Bloch Model Wavefunctions and Pseudopotentials for All Fractional Chern Insulators - Supplemental Material -

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GAUGE FIXING

The connections over the lowest Landau level (LLL) Brillouin zone (BZ) are $A_x^{\mathrm{L}}(\mathbf{k}) = e^{-i2\pi k_y/N_\phi}$, and $A_y^{\mathrm{L}}(\mathbf{k}) = 1$ (superscript 'L' represents LLL). They satisfy the discrete analog of the Coulomb gauge condition [6], i.e. they can be expressed in terms of a "stream function" $\phi_{\mathbf{k}}^{\mathrm{L}} = (k_y + 1/2)^2/(2N_\phi)$ as

$$A_\alpha^{\mathrm{L}}(\mathbf{k}) = \exp\Big(-i2\pi \sum_\beta \varepsilon_{\alpha\beta}\,[\mathrm{d}_\beta \phi^{\mathrm{L}}]_{\mathbf{k}}\Big). \tag{1}$$

Here, d_β is the backward finite difference operator, defined by $[\mathrm{d}_\beta\phi]_{\mathbf{k}} = \phi_{\mathbf{k}} - \phi_{\mathbf{k}-\mathbf{g}_\beta}$, and $\phi_{\mathbf{k}}^{\mathrm{L}}$ satisfies the discrete Poisson equation with curvature as source,

$$[\widetilde{\Delta}\phi^{\mathrm{L}}]_{\mathbf{k}} = 1/N_\phi, \tag{2}$$

with discrete Laplacian $\widetilde{\Delta}$ given by

$$[\widetilde{\Delta}\phi]_{\mathbf{k}} = \sum_{\mathbf{p}}^{\pm\mathbf{g}_x,\pm\mathbf{g}_y} (\phi_{\mathbf{k}+\mathbf{p}} - \phi_{\mathbf{k}}). \tag{3}$$

We impose the same Coulomb gauge condition on the lattice connections, and handle separately the average and the fluctuations of the lattice BZ curvature:

$$A_\alpha^{\mathrm{target}}(\mathbf{k}) = A_\alpha^{\mathrm{L}}(\mathbf{k}+\boldsymbol{\gamma}) \exp\left(-i2\pi\varepsilon_{\alpha\beta}[\mathrm{d}_\beta\phi]_{\mathbf{k}}\right). \tag{4}$$

The non-zero curvature average necessitates the first factor above. The shift $\boldsymbol{\gamma} = \sum_\alpha \gamma_\alpha \mathbf{g}_\alpha$ is determined by $W_x^{\mathrm{lat}} = W_x^{\mathrm{L}}(\gamma_y)$ and $W_y^{\mathrm{lat}} = W_y^{\mathrm{L}}(\gamma_x)$ ('lat' represents lattice), and it accounts for the mismatch in the large Wilson loops between the two systems. The curvature fluctuations are attended by the exponential factor, where the stream function $\phi_{\mathbf{k}}$ satisfies the discrete Poisson equation $[\widetilde{\Delta}\phi]_{\mathbf{k}} = f_{\mathbf{k}} - 1/N_\phi$, with boundary conditions $[\mathrm{d}_\alpha\phi]_{\mathbf{k}} = [\mathrm{d}_\alpha\phi]_{\mathbf{k}-N_\beta\mathbf{g}_\beta}$ (no summation implied) and $\sum_\kappa^{N_x}[\mathrm{d}_y\phi]_{\kappa\mathbf{g}_x} = \sum_\kappa^{N_y}[\mathrm{d}_x\phi]_{\kappa\mathbf{g}_y} = 0$. In plain words, we require that the connection corrections accounting for the curvature fluctuations should be periodic over the lattice BZ [7], and they should not contribute to the large Wilson loops W_α^{lat} which have already been fixed by the $A_\alpha^{\mathrm{L}}(\mathbf{k}+\boldsymbol{\gamma})$ factor.

