EXTREMAL METRICS FOR THE $Q'$-CURVATURE IN THREE DIMENSIONS

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Abstract. We construct contact forms with constant $Q'$-curvature on compact three-dimensional CR manifolds which admit a pseudo-Einstein contact form and satisfy some natural positivity conditions. These contact forms are obtained by minimizing the CR analogue of the $H$-functional from conformal geometry. Two crucial steps are to show that the $P'$-operator can be regarded as an elliptic pseudodifferential operator and to compute the leading order terms of the asymptotic expansion of the Green’s function for $\sqrt{P'}$.

1. Introduction

The geometry of CR manifolds is studied via a choice of contact form and the induced Levi form. A natural question is whether there are preferred choices of contact form. One such choice is a CR Yamabe contact form, which has the property that the pseudohermitian scalar curvature is constant. Such contact forms exist on all compact CR manifolds [10, 13, 14, 24, 26]. Another such choice is a pseudo-Einstein contact form with constant $Q'$-curvature [6, 18]. The primary goal of this article is to show that the latter class of contact forms always exist in dimension three under natural positivity assumptions.

The idea of the $Q'$-curvature arose in the work of Branson, Fontana and Morpurgo [4] on Moser–Trudinger and Beckner–Onofri inequalities on the CR spheres. On any even-dimensional Riemannian manifold $(M^n, g)$, the critical GJMS operator $P_n$ is a conformally covariant differential operator $P_n$ with leading order term $(-\Delta)^{n/2}$ which controls the behavior of the critical $Q$-curvature $Q_n$ within a conformal class (cf. [3]). Specializing to the case of the standard $n$-sphere $(S^n, g_0)$ in even dimensions, Beckner [1] and, via different techniques, Chang and the third-named author [8], used these objects to establish the Beckner–Onofri inequality:

\[ \int_{S^n} w P_n w + 2 \int_{S^n} Q_n w - \frac{2}{n} \left( \int_{S^n} Q_n \right) \log \int_{S^n} e^{nw} \geq 0 \]

for all $w \in W^{n/2, 2}$ and for $Q_n$ an explicit (nonzero) dimensional constant. Moreover, equality holds in (1.1) if and only if $e^{2w} g_0$ is Einstein, or equivalently, if and only if $e^{2w} g_0 = \Phi^* g_0$ for $\Phi$ an element of the conformal transformation group of $S^n$.

Branson, Fontana and Morpurgo investigated to what extent the above discussion holds on the standard CR spheres $(S^{2n+1}, T^{1,0}S^{2n+1}, \theta_0)$. While it has long been known that there is a CR covariant operator $P_n$ with leading order term $(-\Delta_0)^{n+1}$, this operator has an infinite-dimensional kernel, namely the space $\mathcal{P}$ of...
CR pluriharmonic functions [15]. For this reason one does not expect \( P_n \) to give rise to a CR analogue of the sharp Beckner–Onofri inequality (1.1). Instead, Branson, Fontana and Morpurgo [4] observed that there is another operator \( P'_n \) defined only on \( \mathcal{P} \), with all of the desired properties. That is, \( P'_n \) has leading term \((-\Delta)\n^{+1/n} \), is CR covariant, and there is a (nonzero) dimensional constant \( Q'_n \) such that

\[
(1.2) \quad \int_{S^{2n+1}} w P'_nw + 2\int_{S^{2n+1}} Q'_nw - \frac{2}{n+1} \left( \int_{S^{2n+1}} Q'_n \right) \log \int_{S^{2n+1}} e^{(n+1)w} \geq 0
\]

for all \( w \in W^{n+1,2} \cap \mathcal{P} \). Moreover, equality holds in (1.2) if and only if \( e^{2w}\theta_0 \) is pseudo-Einstein and torsion-free, or equivalently, if and only if \( e^{2w}\theta_0 = \Phi^*\theta_0 \) for \( \Phi \) a CR automorphism of \((S^{2n+1}, T^{1,0}S^{2n+1})\).

In light of (1.1), it is natural to seek metrics of constant \( Q \)-curvature within a given conformal class on an even-dimensional Riemannian manifold. This question has been intensively studied in four dimensions. In particular, Chang and the third-named author [8] showed that on any compact Riemannian four-manifold \((M^4, g)\) for which the Paneitz operator \( P_4 \) is nonnegative with trivial kernel and for which \( \int Q_4 < 16\pi^2 \), one can construct a metric \( \hat{g} := e^{2w}g \) for which \( \hat{Q}_4 \) is constant by minimizing the functional

\[
H(w) := \int_M w P_4w + 2\int_M Q_4w - \frac{1}{2} \left( \int_M Q_4 \right) \log \int_M e^{4w}.
\]

This construction, and various modifications of it, have played an important role in studying the geometry of four-manifolds; see [7] for further discussion.

The purpose of this article is to show that one can similarly construct contact forms with constant \( Q \)-curvature on a compact three-dimensional CR manifold under natural positivity assumptions. To explain this, let us first recall the essential features of the \( Q \)-curvature [6]. On any pseudohermitian three-manifold \((M^3, T^{1,0}M, \theta)\), there is a differential operator \( P'_4: \mathcal{P} \to C^\infty(M) \) defined on the space \( \mathcal{P} \) of CR pluriharmonic functions with the properties that \( P'_4 \) has leading term \( \Delta_4^\theta \), is symmetric in the sense that the pairing \((u, v) \mapsto \int u P'_4v \) is symmetric on \( \mathcal{P} \), and satisfies the transformation formula

\[
(1.3) \quad e^{2w} \hat{P}'_4(u) = P'_4(u) \mod \mathcal{P}^\perp
\]

for all \( u \in \mathcal{P} \), where \( w \in C^\infty(M) \) and \( \hat{P}'_4 \) is defined in terms of \( \hat{\theta} = e^w\theta \). The analytic properties of the \( P'\)-operator are improved by projecting onto \( \mathcal{P} \). As we will see, if \( \tau: C^\infty(M) \to \mathcal{P} \) is the orthogonal projection, then the operator \( \tau P'_4: \mathcal{P} \to \mathcal{P} \) is a formally self-adjoint elliptic pseudodifferential operator.

In general, one cannot associate an analogue of the \( Q \)-curvature to \( P'_4 \). However, one can do so when restricting to pseudo-Einstein contact forms. A contact form \( \theta \) on \((M^3, T^{1,0}M)\) is pseudo-Einstein if its scalar curvature \( R \) and torsion \( A_{11} \) satisfy the relation \( \nabla_1 R = i\nabla_1 A_{11} \). This is equivalent to requiring that \( \theta \) is locally volume-normalized with respect to a nonvanishing closed \((2, 0)\)-form [17]; such contact forms always exist on boundaries of domains in \( C^2 \) [11]. For pseudo-Einstein contact forms, one can define a scalar invariant \( Q'_4 \) which satisfies a simple transformation rule in terms of \( P'_4 \) and the CR Paneitz operator \( P_4 \) upon changing the choice of pseudo-Einstein contact forms. In particular, \( \int Q_4' \) is an invariant of the class of pseudo-Einstein contact forms. For boundaries of domains, it is a biholomorphic invariant; indeed, it is the Burns–Epstein invariant [5] [6].
Suppose that $\theta$ is a pseudo-Einstein contact form on $(M^3, T^{1,0}M)$. Then $\hat{\theta} = e^w \theta$ is pseudo-Einstein if and only if $w$ is a CR pluriharmonic function [17]. In particular, it makes sense to consider the transformation formula for the $Q'$-curvature, and one obtains

$$e^{2w} \hat{Q}'_4 = Q'_4 + P'_4(w) \quad \text{mod} \ P'^{-1}$$

(see [6]). It is thus natural to consider the scalar quantity $\overline{Q}'_4 := \tau Q'_4$. In particular, on the standard CR three-sphere, $\overline{P}'_4$ is precisely the operator considered by Branson, Fontana and Morpurgo [4] and $\overline{Q}'_4$ is precisely the constant in (1.2).

We construct contact forms for which $\overline{Q}'_4$ is constant by constructing minimizers of the $II$-functional $II : P \to \mathbb{R}$ given by

$$II(w) = \int_M w \overline{P}'_4 w + 2 \int_M \overline{Q}'_4 w - \left( \int_M \overline{Q}'_4 \right) \log \int_M e^{2w}$$

on a pseudo-Einstein three-manifold $(M^3, T^{1,0}M, \theta)$. Note that, since $II$ is only defined on $P$, the projections in (1.5) can be removed; i.e. we can equivalently define the $II$-functional in terms of $P'_4$ and $Q'_4$. In general the $II$-functional is not bounded below. However, under natural positivity conditions it is bounded below and coercive, in which case we can construct the desired minimizers.

**Theorem 1.1.** Let $(M^3, T^{1,0}M, \theta)$ be a compact, embeddable pseudo-Einstein three-manifold such that the $P'$-operator $\overline{P}'_4$ is nonnegative and $\ker \overline{P}'_4 = \mathbb{R}$. Suppose additionally that

$$\int_M \overline{Q}'_4 \wedge d\theta < 16\pi^2.$$

Then there exists a function $w \in P$ which minimizes the $II$-functional (1.5). Moreover, the contact form $\hat{\theta} := e^w \theta$ is such that $\hat{Q}'_4$ is constant.

