contradiction. \square

Proposition 3.6. *Let Ω be a bounded domain of R^2 . Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Suppose $\text{curl}\vec{F} \neq 0$. Let p_0 be a singular point in Ω . Take $B_{r_1}(p_0) \subset\subset \Omega$ such that $0 < r_1 \leq (\sup_{B_{r_1}(p_0)} |H|)^{-1}$. Then there exists r_2 , $0 < r_2 < r_1$, such that for any nonsingular point $p \in B_{r_2}(p_0) \setminus \{p_0\}$, the characteristic curve Γ_p passing through p must hit a singular point in $B_{r_1}(p_0)$.*

Proof. Suppose the converse is true. Then there exists a sequence of points p_j , converging to p_0 , such that each characteristic curve Γ_{p_j} does not hit any singular point in $B_{r_1}(p_0)$. By Lemma 3.2' (a) we conclude that a subsequence of closed arcs $\subset \Gamma_{p_j}$ converges to a closed arc Γ_∞ while Γ_∞ is a characteristic curve containing p_0 by Lemma 3.2' (b). We have reached a contradiction. \square

Proof. (of Theorem D) Take $r_1 > 0$ such that on $\bar{B}_{r_1}(p)$, $|H| \leq C_1 \ll \frac{1}{r_1}$ for some positive constant C_1 . In $B_{r_1}(p)$ any characteristic curve has to hit two points on the boundary of $B_{r_1}(p)$ if it does not meet p . We claim that there exists $0 < r_0 < r_1$ such that for any $q \in \bar{B}_{r_0}(p) \setminus \{p\}$, the characteristic curve $\tilde{\Gamma}_q$ passing through q has to meet p . Suppose the converse holds. Then we can find a sequence of points q_j approaching p such that all the $\tilde{\Gamma}_{q_j}$'s do not meet p and satisfy the condition in Lemma 3.2'. By Lemma 3.2' (a) we can find a subsequence of closed arcs $\Gamma_{q_j} \subset \tilde{\Gamma}_{q_j} \cap B_{r_2}(p)$ for $0 < r_2 < r_1$, which converges to Γ_∞ . According to Lemma 3.2' (b), Γ_∞ contains p , a singular point. This contradicts another statement of Lemma 3.2' (b) that Γ_∞ contains no singular points. We have proved the first part of the theorem, that is, Γ_q meets p .

Next we observe that $|\theta'(x)| = |-H| \leq C_1$ for $x \in \Gamma_q \cap \bar{B}_{r_1}(p)$, and hence $N^\perp(x) = (\sin\theta(x), -\cos\theta(x))$ is Cauchy in x near p . Therefore the unit tangent vector N^\perp of Γ_q has a limit at p , denoted $v(q)$. Define the map $\psi : q \in \partial B_{r_0}(p) \rightarrow v(q) \in T_p\Omega$. We can extend by a similar proof Theorem B' in [5] for the uniqueness of characteristic curves to include the case that p is a singular point by a similar proof. We can then conclude that ψ is injective. Note that $q \in \partial B_{r_0}(p)$ has two "sides" locally in $\partial B_{r_0}(p)$. When $q' \rightarrow q$ clockwise (counterclockwise, resp.), we write $q' \rightarrow$

q^+ ($q' \rightarrow q^+$, resp.). Observe that $\lim_{q' \rightarrow q^+} v(q')$ exists since characteristic curves do not intersect in $B_{r_0}(p) \setminus \{p\}$ (and hence $\{v(q')\}$ is "ordered"). Let $w := \lim_{q' \rightarrow q^+} v(q')$. Suppose $w \neq v(q)$. From the standard O.D.E. theory (continuous dependence of the solution on initial data), the solution curve of (1.9) with initial tangent w has to meet q . We now have two distinct characteristic curves passing through q , a contradiction, so $w = v(q)$. Similarly we also have $\lim_{q' \rightarrow q^-} v(q') = v(q)$. We have shown that ψ is C^0 .

Take two different points $q_1, q_2 \in \partial B_{r_0}(p)$. Consider the image $\psi([q_1 q_2])$ of ψ from the small (large, resp.) arc $[q_1 q_2]$ formed by q_1, q_2 into the small or large arc $[v(q_1)v(q_2)]$ formed by $v(q_1), v(q_2)$. Since ψ is continuous, the image of a path-connected set is path-connected, so $\psi([q_1 q_2])$ is path-connected. On the other hand, $\psi([q_1 q_2])$ contains $v(q_1)$ and $v(q_2)$, and hence contains $[v(q_1)v(q_2)]$. It follows that $\psi([q_1 q_2]) = [v(q_1)v(q_2)]$ (similar formula holds for another arc). We have shown that ψ is surjective onto the space of unit tangent vectors at p . The continuity of ψ^{-1} follows from the standard O.D.E. theory (continuous dependence of the solution on initial data). We have completed the proof. \square

According to Theorem D and Theorem B (c), it is impossible to have a p -minimal graph over a convex planar domain with two isolated singular points. However, if the domain is not convex, this is possible as shown by the following configuration of characteristic lines and singular points (see Figure 3.5 below: S_1, S_2 are two isolated singular points and the straight lines denote the characteristic lines).

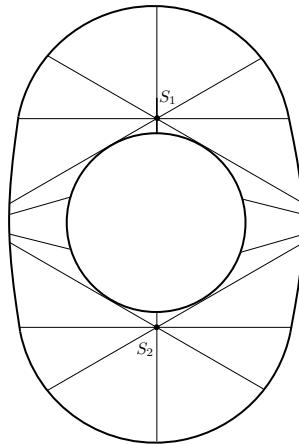


Figure 3.5

4. EXAMPLES

For a C^2 smooth p -minimal surface, we showed ([3]) that, among others, a singular curve (which must be C^1) never has an end (boundary) point. But this situation can occur for a C^1 (hence not C^2) smooth p -minimal surface as shown in the following examples.

Example 4.1. According to Proposition 3.3, any singular point emits at least one characteristic ray. Can we have an example in which a singular point emits exactly one characteristic ray? We are going to construct such an example. First we want the union of the negative x -axis and the origin to be the singular set. Each singular point on the negative x -axis emits two characteristic rays having equal angles with the positive x -axis and the origin emits only one characteristic ray, the positive x -axis (see Figure 4.1).

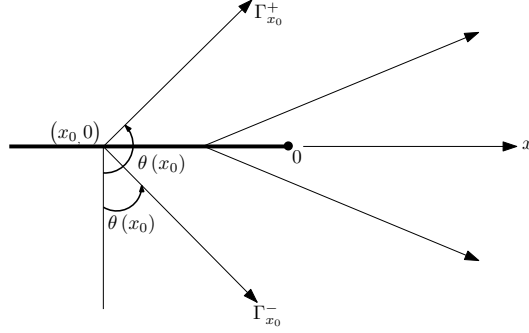


Figure 4.1

Let $\Gamma_{x_0}^+$ ($\Gamma_{x_0}^-$, resp.) denote the characteristic ray emitted from $(x_0, 0)$, $x_0 < 0$, on the upper (lower, resp.) half plane. Describe a point $(x, y) \in \Gamma_{x_0}^+ \cup \{(x_0, 0)\}$ as follows:

$$(4.1) \quad \begin{aligned} x &= x_0 + s \sin \theta(x_0) \\ y &= -s \cos \theta(x_0) \end{aligned}$$

where $s \geq 0$ and $\theta(x_0) := \frac{\pi}{2} + (-, \text{resp.})$ "the angle between $\Gamma_{x_0}^+$ ($\Gamma_{x_0}^-$, resp.) and the positive x -axis" (we choose θ such that the characteristic direction is $N^\perp = (\sin \theta, -\cos \theta)$ accordingly to the previous choice). Integrating $du + xdy - ydx = 0$ along the ray described by (4.1), we determine the u -value uniquely once we know $u(x_0, 0)$. Taking $u(x_0, 0) = 0$, we then obtain

$$(4.2) \quad u(x, y) = -x_0 y.$$

(For characteristic rays on the lower half plane, we use a similar argument to write u as in (4.2)). We require θ to be C^1 in $x_0 \in (-\infty, 0]$ such that $\frac{\pi}{2} < \theta(x_0) < \frac{3\pi}{2}$ for $x_0 \in (-\infty, 0)$, $\theta'(x_0) \leq 0$, $\theta'(0) = a < 0$, and $\lim_{x_0 \rightarrow 0^-} \theta(x_0) = \theta(0) := \frac{\pi}{2}$. It follows from (4.1) that

$$(4.3) \quad \det \frac{\partial(x, y)}{\partial(x_0, s)} = -s\theta'(x_0) - \cos \theta(x_0) > 0$$

unless $x_0 = 0$, where $\theta(0) = \frac{\pi}{2}$. Observe that the C^1 smooth map Ψ_+ from $(-\infty, 0) \times \mathbb{R}^+$ to the upper half plane, defined by $(x_0, s) \rightarrow (x, y)$ according to (4.1), is globally one to one and onto. So Ψ_+^{-1} exists and is C^1 smooth by the inverse function theorem due to (4.3). Similarly for the case of the lower half plane, we take $-\frac{\pi}{2} < \theta(x_0) < \frac{\pi}{2}$ for $x_0 \in (-\infty, 0)$, $\theta'(x_0) \geq 0$, $\theta'(0) (= -a) > 0$, and $\lim_{x_0 \rightarrow 0^-} \theta(x_0) = \theta(0) := \frac{\pi}{2}$. The C^1 smooth map $\Psi_- : (x_0, s) \in (-\infty, 0) \times \mathbb{R}^+ \rightarrow (x, y) \in \text{the lower half plane}$ defined still by (4.1) (but with different range of θ) is globally one

to one and onto. Moreover, the inverse Ψ_-^{-1} is C^1 smooth by the inverse function theorem again since we have

$$\det \frac{\partial(x, y)}{\partial(x_0, s)} = -s\theta'(x_0) - \cos\theta(x_0) < 0$$

for the new range of θ and the nonnegativity of θ' . Thus we can view x_0 as a C^1 smooth function of x and y on the whole plane except the x -axis. Now we claim that the function u defined by

$$(4.4) \quad \begin{aligned} u(x, y) &= -x_0y \text{ for } y \neq 0 \\ u(x, y) &= 0 \text{ for } y = 0 \end{aligned}$$

is C^1 smooth on the whole plane. Moreover, $\Gamma_{x_0}^+$, $\Gamma_{x_0}^-$, and the positive x -axis are characteristic rays of u while $(-\infty, 0] \times \{0\}$ is the singular set of u . For the C^1 smoothness of u , we only need to check whether u_x and u_y are continuous on the x -axis. It is clear that $u_x(x, 0) = 0$. Compute $u_y(x, 0)$ from the definition of u as follows:

$$(4.5) \quad \begin{aligned} u_y(x, 0) &= \lim_{y \rightarrow 0} \frac{u(x, y) - u(x, 0)}{y} \\ &= \lim_{y \rightarrow 0} \frac{-x_0y - 0}{y} = \lim_{y \rightarrow 0} (-x_0(x, y)) \\ &= 0 \text{ if } x \geq 0; = -x \text{ if } x < 0. \end{aligned}$$

On the other hand, we learn from (4.4) and (4.1) that for $y \neq 0$

$$(4.6) \quad \begin{aligned} u_x(x, y) &= -\frac{\partial x_0}{\partial x} y \\ &= \frac{\cos\theta(x_0)}{s\theta'(x_0) + \cos\theta(x_0)} s \cos\theta(x_0). \end{aligned}$$

Observe that

$$(4.7) \quad 0 < \frac{\cos\theta(x_0)}{s\theta'(x_0) + \cos\theta(x_0)} \leq 1$$

for $s \geq 0$, $x_0 < 0$. So from (4.6) and (4.7) we can estimate

$$|u_x(x, y)| \leq s |\cos\theta(x_0)| \rightarrow 0$$

as $(x, y) \rightarrow (\bar{x}, 0)$ since $s \rightarrow 0$ for $\bar{x} \leq 0$ and $\cos\theta(x_0) \rightarrow \cos\theta(0) = \cos\frac{\pi}{2} = 0$ while $s \rightarrow \bar{x}$ for $\bar{x} > 0$. It follows that $u_x(x, y) \rightarrow u_x(\bar{x}, 0) = 0$ as $(x, y) \rightarrow (\bar{x}, 0)$. That is to say, u_x is continuous at the points of the x -axis. Next for $y \neq 0$, we compute

$$(4.8) \quad \begin{aligned} u_y(x, y) &= -\frac{\partial x_0}{\partial y} y - x_0 \\ &= \left(\frac{\sin\theta(x_0)}{s\theta'(x_0) + \cos\theta(x_0)} \right) s \cos\theta(x_0) - x_0 \end{aligned}$$

by (4.4) and (4.1). For $(x, y) \rightarrow (\bar{x}, 0)$ with $\bar{x} \leq 0$, we have $s \rightarrow 0$ and $x_0 \rightarrow \bar{x}$. Therefore $u_y(x, y) \rightarrow 0 - \bar{x} = -\bar{x}$ by (4.8) and (4.7). For $(x, y) \rightarrow (\bar{x}, 0)$ with $\bar{x} > 0$, we have $x_0 \rightarrow 0$, $\theta(x_0) \rightarrow \theta(0) = \frac{\pi}{2}$, $\cos\theta(x_0) \rightarrow 0$, $s \rightarrow \bar{x}$, and $\lim_{(x, y) \rightarrow (\bar{x}, 0^+)} \theta'(x_0) = \theta'(0) = a < 0$, $\lim_{(x, y) \rightarrow (\bar{x}, 0^-)} \theta'(x_0) = \theta'(0) = -a > 0$ by assumption. Observe that $\left(\frac{\sin\theta(x_0)}{s\theta'(x_0) + \cos\theta(x_0)} \right) s$ in (4.8) is bounded in the limit. So $\lim_{(x, y) \rightarrow (\bar{x}, 0)} u_y(x, y) = 0$ for $\bar{x} > 0$. Thus we have shown that u_y is continuous at points of the x -axis in view of (4.5). Altogether on the whole plane u_x and u_y are C^0 and hence u is

C^1 smooth. Next we want to compute $D := \sqrt{(u_x - y)^2 + (u_y + x)^2}$. From (4.6), (4.1), and $u = 0$ on the x -axis we learn that

$$(4.9) \quad \begin{aligned} u_x - y &= \left(\frac{\cos \theta(x_0)}{s\theta'(x_0) + \cos \theta(x_0)} + 1 \right) s \cos \theta(x_0) \quad \text{for } y \neq 0 \\ u_x - y &= 0 \quad \text{for } y = 0. \end{aligned}$$

From (4.8), (4.1), and (4.5) we obtain

$$(4.10) \quad \begin{aligned} u_y + x &= \left(\frac{\cos \theta(x_0)}{s\theta'(x_0) + \cos \theta(x_0)} + 1 \right) s \sin \theta(x_0) \quad \text{for } y \neq 0 \\ u_y + x &= x \quad \text{for } y = 0 \text{ and } x > 0; = 0 \quad \text{for } y = 0 \text{ and } x \leq 0. \end{aligned}$$

Therefore by (4.9) and (4.10) we have

$$(4.11) \quad \begin{aligned} D &= \left(\frac{\cos \theta(x_0)}{s\theta'(x_0) + \cos \theta(x_0)} + 1 \right) s \quad \text{for } y \neq 0 \\ D &= x \quad \text{for } y = 0 \text{ and } x > 0; = 0 \quad \text{for } y = 0 \text{ and } x \leq 0. \end{aligned}$$

It follows that $N^\perp = (u_y + x, -(u_x - y))D^{-1} = (\sin \theta(x_0), -\cos \theta(x_0))$ for $y \neq 0$ and $N^\perp = (1, 0)$ for $y = 0$ and $x > 0$ while $D = 0$ for $y = 0$ and $x \leq 0$. We have shown that $\Gamma_{x_0}^+$, $\Gamma_{x_0}^-$, and the positive x -axis are characteristic rays of u while $(-\infty, 0]$ is the singular set of u . To verify Theorem B (a), we compute

$$(4.12) \quad \begin{aligned} D' &= \frac{\partial D}{\partial s} \\ &= \frac{\cos \theta(x_0)}{s\theta'(x_0) + \cos \theta(x_0)} + 1 - \frac{s\theta'(x_0) \cos \theta(x_0)}{(s\theta'(x_0) + \cos \theta(x_0))^2} \end{aligned}$$

for $y \neq 0$ and

$$D' = \frac{\partial D}{\partial x} = 1$$

for $y = 0$ and $x > 0$ by (4.11). It follows from (4.12) that $D' \rightarrow 2$ as $s \rightarrow 0$ ($D \rightarrow 0$ along $\Gamma_{x_0}^+$ or $\Gamma_{x_0}^-$). Since each singular point on the negative x -axis emits two characteristic rays having equal angles with the positive x -axis, the graph defined by u and restricted to any bounded plane domain is a (C^1) weak solution to (1.1) (with $\vec{F} = (-y, x)$ and $H = 0$) and hence a p -minimizer in view of Theorem 6.3 and Theorem 3.3 in [4].

The following two examples are inspired by [17] (in which Ritoré constructed locally Lipschitz continuous p -minimal graphs $(x, y, u(x, y))$ with finitely many singular half-lines emitted from a singular point).

Example 4.2. We are going to construct a C^1 smooth p -minimal graph $(x, y, u(x, y))$ with two singular half-lines emitted from a singular point (see Figure 4.2 below).