Up to an inconsequential $\mathbf{k}$-independent constant, these conditions allow a *unique* solution

$$\phi_{\mathbf{k}} = \varphi_{\mathbf{k}} + v_y k_x - v_x k_y, \tag{5}$$

with $v_\alpha = \frac{1}{N_\alpha}\sum_{\kappa=0}^{N_\alpha-1}\sum_\beta \varepsilon_{\alpha\beta}[\mathrm{d}_\beta\varphi]_{\kappa\mathbf{g}_\alpha}$, and

$$\varphi_{\mathbf{k}} = \frac{1}{N_x N_y}\sum_{\mathbf{n}\neq 0} \frac{e^{i2\pi(k_x n_x/N_x + k_y n_y/N_y)}}{2\cos(2\pi n_x/N_x) + 2\cos(2\pi n_y/N_y) - 4} \sum_{\mathbf{p}}^{\mathrm{BZ}} e^{-i2\pi(p_x n_x/N_x + p_y n_y/N_y)}\left(f_{\mathbf{p}} - \frac{1}{N_\phi}\right), \tag{6}$$

where $\mathbf{n} \equiv (n_x, n_y)$ runs over $\{[0\,..\,N_x)\times[0\,..\,N_y)\}\backslash(0,0)$.

The connections $A_\alpha^{\mathrm{target}}(\mathbf{k})$ in Eq. (4) are consistent with the actual (fluctuating) curvature over the lattice BZ. Starting from a set of single-particle Bloch states $|\mathbf{k}\rangle$ with an arbitrarily chosen gauge and connections $A_\alpha(\mathbf{k})$, our gauge fixing scheme amounts to the gauge transform $|\mathbf{k}\rangle \to e^{i\zeta_{\mathbf{k}}}|\mathbf{k}\rangle$ that reproduces $A_\alpha^{\mathrm{target}}(\mathbf{k})$,

$$e^{i\zeta_{\mathbf{k}}} = \left[\prod_{\kappa=0}^{k_y-1} R_y(0,\kappa)\right]\left[\prod_{\kappa=0}^{k_x-1} R_x(\kappa, k_y)\right], \tag{7}$$

with $R_\alpha(\mathbf{k}) = A_\alpha^{\mathrm{target}}(\mathbf{k})/A_\alpha(\mathbf{k})$ [8].

EMERGENT PARTICLE-HOLE SYMMETRY AT FILLING $\nu = 2/3$

As noted in the main text, the fractional quantum Hall system has particle-hole symmetry, which is absent in the fractional Chern insulators (FCI) [1, 2]. The anti-unitary particle-hole transformation $\mathcal{P}$ exchanges the band creation and annihilation operators $\psi_{\mathbf{k}} \leftrightarrow \psi_{\mathbf{k}}^\dagger$. The most generic normal-ordered two-body FCI Hamiltonian in the single-band approximation can be written as

$$H = \sum_{\{\mathbf{k}_{1-4}\}}^{\mathrm{BZ}}{}' V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\psi_{\mathbf{k}_1}^\dagger\psi_{\mathbf{k}_2}\psi_{\mathbf{k}_3}^\dagger\psi_{\mathbf{k}_4} - \sum_{\mathbf{p}} V_{\mathbf{kppk}}\psi_{\mathbf{k}}^\dagger\psi_{\mathbf{k}}, \tag{8}$$

where the primed sum is constrained by $\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4 \bmod \mathbf{g}_\alpha$, and the interaction coefficients satisfy $V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}^* = V_{\mathbf{k}_2\mathbf{k}_1\mathbf{k}_4\mathbf{k}_3}$. The particle-hole transformation changes the Hamiltonian by a one-body term plus a constant,

$$\mathcal{P}H\mathcal{P}^{-1} - H = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}\psi_{\mathbf{k}}^\dagger\psi_{\mathbf{k}} + \sum_{\mathbf{pk}}^{\mathrm{BZ}} V_{\mathbf{kppk}}, \tag{9}$$

where the effective dispersion $\varepsilon_{\mathbf{k}}$ is given by

$$\varepsilon_{\mathbf{k}} = \sum_{\mathbf{p}}\Big(V_{\mathbf{pkkp}} + V_{\mathbf{kppk}} - V_{\mathbf{kkpp}} - V_{\mathbf{ppkk}}\Big). \tag{10}$$