The assumptions of Theorem 1.1 can be replaced by the assumptions that the CR Paneitz operator is nonnegative and there exists a pseudo-Einstein contact form with scalar curvature nonnegative but not identically zero. Note that this last assumption implies that the CR Yamabe constant is positive; it would be interesting to know if these conditions are equivalent. Chanillo, Chiu and the third-named author proved [9] that these assumptions imply that $(M^3, T^{1,0}M)$ is embeddable. The first- and third-named authors proved [6] that these assumptions imply both that $\overline{P}'_4 \geq 0$ with $\ker \overline{P}'_4 = \mathbb{R}$ and that $\int \overline{Q}'_4 \leq 16\pi^2$ with equality if and only if $(M^3, T^{1,0}M)$ is CR equivalent to the standard CR three-sphere. Branson, Fontana and Morpurgo proved [4] Theorem 1.1 on the standard CR three-sphere. In summary, Theorem 1.1 implies the following result.

**Corollary 1.2.** Let $(M^3, T^{1,0}M, \theta)$ be a compact pseudo-Einstein manifold with nonnegative CR Paneitz operator which admits a pseudo-Einstein contact form with positive scalar curvature. Then there exists a function $w \in P$ which minimizes the $II$-functional (1.5). Moreover, the contact form $\hat{\theta} := e^w \theta$ is such that $\hat{Q}'_4$ is constant.

Note that the assumptions of Theorem 1.1 are all CR invariant; in particular, if $(M^3, T^{1,0}M)$ is the boundary of a domain in $\mathbb{C}^2$, the assumptions are biholomorphic invariants. Note also that the conclusion that $\hat{Q}'_4$ is constant cannot be strengthened
to the conclusion that $Q'_{4}$ is constant: In Section 3 we classify the contact forms on $S^1 \times S^2$ with its flat CR structure which have $Q'_{4}$ constant, and observe that $Q'_{4}$ is nonconstant for all of them.

The proof of Theorem 1.1 is analogous to the corresponding result in four-dimensional conformal geometry [8], though there are many new difficulties we must overcome. Since we are minimizing within $\mathcal{P}$, there is a Lagrange multiplier in the Euler equation for the $II$-functional which lives in the orthogonal complement $\mathcal{P}^\perp$ to $\mathcal{P}$. This is avoided by working with $\mathcal{P}'_{4}$. The greater difficulty is to show that minimizers for the $II$-functional exist in $W^{2,2} \cap \mathcal{P}$ under the hypotheses of Theorem 1.1. This is achieved by showing that $\mathcal{P}'_{4}$ satisfies a Moser–Trudinger-type inequality with the same constant as on the standard CR three-sphere under the positive assumption on $\mathcal{P}$ and (1.6).

To prove that $\mathcal{P}'_{4}$ satisfies the above Moser–Trudinger-type inequality, we study the asymptotics of the Green’s function of $(\mathcal{P}'_{4})^{1/2}$ in enough detail to apply the general results of Fontana and Morpurgo [12]. To make this precise, we require some more notation. Fix $\zeta \in M$ and let $(z, t)$ be CR normal coordinates in a neighborhood of $\zeta$ such that $(z(\zeta), t(\zeta)) = (0, 0)$. Define $\rho^{4}(z, t) = |z|^4 + t^2$. For $m \in \mathbb{R}$, let

$$ E(\rho^{m}) = \{ g \in C^{\infty}(M \setminus \{\zeta\}) : |\partial_{z}^{p} \partial_{\bar{z}}^{q} g(z, t)| \leq \rho(z, t)^{m-p-q-2} \text{ near } \zeta \}.$$ 

The asymptotics of the Green’s function of $(\mathcal{P}'_{4})^{1/2}$ are as follows.

**Theorem 1.3.** Let $(M^3, T^{1,0}M, \theta)$ be a compact embeddable pseudohermitian manifold such that $P'_{4}$ is nonnegative. Fix $\zeta \in M$ and let $G_{\zeta}$ be the Green’s function for $(\mathcal{P}'_{4})^{1/2}$ with pole at $\zeta$. Then there is a function $B_{\zeta} \in C^{\infty}(M \setminus \{\zeta\})$ such that

$$ B_{\zeta} - \rho^{-2} \in E(\rho^{-1-\epsilon})$$

for all $0 < \epsilon < 1$ and

$$ G_{\zeta} = \tau B_{\zeta} \tau.$$ 

We now outline the main argument used in the proof of Theorem 1.3. Fix a point $\zeta \in M$, the Green’s function of $(\mathcal{P}'_{4})^{1/2}$ at $\zeta$ is given by

$$ G_{\zeta} = (\mathcal{P}'_{4})^{-\frac{1}{2}} \tau \delta_{\zeta} \tau.$$

Using standard argument in spectral theory, we observe that

$$ (\mathcal{P}'_{4})^{-\frac{1}{2}} = c \int_{0}^{\infty} t^{-\frac{1}{2}} (\mathcal{P}'_{4} + t + \pi)^{-1} dt$$

on $(\text{ker } \mathcal{P}') \cap \mathcal{P}$, where $\mathcal{P}$ is the space of $L^{2}$ CR pluriharmonic functions, $\pi : \mathcal{P} \to \text{Ker } \mathcal{P}'_{4}$ is the orthogonal projection, and $c^{-1} = \int_{0}^{\infty} t^{-\frac{1}{2}} (1 + t)^{-1} dt$. Theorem 1.3 then follows from asymptotic expansions for $t^{-\frac{1}{2}} (\mathcal{P}'_{4} + t + \pi)^{-1}$. By using Boutet de Monvel–Sjöstrand’s classical theorem for the Szegő kernel [2], we first show that $\mathcal{P}'_{4} = \tau E_{2}$ for $E_{2}$ a classical elliptic pseudodifferential operator on $M$ of order 2. This allows us to apply classical theory of pseudodifferential operators to find a pseudodifferential operator $G_{t}$ of order $-2$ depending continuously on $t$ such that $(E_{2} + t)G_{t} = I + F_{t}$, where $F_{t}$ is a smoothing operator depending continuously on $t$ and $|F_{t}(x,y)|_{C^{m}(M \times M)} \leq \frac{1}{1+t}$ for all $m \in \mathbb{N}$. Roughly speaking, $\tau G_{t} \tau$ is the leading term of the operator $(\mathcal{P}'_{4} + t + \pi)^{-1}$. By carefully studying the principal
symbol and \( t \)-behavior of \( G_t \), we can show that \( G := \epsilon \int_0^{\infty} t^{-\frac{3}{2}} \tau G_t \tau \) is a smoothing operator of order 2 with \( G \tau \delta \tau = \rho^{-2} \mod \mathcal{E}(\rho^{1-\varepsilon}) \), for every \( \varepsilon > 0 \).

This article is organized as follows. In Section 2 we review some basic concepts from pseudohermitian geometry and the definitions of the \( P^* \)-operator and the \( Q' \)-curvature. In Section 3 we use Theorem 1.3 to show that \( G_{\tau \delta} \) is constant. The remaining sections are devoted to the proof of Theorem 1.3. In Section 6 we review some basic concepts about pseudodifferential operators and Fourier integral operators. In Section 7 we recall some properties of the orthogonal projection \( \tau \) established in [21]. In Section 8 we establish some properties of the principal symbol of \( \tau \Delta_h \tau \). In Section 9 we prove Theorem 1.3.

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## 2. Some pseudohermitian geometry

In this section we summarize some important concepts in pseudohermitian geometry as are needed to study the \( P^* \)-operator and the \( Q' \)-curvature in dimension three.

Let \( M^3 \) be a smooth, oriented (real) three-dimensional manifold. A CR structure on \( M \) is a one-dimensional complex subbundle \( T^{1,0} \subset T\gamma M := TM \otimes \mathbb{C} \) such that \( T^{1,0} \cap T^{0,1} = \{0 \} \) for \( T^{0,1} := \overline{T^{1,0}} \). Let \( H = \text{Re} T^{1,0} \) and let \( J : H \to H \) be the almost complex structure defined by \( J(V + \overline{V}) = i(V - \overline{V}) \).

Let \( \theta \) be a contact form for \( (M^3, T^{1,0}M) \); i.e. \( \theta \) is a nonvanishing real one-form such that \( \ker \theta = H \). Since \( M \) is oriented, a contact form always exists, and is determined up to multiplication by a positive real-valued smooth function. We say that \( (M^3, T^{1,0}M) \) is strictly pseudoconvex if the Levi form \( d\theta(\cdot, J\cdot) \) on \( H \otimes H \) is positive definite for some, and hence any, choice of contact form \( \theta \). We shall always assume that our CR manifolds are strictly pseudoconvex.

A pseudohermitian manifold is a triple \( (M^3, T^{1,0}M, \theta) \) consisting of a CR manifold and a contact form. The Reeb vector field \( T \) is the vector field such that \( \theta(T) = 1 \) and \( d\theta(T, \cdot) = 0 \). A \((1,0)\)-form is a section of \( T^*_{\mathbb{C}}M \) which annihilates \( T^{0,1} \). An admissible coframe is a nonvanishing \((1, 0)\)-form \( \theta^1 \) in an open set \( U \subset M \) such that \( \theta^1(T) = 0 \). Let \( h^1 := \overline{\theta^1} \) be its conjugate. Then \( d\theta = i h_{1\bar{1}} \theta^1 \wedge \theta^1 \) for some positive function \( h_{1\bar{1}} \). The function \( h_{1\bar{1}} \) is equivalent to the Levi form.