Let $\alpha : [0, \infty) \rightarrow [0, \frac{\pi}{4})$ be a C^1 smooth function such that $\alpha(0) = 0$, $\alpha'(0^+) = 1$, and $\alpha'(t) > 0$ for all $t \in (0, \infty)$. Let $\beta : [0, \infty) \rightarrow [0, \infty)$ be a C^1 smooth function such that $\beta(0) = 0$, $\beta'(0^+) = 0$, $\beta'(t) > 0$ for all $t \in (0, \infty)$, and $\beta \rightarrow \infty$ as $t \rightarrow \infty$. We divide the plane into five regions (see Figure 4.3): region I := $\{x > y > 0\}$, region II := $\{y > x > 0\}$, region III := $\{y > -x > 0\}$, region IV := $\{-x > y >$

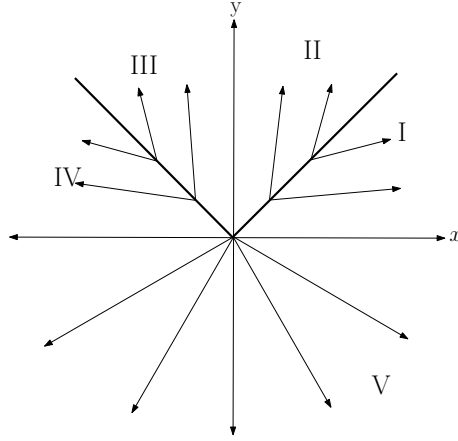


Figure 4.2

0}, and region V := { $y < 0$ }. In region I, we require the characteristic ray emitted from $(\beta(t), \beta(t))$ to be parametrized by

$$(4.13) \quad \begin{aligned} x &= s \cos \alpha(t) + \beta(t) \\ y &= s \sin \alpha(t) + \beta(t) \end{aligned}$$

for $s > 0$. The C^1 smooth map $\Psi : (s, t) \in (0, \infty) \times (0, \infty) \rightarrow (x, y) \in \text{region I}$ is globally one to one and onto, hence Ψ^{-1} exists. We compute the Jacobian matrix of Ψ from (4.13) as follows:

$$(4.14) \quad \begin{aligned} \frac{\partial(x, y)}{\partial(s, t)} &= \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \\ &= \begin{pmatrix} \cos \alpha(t) & -s\alpha'(t) \sin \alpha(t) + \beta'(t) \\ \sin \alpha(t) & s\alpha'(t) \cos \alpha(t) + \beta'(t) \end{pmatrix}. \end{aligned}$$

So we have the Jacobian

$$(4.15) \quad \det \frac{\partial(x, y)}{\partial(s, t)} = s\alpha'(t) + \beta'(t)(\cos \alpha(t) - \sin \alpha(t))$$

which is positive for $(s, t) \in (0, \infty) \times (0, \infty)$. By the inverse function theorem, the map Ψ^{-1} is C^1 smooth and

$$(4.16) \quad \begin{aligned} \frac{\partial(s, t)}{\partial(x, y)} &= \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} \\ &= \frac{1}{s\alpha' + \beta'(\cos \alpha - \sin \alpha)} \begin{pmatrix} s\alpha' \cos \alpha + \beta' & s\alpha' \sin \alpha - \beta' \\ -\sin \alpha & \cos \alpha \end{pmatrix}. \end{aligned}$$

Along a characteristic ray $u(s, t)$ is determined by $u(0, t)$ by integrating $du + xdy - ydx = 0$. We have

$$\begin{aligned} u(s, t) &= u(0, t) + \int_0^s (ydx - xdy) \\ &= u(0, t) + s\beta(t)(\cos \alpha(t) - \sin \alpha(t)). \end{aligned}$$

by (4.13). Set $u(0, t) := 0$. That is, $u(\bar{x}, \bar{x}) := 0$ for $\bar{x} \geq 0$ since $\bar{x} = \beta(t)$ for some $t \geq 0$ (note that (4.13) also describes the boundary of region I by enlarging the domain of (s, t) to $[0, \infty) \times [0, \infty)$). Now it is reasonable to define u in the region I by

$$(4.17) \quad u(s, t) = s\beta(t)(\cos \alpha(t) - \sin \alpha(t)).$$

Similarly in region II (region III, resp.) we have

$$(4.18) \quad \begin{aligned} x &= s \sin \alpha(t) + \beta(t) \\ &\quad (-s \sin \alpha(t) - \beta(t), \text{ resp.}) \\ y &= s \cos \alpha(t) + \beta(t) \\ u &= s\beta(t)(\sin \alpha(t) - \cos \alpha(t)) \\ &\quad (s\beta(t)(\cos \alpha(t) - \sin \alpha(t), \text{ resp.}). \end{aligned}$$

and in region IV we have

$$(4.19) \quad \begin{aligned} x &= -s \cos \alpha(t) - \beta(t) \\ y &= s \sin \alpha(t) + \beta(t) \\ u &= s\beta(t)(\sin \alpha(t) - \cos \alpha(t)). \end{aligned}$$

We set

$$(4.20) \quad u = 0$$

on region V, the x -axis $\{y = 0\}$, the half-lines $\{x = y > 0\}$, $\{x = -y < 0\}$, and the positive y -axis $\{x = 0, y > 0\}$. We can verify that u defined by (4.17), (4.18), (4.19), and (4.20) is C^1 smooth by showing the continuity of u_x and u_y . We leave the details to the reader as an exercise. Moreover, it is a direct verification that the graph defined by u is p -minimal with the expected singular set and characteristic lines.

Since each singular point on the half-lines $\{x = y > 0\}$ and $\{x = -y < 0\}$ emits two characteristic rays having equal angles with the half-line, the graph defined by u and restricted to any bounded planar domain is a (C^1) weak solution to (1.1) (with $\vec{F} = (-y, x)$ and $H = 0$) and hence a p -minimizer in view of Theorem 6.3 and Theorem 3.3 in [4]. On the lower half plane, we can easily see from $u = 0$ that each ray emitted from the origin is characteristic. Compute $D := \sqrt{(u_x - y)^2 + (u_y + x)^2} = \sqrt{y^2 + x^2}$. It follows that along a characteristic ray, we have $N^\perp = \frac{\partial}{\partial r}$ and hence $D' = \frac{\partial D}{\partial r} = 1$ where $r := \sqrt{x^2 + y^2}$. On the positive y -axis (which is a special characteristic ray also emitted from the origin), we have $D = \sqrt{(0 - y)^2 + (0 + 0)^2} = y$, $N^\perp = \frac{\partial}{\partial y}$, and hence $D' = \frac{\partial D}{\partial y} = 1$. On the other hand, we claim that $D' \rightarrow 2$ as its argument tends to a singular point (\bar{x}, \bar{x}) or $(-\bar{x}, \bar{x})$, $\bar{x} > 0$, along a characteristic ray. We check this for the characteristic rays in region I and leave the remaining cases to the reader as an exercise. First observe that from (2.3), (2.1), $\text{curl} \vec{F} = 2$, and $\theta = \alpha + \frac{\pi}{2}$ in region I, we have

$$(4.21) \quad N\alpha = \frac{1}{D}(2 - D').$$

Note that in region I, we have $N := (\cos \theta, \sin \theta) = (-\sin \alpha, \cos \alpha)$. Computing

$$\begin{aligned} N\alpha &= -\alpha' t_x \sin \alpha + \alpha' t_y \cos \alpha \\ &= \frac{\alpha'}{s\alpha' + \beta'(\cos \alpha - \sin \alpha)} \end{aligned}$$

by (4.16), we conclude that $N\alpha$ is bounded as $(x, y) \rightarrow (\bar{x}, \bar{x})$, $\bar{x} > 0$, while $s \rightarrow 0$, $t \rightarrow t_0$ (recall $\beta(t_0) = \bar{x}$) along a characteristic ray in region I. So $D' \rightarrow 2$ from (4.21) since $D \rightarrow 0$ as its argument tends to a singular point.

Example 4.3. Continuing the construction in Example 4.2, we are going to build a C^1 smooth p -minimal graph $(x, y, u(x, y))$ with three singular half-lines emitted from a singular point (see Figure 4.3 below).

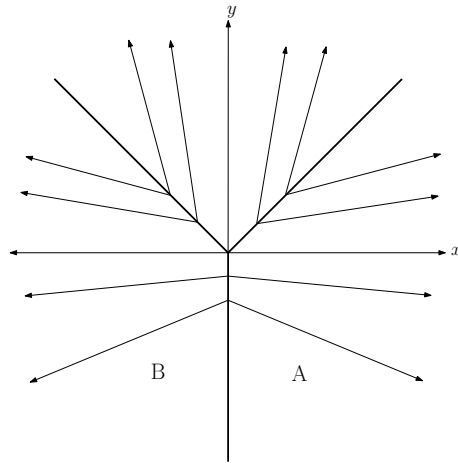


Figure 4.3

On the upper half plane and the x -axis, we define u as in Example 4.2. For the lower half plane, we divide it into two regions $A := \{x > 0, y < 0\}$ and $B := \{x < 0, y < 0\}$. In region A (region B, resp.) we require the characteristic ray emitted from $(0, -\beta(t))$ to be parametrized by

$$\begin{aligned} x &= s \cos \alpha(t) \\ (x &= -s \cos \alpha(t), \text{ resp.}) \\ y &= -s \sin \alpha(t) - \beta(t). \end{aligned}$$

for $(s, t) \in (0, \infty) \times (0, \infty)$. As before we can show that the C^1 smooth map $(s, t) \rightarrow (x, y)$ is a diffeomorphism from $(0, \infty) \times (0, \infty)$ onto region A (region B, resp.). We then integrate $du + xdy - ydx = 0$ along the characteristic rays to get the expected u -value as follows:

$$\begin{aligned} u(s, t) &= u(0, t) + \int_0^s (ydx - xdy) \\ &= u(0, t) - s\beta(t) \cos \alpha(t) \\ (&= u(0, t) + s\beta(t) \cos \alpha(t), \text{ resp.}) \end{aligned}$$

Set $u := 0$ on the negative y -axis. It follows that $u(0, t) = 0$ and hence we define u on region A (region B, resp.) as

$$\begin{aligned} u(s, t) &= -s\beta(t) \cos \alpha(t) \\ &(\quad = s\beta(t) \cos \alpha(t), \text{ resp.}). \end{aligned}$$

It is a direct verification that $u \in C^1$ and the graph defined by u is p -minimal with the expected singular set and characteristic rays. Moreover, restricted to any bounded plane domain, u is a (C^1) weak solution to (1.1) (with $\vec{F} = (-y, x)$ and $H = 0$) and hence a p -minimizer in view of Theorem 6.3 and Theorem 3.3 in [4]. Along a characteristic ray in the upper half plane, we have shown the behavior of D' in Example 4.2. By a similar argument we can show that $D' \rightarrow 2$ as its argument tends to a singular point $(0, \bar{y})$, $\bar{y} < 0$, along a characteristic ray in the lower half plane.

Example 4.4. We are going to construct a C^1 smooth p -minimizer v defined on a bounded plane domain Ω having the line segment $L \equiv \overline{(0, 0), (0, 1)}$ as the singular set (see Figure 4.4). The p -minimizer v is C^∞ smooth on $\Omega \setminus L$ and $C^1 \setminus C^2$ on L .

Let $\alpha, \beta : [0, 1] \rightarrow \mathbb{R}$ be C^∞ smooth functions with the properties:

$$\begin{aligned} (4.22) \quad \alpha(0) &= \alpha\left(\frac{1}{2}\right) = \alpha(1) = 0, \alpha'(0) = \alpha'(1) = 1, \\ \alpha(t) &> 0 \text{ for } 0 < t < \frac{1}{2}, \alpha(t) < 0 \text{ for } \frac{1}{2} < t < 1, |\alpha(t)| < \frac{\pi}{2}, \\ \beta(0) &= 0, \beta\left(\frac{1}{2}\right) = \frac{1}{2}, \beta(1) = 1, \beta'(t) > 0, 0 < t < 1, \\ \beta^{(n)}(0) &= \beta^{(n)}(1) = 0, n = 1, 2, \dots \end{aligned}$$

We then define a p -minimal surface in the parameters s and t as follows (note that $\theta(\tau) = \frac{\pi}{2} + \alpha(\tau)$ in (4.9) of [3]):

$$(4.23) \quad \begin{aligned} x &= s \cos \alpha(t) \\ y &= s \sin \alpha(t) + \beta(t) \\ z &= s\beta(t) \cos \alpha(t) \end{aligned}$$

for $(s, t) \in [0, \infty) \times [0, 1]$. Compute the Jacobian J_φ of the map $\varphi : (s, t) \rightarrow (x, y)$ given by (4.23):

$$(4.24) \quad \begin{aligned} J_\varphi &\equiv \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \cos \alpha(t) & -s\alpha'(t) \sin \alpha(t) \\ \sin \alpha(t) & s\alpha'(t) \cos \alpha(t) + \beta'(t) \end{vmatrix} \\ &= s\alpha'(t) + \beta'(t) \cos \alpha(t). \end{aligned}$$

Observe that $J_\varphi > 0$ in $\{[0, s_0] \times [0, 1]\} \setminus \{(0, 0), (0, 1)\}$ for a small positive number s_0 . Furthermore, for a possibly smaller positive number s_+ , we can show that φ is a C^∞ diffeomorphism (homeomorphism, respectively) from $\{[0, s_+] \times [0, 1]\} \setminus \{(0, 0), (0, 1)\}$ ($[0, s_+] \times [0, 1]$, respectively) into \mathbb{R}^2 in view of the behavior of α near $t = 0, 1$ and the compactness of a closed subinterval away from the boundary points $0, 1$ of $(0, 1)$. So z in (4.23) can be viewed as a function of x, y defining a p -minimal graph $z = u_+(x, y)$ over $\varphi([0, s_+] \times [0, 1]) \subset \mathbb{R}^2$. Similarly we describe another piece of p -minimal surface by

$$(4.25) \quad \begin{aligned} x &= s \cos \alpha(t) \\ y &= -s \sin \alpha(t) + \beta(t) \\ z &= s\beta(t) \cos \alpha(t) \end{aligned}$$

for $(s, t) \in (-\infty, 0] \times [0, 1]$. Computing the Jacobian J_ψ of the map $\psi : (s, t) \rightarrow (x, y)$ given by (4.25) in a similar way as in (4.24), we obtain

$$J_\psi = -s\alpha'(t) + \beta'(t) \cos \alpha(t).$$

Observe that $J_\psi > 0$ in $\{(s_1, 0] \times [0, 1]\} \setminus \{(0, 0), (0, 1)\}$ for some negative number s_1 . For a negative number s_- possibly closer to 0, we can show that ψ is a C^∞ diffeomorphism (homeomorphism, respectively) from $\{(s_-, 0] \times [0, 1]\} \setminus \{(0, 0), (0, 1)\}$ ($(s_-, 0] \times [0, 1]$, respectively) into R^2 . Thus we have another piece of p -minimal graph $z = u_-(x, y)$ defined by (4.25) over $\psi((s_-, 0] \times [0, 1]) \subset R^2$.

Now we define a p -minimal graph $z = u(x, y)$ for $y \leq 0$ and $y \geq 1$ by

$$(4.26) \quad \begin{aligned} u &= 0 \text{ for } y \leq 0 \\ u &= x \text{ for } y \geq 1, \end{aligned}$$

$z = u_+(x, y)$ in $\varphi([0, s_+] \times [0, 1])$, and $z = u_-(x, y)$ in $\psi((s_-, 0] \times [0, 1])$. Observe that u coincides with u_+ (u_- , respectively) on $[0, s_+] \times \{0, 1\}$ ($(s_-, 0] \times \{0, 1\}$, respectively) while $u_+ = u_-$ on $L \equiv \{0\} \times [0, 1]$. Write u_+ (u_- , respectively) = $x\beta(t)$ by (4.23)((4.25), respectively). It follows from $\beta^{(k)}(0) = \beta^{(k)}(1) = 0$ in (4.22) that for all positive integers k ,

$$(4.27) \quad \begin{aligned} \frac{\partial^k u_+}{\partial y^k} &= 0 \text{ on } [0, s_+] \times \{0, 1\} \\ \left(\frac{\partial^k u_-}{\partial y^k} \right) &= 0 \text{ on } (s_-, 0] \times \{0, 1\}, \text{ respectively).} \end{aligned}$$

So u matches with u_+ (u_- , respectively) on $[0, s_+] \times \{0, 1\}$ ($(s_-, 0] \times \{0, 1\}$, respectively) C^∞ smoothly by (4.27). On L (where $x = 0$), we have

$$(4.28) \quad \frac{\partial u_+}{\partial x} = \frac{\partial u_-}{\partial x} = \beta(t).$$

Differentiating the first equation of (4.23) or (4.25) with respect to x at a point on L (where $s = 0$) gives

$$(4.29) \quad \frac{\partial s}{\partial x} = \frac{1}{\cos \alpha(t)}.$$

Differentiating the second equation of (4.23) and (4.25) with respect to x at a point on L (where $s = 0$) $\setminus \{(0, 0), (0, 1)\}$ and substituting (4.29) into the resulting formulas, we obtain

$$(4.30) \quad \frac{\partial t}{\partial x} = \mp \frac{\tan \alpha(t)}{\beta'(t)}$$

for the " \pm sides", respectively. Now computing the second derivative of u_\pm in the x direction on L (where $x = 0$ or $s = 0$), we get

$$(4.31) \quad \frac{\partial^2 u_\pm}{\partial x^2} = 2\beta'(t) \frac{\partial t}{\partial x} = \mp 2 \tan \alpha(t)$$

by (4.30) (for $0 < t < 1$, but the final result also holds for $t = 0$ and 1). It follows from (4.31) and (4.22) that

$$(4.32) \quad \frac{\partial^2 u_+}{\partial x^2} \neq \frac{\partial^2 u_-}{\partial x^2}$$

on $L \setminus (\{0\} \times \{0, \frac{1}{2}, 1\})$ while at $(0, 0)$, $(0, \frac{1}{2})$, and $(0, 1)$, we have

$$(4.33) \quad \frac{\partial^2 u_+}{\partial x^2} = \frac{\partial^2 u_-}{\partial x^2}.$$

We define v to be u , u_+ , u_- on the corresponding domains. Glue a patch of suitable domain around $(0, 0)$ from $\{y < 0\}$ and a patch of suitable domain from $\{y > 1\}$ to $\varphi([0, s_+] \times [0, 1]) \cup \psi((s_-, 0] \times [0, 1])$ to form a C^∞ smooth bounded domain Ω (see Figure 4.4 below).

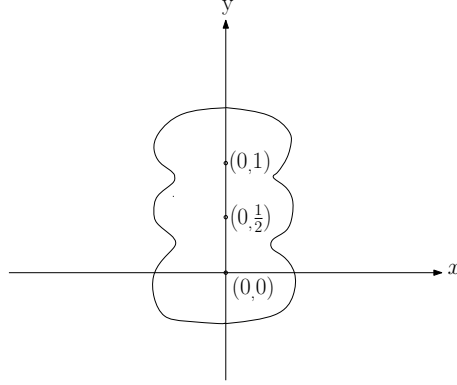


Figure 4.4

Consider v restricted to Ω . From (4.28), (4.32), and from similar arguments as in Example 4.2, we learn that v is $C^1 \setminus C^2$ on L . Moreover, v is C^∞ smooth on $\Omega \setminus L$ by (4.27). In view of the extension theorem (Proposition 3.5 in [3]) of characteristic lines for a C^2 smooth solution and $J_\varphi > 0$ (see (4.24)), $J_\psi > 0$, we can easily show that L is the only singular set in Ω . Observe that 0 and x are solutions to the p -minimal surface equation (see (1.1) with $\vec{F} = (-y, x)$ and $H = 0$). It then follows that v is a p -minimizer in view of Theorem 6.3 and Theorem 3.3 in [4]. We have proved our claim in the beginning of this example.

Example 4.5. According to Theorem C, if a singular point is not isolated, then it emits at least one C^0 singular curve. One may ask if there are only finitely many such singular curves. The configuration of singular lines in Figure 4.5 (b) shows that it is possible to have infinitely many singular curves emitted from a singular point and shrinking to this singular point. On the other hand, it is not possible to have a configuration of singular lines as shown in Figure 4.5 (a), which converges to another singular line of length > 0 . The reason is that any characteristic line emitted from a point in this limit singular line must span an angle, and hence hit an approaching singular line, contradicting Theorem B (c).

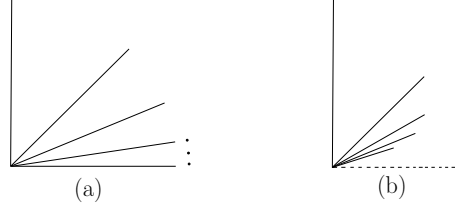


Figure 4.5

5. SIZE AND REGULARITY OF THE SINGULAR SET

In this section we first study the size of the singular set.

Proof. (of Theorem F) It is enough to show that for any $p \in S_{\bar{F}}(u) \cap \Omega$, there exists $r > 0$ such that $B_r(p) \subset \Omega$ and $\mathcal{H}^2(S_{\bar{F}}(u) \cap B_r(p)) = 0$. First take $r_0 > 0$ such that $B_{r_0}(p) \subset \subset \Omega$. Take

$$r = \frac{1}{2} \min\left\{\frac{1}{\max_{\bar{B}_{r_0}(p)} |H|}, r_0\right\}.$$

It is clear that $B_r(p) \subset B_{2r}(p) \subset B_{r_0}(p) \subset \subset \Omega$. Next we are going to show $\mathcal{H}^2(S_{\bar{F}}(u) \cap B_r(p)) = 0$.