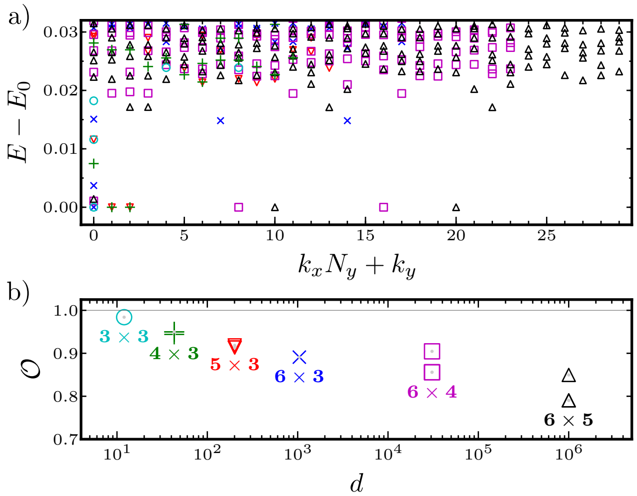

FIG. 1: Panel a) shows the low energy spectrum of the ruby lattice model at filling $\nu = 2/3$ for various system sizes, with energies shifted by E_0, the lowest energy for each system size. Panel b) shows the overlaps $\mathcal{O}$ between our FCI $\nu = 2/3$ Laughlin states and the lowest energy states in the corresponding momentum sectors of the ruby lattice model for various system sizes. The system size (N_x, N_y) represented by each group of markers is annotated in panel b).

We emphasize that $\varepsilon_{\mathbf{k}}$ comes from the interaction and is unrelated to the single-particle dispersion of the Bloch band. In general, $\varepsilon_{\mathbf{k}}$ has a non-trivial $\mathbf{k}$ dependence, and this breaks the particle-hole symmetry of the lattice model. We can apply our construction of FCI model wave functions to test the possible presence of emergent particle-hole symmetry in the lattice models that support a Laughlin-like state. We examine the ruby [3] and the kagome [4] lattice models with Chern number $C = 1$. We focus on the $\nu = 2/3$ filling factor, where the particle-hole conjugate of the $\nu = 1/3$ Laughlin state should appear.

For the ruby lattice model, we observe gapped three-fold ground state in the energy spectrum, as shown in Fig. 1a). A clear energy gap above the three-fold ground state is visible when the number of particles is higher than 12. Using the formalism detailed in the main text, we construct the FCI Laughlin state at filling $\nu = 2/3$. We find reasonable overlaps between these model states and the ground states of the ruby lattice model, as shown in Fig. 1b). Compared with the conjugate states at filling $\nu = 1/3$, the overlap values here are considerably smaller. For the kagome lattice model, we do not observe gapped ground states in the energy spectrum. This model does not exhibit any trace of the particle-hole conjugate Laughlin state. We note that the presence of a robust $\nu = 1/3$ Laughlin state in a Chern insulator does not guarantee the existence of its particle-hole conjugate at $\nu = 2/3$.

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[6] In the continuum limit [5], the exponentiated connections become $A_\alpha(\mathbf{k}) \approx e^{i\mathbf{a}(\mathbf{k})\cdot\mathbf{g}_\alpha}$, where $\mathbf{a}(\mathbf{k}) = -i\langle u_{\mathbf{k}}|\nabla_{\mathbf{k}}|u_{\mathbf{k}}\rangle$ is the Berry connection, with $|u_{\mathbf{k}}\rangle$ being the periodic part of the Bloch state. The Coulomb gauge condition on $\mathbf{a}(\mathbf{k})$ is $\nabla_{\mathbf{k}} \cdot \mathbf{a}(\mathbf{k}) = 0$. This enables one to write the connection in terms of a stream function $\phi(\mathbf{k})$, $\mathbf{a}(\mathbf{k}) = \hat{e}_z \times \nabla_{\mathbf{k}}\phi(\mathbf{k})$. Since $\nabla_{\mathbf{k}} \times \mathbf{a}(\mathbf{k}) = F(\mathbf{k})\hat{e}_z$, $\phi(\mathbf{k})$ satisfies a Poisson equation $\nabla^2_{\mathbf{k}}\phi(\mathbf{k}) = F(\mathbf{k})$, where $F(\mathbf{k})$ is the Berry curvature with the usual normalization $\int \mathrm{d}^2\mathbf{k} F(\mathbf{k}) = 2\pi C$.

[7] The obstruction to simultaneous smoothness and periodicity is manifested in the non-fluctuating part $A^{\mathrm{L}}_\alpha(\mathbf{k} + \boldsymbol{\gamma})$.

[8] Despite the formal similarity of $e^{i\zeta_k}$ expressed as a product of ratios of connections, our gauge choice here is fundamentally different from the "parallel-transport" gauge [5] in the treatment of the curvature *fluctuations*, embodied in the carefully constructed $A^{\mathrm{lat}}_\alpha(\mathbf{k})$.