The connection form \( \omega^1 \) and the torsion form \( \tau^1 \) are determined by an admissible coframe \( \theta^1 \) uniquely determined by
\[
d\theta^1 = \theta^1 \wedge \omega^1 + \theta \wedge \tau^1, \quad \omega_{\bar{1}1} + \omega_{1\bar{1}} = dh_{1\bar{1}},
\]
where we use \( h_{1\bar{1}} \) to raise and lower indices as normal; e.g. \( \tau^1 = h^{1\bar{1}} \tau_{\bar{1}} \) for \( h^{1\bar{1}} = (h_{1\bar{1}})^{-1} \). The connection forms determine the pseudohermitian connection \( \nabla \) by
\[
\nabla Z_1 := \omega^1 \otimes Z_1
\]
for \( \{Z_1, Z_1, T\} \) the dual basis to \( \{\theta^1, \theta^\bar{1}, \theta\} \). The scalar curvature \( R \) of \( \theta \) is given by the expression
\[
d_\omega^{-1} = R\theta^1 \wedge \theta^\bar{1} \bmod \theta.
\]
A (real-valued) function \( w \in C^\infty(M) \) is CR pluriharmonic if locally \( w = \text{Re} f \) for some (complex-valued) function \( f \in C^\infty(M, \mathbb{C}) \) satisfying \( Z_1 f = 0 \). Equivalently, \( w \) is a CR pluriharmonic function if
\[
\nabla_1 \nabla_1 \nabla^1 w + i A_{11} \nabla^1 w = 0
\]
for \( \nabla_1 := \nabla_{Z_1} \) (cf. [27]). We denote by \( \mathcal{P} \) the space of all CR pluriharmonic functions.

Take \( \theta \wedge \text{d} \theta \) to be the volume form on \( M \). This induces a natural inner product \((\cdot, \cdot)\) on \( C^\infty(M) \). Let \( L^2(M) \) and \( \hat{\mathcal{P}} \) denote the completions of \( C^\infty(M) \) and \( \mathcal{P} \), respectively, with respect to this inner product.

The Paneitz operator \( P_4 \) is the differential operator
\[
P_4(w) := 4\nabla^1(\nabla_1 \nabla_1 \nabla^1 w + i A_{11} \nabla^1 w) = \Delta_1^2 w + T^2 - 4 \text{Im} \nabla^1(A_{11} \nabla^1 f)
\]
for \( \Delta_b := \nabla^1 \nabla_1 + \nabla^1 \nabla_\bar{1} \) the sublaplacian. Note in particular that \( \mathcal{P} \subset \ker P_4 \). A key property of the Paneitz operator is that it is CR covariant; if \( \hat{\theta} = e^{w} \theta \), then \( e^{2w} \hat{P}_4 = P_4 \) (cf. [17]).

**Definition 2.1.** Let \( (M^3, T^{1,0}M, \theta) \) be a pseudohermitian manifold. The \( P' \)-operator \( P': \mathcal{P} \rightarrow C^\infty(M) \) is defined by
\[
P'_4 f = 4\Delta_1^2 f - 8 \text{Im} (\nabla^\alpha (A_{\alpha\beta} \nabla^\beta f)) - 4 \text{Re} (\nabla^\alpha (R \nabla_\alpha f)) + \frac{8}{3} \text{Re} W_\alpha \nabla^\alpha f - \frac{4}{3} f \nabla^\alpha W_\alpha
\]
for \( f \in \mathcal{P} \), where \( W_\alpha := \nabla_\alpha R - i \nabla^\beta A_{\alpha\beta} \).

In particular,
\[
P'_4 f = 4\Delta_b^2 f + R\Delta_b f + \Delta_b Rf + (L_1 L_2 + \overline{L_1} \overline{L_2}) f + (L_3 + \overline{L_3}) f + r f, \quad L_1, L_2, L_3 \in C^\infty(M, T^{1,0}M), \quad r \in C^\infty(M), \quad f \in \mathcal{P}.
\]

A key property of the \( P' \)-operator is its conformal covariance: Let \( (M^3, T^{1,0}M, \theta) \) be a pseudohermitian manifold, let \( w \in C^\infty(M) \), and set \( \hat{\theta} = e^{w} \theta \). Then
\[
e^{2w} \hat{P}_4(u) = P'_4(u) + P_4(uw)
\]
for all \( u \in \mathcal{P} \). In particular, since \( P_4 \) is self-adjoint and annihilates CR pluriharmonic functions, (2.3) implies that the \( P' \)-operator is conformally covariant, mod \( \mathcal{P}^\perp \).

A pseudohermitian manifold \((M^3, T^{1,0}M, \theta)\) is pseudo-Einstein if \( W_\alpha = 0 \) for \( W_\alpha \) as in Definition 2.1.

**Definition 2.2.** Let \( (M^3, T^{1,0}M, \theta) \) be a pseudo-Einstein manifold. The \( Q' \)-curvature is
\[
Q'_4 = 2\Delta_b R - 4 |A|^2 + R^2.
\]
A key property of the $Q'$-curvature is its conformal covariance: Let $(M^3, T^{1,0}M, \theta)$ be a pseudo-Einstein manifold, let $w \in \mathcal{P}$, and set $\hat{\theta} = e^{w} \theta$. Hence $\hat{\theta}$ is pseudo-Einstein \cite{17}. Then
\begin{equation}
(2.5) \quad e^{2w} \hat{Q}'_4 = Q'_4 + P'_4(w) + \frac{1}{2} P_4(w^2).
\end{equation}
In particular, $Q'_4$ behaves as the $Q$-curvature for $P'_4$, mod $\mathcal{P}$.\[\]3. The Moser–Trudinger inequality for the $P'$-operator

A key step in our proof of Theorem \ref{mainresult} is to show that the $P'$-operator satisfies the same sharp Moser–Trudinger-type inequality as its counterpart on the sphere. This follows from the asymptotic expansion for the Green’s function of $(\mathcal{P}'_4)^{1/2}$ given in Theorem \ref{asymp} and the general Adams-type theorem of Fontana and Morpurgo \cite{12}.

Given $k \in \mathbb{N}$ and $q > 0$, let $W^{k,q}$ denote the non-isotropic Sobolev space, given by the set of all functions $u$ such that $Z_1Z_2 \cdots Z_j u \in L^q(M)$ for all $Z_j \in C^\infty(M, T^{1,0}M \oplus T^{0,1}M)$, $j = 0, 1, 2, \ldots, k$.

**Theorem 3.1.** Let $(M^3, T^{1,0}M, \theta)$ be a compact pseudo-Einstein three-manifold for which the $P'$-operator is nonnegative with trivial kernel. Then there exists a constant $C$ such that
\begin{equation}
(3.1) \quad \log \int_M e^{2(w-w_0)} \leq C + \frac{1}{16\pi^2} \int_M w \mathcal{P}'_4 w
\end{equation}
for all $w \in W^{2,2} \cap \mathcal{P}$.

**Proof:** From Theorem \ref{asymp} we see that the leading order term of the Green’s function for $\mathcal{P}'_4$ is independent of $(M^3, T^{1,0}M, \theta)$; in particular, it has exactly the same leading order term as the Green’s function for the $P'$-operator on the standard CR three-sphere. Furthermore, the next term in the asymptotic expansion of the Green’s function involves a definite loss of power in the asymptotic coordination $\rho$. Thus, by arguing analogously to the proof of \cite[Theorem 2.1]{4}, we may apply the main result \cite[Theorem 1]{12} to conclude that there is a constant $C > 0$ such that
\begin{equation}
(3.2) \quad \int_M \exp \left( \frac{16\pi^2 (w-w_0)^2}{\int_M \mathcal{P}_4 w} \right) \theta \wedge d\theta \leq C
\end{equation}
for all $f \in W^{2,2} \cap \mathcal{P}$. The desired inequality \ref{mosertrudinger} is an immediate consequence of \ref{finetrems} and the elementary estimate
\[0 \leq 16\pi^2 \left( \int w \mathcal{P}_4 w \right)^{-2} \left( (w - w_0)^2 + \frac{1}{16\pi^2} \int_M w \mathcal{P}'_4 w \right).\]

\[\square\]

**Remark 3.2.** A few comments are in order to explain the above constants. The convention used in \cite{4} is that the sublaplacian is given by $-\text{Re} \nabla^\gamma \nabla_{\gamma}$, which shows that our definition is $-2$ times theirs. With this in mind, their formula \cite[(1.30)]{4} for the $P'$-operator shows that our definition is $4$ times theirs. Finally, they integrate with respect to the Riemannian volume element on $S^3$, regarded as the unit ball in $\mathbb{R}^4$, while we integrate with respect to $\theta \wedge d\theta$ for $\theta = \text{Im} \hat{\mathcal{W}}( |z|^2 - 1)$; in particular, our volume form is $2$ times theirs. Together, these normalizations account for the apparent difference between our constant in \cite{12} and the constant appearing in \cite[(2.11)]{4}. Note that $\theta$ has scalar curvature $R = 2$, and hence $\overline{Q}'_4 = 4$.\[\]
4. Minimizing the Functional $II$

Assuming the results of Section 3, we prove that smooth minimizers of the $II$-functional exist under natural positivity assumptions. We first construct weak minimizers.