By Proposition 3.3' each $q \in S_{\bar{F}}(u) \cap B_r(p)$ emits a characteristic curve Lq which hits the boundary $\partial B_{2r}(p)$ of a bigger ball $B_{2r}(p)$ at q' transversely (noting that $\pm H$ is the line curvature of Lq). Since q' must be nonsingular by Theorem B(c), we can find a small open interval $I_{q'}$, contained in $\partial B_{2r}(p)$ and consisting of nonsingular points, such that the characteristic curves emitted from $I_{q'}$ hit $S_{\bar{F}}(u)$ in a set S_q . Note that each characteristic curve emitted from $I_{q'}$ must hit a singular point in $B_{2r}(p)$ by Lemma 3.2'. Since the number of disjoint open intervals (take the union if two $I_{q'}$ overlap) in $\partial B_{2r}(p)$ is countable, we need only to show $\mathcal{H}^2(S_q) = 0$ (two Lq 's do not hit the same q' ; otherwise, q' becomes singular). Since $I_{q'}$ is transverse to characteristic curves, we may use a parameter τ such that $I_{q'} = \beta(\tau)$ ($\in C^1$ for $\tau \in (\tau_0, \tau_1)$, say) to describe S_q as a (folded) graph $G_q = \{(\tau, \sigma_1(\tau))\}$. Here we parametrize characteristic curves by the arc length σ such that $\sigma = 0$ for $I_{q'}$. We have known that $\sigma_1(\tau)$ is C^0 in τ . Define a map $\varphi : (\tau, \sigma) \rightarrow (x, y)$ such that $(x(\tau, \sigma), y(\tau, \sigma))$ describes a characteristic curve for each τ with the initial data $(x(\tau, 0), y(\tau, 0)) = \beta(\tau)$. Namely we have

$$(5.1) \quad \begin{aligned} \frac{\partial x(\tau, \sigma)}{\partial \sigma} &= \sin \theta(x(\tau, \sigma), y(\tau, \sigma)) \\ \frac{\partial y(\tau, \sigma)}{\partial \sigma} &= -\cos \theta(x(\tau, \sigma), y(\tau, \sigma)) \end{aligned}$$

with $x(\tau, 0) = x_0(\tau)$, $y(\tau, 0) = y_0(\tau)$ where $\beta(\tau) = (x_0(\tau), y_0(\tau))$. Since θ is C^1 smooth by Theorem D in [5], we conclude that the solution to (5.1) is C^1 smooth in the parameter τ for C^1 smooth initial data $x_0(\tau)$, $y_0(\tau)$. Hence the map φ is C^1 smooth (for $\tau \in (\tau_0, \tau_1)$ and $\sigma \in (0, \sigma_1(\tau))$). To extend φ beyond G_q so that $\varphi(G_q) = S_q$, we consider θ as an independent variable and $(x(\tau, \sigma), y(\tau, \sigma))$ to be

the first two components of the unique solution to the following equations:

$$(5.2) \quad \begin{aligned} \frac{dx}{d\sigma} &= \sin \theta, & \frac{dy}{d\sigma} &= -\cos \theta \\ \frac{d\theta}{d\sigma} &= -H(x, y) \end{aligned}$$

with the initial condition $x(0) = x_0(\tau)$, $y(0) = y_0(\tau)$, and $\theta(0) = \theta_0(\tau) \in C^1$. Note that the third equation of (5.2) can be deduced if θ is the angular function associated to the horizontal normal N (see Theorem A in [5]). By Theorem 3.1 on page 95 in [9], $x(\tau, \sigma)$ and $y(\tau, \sigma)$ are C^1 smooth in τ (and σ of course), and apparently they are defined on an open neighborhood of G_q . That is, the map φ extends C^1 smoothly over G_q . Let $J(\varphi)$ denote the Jacobian of φ . Clearly we have

$$|J(\varphi)| \leq C_j$$

on a compact set K_j where $\cup_{j=1}^{\infty} K_j$ exhausts the whole domain. It follows (e.g., 2.10.11 in [6]) that

$$(5.3) \quad \mathcal{H}^2(\varphi(G_q \cap K_j)) \leq C_j \mathcal{H}^2(G_q \cap K_j).$$

On the other hand, we have

$$\mathcal{H}^2(G_q \cap K_j) = \int \mathcal{H}^1((G_q \cap K_j) \cap \{\tau = c\}) dc$$

by Fubini's theorem. But $(G_q \cap K_j) \cap \{\tau = c\}$ consists of only one point $(c, \sigma_1(c))$, and hence $\mathcal{H}^1((G_q \cap K_j) \cap \{\tau = c\}) = 0$. So $\mathcal{H}^2(G_q \cap K_j) = 0$. It follows from (5.3) that $\mathcal{H}^2(\varphi(G_q \cap K_j)) = 0$. We then have

$$\begin{aligned} \mathcal{H}^2(S_q) &= \mathcal{H}^2(\varphi(G_q)) \\ &= \mathcal{H}^2(\varphi(\cup_{j=1}^{\infty} (G_q \cap K_j))) \\ &= \mathcal{H}^2(\cup_{j=1}^{\infty} \varphi(G_q \cap K_j)) \\ &\leq \sum_{j=1}^{\infty} \mathcal{H}^2(\varphi(G_q \cap K_j)) = 0. \end{aligned}$$

Therefore $\mathcal{H}^2(S_q) = 0$. □

From now on, we are going to study the regularity of nondegenerate crack points (see Definition 5.1 and Definition 5.2). Let N_ε denote a mollification of N .

Lemma 5.1. *Suppose $\mathcal{H}^2(K) = 0$ for a subset K of Ω . Let $\theta \in C^0(\Omega \setminus K)$ be a weak solution to (1.11) with $H \in C^0(\Omega)$. Let $\Omega' \subset\subset \Omega$ be a Lipschitzian domain. Then we have*

$$(5.4) \quad \lim_{\varepsilon \rightarrow 0} \oint_{\partial\Omega'} N_\varepsilon \cdot \nu = \int_{\Omega'} H$$

where ν is the outward unit normal to $\partial\Omega'$ (defined a.e. with respect to the boundary measure).

Proof. From the definition of weak solution, we have

$$(5.5) \quad \int_{\Omega} N \cdot \nabla \varphi + \int_{\Omega} H \varphi = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Let φ_ε (V_ε , resp.) denote a mollification of φ (of a vector field V , resp.). Take a bounded domain Ω'' such that $\Omega' \subset \Omega'' \subset\subset \Omega$. Let $\varphi \in C_0^\infty(\Omega'')$. Then from (5.5) we have

$$(5.6) \quad \begin{aligned} 0 &= \int_{\Omega''} N \cdot \nabla \varphi_\varepsilon + \int_{\Omega''} H \varphi_\varepsilon \quad (\varphi_\varepsilon \in C_0^\infty(\Omega'') \text{ for } \varepsilon \text{ small}) \\ &= \int_{\Omega''} N \cdot (\nabla \varphi)_\varepsilon + \int_{\Omega''} H \varphi_\varepsilon \\ &= \int_{\Omega''} N_\varepsilon \cdot \nabla \varphi + \int_{\Omega''} H_\varepsilon \varphi \end{aligned}$$

(note that $|N| = 1$, $H \in L^1_{loc}(\Omega)$, and hence both N and H are in $L^1(\Omega'')$). Here we have used the fact that $\int g f_\varepsilon = \int g_\varepsilon f$ for $g \in L^1$ and $f \in C_0^\infty$ (which can be easily proved by Fubini's theorem). It follows from (5.6) that

$$(5.7) \quad \operatorname{div} N_\varepsilon = H_\varepsilon \text{ in } \Omega''.$$

We can now integrate (5.7) over Ω' and apply the divergence theorem to obtain

$$(5.8) \quad \oint_{\partial\Omega'} N_\varepsilon \cdot \nu = \int_{\Omega'} H_\varepsilon.$$

Since H_ε is bounded on Ω' for ε small, we have $\lim_{\varepsilon \rightarrow 0} \int_{\Omega'} H_\varepsilon = \int_{\Omega'} H$ by Lebesgue's dominated convergence theorem. From this and (5.8), we get (5.4). \square

We remark that Lemma 5.1 holds even if $\mathcal{H}^1(\partial\Omega' \cap K) \neq 0$ where \mathcal{H}^1 denotes the 1-dimensional Hausdorff measure. Since N is not defined on $\partial\Omega' \cap K$, N_ε may not converge to N a.e. on $\partial\Omega'$. Now suppose that we are in the situation of Lemma 5.1 with $H \in C^1(\Omega)$. Take $p \in K$. Suppose

(a) there is an open neighborhood $U \subset \Omega$ of p such that $U \cap K$ is a C^0 curve γ (passing through p) dividing U into two connected regions U^+ and U^- and

(b) for any $q \in \gamma$ there are exactly two characteristic curves Γ_q^+ , Γ_q^- issuing from q , such that $\Gamma_q^+ \subset U^+$ and $\Gamma_q^- \subset U^-$.

Take $q^+ \in \Gamma_q^+ \setminus \{q\}$ ($q^- \in \Gamma_q^- \setminus \{q\}$, resp.). There passes a seed curve (i.e., integral curve of N) γ_+ (γ_- , resp.) each point of which emits a characteristic curve in U^+ (U^- , resp.) hitting a point in a neighborhood of p in γ . We parametrize γ_+ (γ_- , resp.) by the arc-length parameter τ^+ (τ^- , resp.) in some open interval including τ_0^+ (τ_0^- , resp.) so that $\frac{\partial}{\partial \tau^+}$ ($\frac{\partial}{\partial \tau^-}$, resp.) coincides with $\pm N$ and the characteristic curve issuing from $\gamma_+(\tau^+)$ ($\gamma_-(\tau^-)$, resp.) hits γ at $X_+(\tau^+)$ ($X_-(\tau^-)$, resp.). We assume that $X_+(\tau_0^+) = X_-(\tau_0^-)$, denoted as p_0 . Consider the following system of ordinary differential equations:

$$(5.9) \quad \begin{aligned} \frac{dx}{d\sigma} &= \sin \theta, & \frac{dy}{d\sigma} &= -\cos \theta \\ \frac{d\theta}{d\sigma} &= -H. \end{aligned}$$

Note that the first two equations of (5.9) describe the characteristic curves and σ is the unit-speed parameter (on the xy -plane). Let $P_{\pm}(\sigma, \tau^{\pm}) := (x_{\pm}(\sigma, \tau^{\pm}), y_{\pm}(\sigma, \tau^{\pm}))$ where $x_{\pm}(\sigma, \tau^{\pm})$, $y_{\pm}(\sigma, \tau^{\pm})$, and $\theta_{\pm}(\sigma, \tau^{\pm})$ are the solutions to (5.9) with $(x_{\pm}(0, \tau^{\pm}), y_{\pm}(0, \tau^{\pm})) = \gamma_{\pm}(\tau^{\pm})$ and $\theta_{\pm}(0, \tau^{\pm}) = \theta(\gamma_{\pm}(\tau^{\pm}))$. Write

$$(5.10) \quad \begin{aligned} X_+(\tau^+) &= P_+(\sigma_+(\tau^+), \tau^+) \\ X_-(\tau^-) &= P_-(\sigma_-(\tau^-), \tau^-) \end{aligned}$$

for some continuous functions σ_{\pm} .

Definition 5.1. Suppose $H^2(K) = 0$ for a subset K of Ω . Let $\theta \in C^0(\Omega \setminus K)$ be a weak solution to (1.11) with $H \in C^1(\Omega)$. Take $p \in K$. Suppose conditions (a) and (b) above hold. Take $q^{\pm} \in \Gamma_q^{\pm} \setminus \{q\}$. We have seed curves γ_{\pm} , parametrized by τ^{\pm} , $X_{\pm}(\tau^{\pm})$, and $X_+(\tau_0^+) = X_-(\tau_0^-) = p_0$ as above. We call p_0 nondegenerate if the Jacobian of P_{\pm} (see (5.10)) satisfies

$$(5.11) \quad \det \frac{\partial(x_{\pm}(\sigma, \tau^{\pm}), y_{\pm}(\sigma, \tau^{\pm}))}{\partial(\sigma, \tau^{\pm})} \neq 0$$

at $(\sigma_{\pm}(\tau_0^{\pm}), \tau_0^{\pm})$ (note that P_{\pm} is defined on a neighborhood of $(\sigma_{\pm}(\tau_0^{\pm}), \tau_0^{\pm})$ by the basic ODE theory).

Note that Definition 5.1 is independent of the choice of seed curves γ_{\pm} by observing that $(\vec{u} \wedge \vec{v} := u_1v_2 - u_2v_1$ for $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$)

$$(5.12) \quad \begin{aligned} &\frac{\partial P_{\pm}}{\partial \sigma} \wedge \frac{\partial P_{\pm}}{\partial \tau^{\pm}} \\ &= \frac{\partial P_{\pm}}{\partial \tilde{\sigma}} \wedge \left(\frac{\partial P_{\pm}}{\partial \tilde{\tau}^{\pm}} \frac{\partial \tilde{\tau}^{\pm}}{\partial \tau^{\pm}} + \frac{\partial P_{\pm}}{\partial \tilde{\sigma}} \frac{\partial \tilde{\sigma}}{\partial \tau^{\pm}} \right) \\ &= \left(\frac{\partial P_{\pm}}{\partial \tilde{\sigma}} \wedge \frac{\partial P_{\pm}}{\partial \tilde{\tau}^{\pm}} \right) \frac{\partial \tilde{\tau}^{\pm}}{\partial \tau^{\pm}} \end{aligned}$$

where $\tilde{\tau}^{\pm}$ are arc-length parameters for another choice of seed curves $\tilde{\gamma}_{\pm}$ and $\tilde{\sigma}$ is the corresponding unit-speed parameter for characteristic curves. It follows that (5.11) holds with respect to $(\tilde{\sigma}, \tilde{\tau}^{\pm})$ since $\frac{\partial \tilde{\tau}^{\pm}}{\partial \tau^{\pm}} \neq 0$ in the above equality. Now assume that p_0 is nondegenerate. Then by the inverse function theorem, P_{\pm} is a local C^1 diffeomorphism from (σ, τ^{\pm}) to $(x_{\pm}(\sigma, \tau^{\pm}), y_{\pm}(\sigma, \tau^{\pm}))$ near $(\sigma_{\pm}(\tau_0^{\pm}), \tau_0^{\pm})$. So $N_{\pm} := (\cos \theta_{\pm}, \sin \theta_{\pm})$ is well defined near p_0 on the xy -plane. Since γ_{\pm} is C^1 smooth in τ^{\pm} , θ_{\pm} is C^1 smooth in σ and τ^{\pm} by the ODE theory, and hence C^1 smooth near p_0 on the xy -plane. Observe that N_{\pm} satisfies the equation

$$(5.13) \quad \operatorname{div} N_{\pm} = H$$

near p_0 , say, in a neighborhood U' of p_0 since

$$\begin{aligned} \operatorname{div} N_{\pm} &= (\cos \theta_{\pm})_x + (\sin \theta_{\pm})_y \\ &= -(\sin \theta_{\pm})(\theta_{\pm})_x + (\cos \theta_{\pm})(\theta_{\pm})_y \\ &= -\frac{d\theta_{\pm}}{d\sigma} = H \end{aligned}$$

by (5.9).

Lemma 5.2. *Suppose we are in the situation of Definition 5.1. In particular, assume p_0 is nondegenerate. Let N_{\pm} be defined in a neighborhood U' of p_0 , satisfying (5.13) as above. Let $\Upsilon \subset U'$ be a C^1 smooth arc joining p_0 to $\bar{p} \in \gamma \cap U'$. Then*

$$(5.14) \quad \int_{\Upsilon} (N_+ - N_-) \cdot \nu = 0.$$

where ν is the unit normal to Υ such that ν and Υ' are positively oriented.

Proof. Take C^1 smooth arcs $\Upsilon^+ \subset U^+ \cap U'$ and $\Upsilon^- \subset U^- \cap U'$ joining p_0 to \bar{p} (oriented from p_0 towards \bar{p}). Let Ω_+ (Ω_- , resp.) denote the region surrounded by Υ and Υ^+ (Υ^- , resp.). From (5.4) we have

$$(5.15) \quad \begin{aligned} \int_{\Omega_+} H &= \lim_{\varepsilon \rightarrow 0} \oint_{\partial\Omega_+} N_{\varepsilon} \cdot \nu \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Upsilon} N_{\varepsilon} \cdot \nu - \int_{\Upsilon^+} N_+ \cdot \nu \end{aligned}$$

by the Lebesgue Dominated Convergence Theorem since $N_{\varepsilon} \rightarrow N = N_+$ on Υ^+ a.e. (pointwise convergence except 2 end points) and $|N_{\varepsilon}| \leq 2$, say, for ε small. On the other hand, we deduce from (5.13) that

$$(5.16) \quad \begin{aligned} \int_{\Omega_+} H &= \oint_{\partial\Omega_+} N_+ \cdot \nu \\ &= \int_{\Upsilon} N_+ \cdot \nu - \int_{\Upsilon^+} N_+ \cdot \nu. \end{aligned}$$

Comparing (5.15) with (5.16) gives

$$(5.17) \quad \int_{\Upsilon} N_+ \cdot \nu = \lim_{\varepsilon \rightarrow 0} \int_{\Upsilon} N_{\varepsilon} \cdot \nu.$$

Similarly we also have

$$(5.18) \quad \int_{\Upsilon} N_- \cdot \nu = \lim_{\varepsilon \rightarrow 0} \int_{\Upsilon} N_{\varepsilon} \cdot \nu.$$

Now (5.14) follows from (5.17) and (5.18). \square

Definition 5.2. Let $\theta \in C^0(\Omega \setminus K)$ be a weak solution to (1.11) with $H \in C^0(\Omega)$. We call $p \in K$ (where θ is not defined) a crack point if

- (i) there is an open neighborhood $U \subset \Omega$ of p such that $U \cap K$ is a C^0 curve dividing U into two connected regions U^+ , U^- ;
- (ii) $N_{\pm}(p)$ exists as the limit of $N(q)$ for $q \in U^{\pm} \rightarrow p$, resp. and $N_+(p) \neq N_-(p)$.

Corollary 5.3. *Suppose we are in the situation of Lemma 5.2. In particular, assume p_0 is nondegenerate. Moreover, we assume that p_0 is a crack point. Then we have*

$$(5.19) \quad \lim_{\Delta\tau^+ \rightarrow 0^+(0^-, \text{ resp.})} \frac{X_+(\tau_0^+ + \Delta\tau^+) - X_+(\tau_0^+)}{|X_+(\tau_0^+ + \Delta\tau^+) - X_+(\tau_0^+)|} = \pm \frac{N_+ - N_-}{|N_+ - N_-|}(p_0).$$

$$\left(\lim_{\Delta\tau^- \rightarrow 0^+ (0^-, \text{ resp.})} \frac{X_-(\tau_0^- + \Delta\tau^-) - X_-(\tau_0^-)}{|X_-(\tau_0^- + \Delta\tau^-) - X_-(\tau_0^-)|} = \pm \frac{N_+ - N_-}{|N_+ - N_-|}(p_0), \text{ resp.} \right)$$

Proof. Take Υ to be the line segment joining $X_{\pm}(\tau_0^{\pm})$ to $X_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm})$ in (5.14). By the mean-value theorem we then have

$$(5.20) \quad (N_+ - N_-)(\tilde{p}) \cdot (X_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}) - X_{\pm}(\tau_0^{\pm}))^{\perp} = 0$$

for $\tilde{p} \in \Upsilon$. Taking $\Delta\tau^{\pm} \rightarrow 0$ (hence $\tilde{p} \rightarrow p_0$) we get

$$(5.21) \quad \lim_{\Delta\tau^{\pm} \rightarrow 0} \frac{X_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}) - X_{\pm}(\tau_0^{\pm})}{|X_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}) - X_{\pm}(\tau_0^{\pm})|} \cdot (N_{\pm}^{\perp} - N_{\pm}^{\perp})(p_0) = 0.$$

if the limit exists. In case $N_+ \neq N_-$ at p_0 , the limit exists in view of (5.20), and (5.19) follows from (5.21). \square

Proof. (of Theorem G') From (5.10) we have

$$(5.22) \quad \begin{aligned} & \frac{X_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}) - X_{\pm}(\tau_0^{\pm})}{\Delta\tau^{\pm}} \\ &= \frac{\partial P_{\pm}}{\partial \sigma_{\pm}}(\sigma_{\pm}(\tau_0^{\pm} + \Delta\tau_1^{\pm}), \tau_0^{\pm}) \frac{\sigma_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}) - \sigma_{\pm}(\tau_0^{\pm})}{\Delta\tau^{\pm}} \\ & \quad + \frac{\partial P_{\pm}}{\partial \tau^{\pm}}(\sigma_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}), \tau_0^{\pm} + \Delta\tau_2^{\pm}) \end{aligned}$$

where $\Delta\tau_1^{\pm}$ and $\Delta\tau_2^{\pm}$ are numbers between 0 and $\Delta\tau^{\pm}$. Observe that $\frac{\partial P_{\pm}}{\partial \sigma_{\pm}} = N_{\pm}^{\perp}$ from (5.9). Let $\Delta\sigma_{\pm} = \sigma_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}) - \sigma_{\pm}(\tau_0^{\pm})$. Denote

$$(5.23) \quad \begin{aligned} & N_{\pm}^{\perp}(P_{\pm}(\sigma_{\pm}(\tau_0^{\pm} + \Delta\tau_1^{\pm}), \tau_0^{\pm})) \cdot \frac{N_+ - N_-}{|N_+ - N_-|}(p_0), \\ & N_{\pm}^{\perp}(P_{\pm}(\sigma_{\pm}(\tau_0^{\pm} + \Delta\tau_1^{\pm}), \tau_0^{\pm})) \cdot \frac{N_+^{\perp} - N_-^{\perp}}{|N_+^{\perp} - N_-^{\perp}|}(p_0), \\ & \frac{\partial P_{\pm}}{\partial \tau^{\pm}}(\sigma_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}), \tau_0^{\pm} + \Delta\tau_2^{\pm}) \cdot \frac{N_+ - N_-}{|N_+ - N_-|}(p_0), \\ & \frac{\partial P_{\pm}}{\partial \tau^{\pm}}(\sigma_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}), \tau_0^{\pm} + \Delta\tau_2^{\pm}) \cdot \frac{N_+^{\perp} - N_-^{\perp}}{|N_+^{\perp} - N_-^{\perp}|}(p_0) \end{aligned}$$

by A_1, A_2, B_1, B_2 , resp.. So we can write (5.22) as follows:

$$(5.24) \quad \begin{aligned} & \frac{X_{\pm}(\tau_0^{\pm} + \Delta\tau^{\pm}) - X_{\pm}(\tau_0^{\pm})}{\Delta\tau^{\pm}} \\ &= (A_1 \frac{\Delta\sigma_{\pm}}{\Delta\tau^{\pm}} + B_1) \frac{N_+ - N_-}{|N_+ - N_-|}(p_0) + (A_2 \frac{\Delta\sigma_{\pm}}{\Delta\tau^{\pm}} + B_2) \frac{N_+^{\perp} - N_-^{\perp}}{|N_+^{\perp} - N_-^{\perp}|}(p_0). \end{aligned}$$

Observe from (5.23) that

$$\begin{aligned}
(5.25) \quad \lim_{\Delta\tau^\pm \rightarrow 0} A_1 &= (N_\pm^\perp \cdot \frac{N_+ - N_-}{|N_+ - N_-|})(p_0) = \frac{N_-^\perp \cdot N_+}{|N_+ - N_-|}(p_0), \\
\lim_{\Delta\tau^\pm \rightarrow 0} A_2 &= (N_\pm^\perp \cdot \frac{N_+^\perp - N_-^\perp}{|N_+^\perp - N_-^\perp|})(p_0) = \pm \frac{1 - N_+ \cdot N_-}{|N_+ - N_-|}(p_0), \\
\lim_{\Delta\tau^\pm \rightarrow 0} B_1 &= \frac{\partial P_\pm}{\partial \tau^\pm}(p_0) \cdot \frac{N_+ - N_-}{|N_+ - N_-|}(p_0), \\
\lim_{\Delta\tau^\pm \rightarrow 0} B_2 &= \frac{\partial P_\pm}{\partial \tau^\pm}(p_0) \cdot \frac{N_+^\perp - N_-^\perp}{|N_+^\perp - N_-^\perp|}(p_0).
\end{aligned}$$

Comparing (5.24) with (5.19), we obtain

$$(5.26) \quad \lim_{\Delta\tau^\pm \rightarrow 0} \frac{A_2 \frac{\Delta\sigma_\pm}{\Delta\tau^\pm} + B_2}{A_1 \frac{\Delta\sigma_\pm}{\Delta\tau^\pm} + B_1} = 0.$$

Note that $N_+ \neq N_-$ at p_0 by the assumption that p_0 is a crack point. It follows from (5.25) that $\lim_{\Delta\tau^\pm \rightarrow 0} A_2 \neq 0$. Hence from (5.26) we conclude that the limit of $\frac{\Delta\sigma_\pm}{\Delta\tau^\pm}$ exists and

$$(5.27) \quad \lim_{\Delta\tau^\pm \rightarrow 0} \frac{\Delta\sigma_\pm}{\Delta\tau^\pm} = -\frac{\lim_{\Delta\tau^\pm \rightarrow 0} B_2}{\lim_{\Delta\tau^\pm \rightarrow 0} A_2}$$

since all the limits of A_1, A_2, B_1, B_2 , as $\Delta\tau^\pm \rightarrow 0$ exist by (5.25). From (5.24) we can then compute

$$\begin{aligned}
(5.28) \quad \lim_{\Delta\tau^\pm \rightarrow 0} \frac{X_\pm(\tau_0^\pm + \Delta\tau^\pm) - X_\pm(\tau_0^\pm)}{\Delta\tau^\pm} &= \left[\lim_{\Delta\tau^\pm \rightarrow 0} A_1 \left(-\frac{\lim_{\Delta\tau^\pm \rightarrow 0} B_2}{\lim_{\Delta\tau^\pm \rightarrow 0} A_2} \right) + \lim_{\Delta\tau^\pm \rightarrow 0} B_1 \right] \frac{N_+ - N_-}{|N_+ - N_-|}(p_0) \\
&+ \left[\lim_{\Delta\tau^\pm \rightarrow 0} A_2 \left(-\frac{\lim_{\Delta\tau^\pm \rightarrow 0} B_2}{\lim_{\Delta\tau^\pm \rightarrow 0} A_2} \right) + \lim_{\Delta\tau^\pm \rightarrow 0} B_2 \right] \frac{N_+^\perp - N_-^\perp}{|N_+^\perp - N_-^\perp|}(p_0) \\
&= \left[\lim_{\Delta\tau^\pm \rightarrow 0} A_1 \left(-\frac{\lim_{\Delta\tau^\pm \rightarrow 0} B_2}{\lim_{\Delta\tau^\pm \rightarrow 0} A_2} \right) + \lim_{\Delta\tau^\pm \rightarrow 0} B_1 \right] \frac{N_+ - N_-}{|N_+ - N_-|}(p_0)
\end{aligned}$$

by (5.27). (a) follows from (5.28) since all the quantities in the formula are continuous at p_0 . From (5.11) and $\frac{\partial P_\pm}{\partial \sigma_\pm} = N_\pm^\perp$, we write

$$(5.29) \quad \frac{\partial P_\pm}{\partial \tau^\pm}(p_0) = \lambda_\pm(p_0)N_\pm(p_0) + \mu_\pm(p_0)N_\pm^\perp(p_0)$$

for $\lambda_\pm(p_0), \mu_\pm(p_0) \in \mathbb{R}$ and $\lambda_\pm(p_0) \neq 0$. Substituting (5.29) into (5.25), we compute

$$\begin{aligned}
(5.30) \quad \lim_{\Delta\tau^\pm \rightarrow 0} A_1 \left(-\frac{\lim_{\Delta\tau^\pm \rightarrow 0} B_2}{\lim_{\Delta\tau^\pm \rightarrow 0} A_2} \right) + \lim_{\Delta\tau^\pm \rightarrow 0} B_1 &= \pm \lambda_\pm(p_0) \left[\frac{|N_+ - N_-|}{1 - N_+ \cdot N_-} \right](p_0) \neq 0
\end{aligned}$$

where we have used the identity $(1 - N_+ \cdot N_-)^2 + (N_+ \cdot N_-^\perp)^2 = |N_+ - N_-|^2$ (the term involving $\mu_\pm(p_0)$ vanishes). Now (b) follows from (5.28) in view of (5.30). \square

In the above proof, we observe from (5.29), (5.11), and $\frac{\partial P_{\pm}}{\partial \sigma_{\pm}} = N_{\pm}^{\perp}$ that nondegeneracy of p_0 is equivalent to the condition

$$(5.31) \quad \lambda_{\pm}(p_0) = \frac{\partial P_{\pm}}{\partial \tau^{\pm}}(p_0) \cdot N_{\pm}(p_0) \neq 0$$

for both "+" and "-" (the expanding rate of characteristic curves).

Proof. (of Theorem G) Let p_0 be a nondegenerate singular point. By Theorem B (b) the directions of N_{+}^{\perp} and N_{-}^{\perp} at p_0 must point inwards (outwards, resp.) of U^{+} and U^{-} , resp. if $\text{curl } \vec{F}(p_0) > 0$ ($\text{curl } \vec{F}(p_0) < 0$, resp.), where U^{+} and U^{-} are the regions in which a singular curve γ passing through p_0 divides a small neighborhood U of p_0 . Let $\Gamma_{+} \subset U^{+}$ ($\Gamma_{-} \subset U^{-}$, resp.) denote the characteristic curve meeting p_0 with tangent vector $N_{+}^{\perp}(p_0)$ ($N_{-}^{\perp}(p_0)$, resp.) at p_0 . Suppose that $N_{+}(p_0) = N_{-}(p_0)$ (and hence $N_{+}^{\perp}(p_0) = N_{-}^{\perp}(p_0)$). By the uniqueness of the characteristic curves (extending Theorem B (b) or Theorem B' in [5] to the case that p is singular by a similar argument), Γ_{+} must coincide with Γ_{-} near p_0 , a contradiction due to $U^{+} \cap U^{-}$ being empty. We have shown $N_{+}(p_0) \neq N_{-}(p_0)$. I.e. p_0 is a crack point. Now (a), (b) follow from (a), (b) of Theorem G', resp.. \square

In the above proof of Theorem G we showed that a singular point is a crack point in a certain situation. We now want to prove the converse. Let γ be a C^1 smooth curve dividing a planar domain U into two connected regions U^{+} and U^{-} . Let $\tilde{u} \in C^1(U \setminus \gamma) \cap C^0(U)$ be such that $U \setminus \gamma$ is a nonsingular domain. Moreover, suppose \tilde{u} is a weak solution to (1.1) with $\vec{F} \in C^1(U)$, $\text{curl } \vec{F} \neq 0$ and $H \in C^1(U)$ in the sense that for any $\varphi \in C_0^{\infty}(U)$, there holds

$$(5.32) \quad \int_{U \setminus \gamma} \frac{\nabla \tilde{u} + \vec{F}}{|\nabla \tilde{u} + \vec{F}|} \cdot \nabla \varphi + \int_U H \varphi = 0.$$

Take $p_0 \in \gamma$. Suppose that near p_0 we are in the situation of Definition 5.1. Namely, we have the seed curves γ_{\pm} parametrized by τ^{\pm} and the characteristic curves issuing from γ_{\pm} hit γ . By adding the u -variable to (5.9), we consider

$$(5.33) \quad \begin{aligned} \frac{dx}{d\sigma} &= \sin \theta, & \frac{dy}{d\sigma} &= -\cos \theta \\ \frac{d\theta}{d\sigma} &= -H, & \frac{du}{d\sigma} &= -F_1 \sin \theta + F_2 \cos \theta \end{aligned}$$

where $\vec{F} = (F_1, F_2)$. Let $(x_{\pm}(\sigma, \tau^{\pm}), y_{\pm}(\sigma, \tau^{\pm}), \theta_{\pm}(\sigma, \tau^{\pm}), u_{\pm}(\sigma, \tau^{\pm}))$ be the solution to (5.33) with the initial data $(x_{\pm}(0, \tau^{\pm}), y_{\pm}(0, \tau^{\pm})) = \gamma_{\pm}(\tau^{\pm})$, $\theta_{\pm}(0, \tau^{\pm}) = \tilde{\theta}(\gamma_{\pm}(\tau^{\pm}))$, and $u_{\pm}(0, \tau^{\pm}) = u(\gamma_{\pm}(\tau^{\pm}))$. Here we write $\frac{\nabla \tilde{u} + \vec{F}}{|\nabla \tilde{u} + \vec{F}|} = (\cos \tilde{\theta}, \sin \tilde{\theta})$ near p_0 . Suppose that p_0 is nondegenerate. Then there is a diffeomorphism between (σ, τ^{\pm}) and (x, y) near p_0 as shown before. By the basic ODE theory, the solution $(x_{\pm}(\sigma, \tau^{\pm}), y_{\pm}(\sigma, \tau^{\pm}), \theta_{\pm}(\sigma, \tau^{\pm}), u_{\pm}(\sigma, \tau^{\pm}))$ is C^1 smooth in (σ, τ^{\pm}) and is defined near p_0 on the xy -plane. Recall that N_{\pm} is defined to be $(\cos \theta_{\pm}, \sin \theta_{\pm})$ and we call p_0 a crack point if $N_{+}(p_0) \neq N_{-}(p_0)$. Note that $dz + F_1 dx + F_2 dy$ is a contact form in \mathbf{H}_1 due to the condition $\text{curl } \vec{F} \neq 0$. From Theorem A in [5] and $du + F_1 dx + F_2 dy = 0$ along the characteristic curves, we have $\theta_{\pm} = \tilde{\theta}$, $u_{\pm} = \tilde{u}$ on U^{\pm} by

the uniqueness of solutions to the ODE system (5.33) with the same initial data. It follows that $N_{\pm} = \frac{\nabla u_{\pm} + \vec{F}}{D_{\pm}}$ where $D_{\pm} := |\nabla u_{\pm} + \vec{F}|$.

Theorem 5.4. *Let γ be a C^1 smooth curve dividing a planar domain U into two connected regions U^+ and U^- . Let $\tilde{u} \in C^1(U \setminus \gamma) \cap C^0(U)$ be such that $U \setminus \gamma$ is a nonsingular domain. Moreover, \tilde{u} is a weak solution to (1.1) with $\vec{F} \in C^1(U)$, $\text{curl} \vec{F} \neq 0$ and $H \in C^1(U)$. Let $p_0 \in \gamma$ be a nondegenerate crack point. Then $\tilde{u} \in C^1(V)$ for some neighborhood $V \subset U$ of p_0 and p_0 is a singular point of \tilde{u} . That is, $\nabla \tilde{u} + \vec{F} = 0$ at p_0 .*

Proof. Observe that along a characteristic curve, there holds $\frac{d\tilde{u}(x(\sigma), y(\sigma))}{d\sigma} + F_1 \frac{dx}{d\sigma} + F_2 \frac{dy}{d\sigma} = 0$ on $U \setminus \gamma = U^+ \cup U^-$ for $\tilde{u} \in C^1(U \setminus \gamma)$. It follows from the uniqueness of solutions to the ODE system (5.33) with the same initial data and $\tilde{u} \in C^0(U)$ that

$$(5.34) \quad \begin{aligned} u_+ &= \tilde{u} \text{ on } U^+ \cup \gamma \\ u_- &= \tilde{u} \text{ on } U^- \cup \gamma. \end{aligned}$$

So $u_+ = u_-$ on γ . Take a unit-speed parameter ς for γ ($\in C^1$) and note that both u_+ and u_- are defined and C^1 smooth in a small neighborhood V of p_0 . We can choose V such that each $p \in \gamma \cap V$ is a nondegenerate crack point. Therefore $\frac{du_{\pm}}{d\varsigma}$ exists and

$$(5.35) \quad \frac{du_+}{d\varsigma} = \frac{du_-}{d\varsigma}$$

on $\gamma \cap V$. Compute

$$(5.36) \quad \begin{aligned} &\frac{du_{\pm}}{d\varsigma} + F_1 \frac{dx}{d\varsigma} + F_2 \frac{dy}{d\varsigma} \\ &= (\nabla u_{\pm} + \vec{F}) \cdot \frac{d\gamma}{d\varsigma} \end{aligned}$$

where $\frac{d\gamma}{d\varsigma} = (\frac{dx}{d\varsigma}, \frac{dy}{d\varsigma})$. From (5.35) and (5.36) we obtain

$$(5.37) \quad D_+ N_+ \cdot \frac{d\gamma}{d\varsigma} = D_- N_- \cdot \frac{d\gamma}{d\varsigma}.$$

Here we recall that $D_{\pm} := |\nabla u_{\pm} + \vec{F}|$ and $N_{\pm} = \frac{\nabla u_{\pm} + \vec{F}}{D_{\pm}}$. From Theorem G' (b) (equal-angle condition) (see (5.28) and (5.30)) and $N_+(p) \neq N_-(p)$ for each $p \in \gamma \cap V$, we obtain

$$(5.38) \quad -N_+ + N_- = (N_+^{\perp} - N_-^{\perp})^{\perp} \parallel \frac{d\gamma}{d\varsigma}.$$

It follows from the identity $(N_+ + N_-) \cdot (N_+ - N_-) = 0$ and (5.38) that

$$(N_+ + N_-) \cdot \frac{d\gamma}{d\varsigma} = 0$$

and hence

$$(5.39) \quad N_+ \cdot \frac{d\gamma}{d\varsigma} \text{ and } N_- \cdot \frac{d\gamma}{d\varsigma} \text{ have different sign}$$

at each $p \in \gamma \cap V$. Therefore we have $D_{\pm} = 0$ on $\gamma \cap V$ in view of $D_{\pm} \geq 0$, (5.37) and (5.39). It follows that $\nabla u_{\pm} + \vec{F} = 0$, and hence $\nabla u_+ = \nabla u_-$ at each $p \in \gamma \cap V$. This implies that $\nabla \tilde{u}$ exists and equals $\nabla u_+ = \nabla u_-$ at each $p \in \gamma \cap V$ by (5.34). Since $u_+, u_- \in C^1(V)$, we see that $\nabla \tilde{u}$ is continuous at each $p \in \gamma \cap V$, and hence $\tilde{u} \in C^1(V)$. From $\nabla \tilde{u} + \vec{F} = \nabla u_+ + \vec{F} = 0$ on $\gamma \cap V$, we learn that in particular, p_0 is a singular point of \tilde{u} . \square

To illustrate Theorem G, we consider the case of a p -minimal graph over a planar domain Ω , defined by $u \in C^1(\Omega)$. In this case we have some interesting formulas such as (5.49) and (5.50). From (1.7) we can deduce that if $D' > 1$ at an initial point, then

$$(5.40) \quad D' = \frac{4 + cD^2 \pm \sqrt{(4 + cD^2)cD^2}}{2}.$$

Meanwhile, if $D' < 1$ at an initial point, we obtain

$$(5.41) \quad D' = \frac{4 - cD^2 \pm \sqrt{(4 - cD^2)(-cD^2)}}{2}.$$

Since $c > 0$, we must have $4 - cD^2 \leq 0$ in this case. It follows that $\sqrt{\frac{4}{c}} \leq D$. So D never reaches 0 along a characteristic line if $D' < 1$ at an initial point. Observe that

$$(5.42) \quad \frac{2}{4 + cD^2 \pm \sqrt{(4 + cD^2)cD^2}} = \frac{1}{2} \mp \frac{1}{2} \frac{\sqrt{c}D}{\sqrt{4 + cD^2}}.$$

When $H = 0$, we can take $f \equiv 1$ so that the parameter s in Theorem C of [5] is just the arc length σ along characteristic lines. From (5.40) and (5.42) we can integrate and obtain

$$(5.43) \quad \frac{1}{2} \left(D \mp \sqrt{D^2 + \frac{4}{c}} \right) \Big|_{\sigma_0}^{\sigma_1} = \sigma_1 - \sigma_0.$$

Suppose $D(0) = 0$. Taking $\sigma_0 = 0$, $\sigma_1 = s$ in (5.43) we obtain

$$(5.44) \quad D(s) = s + a - \frac{a^2}{s + a}$$

where $a = \mp \sqrt{\frac{1}{c}}$. By (5.44) we compute

$$(5.45) \quad D'(s) = 1 + \frac{a^2}{(s + a)^2}.$$

It follows from (5.45) that $D' \rightarrow 2$ as $s \rightarrow 0$ for $a \neq 0$ while $D' \equiv 1$ for $a = 0$. This verifies Theorem B (a) for the case of p -minimal graphs.