**Theorem 4.1.** Let $(M^3, T^{1.0}M, \theta)$ be a compact pseudo-Einstein three-manifold such that $\int \mathcal{Q}_4 < 16\pi^2$. Suppose additionally that the $P_4'$-operator is nonnegative with $\ker P_4' = \mathbb{R}$. Then

$$\inf_{w \in W^{2,2} \cap \mathcal{P}} II[w]$$

is obtained by some function $w \in W^{2,2} \cap \mathcal{P}$.

**Proof.** Denote $k = \int \mathcal{Q}_4$. Recall that

$$II[w] = (P_4'w, w) + 2 \int_M \mathcal{Q}_4(w - w_0) - k \log \int_M e^{2(w - w_0)}$$

for $w_0 = \bar{w}$ the average value of $M$. If $k \leq 0$, it follows immediately that

$$II[w] \geq (P_4'w, w) + 2 \int_M \mathcal{Q}_4(w - w_0),$$

while if $k > 0$, Theorem 3.1 implies that

$$II[w] \geq \left(1 - \frac{k}{16\pi^2}\right) (P_4'w, w) + 2 \int_M \mathcal{Q}_4(w - w_0) - KC.$$

Together, these estimates imply that

$$II[w] \geq \left(1 - \frac{k^+}{16\pi^2}\right) (P_4'w, w) + 2 \int_M \mathcal{Q}_4(w - w_0) - C$$

for $k^+ = \max\{0, k\}$ and $C$ a positive constant depending only on $(M^3, T^{1.0}M, \theta)$.

Denote by $\lambda_1 = \lambda_1(P_4')$ the first nonzero eigenvalue

$$\lambda_1(P_4') = \inf \left\{ \frac{(P_4'w, w)}{\|w\|_2^2} : w \in W^{2,2} \cap \mathcal{P}, \int_M w = 0 \right\}$$

of $P_4'$. By assumption, $\lambda_1 > 0$. Together with (4.1), this shows that there are positive constants $c_1, c_2$ depending only on $(M^3, T^{1.0}M, \theta)$ such that

$$II[w] \geq c_1\|w - w_0\|_2^2 - c_2.$$  

In particular, $II$ is bounded below.

Let $\{w_k\} \subset \mathcal{P}$ be a minimizing sequence of $II$, normalized so that $\|w_k\|_2 = 1$ for all $k \in \mathbb{N}$. Using (4.1) and the local formula (2.1) for $P_4'$, it is easily seen that there is a positive constant $c_3$ depending only on $(M^3, T^{1.0}M, \theta)$ such that

$$\left(1 - \frac{k^+}{16\pi^2}\right) \int_M (\Delta bw_k)^2 \leq c_3 \int_M R|\nabla b w_k|^2 + c_3 \int_M \text{Im} A_{\alpha\beta} \nabla^\alpha w_k \nabla^\beta w_k$$

$$+ 2 \left| \int_M P_4'(w_k - (w_k)_0) \right| + c_3.$$  

(4.3)

On the other hand, given any $\varepsilon > 0$, it holds that

$$\int_M |\nabla b w_k|^2 = -\int_M w_k \Delta b w_k \leq \varepsilon \int_M (\Delta b w_k)^2 + \frac{1}{4\varepsilon} \|w_k - (w_k)_0\|_2^2.$$
We may thus combine (4.2) and (4.3) to conclude that \( \{ w_k \} \) is uniformly bounded in \( W^{2,2} \cap P \). Thus, by choosing a subsequence if necessary, we see that \( w_k \) converges weakly in \( W^{2,2} \cap P \) to a minimizer \( w \in W^{2,2} \cap P \) of \( II \).

We next show that weak critical points of the \( II \)-functional are smooth.

**Theorem 4.2.** Let \((M^3, T^{1,0} M, \theta)\) be a compact three-dimensional pseudo-Einstein manifold. Suppose that \( w \in W^{2,2} \cap P \) is a critical point of the \( II \)-functional. Then \( w \) is smooth, and moreover, the contact form \( \hat{\theta} := e^w \theta \) is such that \( \hat{Q}'_4 \) is constant.

**Proof.** It is readily seen that \( w \) is a critical point of the \( II \)-functional if and only if

\[
P'_4 w + Q'_4 = \lambda e^{2w} \mod P^\perp.
\]

In particular, if \( w \) is smooth, then (2.5) implies that \( \hat{Q}'_4 \) is constant. Now, we prove that \( w \) is smooth. Fix \( \ell \in \mathbb{N} \) sufficiently large and let \( B_\ell \) and \( C_\ell \) be as in Theorem 9.17. From (4.4), we have

\[
\tau B_\ell \tau (e^{2w}) = \tau B_\ell P'_4 w + \tau B_\ell \tau Q'_4 = w + \tau C_\ell w + \tau B_\ell \tau Q'_4.
\]

Note that

\[
\tau C_\ell w + \tau B_\ell \tau Q'_4 \in C^\ell(M).
\]

Since \( w \in W^{2,2} \), we have \( \Delta_b w \in L^2(M) \). From Theorem 9.18, we conclude that \( e^{c|w|^2} \in L^1(M), \quad c > 0, \)

and hence

\[
\lambda e^{2w} \in L^q(M), \quad \forall q > 1.
\]

Since \( \tau B_\ell \tau \) is a smoothing operator of order \( 4 - \varepsilon \) for all \( 0 < \varepsilon < 1 \), it holds that (see [23, Proposition 2.7])

\[
\tau B_\ell \tau : W^{k,q} \to W^{k+1,q}, \quad \text{for all } q > 1 \text{ and all } k \in \mathbb{N}_0.
\]

From (4.5), (4.7) and (4.8), we obtain that

\[
w \in W^{1,q}, \quad \text{for all } q > 1.
\]

From (4.7) and (4.9) it is easy to see that \( \lambda e^{2w} \in W^{1,q} \) for all \( q > 1 \). From this, (4.5) and (4.9) we conclude that \( w + \tau C_\ell w + \tau B_\ell \tau Q'_4 \in W^{2,q} \) for all \( q > 1 \). Continuing in this way, we deduce that \( w + \tau C_\ell w + \tau B_\ell \tau Q'_4 \in W^{k,q} \) for all \( q > 1 \) and all \( k \in \mathbb{N}_0 \) with \( k \leq \ell \). Thus, \( w \in W^{\ell,q} \), for all \( q > 1 \). Since \( \ell \) is arbitrary, we deduce that \( w \) is smooth. □

**Proof of Theorem 1.1.** By Theorem 4.1, there is a minimizer \( w \in W^{2,2} \cap P \) of the \( II \)-functional. By Theorem 4.2, \( w \) is smooth and the contact form \( \hat{\theta} := e^w \theta \) is such that \( \hat{Q}'_4 \) is constant, as desired. □
5. AN EXAMPLE

Here we provide an example to show that minimizers of the \( H \)-functional, while they have \( \overline{Q}_4 \) constant, need not have \( Q'_4 \)-constant. More precisely, we will prove the following theorem.

**Theorem 5.1.** Let \( \Gamma \) be a nontrivial dilation of the Heisenberg group \( \mathbb{H}^1 \) which fixes the origin \( 0 \in \mathbb{H}^1 \). Then \( S^1 \times S^2 = (\mathbb{H}^1 \setminus \{0\})/\Gamma \) with its standard CR structure is such that the minimizer of the \( H \)-functional is unique up to an additive constant, and moreover, the corresponding contact form \( \hat{\theta} \) has \( \overline{Q}_4 \equiv 0 \) but \( Q'_4 \neq 0 \).

**Proof.** Let \( \rho(z, t) = (|z|^4 + t^2)^{1/4} \) be the usual pseudo-distance on \( \mathbb{H}^1 \). It is straightforward to check that the contact form \( \theta_1 = \rho^{-4} \theta_0 \) on \( \mathbb{H}^1 \setminus \{0\} \) is such that \( \theta_1 = \Phi^{\ast} \theta_0 \) for \( \Phi(z, t) \) the CR inversion through the pseudo-sphere \( \rho^{-1}(1) \). In particular, \( \theta_1 \) is flat, and hence \( \log \rho \in \mathcal{P} \). From (2.5) it follows that

\[
P'_4 \log \rho^{-4} + \frac{1}{2} P_4 \log^2 \rho^{-4} = 0.
\]

Consider now the contact form \( \theta := \rho^{-2} \theta_0 \). It is clear that \( \theta \) is invariant under the action of \( \Gamma \), and hence \( \theta \) descends to a well-defined contact form on \( S^1 \times S^2 \). Since \( \log \rho \in \mathcal{P} \), we know that \( \theta \) is pseudo-Einstein. From (2.5) we see that the \( Q' \)-curvature \( Q'_4 \) of \( \theta \) is

\[
\rho^{-4} Q'_4 = P'_4 \log \rho^{-2} + \frac{1}{2} P_4 \log^2 \rho^{-2} = -\frac{1}{2} P_4 \log^2 \rho^{-2},
\]

where the second equality uses (2.1). Since the Paneitz operator \( P_4 \) is self-adjoint and \( \mathcal{P} \subset \ker P_4 \), it follows that \( Q'_4 \) is orthogonal, with respect to \( \theta \wedge d\theta \), to the CR pluriharmonic functions. In particular, \( \overline{Q}_4 \equiv 0 \). Furthermore, one can compute directly from (5.2) that

\[
Q'_4 = 8 \frac{|z|^4 - t^2}{|z|^4 + t^2},
\]

which is clearly not identically zero.