If we start with points on a C^1 smooth curve $\beta(\tau)$ transverse to the characteristic lines, described by $\sigma_0 = 0$, say, then we can describe the first (singular) points where the characteristic lines hit the singular set by $\sigma_1 = \sigma_1(\tau)$. Noting that $D = 0$ at the points described by $(\tau, \sigma_1(\tau))$, we have

$$(5.46) \quad \sigma_1(\tau) = \mp \frac{1}{\sqrt{c(\tau)}} - \frac{1}{2} D(\beta(\tau)) \pm \frac{1}{2} \sqrt{D(\beta(\tau))^2 + \frac{4}{c(\tau)}}$$

by (5.43). Here we have written $c = c(\tau)$. Since D and D' are continuous by Theorem D of [5], we conclude that $c = c(\tau)$ is continuous in τ by (1.7). It follows from (5.46) that $\sigma_1(\tau)$ is continuous in τ . For general $H \in C^1$ and $\vec{F} \in C^1$, we cannot expect to have an explicit formula for $\sigma_1(\tau)$ (replace characteristic lines by characteristic curves in the definition) like (5.46). But still $\sigma_1(\tau)$ is C^0 in τ by Lemma 3.2'.

Let $p \in S(u)$ be a singular point. Suppose we are in the situation described in Definition 5.1 with K replaced by $S(u)$. So we have γ_{\pm} , parametrized by τ^{\pm} , $X_{\pm}(\tau^{\pm})$, and $X_+(\tau_0^+) = X_-(\tau_0^-) = p_0$. Since the characteristic curves are straight lines in this case (θ being constant along a characteristic curve due to $\frac{d\theta}{d\sigma} = -H = 0$ by (5.9)), we can write

$$(5.47) \quad \begin{aligned} X_+(\tau^+) &= \gamma_+(\tau^+) + \sigma_+(\tau^+)N_+(\tau^+) \\ X_-(\tau^-) &= \gamma_-(\tau^-) + \sigma_-(\tau^-)N_-(\tau^-) \end{aligned}$$

for some real functions $\sigma_{\pm} (\in C^0$ since X_{\pm} , γ_{\pm} , and N_{\pm}^{\perp} are C^0), where $N_{\pm}^{\perp}(\tau^{\pm}) := (\sin \theta_{\pm}(\tau^{\pm}), -\cos \theta_{\pm}(\tau^{\pm}))$ for some angular functions θ_{\pm} (cf. (5.10)). It follows from (5.47) and (5.10) that

$$(5.48) \quad \begin{aligned} \frac{\partial P_{\pm}}{\partial \tau^{\pm}} &= \frac{\partial \gamma_{\pm}}{\partial \tau^{\pm}} + \sigma_{\pm} \frac{\partial N_{\pm}^{\perp}}{\partial \tau^{\pm}} \\ &= N_{\pm} + \sigma_{\pm} \theta'_{\pm} N_{\pm} \end{aligned}$$

(noting that $\theta_{\pm} \in C^1$ by the ODE theory and $\theta \in C^1$ by Theorem D in [5]). From (5.31) and (5.48) we have

$$(5.49) \quad \begin{aligned} \lambda_{\pm}(p_0) &= \frac{\partial P_{\pm}}{\partial \tau^{\pm}}(p_0) \cdot N_{\pm}(p_0) \\ &= 1 + \sigma_{\pm}(\tau_0^{\pm}) \theta'_{\pm}(\tau_0^{\pm}). \end{aligned}$$

Recall that p_0 is nondegenerate if and only if both $\lambda_+(p_0) \neq 0$ and $\lambda_-(p_0) \neq 0$. Since $\frac{\partial P_{\pm}}{\partial \sigma_{\pm}} = N_{\pm}^{\perp}$ and hence $\frac{\partial P_{\pm}}{\partial \tau^{\pm}} \cdot N_{\pm} = \frac{\partial P_{\pm}}{\partial \sigma_{\pm}} \wedge \frac{\partial P_{\pm}}{\partial \tau^{\pm}}$, we learn that $\lambda_{\pm}(p_0) \neq 0$ is independent of the choice of seed curves by (5.12).

Proposition 5.5. *Consider a p -minimal graph over a planar domain Ω , defined by $u \in C^1(\Omega)$. Suppose we are in the situation described in Definition 5.1 with K replaced by $S(u)$. Suppose further $X_+(\tau^+) = X_-(\tau^-)$. Then we have*

$$(5.50) \quad \sigma_+(\tau^+) - \sigma_+(\tau_0^+) = \sigma_-(\tau^-) - \sigma_-(\tau_0^-).$$

Proof. Let $\Gamma_{\tau^+}^+$ ($\Gamma_{\tau_0^+}^+$, $\Gamma_{\tau^-}^-$, $\Gamma_{\tau_0^-}^-$, resp.) denote the characteristic line (segment) which connect $\gamma_+(\tau^+)$ ($\gamma_+(\tau_0^+)$, $\gamma_-(\tau^-)$, $\gamma_-(\tau_0^-)$, resp.) with $X_+(\tau^+)$ ($X_+(\tau_0^+)$, $X_-(\tau^-)$, $X_-(\tau_0^-)$, resp.) (see Figure 5.1).

Let $\hat{\Omega}$ denote the region surrounded by $\Gamma_{\tau_0^+}^+$, $\Gamma_{\tau_0^-}^-$, segment($\gamma_-(\tau_0^-)\gamma_-(\tau^-)$), $\Gamma_{\tau^-}^-$, $\Gamma_{\tau^+}^+$, and segment($\gamma_+(\tau^+)\gamma_+(\tau_0^+)$). By Theorem B (b) (the unit outward normal)

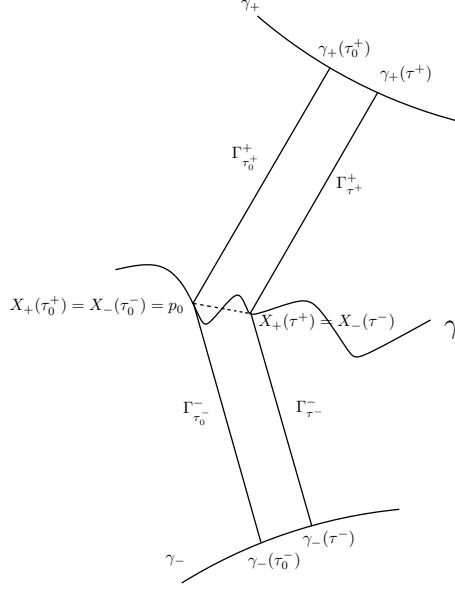


Figure 5.1

$\nu = +N$ along $\Gamma_{\tau_0^+}^+$ and $\Gamma_{\tau^-}^-$ while $\nu = -N$ along $\Gamma_{\tau_0^-}^-$ and $\Gamma_{\tau^+}^+$. We can now apply Lemma 5.1 with $\vec{F} = (-y, x)$ and $H = 0$ to obtain

$$\begin{aligned}
 (5.51) \quad 0 &= \oint_{\partial\hat{\Omega}} N \cdot \nu = \int_{\Gamma_{\tau_0^+}^+} N \cdot N \\
 &\quad + \int_{\Gamma_{\tau_0^-}^-} N \cdot (-N) + \int_{\Gamma_{\tau^-}^-} N \cdot N + \int_{\Gamma_{\tau^+}^+} N \cdot (-N) \\
 &= \sigma_+(\tau_0^+) - \sigma_-(\tau_0^-) + \sigma_-(\tau^-) - \sigma_+(\tau^+).
 \end{aligned}$$

Note that $\nu = \pm N^\perp$ and hence $N \cdot \nu = 0$ along the seed curves γ_+ and γ_- . Now (5.50) follows from (5.51). \square

We can interpret the quantity $\lambda_\pm(p_0) = 1 + \sigma_\pm(\tau_0^\pm)\theta'_\pm(\tau_0^\pm)$ in (5.49) for a nonsingular point p_0 in terms of an integrating factor for solving the t -coordinate in Theorem C of [5]. Recall that the integrating factor gD for solving t such that $\nabla t = gDN$ satisfies

$$(5.52) \quad N^\perp g + \frac{(\text{curl}\vec{F})g}{D} = 0.$$

Let $\tilde{g} := gD$. A direct computation shows that

$$(5.53) \quad N^\perp \tilde{g} = \tilde{g} \left(\frac{N^\perp D - \text{curl}\vec{F}}{D} \right).$$

By (1.13) and $N^\perp = f \frac{\partial}{\partial s}$ in [5] we can reduce (5.53) to

$$(5.54) \quad f \frac{\partial \tilde{g}}{\partial s} + \tilde{g}^2 \frac{\partial \theta}{\partial t} = 0.$$

We can take $f \equiv 1$ (satisfying the first equation of (1.10) in [5]) for the case $H = 0$. Observe that $\theta \in C^1$ by Theorem D in [5] and $\partial_t(\partial_s \theta)$ exists and is continuous by (1.12) of [5] for $H \in C^1$, say (f is known to be C^1 smooth in t). By Lemma 5.4 in [5] we have the existence of $\partial_s(\partial_t \theta)$ and

$$(5.55) \quad \begin{aligned} \partial_s(\partial_t \theta) &= \partial_t(\partial_s \theta) = \partial_t\left(-\frac{H}{f}\right) \\ &(\quad = 0 \text{ if } H = 0). \end{aligned}$$

So $\partial_t \theta$ is independent of s for the case $H = 0$ by (5.55). We can now solve \tilde{g} for (5.54) in the case of $H = 0$ (in which $f \equiv 1$) to get

$$\tilde{g}^{-1}(s, t) = 1 + s \frac{\partial \theta}{\partial t}(0, t).$$

Here we have taken $\tilde{g}(0, t) = 1$ for which $\partial_t = \partial_\tau$ at $s = 0$.

6. THE LOCAL THEORY OF SURFACES WITH PRESCRIBED p -MEAN CURVATURE

We are going to prove Theorem H. We have in mind that V plays the role of N^\perp . We need to find N in the ξ, η coordinates. Recall that on the xy -plane, $\operatorname{div} DN^\perp = \operatorname{div}(u_y + x, -(u_x - y)) = 2$ for $N = \frac{(u_x - y, u_y + x)}{D}$ and $D = [(u_x - y)^2 + (u_y + x)^2]^{1/2}$. It follows that $N^\perp(D) = 2 - N(\theta)D$ if we write $N = (\cos \theta, \sin \theta)$ while $N^\perp = (\sin \theta, -\cos \theta)$. Therefore we have

$$(6.1) \quad N(\theta) = \frac{2 - N^\perp(D)}{D}.$$

We compute the commutator of N and N^\perp :

$$(6.2) \quad \begin{aligned} [N, N^\perp] &= (\theta_x \cos^2 \theta + \theta_y \sin \theta \cos \theta) \partial_x + (\theta_x \cos \theta \sin \theta + \theta_y \sin^2 \theta) \partial_y \\ &\quad - (\theta_x \sin \theta (-\sin \theta) + \theta_y \cos \theta \sin \theta) \partial_x - (\theta_x \cos \theta \sin \theta - \theta_y \cos^2 \theta) \partial_y \\ &= \theta_x \partial_x + \theta_y \partial_y = \nabla \theta. \end{aligned}$$

We can express

$$(6.3) \quad \begin{aligned} \nabla \theta &= N(\theta)N + N^\perp(\theta)N^\perp \\ &= \left(\frac{2 - N^\perp(D)}{D}\right)N - HN^\perp \end{aligned}$$

by (6.1) and (2.23) in [3].

Proof. (of Theorem H) Without specifying the regularity, we mean C^∞ smoothness for each quantity in the following argument. Note that $L_V P = [V, P]$ and we (having $P = N$ in mind) obtain equation (1.13) in view of (6.2) and (6.3). Next we can find a solution $f > 0$ ($g > 0$, resp.) to the equation

$$(6.4) \quad P(f) + Hf = 0 \quad (V(g) + \frac{2g}{D} = 0, \text{ resp.})$$

in a small neighborhood of p in the $\xi\eta$ -plane by assigning any positive initial value of f (g , resp.) along an integral curve of V (P , resp.) through p . Now we want to solve in s and t for the following equations:

$$(6.5) \quad V(s) = f, \quad P(s) = 0;$$

$$(6.6) \quad V(t) = 0, \quad P(t) = gD.$$

From (1.13) we need to check whether the integrability condition for (6.5) ((6.6), resp.) $-\left(\frac{2-V(D)}{D}\right)P(s) + HV(s) = VP(s) - PV(s) = 0 - P(f) \left(-\left(\frac{2-V(D)}{D}\right)P(t) + HV(t) = VP(t) - PV(t) = V(gD)$, resp.) holds by Frobenius' theorem. A direct computation shows that (6.4) makes these integrability conditions hold. Therefore (6.5) and (6.6) are solvable. Since V and P are transversal, s and t form a local coordinate system by (6.5) and (6.6) (note that f , g , and D are all positive). Moreover, (V, P) has the same orientation as (∂_s, ∂_t) .

Next we want to find local coordinate functions x and y such that

$$(6.7) \quad \frac{ds^2}{f^2} + \frac{dt^2}{g^2D^2} = dx^2 + dy^2$$

(in view of (1.11) in [5]). By the fundamental theorem of Riemannian geometry, we need to check if the Gaussian curvature K of the metric $\frac{ds^2}{f^2} + \frac{dt^2}{g^2D^2}$ equals zero. For a metric of the form $Eds^2 + Gdt^2$ (orthogonal parametrization), we have

$$(6.8) \quad K = -\frac{1}{2A}[\partial_s\left(\frac{G_s}{A}\right) + \partial_t\left(\frac{E_t}{A}\right)]$$

where $A = \sqrt{EG}$. Substituting $E = \frac{1}{f^2}$ and $G = \frac{1}{g^2D^2}$ into (6.8) we have $A = \frac{1}{fgD}$ and

$$(6.9) \quad \begin{aligned} K &= -\frac{fgD}{2}[\partial_s\left(\frac{\left(\frac{1}{g^2D^2}\right)_s}{\frac{1}{fgD}}\right) + \partial_t\left(\frac{\left(\frac{1}{f^2}\right)_t}{\frac{1}{fgD}}\right)] \\ &= -\frac{fgD}{2}[\partial_s(-2g^{-2}D^{-1}fg_s - 2g^{-1}D^{-2}fD_s) + \partial_t(-2f^{-2}gDf_t)]. \end{aligned}$$

From (6.5) and (6.6) we can easily relate ∂_s, ∂_t to V, P as follows:

$$(6.10) \quad V = f\frac{\partial}{\partial s}, \quad P = gD\frac{\partial}{\partial t}.$$

It follows from (6.10) and (6.4) that

$$(6.11) \quad \begin{aligned} fg_s &= Vg = -\frac{2g}{D}, \quad fD_s = V(D), \quad \text{and} \\ gDf_t &= Pf = -Hf. \end{aligned}$$

Substituting (6.11) into (6.9), we obtain

$$(6.12) \quad \begin{aligned} K &= -\frac{fgD}{2}[\partial_s(4g^{-1}D^{-2} - 2g^{-1}D^{-2}D') + \partial_t(2Hf^{-1})] \\ &= -\frac{gD}{2}\{[4(-1)g^{-2}V(g)D^{-2} + 4g^{-1}(-2)D^{-3}D'](1 - \frac{1}{2}D') \\ &\quad + 4g^{-1}D^{-2}(-\frac{1}{2}D'')\} - \frac{f}{2}(2H(-1)f^{-2}N(f)) - P(H) \\ &= D^{-1}D'' - 2D^{-2}(D' - 1)(D' - 2) - H^2 - P(H) \end{aligned}$$

by (6.10) and (6.11). Comparing (6.12) with the condition (1.12), we can finally conclude that $K = 0$. So we have proved the existence of local coordinates x and y such that (6.7) holds. Moreover, we can find x, y such that (∂_x, ∂_y) has the same orientation as $(\partial_\xi, \partial_\eta)$.

Observe that both V and P are unit vectors and orthogonal with respect to the metric (6.7). So we can write

$$(6.13) \quad V(x) = \sin \theta, \quad V(y) = -\cos \theta$$

for some function θ locally near p . It follows from the orthonormality and the arrangement of orientation that

$$(6.14) \quad P(x) = \cos \theta, \quad P(y) = \sin \theta.$$

So we have $N = P$ and $N^\perp = V$. With N^\perp replaced by V in (6.2) and (6.3), we get

$$(6.15) \quad [V, N] = -N(\theta)N - V(\theta)V$$

by (6.13) and (6.14). Comparing (6.15) with (1.13) (P replaced by N), we obtain

$$(6.16) \quad \begin{aligned} N(\theta) &= \frac{2 - V(D)}{D} \text{ and} \\ V(\theta) &= -H. \end{aligned}$$

We have proved (1.15) and (1.14).

Next we are going to prove (2). Take a local integral curve ℓ of N through p (may assume $x(p) = y(p) = 0$). Choose a C^∞ smooth function u_0 along ℓ with $N(u_0) + (-y, x) \cdot N = D$ where " \cdot " denotes the standard planar inner product. For any point $q \in \ell$, there passes an integral (characteristic) curve Γ_q of V . Define the value of u on Γ_q by integrating the contact form

$$(6.17) \quad du + xdy - ydx = 0$$

along Γ_q . That is, at $\zeta \in \Gamma_q$, we define

$$u(\zeta) = u_0(q) + \int_q^\zeta (ydx - xdy)$$

where the integral means the line integral from q to ζ along Γ_q . Thus $u = u(x, y)$ is defined in an open neighborhood of p and is a C^∞ smooth function. Writing (6.17) as $(u_x - y)dx + (u_y + x)dy = 0$, we obtain

$$[\nabla u + (-y, x)] \cdot V = 0.$$

It follows that

$$(6.18) \quad \nabla u + (-y, x) = \tilde{D}N$$

for some function \tilde{D} by the orthonormality of V and N . Taking the inner product of (6.18) and N gives

$$(6.19) \quad \tilde{D} = N(u) + (-y, x) \cdot N = D \text{ along } \ell.$$

Since $D > 0$, we may assume that $\tilde{D} > 0$ as well in a small neighborhood of p . From (6.18) we have $(u_y, -u_x) + (x, y) = (\nabla u + (-y, x))^\perp = \tilde{D}V$ (recall that $\vec{G}^\perp := (G_2, -G_1)$ for $\vec{G} = (G_1, G_2)$). Applying the divergence operator to both sides and expressing $V = (\sin \theta, -\cos \theta)$, we obtain

$$(6.20) \quad \begin{aligned} 2 &= \operatorname{div}(\tilde{D}V) \\ &= V(\tilde{D}) + \tilde{D}N(\theta). \end{aligned}$$

It follows that

$$(6.21) \quad \frac{2 - V(\tilde{D})}{\tilde{D}} = N(\theta).$$

Comparing (6.21) with (6.16) we get

$$(6.22) \quad V(\tilde{D}) = V(D) \text{ along } \ell$$

in view of (6.19). Now applying V to (6.21), we compute

$$(6.23) \quad \begin{aligned} V\left(\frac{2 - V(\tilde{D})}{\tilde{D}}\right) &= V(N(\theta)) \\ &= [V, N](\theta) + N(V(\theta)) \\ &= -N(\theta)^2 - V(\theta)^2 - N(H) \end{aligned}$$

by (6.15) and $V(\theta) = -H$, the second equation of (6.16). Substituting (6.21) and $V(\theta) = -H$ into (6.23) we finally obtain

$$\tilde{D}\tilde{D}'' = 2(\tilde{D}' - 1)(\tilde{D}' - 2) + (H^2 + N(H))\tilde{D}^2$$

in which we denote $V(\tilde{D})$, $V(V(\tilde{D}))$ by \tilde{D}' , \tilde{D}'' , resp.. That is, \tilde{D} satisfies the same equation as D does in (1.12) (noting that $P = N$). Since \tilde{D} and D satisfy the same initial data by (6.19) and (6.22), we can conclude that $\tilde{D} = D$ in a small neighborhood of p by the uniqueness of the solution to an ordinary differential equation of second order. Substituting $\tilde{D} = D$ into (6.18) we have

$$N = \frac{\nabla u + (-y, x)}{D}$$

and hence $D = |\nabla u + (-y, x)| = \sqrt{(u_x - y)^2 + (u_y + x)^2}$. (1.16) follows from $V(\theta) = -H$ and observing that $\operatorname{div} N = -V(\theta)$. (1.17) is simply (6.20) with $\tilde{D} = D$. \square

7. INDEX OF THE SINGULAR SET

In this section we are going to show some global results about the singular set.