Finally, using Lee’s formula for the change of the scalar curvature under a conformal change of contact form [27] Lemma 2.4], we compute that the scalar curvature \( R \) of \( \theta \) is

\[
R = 2 \frac{|z|^2}{\rho^2}.
\]

Since this is nonnegative and \( \theta \) is pseudo-Einstein, \( \overline{P}'_4 \) is nonnegative with trivial kernel [6] Proposition 4.9]. Now, if \( \hat{\theta} = e^{u} \theta \) is a pseudo-Einstein contact form on \( S^1 \times S^2 \) for which \( \hat{Q}'_4 \equiv 0 \), the transformation formula (2.5) implies that \( \overline{P}'_4 u \equiv 0 \), whence \( u \) is constant, as desired. \( \square \)

6. PRELIMINARIES FOR PSEUDODIFFERENTIAL OPERATORS

We shall use the following notations: \( \mathbb{R} \) is the set of real numbers, \( \mathbb{R}_+ := \{ x \in \mathbb{R}; x > 0 \} \), \( \mathbb{R}_0 := \{ x \in \mathbb{R}; x \geq 0 \} \), \( \mathbb{N} = \{ 1, 2, \ldots \} \), and \( \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \} \). An element \( \alpha = (\alpha_1, \ldots, \alpha_n) \) of \( \mathbb{N}_0^n \) is a multi-index, the size of \( \alpha \) is \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), and the length of \( \alpha \) is \( l(\alpha) = n \). For \( m \in \mathbb{N} \), we write \( \alpha \in \{ 1, \ldots, m \}^n \) if \( \alpha_j \in \{ 1, \ldots, m \} \) for all \( j = 1, \ldots, n \). We say that \( \alpha \) is strictly increasing if \( \alpha_1 < \alpha_2 < \cdots < \alpha_n \).
Given a multi-index $\alpha$, we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n)$; we write $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ for $\partial_{x_j} = \frac{\partial}{\partial x_j}$ and $\partial_x^\alpha = \frac{\partial^{\alpha}}{\partial x^\alpha}$; we write $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ for $D_x = \frac{i}{2} \partial_x$ and $D_{x_j} = \frac{1}{i} \partial_{x_j}$.

Let $z = (z_1, \ldots, z_n)$, $z_j = x_{2j-1} + i x_{2j}$, $j = 1, \ldots, n$, be coordinates of $\mathbb{C}^n$. Given a multi-index $\alpha$, we write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $\overline{z}^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$; we write $\frac{\partial z^\alpha}{\partial z^\alpha} = \frac{\partial}{\partial z^\alpha} = \frac{\partial}{\partial z_1^{\alpha_1}} \cdots \frac{\partial}{\partial z_n^{\alpha_n}}$, where $\partial_{z_j} = \frac{1}{2} (\frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}})$ for all $j = 1, \ldots, n$; similarly, we write $\frac{\partial \overline{z}^\alpha}{\partial \overline{z}^\alpha} = \frac{\partial}{\partial \overline{z}^\alpha} = \frac{\partial}{\partial \overline{z}_1^{\alpha_1}} \cdots \frac{\partial}{\partial \overline{z}_n^{\alpha_n}}$, where $\partial_{\overline{z}_j} = \frac{1}{2} (\frac{\partial}{\partial x_{2j-1}} + i \frac{\partial}{\partial x_{2j}})$ for all $j = 1, \ldots, n$.

Let $M$ be a smooth manifold. We denote by $\langle \cdot, \cdot \rangle$ the pointwise duality between $TM$ and $T^*M$. We extend $\langle \cdot, \cdot \rangle$ bilinearly to $T_C M \times T_C^* M$. Let $E$ be a $C^\infty$ vector bundle over $M$. The fiber of $E$ at $x \in M$ are denoted by $E_x$. Let $Y \subset M$ be an open set. The spaces of smooth sections of $E$ over $Y$ and distributional sections of $E$ over $Y$ are denoted by $C^\infty(Y, E)$ and $\mathcal{D}'(Y, E)$, respectively. Let $\mathcal{D}'(Y, E)$ be the subspace of $\mathcal{D}'(Y, E)$ whose elements have compact support in $Y$. For $m \in \mathbb{R}$, let $H^m(Y, E)$ denote the Sobolev space of order $m$ of sections of $E$ over $Y$. Put

$$H^m_{loc}(Y, E) = \{ u \in \mathcal{D}'(Y, E) : \varphi u \in H^m(Y, E) \text{ for all } \varphi \in C^\infty(Y) \},$$

$$H^m(\text{comp})(Y, E) = H^m_{loc}(Y, E) \cap \mathcal{D}'(Y, E).$$

Fix a smooth density of integration on $M$. If $A : C^\infty_0(M, E) \to \mathcal{D}'(M, F)$ is continuous, we write $A(x, y)$ to denote the distributional kernel of $A$. The following two statements are equivalent:

(a) $A$ is continuous as a mapping from $\mathcal{D}'(M, E)$ to $C^\infty(M, F)$.
(b) $A(x, y) \in C^\infty(M \times M, E_y \boxtimes F_x)$.

If $A$ satisfies (a) or (b), we say that $A$ is smoothing. Let $B : C^\infty_0(M, E) \to \mathcal{D}'(M, F)$ be a continuous operator. We write $A \equiv B$ if $A - B$ is a smoothing operator.

Let $H(x, y) \in \mathcal{D}'(M \times M, E_y \boxtimes F_x)$. We also denote by $h$ the unique continuous operator $H : C^\infty_0(M, E) \to \mathcal{D}'(M, F)$ with distribution kernel $H(x, y)$. We henceforth identify $H$ with $H(x, y)$.

Recall the Hörmander symbol spaces:

**Definition 6.1.** Let $M \subset \mathbb{R}^N$ be an open set and let $m \in \mathbb{R}$. $S^m_{1,0}(M \times \mathbb{R}^N)$ is the space of all $a \in C^\infty(M \times \mathbb{R}^N)$ such that for all compact $K \subset M$ and all $\alpha \in \mathbb{N}_0^N$, $\beta \in \mathbb{N}_0$, there is a constant $C > 0$ such that

$$\left| \partial_x^\alpha \partial_\theta^\beta a(x, \theta) \right| \leq C(1 + |\theta|)^{m - |\beta|} \text{ for all } (x, \theta) \in K \times \mathbb{R}^N.$$ 

Denote

$$S^{-\infty}(M \times \mathbb{R}^N) := \bigcap_{m \in \mathbb{R}} S^m_{1,0}(M \times \mathbb{R}^N).$$

Let $a_j \in S^m_{1,0}(M \times \mathbb{R}^N)$ for $j \in \mathbb{N}_0$ with $m_j \to -\infty$ as $j \to \infty$. Then there exists $a \in S^m_{1,0}(M \times \mathbb{R}^N)$, unique modulo $S^{-\infty}(M \times \mathbb{R}^N)$, such that $a - \sum_{j=0}^{k-1} a_j \in S^m_{1,0}(M \times \mathbb{R}^N)$ for all $k \in \{1, 2, \ldots\}$.

If $a$ and $a_j$ have the properties above, we write $a \sim \sum_{j=0}^{\infty} a_j$ in $S^m_{1,0}(M \times \mathbb{R}^N)$. Let $S^m_{11}(M \times \mathbb{R}^N)$ be the space of all symbols $a(x, \theta) \in S^m_{1,0}(M \times \mathbb{R}^N)$ with

$$a(x, \theta) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \theta) \text{ in } S^m_{1,0}(M \times \mathbb{R}^N),$$

where
with \( a_k(x, \theta) \in \mathcal{C}^\infty(M \times \mathbb{R}^N) \) positively homogeneous of degree \( k \) in \( \theta \); that is, \( a_k(x, \lambda \theta) = \lambda^k a_k(x, \theta) \) for all \( \lambda \geq 1 \) and all \( |\theta| \geq 1 \).

By using partition of unity, we extend the definitions above to the cases when \( M \) is a smooth manifold and when we replace \( M \times \mathbb{R}^N \) by \( T^*M \).

Let \( \Omega \subset M^3 \) be an open coordinate patch. Let \( a(x, \xi) \in S^k_{1,0}(T^*\Omega) \). We define

\[
A(x, y) = \frac{1}{(2\pi)^3} \int e^{i<x-y, \xi>} a(x, \xi) d\xi
\]

as an oscillatory integral. One can show that

\[
A: \mathcal{C}^\infty_0(\Omega) \rightarrow \mathcal{C}^\infty(\Omega)
\]

is continuous and has a unique continuous extension \( A: \mathcal{E}'(\Omega) \rightarrow \mathcal{D}'(\Omega) \).

**Definition 6.2.** Let \( k \in \mathbb{R} \). A classical pseudodifferential operator of order \( k \) on \( M \) is a continuous linear map \( A: \mathcal{C}^\infty(M) \rightarrow \mathcal{D}'(M) \) such that on every open coordinate patch \( \Omega \), if we consider \( A \) as a continuous operator

\[
A: \mathcal{C}^\infty_0(\Omega) \rightarrow \mathcal{C}^\infty(\Omega),
\]

then the distributional kernel of \( A \) is

\[
A(x, y) = \frac{1}{(2\pi)^3} \int e^{i<x-y, \xi>} a(x, \xi) d\xi
\]

with \( a \in S^k_{1,0}(T^*\Omega) \). We call \( a(x, \xi) \) the symbol of \( A \). We write \( L^k_\mathcal{H}(M) \) to denote the space of classical pseudodifferential operators of order \( k \) on \( M \).