Lemma 7.1. *Let Ω be a bounded planar domain. Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Assume further $N^\perp(\operatorname{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω and $\operatorname{curl}\vec{F} \neq 0$. Suppose the singular set $S_{\vec{F}}(u) \subset \Omega$ is compact. Then $\# \pi_0(S_{\vec{F}}(u)) < \infty$, i.e. the number of the connected components of $S_{\vec{F}}(u)$ is finite.*

Proof. Suppose $\# \pi_0(S_{\vec{F}}(u)) = \infty$. Since Ω is bounded, there exist a sequence of distinct connected components S_j , $j = 1, 2, \dots$ and a sequence of points $q_j \in S_j$ converging to $q_\infty \in \Omega$. It follows from the compactness of $S_{\vec{F}}(u)$ that $q_\infty \in \Omega$ and hence $q_\infty \in S_{\vec{F}}(u)$ by closedness. From Theorem C (b) there exists a neighborhood $U \subset \Omega$ of q_∞ such that $U \cap S_{\vec{F}}(u)$ is path-connected. This implies that $S_K = S_{K+1} = S_{K+2} = \dots$ for some large K . We have reached a contradiction. \square

In the following we want to use "step functions" to approximate a C^0 singular curve. Let $\beta : [0, \tilde{\tau}] \rightarrow \Omega$ be a nonsingular C^1 smooth curve which is transverse to the characteristic curves $\Gamma(\beta(\tau))$ issuing from $\beta(\tau)$ for all $\tau \in [0, \tilde{\tau}]$ ($\beta(0) = \beta(\tilde{\tau})$ if β is closed). Suppose each $\Gamma(\beta(\tau))$ hits a singular point $\mathfrak{s}(\tau)$ (see Figure 7.1).

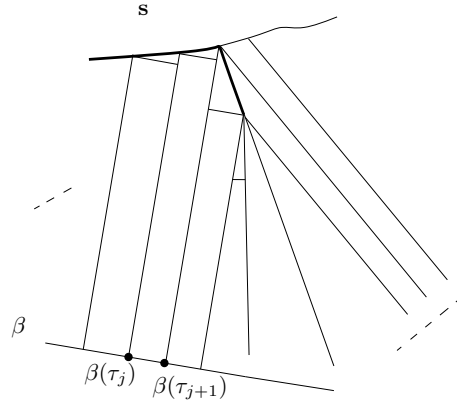


Figure 7.1

Lemma 7.2. *Let Ω be a bounded planar domain. Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω and $\text{curl}\vec{F} \neq 0$. Suppose each of the characteristic curves passing through a nonsingular C^1 smooth curve $\beta(\tau)$ hits a singular point $\mathfrak{s}(\tau)$ as above. Then \mathfrak{s} is continuous.*

Proof. Suppose \mathfrak{s} is not continuous at $\tau_0 \in [0, \tilde{\tau}]$. Then we can find a sequence $\tau_j \in [0, \tilde{\tau}]$, converging to τ_0 , such that the characteristic curves $\Gamma(\beta(\tau_j))$ converge (C^2 for any compact parameter interval) to a characteristic curve Γ_∞ and either Γ_∞ contains $\mathfrak{s}(\tau_0)$ or $\Gamma(\beta(\tau_0)) \cup \{\beta(\tau_0)\}$ contains $\bar{\Gamma}_\infty$. Both cases contradict Lemma 3.2' (b) by noting that $\bar{\Gamma}_\infty$ contains a singular end point. \square

We remark that the result in Lemma 7.2 has been used in proving Lemma 3.5'. Recall that in Section 5 the map $P : (\sigma, \tau) \rightarrow (x, y)$ describes $\Gamma(\beta(\tau))$ with $P(0, \tau) = \beta(\tau)$. Take a partition of $[0, \tilde{\tau}]$, $0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_\kappa = \tilde{\tau}$. Let $\bar{\sigma}_j$ be such a number that either $P(\bar{\sigma}_j, \tau_j)$ meets $\mathfrak{s}(\tau_j)$ or $P(\bar{\sigma}_j, \tau_{j+1})$ meets $\mathfrak{s}(\tau_{j+1})$ for $j = 0, 1, \dots, \kappa - 1$. Let $\sigma_1(\tau)$ denote the length of $\Gamma(\beta(\tau))$ from $\beta(\tau)$ to $\mathfrak{s}(\tau)$. Note that \mathfrak{s} and hence σ_1 is C^0 by Lemma 7.2 under the assumption there and P is C^1 smooth for $\tau \in [0, \tilde{\tau}]$ and $\sigma \in [0, \sigma_1(\tau))$ originally, but in fact P extends C^1 smoothly over $\{(\sigma_1(\tau), \tau)\}$ if $H \in C^1(\Omega)$ besides the conditions in Lemma 7.2. See the argument in the proof of Theorem F in Section 5. So there is small $\delta > 0$ such that

$$(7.1) \quad \begin{aligned} & \sup_{\max_j |\tau_{j+1} - \tau_j| \leq \delta} \sum_{j=0}^{\kappa-1} |P(\bar{\sigma}_j, \tau_{j+1}) - P(\bar{\sigma}_j, \tau_j)| \\ & \leq \tilde{\tau} \sup_{(\sigma, \tau) \in K} \left| \frac{\partial P}{\partial \tau}(\sigma, \tau) \right| \end{aligned}$$

by the mean value theorem, where $K = K(\delta)$ is a compact set in the (σ, τ) -plane, containing $\{(\sigma_1(\tau), \tau) : \tau \in [0, \tilde{\tau}]\}$.

We remark that although the C^0 singular curve $\mathfrak{s}(\tau)$ may not be rectifiable, the sum of the length of "steps" that approximate \mathfrak{s} is bounded by (7.1). Let $\Theta_{\vec{F}} := du + F_1 dx + F_2 dy = (u_x + F_1) dx + (u_y + F_2) dy$. Let $\text{Area}(\beta, \mathfrak{s})$ denote the area of region $R(\beta, \mathfrak{s})$ surrounded by $\Gamma(\beta(\tilde{\tau}))$, \mathfrak{s} , $\Gamma(\beta(0))$, and β (see Figure 7.1 in which \int_{β} means the integration from $\beta(0)$ to $\beta(\tilde{\tau})$).

Lemma 7.3. *Suppose we are in the situation of Lemma 7.2. Assume further $H \in C^1(\Omega)$. Then if $0 < C_1 \leq |\text{curl} \vec{F}| \leq C_2$ on region $R(\beta, \mathfrak{s})$ we have*

$$(7.2) \quad C_1 \text{Area}(\beta, \mathfrak{s}) \leq \left| \int_{\beta} \Theta_{\vec{F}} \right| \leq C_2 \text{Area}(\beta, \mathfrak{s}).$$

Proof. Let L_j denote the line segment from $P(\bar{\sigma}_j, \tau_{j+1})$ to $P(\bar{\sigma}_j, \tau_j)$. Let ω_j denote the region surrounded by $\beta([\tau_j, \tau_{j+1}])$, $P([0, \bar{\sigma}_j], \tau_{j+1})$, L_j , and $P([0, \bar{\sigma}_j], \tau_j)$. By Green's theorem we have

$$(7.3) \quad \begin{aligned} \int_{\omega_j} \text{curl} \vec{F} dx \wedge dy &= \oint_{\partial \omega_j} \Theta_{\vec{F}} \\ &= \int_{L_j} \Theta_{\vec{F}} + \int_{\beta([\tau_j, \tau_{j+1}])} \Theta_{\vec{F}} \end{aligned}$$

since $\Theta_{\vec{F}} = 0$ when acting on $N^\perp(u)$. Summing (7.3) over j we get

$$(7.4) \quad \sum_{j=0}^{\kappa-1} \int_{\omega_j} \text{curl} \vec{F} dx \wedge dy - \int_{\beta} \Theta_{\vec{F}} = \sum_{j=0}^{\kappa-1} \int_{L_j} \Theta_{\vec{F}}.$$

Note that $D := \sqrt{(u_x + F_1)^2 + (u_y + F_2)^2} = 0$ on \mathfrak{s} (consisting of singular points) and D is continuous. Therefore given $\varepsilon > 0$, we have $D(q) < \varepsilon$ for $q \in \cup_{j=0}^{\kappa-1} L_j$ when κ is large and $\max_{0 \leq j \leq \kappa-1} |\tau_{j+1} - \tau_j|$ is small enough. We can then estimate

$$(7.5) \quad \begin{aligned} \left| \sum_{j=0}^{\kappa-1} \int_{L_j} \Theta_{\vec{F}} \right| &= \left| \int_{\cup_{j=0}^{\kappa-1} L_j} \Theta_{\vec{F}} \right| \\ &\leq \int_{\cup_{j=0}^{\kappa-1} L_j} D d\bar{s} \quad (\bar{s} : \text{arc-length parameter}) \\ &\leq \varepsilon \tilde{\tau} \sup_{(\sigma, \tau) \in K} \left| \frac{\partial P}{\partial \tau}(\sigma, \tau) \right| \end{aligned}$$

by (7.1). Now (7.2) follows from (7.4) and (7.5) as $\varepsilon \rightarrow 0$. □

Lemma 7.4. *Let Ω be a bounded domain of R^2 . Let $u \in C^1(\Omega)$ be a weak solution to (1.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^1(\Omega)$. Assume further $N^\perp(\text{curl}\vec{F})$ and $N(H)$ exist and are continuous (extended over singular points) in Ω . Let \hat{S} be a connected component of $S_{\vec{F}}(u)$. Suppose $\text{curl}\vec{F} > 0$ ($\text{curl}\vec{F} < 0$, resp.) and \hat{S} is compact and path-connected. Then for any $\varepsilon > 0$, there exists a simple closed C^0 curve γ_ε in Ω such that*

- (a) $\text{dist}(\gamma_\varepsilon, \hat{S}) < \varepsilon$ where $\text{dist}(\gamma_\varepsilon, \hat{S})$ denotes the distance between γ_ε and \hat{S} .
- (b) $N^\perp(u)$ ($-N^\perp(u)$, resp.) points outward along γ_ε , i.e., the characteristic curve issuing from $q \in \gamma_\varepsilon$ with tangent $-N^\perp(u)$ ($N^\perp(u)$, resp.) lies in the bounded domain, denoted as $\Omega_{\gamma_\varepsilon}$, surrounded by γ_ε .
- (c) $\Omega_{\gamma_\varepsilon} \subset \Omega$ and $\text{Area}(\Omega_{\gamma_\varepsilon}) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. Let $r_0 := (2\max_{p \in \hat{S}} |H(p)|)^{-1}$. Let $\hat{\Omega}_{r_1} := \{p \in \Omega \mid 0 < \text{dist}(p, \hat{S}) < r_1\}$. We claim the existence of r_1 , $0 < r_1 < r_0$, such that for any r_2 , $0 < r_2 \leq r_1$, there hold

- (1) $\hat{\Omega}_{r_2} \subset \subset \Omega$,
- (2) $\hat{\Omega}_{r_2} \cap S_{\vec{F}}(u) = \emptyset$ (Note that $\hat{\Omega}_{r_2}$ does not contain \hat{S} by definition), and
- (3) the characteristic curve Γ_p passing through p hits \hat{S} for any $p \in \hat{\Omega}_{r_2}$.

Conditions (1) and (2) are easily achieved. Suppose condition (3) fails. That means we can find a sequence p_j such that $\text{dist}(p_j, \hat{S}) \rightarrow 0$ while Γ_{p_j} does not hit \hat{S} for each j . Since \hat{S} is compact, we end up finding a subsequence, still denoted as p_j , converging to $p_\infty \in \hat{S}$. On the other hand, Γ_{p_j} converges to a curve Γ_∞ by Lemma 3.2' (a) and Γ_∞ contains no singular points by Lemma 3.2' (b). But it is clear that the singular point $p_\infty \in \Gamma_\infty$. We have reached a contradiction.

Let $\mathfrak{s}(p) \in \hat{S}$ be the point at which Γ_p hits \hat{S} . Let $\hat{\Gamma}_p \subset \Gamma_p$ denote the part from p to $\mathfrak{s}(p)$, not including end points p and $\mathfrak{s}(p)$. By (3) and the choice of r_0 (recall that the curvature of a characteristic curve is $-H$), we conclude

$$\hat{\Omega}_{r_2} = \cup_{p \in l(r_2)} \hat{\Gamma}_p$$

where $l(r_2) := \{p \in \Omega \mid \text{dist}(p, \hat{S}) = r_2\}$. Let $\gamma^{r_2} := \{q \in \hat{\Omega}_{r_2} \mid \sigma_1(q) := \text{the length of } \hat{\Gamma}_q = \frac{r_2}{2}\}$. We claim that γ^{r_2} is a C^0 curve. Take $q_0 \in \gamma^{r_2}$. There is a point $p_0 \in l(r_2)$ such that $\hat{\Gamma}_{p_0} \supset \hat{\Gamma}_{q_0}$. Take a C^1 smooth nonsingular curve $\beta = \beta(\tau) \subset \hat{\Omega}_{r_2} \cup l(r_2)$ for $\tau \in (\tau_0 - \delta, \tau_0 + \delta)$ with $\beta(\tau_0) = p_0$, which is transversal to Γ_{p_0} (a circular arc $\subset \partial B_{r_2}(\mathfrak{s}_0)$ where $\mathfrak{s}_0 \in \hat{S}$ and $\text{dist}(p_0, \mathfrak{s}_0) = r_2$, passing through p_0 , will serve as such β). Let $\mathfrak{s}(\tau) \in \hat{S}$ be the point which $\Gamma_{\beta(\tau)}$ hits. It follows that \mathfrak{s} and hence σ_1 is C^0 by Lemma 7.2. Now any point in γ^{r_2} near q_0 is the intersection of $\hat{\Gamma}_{\beta(\tau)}$ and γ^{r_2} for a unique τ near τ_0 . We can therefore parametrize γ^{r_2} near q_0 by τ , denoted as $\gamma^{r_2}(\tau)$. We compute

$$\begin{aligned} \frac{r_2}{2} &= \lim_{\tau_j \rightarrow \tau_0} \sigma_1(\gamma^{r_2}(\tau_j)) \\ &= \sigma_1(q'_0) \end{aligned}$$

by the continuity of σ_1 if $\gamma^{r_2}(\tau_j) \rightarrow q'_0$ as $\tau_j \rightarrow \tau_0$. Observe that $q'_0 \in \Gamma_{p_0}$ and hence $q'_0 = q_0$ since $\sigma_1(q_0) = \frac{r_2}{2}$ too. So $\tau \rightarrow \gamma^{r_2}(\tau)$ is continuous. In fact, it is a homeomorphism near τ_0 . We have shown that γ^{r_2} is a C^0 curve (in R^2 as a submanifold). Since \hat{S} is compact, γ^{r_2} is bounded and hence any connected component (still denoted as γ^{r_2}) of γ^{r_2} must be simple closed. Given $\varepsilon > 0$, we

take $\gamma_\varepsilon = \gamma^{\min\{\varepsilon, r_1\}}$. (a) follows from the fact that $\sigma_1(q) \geq \text{dist}(q, \mathfrak{s}(q))$. Let $\Omega_{\gamma_\varepsilon}$ denote the bounded domain surrounded by (a connected component of) γ_ε . Suppose the characteristic curve issuing from $q \in \gamma_\varepsilon$ with tangent $N^\perp(u)$ lies in $\Omega_{\gamma_\varepsilon}$ (we are assuming $\text{curl}\vec{F} > 0$). Then the characteristic curve issuing from $q \in \gamma_\varepsilon$ with tangent $-N^\perp(u)$ hits a singular point $\mathfrak{s}(q) \in \hat{S}$ (we are assuming $\text{curl}\vec{F} > 0$, so $N^\perp(u)$ points away from \hat{S} according to Theorem B (b)). From (7.2) with $\beta = \gamma_\varepsilon$ in Lemma 7.3, we have

$$(7.6) \quad C_1 \text{Area}(\gamma_\varepsilon, \mathfrak{s}) \leq \left| \oint_{\gamma_\varepsilon} \Theta_{\vec{F}} \right| \leq C_2 \text{Area}(\gamma_\varepsilon, \mathfrak{s}).$$

Note that we can take C_1 and C_2 to be independent of ε .

On the other hand, we compute

$$(7.7) \quad \begin{aligned} \oint_{\gamma_\varepsilon} \Theta_{\vec{F}} &= \int_{\Omega_{\gamma_\varepsilon}} d\Theta_{\vec{F}} \\ &= \int_{\Omega_{\gamma_\varepsilon}} \text{curl}\vec{F} dx \wedge dy \geq C_1 \text{Area}(\Omega_{\gamma_\varepsilon}) \end{aligned}$$

by Green's theorem. Choosing ε small so that $\text{Area}(\gamma_\varepsilon, \mathfrak{s})$ (close to 0) $\ll \frac{C_1}{C_2} \text{Area}(\Omega_{\gamma_\varepsilon})$ (note that $\text{Area}(\Omega_{\gamma_\varepsilon})$ is nonincreasing in ε), we reach a contradiction by substituting (7.7) into (7.6). So \hat{S} must lie inside $\Omega_{\gamma_\varepsilon}$ and it is impossible for γ_ε to have more than one connected components. We have proved (b). Since $\text{Area}(\gamma_\varepsilon, \mathfrak{s}) \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \text{Area}(\Omega_{\gamma_\varepsilon}) = 0$$

by (7.6) and (7.7). That $\Omega_{\gamma_\varepsilon} \subset \Omega$ follows by observing that $\Omega_{\gamma_\varepsilon} \subset$ the closure of $\hat{\Omega}_{\min\{\varepsilon, r_1\}} \subset \Omega$. We have proved (c). We can deal with the case $\text{curl}\vec{F} < 0$ similarly. \square

Proof. (of Theorem I) By Lemma 7.1 we have $\# \pi_0(S_{\vec{F}}(u)) < \infty$. Let \hat{S} be a connected component of $S_{\vec{F}}(u)$. We claim that \hat{S} is compact and path-connected. By Theorem C (b) \hat{S} is closed (in Ω and R^2), and hence compact since $S_{\vec{F}}(u)$ is compact. Suppose that \hat{S} is decomposed as the union of path-connected components \hat{S}_j , $j = 1, 2, \dots$. Each \hat{S}_j is closed by Theorem C (b), and hence compact since \hat{S} is compact. Similarly we show that $\hat{S}' := \cup_{j \geq 2} \hat{S}_j$ is compact. Since we can separate two nonintersecting compact sets by two nonintersecting open sets containing them respectively, \hat{S}' must be empty (otherwise contradicting the connectedness of \hat{S}) and hence $\hat{S} = \hat{S}_1$ is path-connected. Let γ_ε be a simple closed C^0 curve around \hat{S} in Lemma 7.4.

Recall (see Section 1) that we denote the line field (1-dimensional distribution) defined by the tangent lines (the lines having the direction $\pm N^\perp(u)$) of the characteristic curves by \mathcal{D} . Let P^1 denote the projective line consisting of all lines through the origin of R^2 . We define a homeomorphism $\varsigma: P^1 \rightarrow S^1$ by noting that P^1 is the same as a semi-circle with end points identified. Let \approx denote a homeomorphism. We define $\text{index}(\gamma_\varepsilon, \mathcal{D})$ to be half the degree of the map $S^1 \approx \gamma_\varepsilon \rightarrow P^1$, defined

by $q \in \gamma_\varepsilon \rightarrow \mathcal{D}(q) \in P^1$, composed with ς (cf. p.325 in [19] for the index of a line field at a point). It follows from Lemma 7.4 that

$$(7.8) \quad \text{index}(\gamma_\varepsilon, \mathcal{D}) = 1$$

Namely, the connected component \hat{S} of the singular set, surrounded by γ_ε , has the index contribution 1.

Next let us investigate the index contribution of the boundary curves. We consider another copy of the domain Ω , denoted as Ω' . Denote the corresponding line field, boundary curves of Ω' by \mathcal{D}' , C'_j , $1 \leq j \leq l$, resp.. We glue C'_j with C_j for all j to get a closed surface Σ . Consider the line field $\hat{\mathcal{D}}$ on Σ , obtained from \mathcal{D} and \mathcal{D}' (smoothing it along $\partial\Omega = \partial\Omega'$ so that the topological type of the line field does not change). Therefore the index sum of $C'_j = C_j \subset \Sigma$ with respect to $\hat{\mathcal{D}}$ equals 2 times $\text{index}(C_j; u)$ according to (1.18) and (1.19). Together with (7.8) we have

$$(7.9) \quad \chi(\Sigma) = 2 \# \pi_0(S_{\bar{F}}(u)) + 2 \sum_{j=1}^l \text{index}(C_j; u)$$

by the Hopf index theorem for a closed surface. Now (1.20) follows from $\chi(\Omega) = \frac{1}{2}\chi(\Sigma)$ and (7.9). □

We remark that (7.8) is the key to the proof of Theorem I.

Proof. (of Corollary J) Observe that $\partial\Omega$ contains no singular point by assumption. So either the characteristic line field is transversal at each point of $\partial\Omega$ or there is a boundary point at which the characteristic line is tangent to $\partial\Omega$. In the latter case, Ω must be foliated by the characteristic lines in view of Lemma 3.2 (here we use the convexity of Ω). This contradicts the assumption that $S(u)$ is nonempty in Ω . In the former case, the index of $\partial\Omega$ is zero. Observe that a convex domain is contractible to a point, and hence $\chi(\Omega) = 1$. Now $\# \pi_0(S(u)) = 1$ follows from (1.20). □

Note that in Corollary J, if the singular set touches the boundary, we can have more than one connected components of the singular set. For instance, take a circular disc covering more than one singular line segments in Figure 4.5 (b). The boundary having no singular point can be obtained by the graph restricted to it being a nonlegendrian (nonhorizontal) curve in \mathbf{H}_1 .

Example 7.1. Look at Figure 7.2 (note that Ω is the unbounded region): Let \mathcal{D}_{U_j} denote the line field in a small neighborhood U_j of p_j as shown in Figure 7.2 for $j = 1, 2, 3, 4$. According to the definition (1.18) (see Figure 7.3), we have

$$\begin{aligned} \text{index}(p_1, \mathcal{D}_{U_1}) &= -\frac{1}{2}, \quad \text{index}(p_2, \mathcal{D}_{U_2}) = +\frac{1}{2}, \\ \text{index}(p_3, \mathcal{D}_{U_3}) &= \text{index}(p_4, \mathcal{D}_{U_4}) = 0. \end{aligned}$$

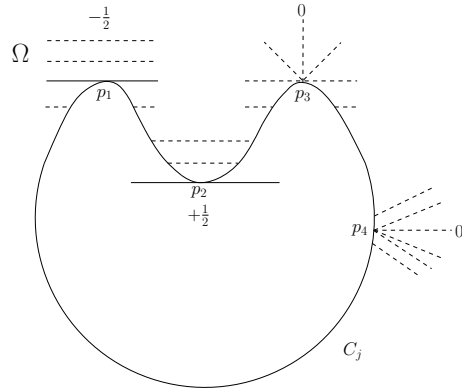


Figure 7.2

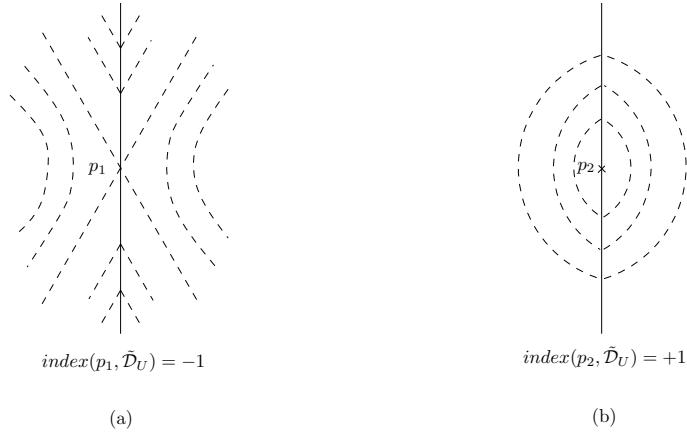


Figure 7.3

Example 7.2. Let Ω be a C^1 smooth, bounded planar domain with boundary curves C_1 , C_2 , and C_3 as shown in Figure 7.4.

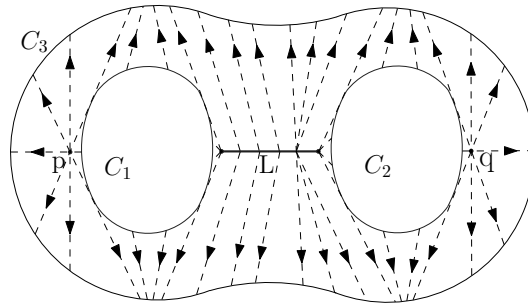


Figure 7.4

Suppose that a C^1 smooth p -minimal graph defined by u over Ω has its singular set $S(u) = \{p\} \cup L \cup \{q\}$ where L is a closed line segment. The dotted lines in Figure 7.4 denote the characteristic lines which are tangent to $\partial\Omega$ at finitely many points, satisfying the situation to which we can apply Theorem I. There are four points on C_1 each of which has index contribution $-\frac{1}{2}$. Hence by (1.19) we have

$$(7.10) \quad \text{index}(C_1; u) = 4 \cdot \left(-\frac{1}{2}\right) = -2.$$

Similarly we can easily compute

$$(7.11) \quad \begin{aligned} \text{index}(C_2; u) &= 4 \cdot \left(-\frac{1}{2}\right) = -2, \\ \text{index}(C_3; u) &= 0. \end{aligned}$$

On the other hand, we have $\#\pi_0(S(u)) = 3$. So together with (7.10) and (7.11) we have

$$\begin{aligned} \#\pi_0(S(u)) + \sum_{j=1}^3 \text{index}(C_j; u) \\ = 3 + (-2) + (-2) + 0 = -1 \end{aligned}$$

which equals the Euler characteristic number $\chi(\Omega)$ of Ω . We have verified Theorem I for this specific example.

Example 7.3. We consider closed surfaces of bounded p -mean curvature in the Heisenberg group \mathbf{H}_1 . Let \mathfrak{S}^2 denote a Pansu sphere ([14]), a sphere of nonzero constant p -mean curvature in \mathbf{H}_1 . There are exactly two singular points on \mathfrak{S}^2 , denoted as N,S. Take a short slit along each of $g+1$ characteristic curves joining N to S (see Figure 7.5 for $g=2$).

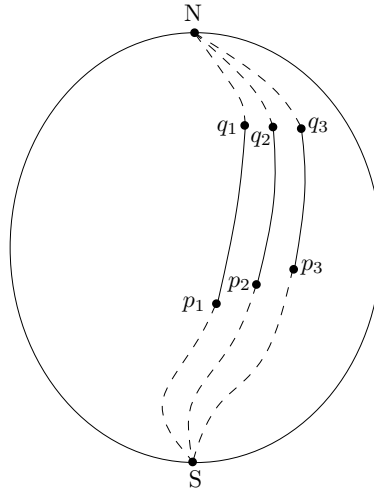


Figure 7.5

Take the 2-fold branched cover Σ_g of \mathfrak{S}^2 with branch locus consisting of the $2(g+1)$ points, $p_j, q_j, j = 1, \dots, g+1$, which are end points of the slits. From

standard topological arguments we learn that Σ_g is a closed surface of genus g . Let $\hat{\varphi} : \Sigma_g \rightarrow \mathfrak{S}^2 \subset \mathbf{H}_1$ denote this 2-fold branched covering map. Consider the characteristic line field on Σ_g . Observe that the index at a branch point is -1 while the index at N or S (two copies) is $+1$. The total index count is correct since $(-1) \cdot 2(g+1) + (+1) \cdot 4 = 2 - 2g$, the Euler characteristic of Σ_g . We can deform the surface by moving the top copy of the Pansu sphere a little bit away from the bottom copy (only do this away from the branch cuts) to keep the p -mean curvature bounded while the only self intersection of this new surface is the original branch cuts.

We can consider the problem of minimizing the p -area among mappings from genus g surfaces to \mathbf{H}_1 to enclose a fixed volume. Observe that $\hat{\varphi}$ is not the minimizer, but only a critical point.

8. GENERALIZATION TO PSEUDOHERMITIAN MANIFOLDS

Let Σ be a (say, C^∞) surface of a pseudohermitian 3-manifold (M, J, Θ) (see, e.g., [3] for the definition of pseudohermitian 3-manifolds). Let ψ be a defining function of Σ . Let \hat{e}_1, \hat{e}_2 (\hat{e}^1, \hat{e}^2 , resp.) denote a local orthonormal basis (dual coframe, resp.) of the contact bundle with respect to the Levi metric $\frac{1}{2}d\Theta(\cdot, J\cdot)$. Write

$$(8.1) \quad \frac{\nabla_b \psi}{|\nabla_b \psi|} = \frac{\hat{e}_1 \psi}{|\nabla_b \psi|} \hat{e}_1 + \frac{\hat{e}_2 \psi}{|\nabla_b \psi|} \hat{e}_2 = (\cos \theta) \hat{e}_1 + (\sin \theta) \hat{e}_2$$

for some angular function θ , where $|\nabla_b \psi| = \sqrt{(\hat{e}_1 \psi)^2 + (\hat{e}_2 \psi)^2}$. Let $D := |\nabla_b \psi|$. Recall ([3]) that associated to the nonsingular part ($D \neq 0$) of Σ , we have the characteristic vector field e_1 , the Legendrian normal e_2 , and the dual coframe e^1, e^2 . Since $e_1 \psi = 0$, we then have

$$(8.2) \quad \begin{aligned} D &= |\nabla_b \psi| = \sqrt{(\hat{e}_1 \psi)^2 + (\hat{e}_2 \psi)^2} \\ &= \sqrt{(e_1 \psi)^2 + (e_2 \psi)^2} = |e_2 \psi| = e_2 \psi \end{aligned}$$

by changing the sign of ψ if necessary. Let T denote the Reeb vector field with respect to Θ . On Σ we compute

$$(8.3) \quad \begin{aligned} 0 &= d\psi = (e_1 \psi) e^1 + (e_2 \psi) e^2 + (T\psi) \Theta \\ &= (e_2 \psi) e^2 + (T\psi) \Theta \\ &= D e^2 + (T\psi) \Theta \end{aligned}$$

by (8.2). Taking the exterior differentiation of (8.3) gives

$$(8.4) \quad e_1(D) e^1 \wedge e^2 + D d e^2 = -e_1(T\psi) e^1 \wedge \Theta - T\psi d\Theta.$$

Here we have used the fact that $e^2 \wedge \Theta = 0$ on Σ . Formulas (A.3r) in [3] when restricted to Σ read

$$(8.5) \quad \begin{aligned} d e^1 &= (H\alpha - \operatorname{Re} A_1^1) e^1 \wedge \Theta \\ d e^2 &= (\omega(\alpha e_2 + T) - \operatorname{Im} A_1^1) e^1 \wedge \Theta \end{aligned}$$

(recall that $\omega_1^1 = i\omega$, $\omega(e_1) = H$, the p -mean curvature, α is defined so that $\alpha e_2 + T \in T\Sigma$, and $e^1 \wedge e^2 = \alpha e^1 \wedge \Theta$ on Σ). Substituting the second equality of (8.5) into (8.4) and noting that $d\Theta = 2e^1 \wedge e^2 = 2\alpha e^1 \wedge \Theta$ on Σ , we obtain

$$(8.6) \quad e_1(D)\alpha + D(\omega(\alpha e_2 + T) - \operatorname{Im} A_1^1) = -e_1(T\psi) - 2\alpha T\psi.$$

If $T\psi = 0$, $T \in T\Sigma$, so it follows that $\alpha = 0$. Let us assume $T\psi \neq 0$. We may suppose $T\psi > 0$ (otherwise change Θ to $-\Theta$). Adjust ψ such that $T\psi = 1$ on Σ while $e_2\psi$ is positive (choose for example $\frac{\psi}{T\psi}$). Under the condition $T\psi = 1$ we can reduce (8.6) to

$$(8.7) \quad e_1(D)\alpha + D(\omega(\alpha e_2 + T) - \text{Im } A_1^1) = -2\alpha.$$

Since $(\alpha e_2 + T)\psi = 0$ and $T\psi = 1$, we get

$$(8.8) \quad \alpha = \frac{-1}{e_2\psi} = -\frac{1}{D}.$$

Write

$$(8.9) \quad \begin{aligned} \nabla_b \psi &= (\hat{e}_1\psi)\hat{e}_1 + (\hat{e}_2\psi)\hat{e}_2 = D(\cos\theta\hat{e}_1 + \sin\theta\hat{e}_2) \\ &= (e_1\psi)e_1 + (e_2\psi)e_2 = (e_2\psi)e_2 = De_2. \end{aligned}$$

It follows that

$$(8.10) \quad \begin{aligned} e_2 &= \cos\theta\hat{e}_1 + \sin\theta\hat{e}_2, \text{ and} \\ e_1 &= \sin\theta\hat{e}_1 - \cos\theta\hat{e}_2, \quad e^1 = \sin\theta\hat{e}^1 - \cos\theta\hat{e}^2 \\ e^2 &= \cos\theta\hat{e}^1 + \sin\theta\hat{e}^2. \end{aligned}$$

From (8.10) we learn that

$$(8.11) \quad \theta^1 = e^{i(\frac{\pi}{2}-\theta)}\hat{\theta}^1.$$

Therefore $A_{11} = e^{-2i(\frac{\pi}{2}-\theta)}\hat{A}_{11} = -e^{2i\theta}\hat{A}_{11}$ and hence

$$(8.12) \quad \begin{aligned} \text{Re } A_{11} &= \sin 2\theta \text{Im } \hat{A}_{11} - \cos 2\theta \text{Re } \hat{A}_{11} \\ \text{Im } A_{11} &= -\sin 2\theta \text{Re } \hat{A}_{11} - \cos 2\theta \text{Im } \hat{A}_{11}. \end{aligned}$$

The connection forms change according to $\omega_1^1 = \hat{\omega}_1^1 - id(\frac{\pi}{2} - \theta)$. It follows that

$$(8.13) \quad \omega = \hat{\omega} + d\theta.$$

Let

$$(8.14) \quad v_2 := \alpha e_2 + T.$$

From (8.8), (8.13), and 8.14, we can rewrite (8.7) as

$$(8.15) \quad e_1(D) = D^2(v_2(\theta) + \hat{\omega}(v_2) - \text{Im } A_1^1) - 2.$$

Taking the derivative of (8.15) in the direction of e_1 , we have

$$(8.16) \quad \begin{aligned} e_1^2(D) &= 2De_1(D)(v_2(\theta) + \hat{\omega}(v_2) - \text{Im } A_1^1) \\ &\quad + D^2\{e_1(v_2(\theta)) + e_1(\hat{\omega}(v_2) - \text{Im } A_1^1)\}. \end{aligned}$$

On the other hand, we write

$$(8.17) \quad e_1(v_2(\theta)) = v_2(e_1(\theta)) + [e_1, v_2](\theta)$$

and observe that

$$(8.18) \quad e_1(\theta) = \omega(e_1) - \hat{\omega}(e_1) = H - \hat{\omega}(e_1)$$

by applying (8.13) to e_1 . Making use of (A.6r) and (A.7r) in [3], we have

$$(8.19) \quad \begin{aligned} [e_1, v_2] &= \alpha[e_1, e_2] + e_1(\alpha)e_2 + [e_1, T] \\ &= (-\alpha H + \text{Re } A_{11})e_1 + (-\omega(v_2) + e_1(\alpha) - \text{Im } A_{11})e_2 - 2\alpha T \\ &= (-\alpha H + \text{Re } A_{11})e_1 - 2\alpha v_2. \end{aligned}$$

For the last equality of (8.19) we have used the integrability condition for α :

$$(8.20) \quad -\omega(v_2) + e_1(\alpha) - \operatorname{Im} A_{11} = -2\alpha^2$$

(in order to make $[e_1, v_2] \in T\Sigma$). By (8.17), (8.18), (8.19), and (8.15), we obtain

$$(8.21) \quad \begin{aligned} e_1(v_2(\theta)) &= v_2(H - \hat{\omega}(e_1)) \\ &\quad + (-\alpha H + \operatorname{Re} A_{11})e_1(\theta) - 2\alpha v_2(\theta) \\ &= (v_2 - \alpha H + \operatorname{Re} A_{11})(H - \hat{\omega}(e_1)) \\ &\quad - 2\alpha \left[\frac{e_1(D) + 2}{D^2} - \hat{\omega}(v_2) - \operatorname{Im} A_{11} \right]. \end{aligned}$$

Note that $\operatorname{Im} A_{\bar{1}}^1 = \operatorname{Im} A_{\bar{1}\bar{1}} = -\operatorname{Im} A_{11}$. Next we observe from (8.19), (A.5r), and (A.5) in [3] that

$$(8.22) \quad \begin{aligned} e_1(\hat{\omega}(v_2)) - v_2(\hat{\omega}(e_1)) - (-\alpha H + \operatorname{Re} A_{11})\hat{\omega}(e_1) + 2\alpha\hat{\omega}(v_2) \\ &= e_1(\hat{\omega}(v_2)) - v_2(\hat{\omega}(e_1)) - \hat{\omega}([e_1, v_2]) \\ &= d\hat{\omega}(e_1, v_2) \\ &= -2\alpha W + 2\operatorname{Im} A_{11, \bar{1}}. \end{aligned}$$

For the last equality of (8.22) we have used the transformation law

$$\begin{aligned} \operatorname{Im} A_{11, \bar{1}} &= \sin \theta \operatorname{Im} \hat{A}_{11, \bar{1}} - \cos \theta \operatorname{Re} \hat{A}_{11, \bar{1}} \\ (\operatorname{Re} A_{11, \bar{1}} &= \sin \theta \operatorname{Re} \hat{A}_{11, \bar{1}} + \cos \theta \operatorname{Im} \hat{A}_{11, \bar{1}}) \end{aligned}$$

under the coframe change (8.11). Denote $\frac{v_2}{\alpha} = e_2 + \frac{T}{\alpha}$ by \tilde{e}_2 . Substituting

$$v_2(\theta) + \hat{\omega}(v_2) - \operatorname{Im} A_{\bar{1}}^1 = \frac{e_1(D) + 2}{D^2}$$

from (8.15) and (8.21) into (8.16), we finally reach

$$(8.23) \quad \begin{aligned} e_1^2(D) \\ &= \frac{2(e_1(D) + 1)(e_1(D) + 2)}{D} + D[-\tilde{e}_2(H) + H^2] \\ &\quad + D^2 \{ 2D^{-1}W + 2\operatorname{Im} A_{11, \bar{1}} + (e_1 - 2D^{-1})(\operatorname{Im} A_{11}) + (\operatorname{Re} A_{11})H \} \end{aligned}$$

in view of (8.22) and $\alpha = -\frac{1}{D}$ by (8.8). Note that for a graph over the xy -plane in the 3-dimensional Heisenberg group, $e_1(D) = -N^\perp(D)$ and $\tilde{e}_2(H) = -N(H)$ for H being a function of x and y . So (8.23) is reduced to previous formula (1.3) with $\vec{F} = (-y, x)$.