7. **The distributional kernel of \( \tau \)**

In this section, we review some results in [21] about the orthogonal projection \( \tau: L^2 \rightarrow L^2 \cap \mathcal{P} \) which are needed in the proof of our main result.

Let \( \langle \cdot, \cdot \rangle \) be the Hermitian inner product on \( T^*_C M \) given by

\[
\langle Z_1 | Z_2 \rangle = \frac{1}{2i} \langle d\theta, Z_1 \wedge \overline{Z}_2 \rangle \quad \text{for all } Z_1, Z_2 \in T^{1,0} \Omega.
\]

The Hermitian metric \( \langle \cdot, \cdot \rangle \) on \( T^*_C M \) induces a Hermitian metric \( \langle \cdot, \cdot \rangle \) on \( T^*_C \Omega \). Take \( \theta \wedge d\theta \) to be the volume form on \( M \), we then get natural inner product on \( \Omega^{0,1}(M) := \mathcal{C}^\infty(M, T^{0,1} \Omega) \) induced by \( \theta \wedge d\theta \) and \( \langle \cdot, \cdot \rangle \), where \( T^{0,1} \Omega \) denotes the bundle of \( (0, 1) \) forms of \( M \). We denote this inner product by \( \langle \cdot, \cdot \rangle \) and denote the corresponding norm by \( \| \cdot \| \).

Take \( L^2_{(0,1)}(M) \) denote the completion of \( \Omega^{0,1}(M) \) with respect to \( \langle \cdot, \cdot \rangle \). Let \( \overline{\partial}_b: C^\infty(M) \rightarrow \Omega^{0,1}(M) \) be the tangential Cauchy-Riemann operator. We extend \( \overline{\partial}_b \) to \( L^2 \) by \( \overline{\partial}_b : \text{Dom} \overline{\partial}_b \rightarrow L^2_{(0,1)}(M) \), where

\[
\text{Dom} \overline{\partial}_b := \left\{ u \in L^2(M) : \overline{\partial}_b u \in L^2_{(0,1)}(M) \right\}.
\]

Let \( \overline{\partial}_b : \text{Dom} \overline{\partial}_b \rightarrow L^2(M) \) be the \( L^2 \) adjoint of \( \overline{\partial}_b \). The Kohn Laplacian is given by

\[
\Box_b := \overline{\partial}_b \overline{\partial}_b : \text{Dom} \Box_b \rightarrow L^2(M),
\]

(7.1)

\[
\text{Dom} \Box_b = \left\{ u \in L^2(M) : u \in \text{Dom} \overline{\partial}_b, \overline{\partial}_b u \in \text{Dom} \overline{\partial}_b \right\}.
\]

Note that \( \Box_b \) is self-adjoint.
The orthogonal projection $S: L^2(M) \to \ker \overline{\partial}_b = \Ker \Box_b$ is the Szegő projection. From now on, we assume that $M$ is embeddable. The following facts are shown by the second-named author; see [21, Theorem 1.2 and Remark 1.4].

**Theorem 7.1.** With the assumptions and notations above, we have

\begin{equation}
\tau = S + \overline{S} + F,
\end{equation}

where $F$ is a smoothing operator. Moreover, the kernel $\tau(x, y) \in \mathcal{D}'(M \times M)$ of $\tau$ satisfies

\begin{equation}
\tau(x, y) \equiv \int_0^\infty e^{i\varphi(x,y)s} a(x, y, s)ds + \int_0^\infty e^{-i\varphi(x,y)s} \pi(x, y, s)ds,
\end{equation}

where

\begin{align}
a(x, y, s) &\in S^1_{\mathrm{cl}} (M \times M \times (0, \infty)), \\
(a(x, y, s)) &\sim \sum_{j=0}^{\infty} a_j(x, y)s^{1-j} \text{ in } S^1_{1,0} (M \times M \times (0, \infty)), \\
a_j(x, y) &\in C^\infty (M \times M) \text{ for all } j \in \mathbb{N}_0, \\
a_0(x, x) &= \frac{1}{2\pi^n} \text{ for all } x \in M,
\end{align}

and

\begin{align}
\varphi &\in C^\infty (M \times M), \quad \text{Im} \varphi(x, y) \geq 0, \quad d_x \varphi|_{x=y} = -\theta(x), \\
\varphi(x, y) &= -\overline{\varphi}(y, x), \\
\varphi(x, y) &= 0 \text{ if and only if } x = y, \\
\sigma\Box_b (x, \varphi'(x, y)) &\text{ vanishes to infinite order on } x = y.
\end{align}

Here $\sigma\Box_b$ denotes the principal symbol of $\Box_b$.

We need the following fact about the Szegő kernel (cf. [2, 20]).

**Theorem 7.2.** With the assumptions and notations above, the distributional kernel of $S$ satisfies

\begin{equation}
S(x, y) \equiv \int_0^\infty e^{i\varphi(x,y)s} a(x, y, s)ds
\end{equation}

where $\varphi(x, y) \in C^\infty (M \times M)$ and $a(x, y, s) \in S^1_{\mathrm{cl}} (M \times M \times (0, \infty))$ are as in Theorem 7.1.

8. The principal symbol of $\tau\Delta_b$ on $\mathcal{P}$

It is well-known that $\Delta_b$ is a subelliptic operator. However, if we restrict $\Delta_b$ to $\mathcal{P}$, it is equivalent to an elliptic pseudodifferential operator.

**Theorem 8.1.** There is a classical elliptic pseudodifferential operator $E_1 \in L^1_{\mathrm{cl}}(M)$ with real-valued principal symbol such that

\[ \tau\Delta_b \tau = \tau E_1 \tau \text{ on } \mathcal{D}'(M). \]

In particular, $\tau\Delta_b = \tau E_1$ on $\mathcal{P}$.

The proof of Theorem 8.1 requires many ingredients. First, we have the following immediate consequence of the commutator formulae proven by Lee [27].

**Lemma 8.2.** It holds that $\overline{\Delta}_b = \Box_b + 2iT + L$ for some $L \in C^\infty (M, T^{1,0}M \oplus T^0,1M)$. 
We need the following result given in [22, Lemma 5.7]

**Lemma 8.3.** Let $A, B : C^\infty_0(M) \to \mathcal{D}'(M)$ be continuous operators such that the kernels of $A$ and $B$ satisfy

\[
A(x, y) = \int_0^\infty e^{i\varphi(x,y)s}\alpha(x, y, s) ds, \quad \alpha(x, y, s) \in S^m_{cl}(M \times M \times \mathbb{R}^+),
\]

\[
B(x, y) = \int_0^\infty e^{-i\varphi(x,y)s}\beta(x, y, s) ds, \quad \beta(x, y, s) \in S^k_{cl}(M \times M \times \mathbb{R}^+)
\]

for some $m, k \in \mathbb{Z}$, where $\varphi(x, y) \in C^\infty(M \times M)$ is as in Theorem 7.1. Then, $A \circ B \equiv 0$, $B \circ A \equiv 0$.

To proceed, set

\[(8.1) \quad \Sigma^- = \{(x, \lambda \theta(x)) \in T^*M; \lambda < 0\}, \quad \Sigma^+ = \{(x, \lambda \theta(x)) \in T^*M; \lambda > 0\}.
\]

Let $\sigma_{\Box_b}(x, \xi)$ and $\sigma_{2iT}(x, \xi)$ be the principal symbols of $\Box_b$ and $2iT$, respectively. It is easy to see that $\sigma_{\Box_b}(x, \xi) = 0$ for all $(x, \xi) \in \Sigma^- \cup \Sigma^+$; that $\sigma_{2iT}(x, \xi) > 0$ for all $(x, \xi) \in \Sigma^-$; and that $\sigma_{2iT}(x, \xi) < 0$ for all $(x, \xi) \in \Sigma^+$. For $(x, \xi) \in T^*M$, we write $|\xi|$ to denote the point norm of the cotangent vector $\xi \in T^*_xM$. Take $\chi_0, \chi_1 \in C^\infty(T^*M, [0, 1])$ such that

1. $\chi_0 = 1$ in a small neighbourhood of $\Sigma^- \cap \{(x, \xi) \in T^*M; |\xi| \geq 1\}$,
2. $\chi_1 = 1$ in a small neighbourhood of $\Sigma^+ \cap \{(x, \xi) \in T^*M; |\xi| \geq 1\}$,
3. $\text{supp}\chi_0 \cap \text{supp}\chi_1 = \emptyset$,
4. $\sigma_{2iT}(x, \xi) > 0$ for all $(x, \xi) \in \text{supp}\chi_0$,
5. $\sigma_{2iT}(x, \xi) < 0$ for all $(x, \xi) \in \text{supp}\chi_1$, and
6. $\chi_0, \chi_1$ are positively homogeneous of degree zero in the sense that

\[\chi_0(x, \lambda \xi) = \chi_0(x, \xi), \quad \chi_1(x, \lambda \xi) = \chi_1(x, \xi) \quad \text{for all } \lambda \geq 1 \text{ and } |\xi| \geq 1.
\]

Define

\[(8.2) \quad q(x, \xi) = (1 - \chi_0(x, \xi) - \chi_1(x, \xi))\sqrt{\sigma_{\Box_b}(x, \xi)} + \chi_0(x, \xi)\sigma_{2iT}(x, \xi) - \chi_1(x, \xi)\sigma_{2iT}(x, \xi).
\]