Example 8.1. We want to show that $\operatorname{div} \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}$ ($u \in C^2$, say) is proportional to the p -mean curvature of a certain pseudohermitian structure. Let $\Theta_{\vec{F}} := dz + F_1 dx + F_2 dy$ where $\vec{F} = (F_1, F_2)$. Then we have

$$d\Theta_{\vec{F}} = (\operatorname{curl} \vec{F}) dx \wedge dy$$

where $\operatorname{curl} \vec{F} := \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$. We require that $\operatorname{curl} \vec{F} \neq 0$. Without loss of generality we may assume $\operatorname{curl} \vec{F} > 0$. We have $\Theta_{\vec{F}} \wedge d\Theta_{\vec{F}} \neq 0$. That is to say, $\Theta_{\vec{F}}$ is a contact

form. Let

$$(8.24) \quad \begin{aligned} \hat{e}_1 &= \frac{1}{\sqrt{\text{curl}\vec{F}/2}} \left(\frac{\partial}{\partial x} - F_1 \frac{\partial}{\partial z} \right) \\ \hat{e}_2 &= \frac{1}{\sqrt{\text{curl}\vec{F}/2}} \left(\frac{\partial}{\partial y} - F_2 \frac{\partial}{\partial z} \right). \end{aligned}$$

It is clear that $\hat{e}_1, \hat{e}_2 \in \text{Ker}\Theta_{\vec{F}}$ form a basis. Define the CR structure $J_{\vec{F}}$ on this basis by $J_{\vec{F}}(\hat{e}_1) := \hat{e}_2$ and $J_{\vec{F}}(\hat{e}_2) := -\hat{e}_1$. It follows that \hat{e}_1, \hat{e}_2 form an orthonormal basis with respect to the Levi metric $\frac{1}{2}d\Theta_{\vec{F}}(\cdot, J_{\vec{F}}\cdot)$. Let $\psi := z - u(x, y)$ be a defining function. Then we have

$$(8.25) \quad \begin{aligned} \hat{e}_1\psi &= \frac{1}{\sqrt{\text{curl}\vec{F}/2}} (-u_x - F_1) \\ \hat{e}_2\psi &= \frac{1}{\sqrt{\text{curl}\vec{F}/2}} (-u_y - F_2). \end{aligned}$$

So from $\nabla_b\psi = (\hat{e}_1\psi)\hat{e}_1 + (\hat{e}_2\psi)\hat{e}_2$, we have

$$(8.26) \quad \begin{aligned} |\nabla_b\psi| &= \sqrt{(\hat{e}_1\psi)^2 + (\hat{e}_2\psi)^2} \\ &= \frac{1}{\sqrt{\text{curl}\vec{F}/2}} |\nabla u + \vec{F}| \end{aligned}$$

by (8.25). Now we can compute the p -mean curvature $H_{J_{\vec{F}}, \Theta_{\vec{F}}}$ of the graph $z = u(x, y)$ with respect to the pseudohermitian structure $(J_{\vec{F}}, \Theta_{\vec{F}})$ according to formula ($pMCE$) in [3]:

$$\begin{aligned} H_{J_{\vec{F}}, \Theta_{\vec{F}}} &= -\text{div}_b \frac{\nabla_b\psi}{|\nabla_b\psi|} \\ &= -\hat{e}_1 \left(\frac{\hat{e}_1\psi}{|\nabla_b\psi|} \right) - \hat{e}_2 \left(\frac{\hat{e}_2\psi}{|\nabla_b\psi|} \right) \\ &= \frac{1}{\sqrt{\text{curl}\vec{F}/2}} \text{div} \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|} \end{aligned}$$

by (8.25) and (8.26).

We may write equation (8.23) in the form of (9.1) (see the Appendix). By Theorem A.1 we can then conclude

Theorem B'. *Let Σ be a (say, C^∞) surface of a pseudohermitian 3-manifold (M, J, Θ) , defined by $\{\psi = 0\}$. Suppose that $T\psi \neq 0$. Then either $e_1(D)$ tends to -1 or $e_1(D)$ tends to -2 as the argument tends to a singular point along a characteristic curve*

9. APPENDIX: A GENERALIZED ODE

We generalize the result (1.8) for equation (1.3) in this section.

Theorem A.1. *Let $v \in C^2(0, \rho_0)$, $E_1, l, m \in C^0[0, \rho_0]$ ($\rho_0 > 0$) be real functions such that $v(\rho) > 0$ and v is bounded on $(0, \rho_0)$, $E_1(0) > 0$, $l(0) < m(0)$. Let $E_2 = E_2(\rho, v)$ be a real function continuous in $(\rho, v) \in [0, \rho_0) \times [0, \infty)$ with $E_2(0, 0) = 0$. Consider the following ODE (which generalizes (1.3)):*

$$(9.1) \quad v(\rho)v''(\rho) = E_1(\rho)(v'(\rho) - l(\rho))(v'(\rho) - m(\rho)) + E_2(\rho, v(\rho)).$$

Then there hold

- (a) The limit of $v(\rho)$ and $v'(\rho)$, resp. exists as $\rho \rightarrow 0$.
- (b) In case $\lim_{\rho \rightarrow 0} v(\rho) = 0$, we have either

$$(9.2) \quad \lim_{\rho \rightarrow 0} v'(\rho) = l(0) \quad \text{or} \quad \lim_{\rho \rightarrow 0} v'(\rho) = m(0).$$

Lemma A.2. *Suppose we are in the situation of Theorem A.1 (excluding the condition $v(\rho) > 0$ for $0 < \rho < \rho_0$ and replacing $E_1(0) > 0$ by $E_1(0) \neq 0$). Then for any $0 < \varepsilon < \frac{m(0)-l(0)}{3}$ there exists $\delta = \delta(\varepsilon)$ ($< \rho_0$) such that for any $0 < \rho < \delta$ there holds*

$$(9.3) \quad |v(\rho)v''(\rho)| \geq \frac{|E_1(0)|}{16} |(v'(\rho) - l(0))(v'(\rho) - m(0))|.$$

for $v'(\rho) \in (-\infty, l(0) - \varepsilon] \cup [l(0) + \varepsilon, m(0) - \varepsilon] \cup [m(0) + \varepsilon, \infty)$.

Proof. Write $E_1(\rho) = E_1(0) + E_1(\rho) - E_1(0)$. There exists $\delta_1 > 0$ such that $|E_1(\rho) - E_1(0)| \leq \frac{|E_1(0)|}{2}$ for $0 < \rho < \delta_1$ by the continuity of E_1 and the assumption $E_1(0) \neq 0$. It follows that

$$(9.4) \quad |E_1(\rho)| \geq \frac{|E_1(0)|}{2}$$

for $0 < \rho < \delta_1$. Choose $\delta_2 > 0$ such that $|l(\rho) - l(0)| \leq \frac{\varepsilon}{2}$ for $0 < \rho < \delta_2$. So we have

$$(9.5) \quad \begin{aligned} |v'(\rho) - l(\rho)| &= |v'(\rho) - l(0) + l(0) - l(\rho)| \\ &\geq \frac{|v'(\rho) - l(0)|}{2} + \frac{|v'(\rho) - l(0)|}{2} - |l(\rho) - l(0)| \\ &\geq \frac{|v'(\rho) - l(0)|}{2} \end{aligned}$$

for $0 < \rho < \delta_2$ since $|v'(\rho) - l(0)| \geq \varepsilon$ by assumption. Similarly we can choose $\delta_3 > 0$ such that

$$(9.6) \quad |v'(\rho) - m(\rho)| \geq \frac{|v'(\rho) - m(0)|}{2}$$

for $0 < \rho < \delta_3$. By continuity and $E_2(0, 0) = 0$, we can find $\delta_4 > 0$ such that

$$(9.7) \quad |E_2(\rho, v(\rho))| \leq \frac{|E_1(0)|}{16} |(v'(\rho) - l(0))(v'(\rho) - m(0))|$$

for $0 < \rho < \delta_4$. Note that the right-hand side of (9.7) is greater or equal to $\frac{|E_1(0)|}{16}\varepsilon^2 > 0$. From (9.1), (9.4), (9.5), (9.6), and (9.7), we have

$$\begin{aligned} |v(\rho)v''(\rho)| &= |E_1(\rho)(v'(\rho) - l(\rho))(v'(\rho) - m(\rho)) + E_2(\rho, v(\rho))| \\ &\geq |E_1(\rho)||v'(\rho) - l(\rho)||v'(\rho) - m(\rho)| - |E_2(\rho, v(\rho))| \\ &\geq \frac{|E_1(0)|}{16}|(v'(\rho) - l(0))(v'(\rho) - m(0))| \end{aligned}$$

for $0 < \rho < \delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \rho_0\}$. We have proved (9.3). \square

Proof. (of Theorem A.1) Suppose that $\lim_{\rho \rightarrow 0} v(\rho)$ does not exist. Then $a := \liminf_{\rho \rightarrow 0} v(\rho) < \limsup_{\rho \rightarrow 0} v(\rho) := b$, $a, b \in \mathbb{R}$ since v is bounded by assumption. There exist sequences $\bar{\rho}_j$ and ρ_j such that $v(\bar{\rho}_j) \rightarrow b$ as $\bar{\rho}_j \rightarrow 0$ while $v(\rho_j) \rightarrow a$ as $\rho_j \rightarrow 0$. Now as both $\bar{\rho}_j$ and ρ_k tend to 0, we have

$$\begin{aligned} v'(\xi_{jk}) &= \frac{v(\bar{\rho}_j) - v(\rho_k)}{\bar{\rho}_j - \rho_k} \text{ for } \xi_{jk} \text{ between } \bar{\rho}_j \text{ and } \rho_k \\ &\text{approximates } \frac{b - a}{\bar{\rho}_j - \rho_k} \end{aligned}$$

which goes to $+\infty$ or $-\infty$ depending on $\bar{\rho}_j - \rho_k$ is positive or negative, resp.. On the other hand, we always have $v'' > 0$ according to (9.1) for $|v'|$ large. We claim that it is impossible for v' to change drastically from positive large to negative large while v'' is positive. Suppose that $v'(\xi_{j_1 k_1})$ is positively large while $v'(\xi_{j_2 k_2})$ is negatively large for $0 < \xi_{j_1 k_1} < \xi_{j_2 k_2}$. Then there exists $\hat{\xi}$, $\xi_{j_1 k_1} < \hat{\xi} < \xi_{j_2 k_2}$, such that $v'(\hat{\xi}) > v'(\xi_{j_1 k_1})$ and $v''(\hat{\xi}) = 0$ since $v''(\xi_{j_1 k_1}) > 0$. But applying (9.1) to $\rho = \hat{\xi}$, we get $0 = a$ positive large number, a contradiction. We have proved the existence of $\lim_{\rho \rightarrow 0} v(\rho)$.

Suppose $\lim_{\rho \rightarrow 0} v(\rho) = 0$ and (9.2) does not hold. Then there exists a sequence of $\rho_j \rightarrow 0$ such that

$$(9.8) \quad v'(\rho_j) \in (-\infty, l(0) - \varepsilon] \cup [l(0) + \varepsilon, m(0) - \varepsilon] \cup [m(0) + \varepsilon, \infty)$$

for a given $0 < \varepsilon < \frac{m(0) - l(0)}{3}$. By Lemma A.2 (may assume $0 < \rho_j < \delta$), we have

$$(9.9) \quad |v(\rho_j)v''(\rho_j)| \geq \frac{|E_1(0)|}{16}|(v'(\rho_j) - l(0))(v'(\rho_j) - m(0))|.$$

Since $v(\rho_j) \rightarrow 0$ as $\rho_j \rightarrow 0$, $|v''(\rho_j)|$ must tend to infinity in view of (9.9) and (9.8). We may assume either a subsequence of $v''(\rho_j)$ goes to $+\infty$ or a subsequence of $v''(\rho_j)$ goes to $-\infty$. Still denote the subsequence by $v''(\rho_j)$. Observe from (9.1) and $v(\rho) > 0$ for $(0 <) \rho$ small that

$$(9.10) \quad \begin{aligned} v''(\rho) \text{ has the same sign } (> 0 \text{ or } < 0) \text{ for all } \rho \\ \text{small enough so that } v'(\rho) \in (-\infty, l(0) - \varepsilon] \end{aligned}$$

($[l(0) + \varepsilon, m(0) - \varepsilon]$ or $[m(0) + \varepsilon, \infty)$, resp.). Now suppose $v''(\rho_j) \rightarrow -\infty$ as $\rho_j \rightarrow 0$. Clearly v' is increasing at ρ_j . Let $\bar{c} := \limsup v'(\rho_j) \in (-\infty, l(0) - \varepsilon] \cup [l(0) + \varepsilon, m(0) - \varepsilon] \cup [m(0) + \varepsilon, \infty]$ by (9.8). Then in view of (9.10) we can easily show that as $\rho \rightarrow 0$ $v'(\rho)$ increases and converges to \bar{c} (in particular, $\bar{c} \neq l(0) + \varepsilon$ and $\bar{c} \neq m(0) + \varepsilon$). So there exists $\hat{\rho} > 0$ (may depend on \bar{c}) such that $v'(\hat{\rho}) \in$

$(-\infty, l(0) - \varepsilon] \cup [l(0) + \varepsilon, m(0) - \varepsilon] \cup [m(0) + \varepsilon, \infty)$, $v''(\rho) < 0$ for all $0 < \rho < \hat{\rho}$, and

$$(9.11) \quad v'(\rho)v''(\rho) \text{ has the same sign for all } 0 < \rho < \hat{\rho}.$$

By the assumption $E_1(0) > 0$ and $u'' < 0$ for all $0 < \rho < \hat{\rho}$, we have

$$(9.12) \quad \bar{c} \notin (-\infty, l(0) - \varepsilon] \cup [m(0) + \varepsilon, \infty]$$

in view of equation (9.1).

On the other hand, we deduce from (9.3) that

$$(9.13) \quad \begin{aligned} & \left| \frac{v'(\rho)v''(\rho)}{(v'(\rho) - l(0))(v'(\rho) - m(0))} \right| \\ & \geq \frac{|E_1(0)|}{16} \left| \frac{v'(\rho)}{v(\rho)} \right| = \frac{|E_1(0)|}{16} |(\log v)'(\rho)|. \end{aligned}$$

We express the left side of (9.13) as follows:

$$(9.14) \quad \begin{aligned} & \frac{v'(\rho)v''(\rho)}{(v'(\rho) - l(0))(v'(\rho) - m(0))} \\ & = \frac{\alpha(v'(\rho) - l(0))'(\rho)}{v'(\rho) - l(0)} + \frac{\beta(v'(\rho) - m(0))'(\rho)}{v'(\rho) - m(0)} \\ & = (\log |v'(\rho) - l(0)|^\alpha |v'(\rho) - m(0)|^\beta)'(\rho) \end{aligned}$$

where $\alpha := \frac{-l(0)}{m(0)-l(0)}$, $\beta := \frac{m(0)}{m(0)-l(0)}$. Now substituting (9.14) into (9.13) and integrating (9.13) over $\rho \in (0, \hat{\rho})$, we obtain

$$(9.15) \quad \begin{aligned} & \int_0^{\hat{\rho}} |(\log |v'(\rho) - l(0)|^\alpha |v'(\rho) - m(0)|^\beta)'(\rho)| d\rho \\ & \geq \frac{|E_1(0)|}{16} \int_0^{\hat{\rho}} |(\log v)'(\rho)| d\rho \\ & \geq \frac{|E_1(0)|}{16} \left| \int_0^{\hat{\rho}} (\log v)'(\rho) d\rho \right| \\ & = \frac{|E_1(0)|}{16} |(\log v(\hat{\rho}) - \log v(0))| = +\infty. \end{aligned}$$

On the other hand, either $(\log |v'(\rho) - l(0)|^\alpha |v'(\rho) - m(0)|^\beta)'(\rho)$ is positive for all $0 < \rho < \hat{\rho}$ or negative for all $0 < \rho < \hat{\rho}$ by (9.11) and (9.14). But we then have

$$(9.16) \quad \begin{aligned} & \int_0^{\hat{\rho}} (\log |v'(\rho) - l(0)|^\alpha |v'(\rho) - m(0)|^\beta)'(\rho) d\rho \\ & = \log(|v'(\hat{\rho}) - l(0)|^\alpha |v'(\hat{\rho}) - m(0)|^\beta) - \log(|c - l(0)|^\alpha |c - m(0)|^\beta), \end{aligned}$$

a finite number ($\bar{c} \neq +\infty$ by (9.12)), contradicting (9.15). For the situation that $v''(\rho_j) \rightarrow +\infty$ as $\rho_j \rightarrow 0$, we have a similar reasoning with v' being decreasing at ρ_j and $\underline{c} := \liminf v'(\rho_j) \in [-\infty, l(0) - \varepsilon] \cup [l(0) + \varepsilon, m(0) - \varepsilon] \cup [m(0) + \varepsilon, \infty)$ by (9.8). Then in view of (9.10) we can also show that as $\rho \rightarrow 0$ $v'(\rho)$ decreases and converges to \underline{c} (in particular, $\underline{c} \neq l(0) - \varepsilon$ and $\underline{c} \neq m(0) - \varepsilon$). Since $v(\rho) > 0$ for $0 < \rho < \rho_0$, we can find a sequence of $a_j \rightarrow 0$ such that $v'(a_j) > 0$. This property implies that $\underline{c} \geq 0$ (in particular, $\underline{c} = -\infty$ is excluded). By a similar argument as in (9.13)-(9.16) we finally reach a contradiction again. We have proved (9.2), hence (b).

Now suppose $\lim_{\rho \rightarrow 0} v(\rho) := v(0) \neq 0$ (so $v(0) > 0$ since $v > 0$ in $(0, \rho_0)$). We still want to prove the existence of $\lim_{\rho \rightarrow 0} v'(\rho)$. If v' is bounded in $(0, \hat{\rho}_0)$ for $0 < \hat{\rho}_0 < \rho_0$, then v'' is bounded in $(0, \bar{\rho}_0)$ for $\bar{\rho}_0$ small, $0 < \bar{\rho}_0 < \hat{\rho}_0$ by (9.1). It follows that v' is Cauchy in $(0, \bar{\rho}_0)$ since

$$\begin{aligned} |v'(\tilde{\rho}) - v'(\bar{\rho})| &= \left| \int_{\bar{\rho}}^{\tilde{\rho}} v''(\rho) d\rho \right| \\ &\leq \int_{\bar{\rho}}^{\tilde{\rho}} |v''(\rho)| d\rho \leq C_1 |\tilde{\rho} - \bar{\rho}| \end{aligned}$$

for $0 < \tilde{\rho} < \bar{\rho} < \bar{\rho}_0$, where $|v''(\rho)| \leq C_1$, a positive constant independent of ρ , for $\rho \in (0, \bar{\rho}_0)$. So $\lim_{\rho \rightarrow 0} v'(\rho)$ exists. On the other hand, if v' is not bounded near 0, then there exists a sequence $\rho_j \rightarrow 0$ such that $\lim_{\rho_j \rightarrow 0} v'(\rho_j) = -\infty$ ($+\infty$ is impossible by a similar argument as in the first paragraph of the proof since $v''(\rho_j) > 0$). In fact we can easily show that $v'(\rho)$ is monotonically decreasing to $-\infty$ as $\rho \rightarrow 0$ since $v'' > 0$ for $|v'|$ large by (9.1). We can find $\rho'_0 > 0$ small so that v' is negatively large in $(0, \rho'_0]$ and there holds

$$(9.17) \quad v(\rho)v''(\rho) \leq C_2(v'(\rho))^2$$

for $\rho \in (0, \rho'_0]$ and some positive constant C_2 independent of ρ . Dividing (9.17) by $-vv' > 0$ (noting that $v' < 0$ in $(0, \rho'_0]$) we obtain

$$(9.18) \quad \begin{aligned} -(\log(-v'))' &= -\frac{v''}{v'} \\ &\leq C_2\left(-\frac{v'}{v}\right)' = -C_2(\log v)'. \end{aligned}$$

Integrating (9.18) from ε , $0 < \varepsilon < \rho'_0$ to ρ'_0 we get

$$(9.19) \quad -\log(-v'(\rho'_0)) + \log(-v'(\varepsilon)) \leq C_2\{-\log v(\rho'_0) + \log v(\varepsilon)\}.$$

Letting $\varepsilon \rightarrow 0$ in (9.19) we reach $+\infty \leq C_2\{-\log v(\rho'_0) + \log v(0)\}$ (note that $v(0) > 0$), a contradiction. We have proved the existence of $\lim_{\rho \rightarrow 0} v'(\rho)$ and completed the proof of (a). \square

We remark that Theorem A.1 can be applied to show a generalized version of Theorem B (a) (see Theorem B' in Section 8). Since $v(0)$ is not necessarily 0, we may also apply Theorem A.1 to the piecewise C^1 case in which $\{\rho = 0\}$ corresponds to a nonsmooth edge.

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