Note that $\sigma_{\Box_b}(x, \xi) > 0$ for all $(x, \xi) \notin \Sigma^- \cup \Sigma^+$. From this observation, it is easy to see that $q(x, \xi) \geq c|\xi|$ for all $(x, \xi) \in T^*M$ with $|\xi| \geq 1$, where $c > 0$ is a constant. Let $\tilde{E}_1 \in L^1_c(M)$ with symbol $q(x, \xi) \in C^\infty(T^*M)$. Then $\tilde{E}_1$ is a classical elliptic pseudodifferential operator. It is known that (see [20]) $WF'(S) = \text{diag}(\Sigma^- \times \Sigma^-)$ and $WF'(\tilde{S}) = \text{diag}(\Sigma^+ \times \Sigma^+)$, where

\[
WF'(S) = \{(x, \xi, y, \eta) \in T^*M \times T^*M; (x, \xi, y, -\eta) \in WF(S)\}
\]

and $WF(S)$ denotes the wave front set of $S$ in the sense of Hörmander [19, Chapter 8]. Recall that $S$ denotes the Szegö projection. From this observation and (8.2), it is not difficult to see that

\[(8.3) \quad S\tilde{E}_1 \equiv S(2iT), \quad \tilde{E}_1S \equiv (2iT)S, \quad \tilde{S}\tilde{E}_1 \equiv \tilde{S}(-2iT), \quad \tilde{E}_1\tilde{S} \equiv (-2iT)\tilde{S}.
\]

Alternatively, (8.3) can be checked directly from the fact that $d_x\varphi|_{x=y} = -\theta(x)$.

Now, we can prove the following theorem.

**Theorem 8.4.** With the notations above, there is an $\tilde{E}_0 \in L^0_c(M)$ such that

\[S\Delta_bS \equiv S(\tilde{E}_1 + \tilde{E}_0)S \quad \text{and} \quad \tilde{S}\Delta_b\tilde{S} \equiv \tilde{S}(\tilde{E}_1 + \tilde{E}_0)\tilde{S}.
\]
Proof. From Lemma 8.2 and the observation that $\Box_b S = 0$, we have
\begin{equation}
S\Delta_b S = S(2T + L)S = S\overline{E}_1 S + SLS + F_0,
\end{equation}
where $F_0 \equiv 0$. We write $L = U + \nabla$ for $U, V \in C^\infty(M, T^1,0 M)$. Since $\overline{\partial}_b S = 0$, we have
\begin{equation}
S\overline{\nabla} S = 0.
\end{equation}

Now,
\begin{equation}
(SUS)^* = SU^* S = S(-\overline{\nabla} + r) S = S\tau S,
\end{equation}
where $(SUS)^*$ and $U^*$ are the adjoints of $SUS$ and $S$ respectively and $r \in C^\infty(M)$. Hence,
\begin{equation}
SUS = S\tau S.
\end{equation}

From (8.4), (8.5) and (8.6), we conclude that
\begin{equation}
S\Delta_b S = S(\overline{E}_1 + g_0) S + F_0,
\end{equation}
where $g_0 \in C^\infty(M)$, $F_0 \equiv 0$. Similarly,
\begin{equation}
\overline{S}\Delta_b \overline{S} = \overline{S}(\overline{E}_1 + g_1) \overline{S} + F_1,
\end{equation}
where $g_1 \in C^\infty(M)$ and $F_1 \equiv 0$. Put
\begin{equation}
\overline{E}_0 = \chi_0(x, \xi) g_0 + \chi_1(x, \xi) g_1,
\end{equation}
where $\chi_0, \chi_1$ are as in (8.2). As in the discussion before (8.3), we have
\begin{equation}
Sg_0 S \equiv S\overline{E}_0 S, \overline{S}g_1 \overline{S} \equiv \overline{S}\overline{E}_0 \overline{S}.
\end{equation}
The desired conclusion follows from (8.7), (8.8) and (8.9).\qed

Proof of Theorem 8.1. From Theorem 8.4 and (7.2), we have
\begin{equation}
\tau \Delta_b \tau = (S + \overline{S})\Delta_b (S + \overline{S}) + G_0
\end{equation}
\begin{equation}
= S\Delta_b S + \overline{S}\Delta_b \overline{S} + S\Delta_b \overline{S} + \overline{S}\Delta_b S + G_0
\end{equation}
\begin{equation}
= (S + \overline{S})(\overline{E}_1 + \overline{E}_0)(S + \overline{S}) - S(\overline{E}_1 + \overline{E}_0)\overline{S} - \overline{S}(\overline{E}_1 + \overline{E}_0)S
\end{equation}
\begin{equation}
+ S\Delta_b \overline{S} + \overline{S}\Delta_b S + G_0
\end{equation}
\begin{equation}
= \tau(\overline{E}_1 + \overline{E}_0) \tau - S(\overline{E}_1 + \overline{E}_0)\overline{S} - \overline{S}(\overline{E}_1 + \overline{E}_0)S + S\Delta_b \overline{S} + \overline{S}\Delta_b S + G_2,
\end{equation}
where $G_0, G_1, G_2$ are smoothing operators. In view of Lemma 8.3 and Theorem 7.2 we see that $S(\overline{E}_1 + \overline{E}_0)\overline{S}, \overline{S}(\overline{E}_1 + \overline{E}_0)S, S\Delta_b \overline{S}$ and $\overline{S}\Delta_b S$ are smoothing. From this and (8.10), we get
\begin{equation}
\tau \Delta_b \tau = \tau(\overline{E}_1 + \overline{E}_0) \tau + G,
\end{equation}
where $G$ is smoothing. Hence,
\begin{equation}
\tau \Delta_b \tau = \tau^2 \Delta_b \tau^2 = \tau^2(\overline{E}_1 + \overline{E}_0)^2 + \tau G \tau = \tau(\overline{E}_1 + \overline{E}_0 + G) \tau.
\end{equation}

Put $E_1 = \overline{E}_1 + \overline{E}_0 + G \in L^1_{cl}(M)$. From (8.11), we get $\tau \Delta_b \tau = \tau E_1 \tau$. The theorem follows.\qed
9. The Green’s function of square root of $\mathcal{P}'_4$

In this section, we will prove Theorem 1.3. First, we can repeat the proof of Theorem 8.1 with minor change and get the following result.

**Theorem 9.1.** We have

$$\mathcal{P}'_4 = \tau((2E_1)^2 + \hat{E}_1)$$ on $\mathcal{P}$,

where $E_1 \in L^1_{\text{cl}}(M)$ is as in Theorem 8.1 and $\hat{E}_1 \in L^1_{\text{cl}}(M)$.

In particular, $\mathcal{P}'_4$ is an elliptic pseudodifferential operator on $\mathcal{P}$. Standard arguments for elliptic operators imply that the spectrum $\text{Spec} \mathcal{P}'_4$ of $\mathcal{P}'_4$ is a discrete subset of $(-\infty, \infty)$ such that every $\lambda \in \text{Spec} \mathcal{P}'_4$ is an eigenvalue of $\mathcal{P}'_4$ and the eigenspace

$$\mathcal{E}_\lambda(\mathcal{P}'_4) := \{u \in \text{Dom} \mathcal{P}'_4: \mathcal{P}'_4 u = \lambda u\}$$

is a finite dimensional subspace of $\mathcal{P}$.

Let

$$\pi: \mathcal{P} \to \text{Ker} \mathcal{P}'_4$$

be the orthogonal projection. Let $\{g_1, g_2, \ldots, g_d\} \subset \mathcal{P}$ be an orthonormal frame for $\text{Ker} \mathcal{P}'_4$, where $d \in \mathbb{N}_0$. Then

$$\pi(x, y) = \sum_{j=1}^{d} g_j(x)g_j(y) \in C^\infty(M \times M).$$

From (9.1), we can extend $\pi$ to $\mathcal{D}'(M)$ as a smoothing operator on $M$.

Assume that $\mathcal{P}'_4$ is nonnegative. Then $\text{Spec} \mathcal{P}'_4 \subset [0, \infty)$ and $\mathcal{P}'_4$ has a well-defined square root

$$\mathcal{P}'_4^{1/2}: \text{Dom} \mathcal{P}'_4^{1/2} \subset \hat{\mathcal{P}} \to \hat{\mathcal{P}}.$$ 

Note that $\text{Dom} \mathcal{P}'_4^{1/2} = \text{Dom} \mathcal{P}'_4$. We write

$$\mathcal{P}'_4^{-1/2}: \hat{\mathcal{P}} \to \text{Dom} \mathcal{P}'_4^{1/2}$$

to denote the Green’s function of $(\mathcal{P}'_4)^{1/2}$. That is,

$$(\mathcal{P}'_4)^{1/2} \circ (\mathcal{P}'_4)^{-1/2} + \pi = I$$ on $\hat{\mathcal{P}}$,

$$(\mathcal{P}'_4)^{-1/2} \circ (\mathcal{P}'_4)^{1/2} + \pi = I$$ on $\text{Dom} \mathcal{P}'_4^{1/2}$.

For every $t > 0$, the operator

$$\mathcal{P}'_4 + t + \pi: \text{Dom} \mathcal{P}'_4 \to \hat{\mathcal{P}}$$

has a continuous inverse

$$(\mathcal{P}'_4 + t + \pi)^{-1}: \hat{\mathcal{P}} \to \hat{\mathcal{P}}$$

and the operator $(\mathcal{P}'_4 + t + \pi)^{-1}$ depends continuously on $t$. Let $\lambda_1 > 0$ be the first non-zero eigenvalue of $\mathcal{P}'_4$. Then

$$\left\| (\mathcal{P}'_4 + t + \pi)^{-1} u \right\| \leq \frac{1}{\lambda_1 + t} \| (I - \pi) u \| + \frac{1}{1 + t} \| \pi u \|$$

$$\leq \frac{1}{\min\{\lambda_1, 1\} + t} \| u \|$$

(9.3)
for all \( u \in \mathcal{P} \). \((\mathcal{P}_4')^{−1/2}\) can be understood as follows.

**Lemma 9.2.** On \( \mathcal{P} \cap (\text{Ker} \mathcal{P}_4')\), we have

\[
(\mathcal{P}_4')^{-\frac{1}{2}} = c \int_0^\infty t^{-\frac{1}{2}}(\mathcal{P}_4' + t + \pi)^{-1} dt,
\]

where \( c^{-1} = \int_0^\infty t^{-\frac{1}{2}}(1 + t)^{-1} dt \).

**Proof.** Fix a positive eigenvalue \( \lambda \in \text{Spec} \mathcal{P}_4' \). Let \( u \in \mathcal{E}_\lambda(\mathcal{P}_4') \). Then,

\[
(\mathcal{P}_4')^{-\frac{1}{2}} u = \frac{1}{\sqrt{\lambda}} u.
\]

We compute that

\[
c \left( \int_0^\infty t^{-\frac{1}{2}}(\mathcal{P}_4' + t + \pi)^{-1} dt \right) u = c u \int_0^\infty t^{-\frac{1}{2}} \frac{1}{\lambda + t} dt = \frac{1}{\sqrt{\lambda}} u.
\]

Hence the conclusion is true on \( \mathcal{E}_\lambda(\mathcal{P}_4') \) for all \( \lambda \in \text{Spec} \mathcal{P}_4' \).

Let \( u \in \mathcal{P} \cap (\text{Ker} \mathcal{P}_4') \). For each \( N \in \mathbb{N} \), let \( u_N \) be the orthogonal projection of \( u \) onto \( \bigoplus_{\lambda \leq N} \mathcal{E}_\lambda(\mathcal{P}_4') \). It follows that \( u_N \to u \) and that \( (\mathcal{P}_4')^{-\frac{1}{2}} u_N \to (\mathcal{P}_4')^{-\frac{1}{2}} u \).

From (9.3), we have

\[
c \left( \int_0^\infty t^{-\frac{1}{2}}(\mathcal{P}_4' + t + \pi)^{-1} dt \right) u_N \to c \left( \int_0^\infty t^{-\frac{1}{2}}(\mathcal{P}_4' + t + \pi)^{-1} dt \right) u
\]

in \( \mathcal{P} \) as \( N \to \infty \). Together these observations yield the result. \( \square \)

To proceed, we require some additional symbol spaces.

**Definition 9.3.** Let \( m \) be real number. The class \( S^m_{1,0,d}(T^* M, \mathbb{R}^+) \) consists of all functions \( a(x, \xi, t) \in C^\infty(T^* M \times \mathbb{R}^+) \) such that for arbitrary multi-indices \( \alpha, \beta \in \mathbb{N}_0^3 \), and for any compact set \( K \subset M \) there exists \( C_{\alpha, \beta, K} > 0 \) such that

\[
| \partial^\alpha_x \partial^\beta_\xi a(x, \xi, t) | \leq C_{\alpha, \beta, K}(1 + |\xi| + |t|^{\frac{1}{2}})^{|m| - |\beta|}
\]

for all \( (x, \xi) \in T^* K, t \in \mathbb{R}^+ \). Denote

\[
S^{-\infty}(T^* M, \mathbb{R}^+) = \bigcap_{m \in \mathbb{R}} S^m_{1,0,d}(T^* M, \mathbb{R}^+).
\]

Let \( a_j \in S^{m_j}_{1,0,d}(T^* M, \mathbb{R}^+) \) for \( j \in \{0, 1, 2, \ldots \} \) with \( m_j \to -\infty \) as \( j \to \infty \). Then there exists \( a \in S^m_{1,0,d}(T^* M, \mathbb{R}^+) \), unique modulo \( S^{-\infty}(T^* M, \mathbb{R}^+) \), such that

\[
a - \sum_{j=1}^{k-1} a_j \in S^m_{1,0,d}(T^* M, \mathbb{R}^+) \text{ for } k \in \{0, 1, 2, \ldots \}.
\]

If \( a \) and \( a_j \) have the properties above, we write

\[
a \sim \sum_{j=0} a_j \text{ in } S^{m_0}_{1,0,d}(T^* M, \mathbb{R}^+).
\]

Let \( S^m_{1,d,d}(T^* M, \mathbb{R}^+) \) be the space of all symbols \( a(x, \xi, t) \in S^m_{1,0,d}(T^* M, \mathbb{R}^+) \) with

\[
a(x, \xi, t) \sim \sum_{j=0} a_{m-j}(x, \xi, t)
\]

in \( S^m_{1,0,d}(T^* M, \mathbb{R}^+) \), where \( a_{m-j}(x, \xi, t) \) is positively homogeneous of degree \( m - j \) in \( (\xi, t^{\frac{1}{2}}) \); i.e.

\[
a_{m-j}(x, \lambda \xi, \lambda^d t) = \lambda^{m-j} a_{m-j}(x, \xi, t), \quad \text{for } t \in \mathbb{R}^+, \lambda \geq 1, |\xi| \geq 1.
\]
Let \( a(x, \xi, t) \in S^0_{cl,d}(T^*M, \mathbb{R}^+) \). We construct a pseudodifferential operator \( P_t \), depending smoothly on \( t \), by
\[
(P_t u)(x) = \frac{1}{(2\pi)^d} \int e^{i<x-y, \xi>} a(x, \xi, t) u(y) dy d\xi \quad \text{for all } u \in C^\infty(M).
\]
We call \( a(x, \xi, t) \) the symbol of \( P_t \) and \( a_m(x, \xi, t) \) the principal symbol of \( P_t \). In this case, we will write \( P_t \in L^m_{cl,d}(M, \mathbb{R}^+) \).

Let \( P_t \in L^2_{cl,d}(M, \mathbb{R}^+) \). Then \( P_t : H^s(M) \rightarrow H^{s+2}(M) \) is continuous for all \( s \in \mathbb{Z} \) and all \( t \in \mathbb{R}^+ \). Let \( f(t) \) be a strictly positive continuous function. We write
\[
P_t = O(f(t)) : H^{s_1}(M) \rightarrow H^{s_2}(M), \quad s_1, s_2 \in \mathbb{Z},
\]
if \( \|P_t u\|_{s_2} \leq C f(t) \|u\|_{s_1} \) for all \( u \in H^{s_1}(M) \) and all \( t \in \mathbb{R}^+ \), where \( \|\cdot\|_s \) denotes the standard Sobolev norm of order \( s \) and \( C > 0 \) is a constant independent of \( t \).

We return to our situation. Put
\[
E_2 = (2E_1)^2 + \hat{E}_1,
\]
where \( E_1, \hat{E}_1 \in L^2_{cl,d}(M) \) are as in Theorem \( \text{[11]} \). Let \( e_2(x, \xi) \in S^0_{cl}(T^*M) \) be the principal symbol of \( E_2 \). The following is well-known \( \text{[30, Chapter 2]} \).

**Theorem 9.4.** There exists \( G_t \in L^2_{cl,d}(M, \mathbb{R}^+) \) depending continuously on \( t \) in \( L^2(M) \) such that
\[
\begin{align*}
G_t &= O\left(\frac{1}{1+t}\right) : H^s(M) \rightarrow H^s(M) \quad \text{for all } s \in \mathbb{Z}, \\
(9.6) \quad G_t &= O\left(\frac{1}{\sqrt{1+t}}\right) : H^s(M) \rightarrow H^{s+1}(M) \quad \text{for all } s \in \mathbb{Z}, \\
(9.7) \quad G_t &= O(1) : H^s(M) \rightarrow H^{s+2}(M) \quad \text{for all } s \in \mathbb{Z}, \\
(9.8) \quad g_0(x, \xi, t) &= \frac{1}{e_2(x, \xi) + t} \quad \text{for all } |\xi| \geq 1, \\
(9.9) \quad (E_2 + t) G_t &= I + F_t \quad \text{for all } t > 0,
\end{align*}
\]
where \( g_0(x, \xi, t) \) denotes the principal symbol of \( G_t \) and \( F_t \) is a smoothing operator on \( M \) depending smoothly on \( t \) with the property that for all \( m \in \mathbb{N}_0 \), there is a constant \( C_m > 0 \) such that for all \( t \in \mathbb{R}^+ \),
\[
|F_t(x,y)|_{C^m(M \times M)} \leq C_m \frac{1}{1+t}.
\]
Moreover, in local coordinates \( x \), let \( g(x, \xi, t) \) denote the full symbol of \( G_t \). Then, for every \( \alpha, \beta \in \mathbb{N}_0^3 \), there is a constant \( C_{\alpha,\beta} > 0 \), independent of \( t \), such that
\[
\left| \partial_\xi^\alpha \partial_x^\beta g(x, \xi, t) \right| \leq C_{\alpha, \beta} \frac{1}{\sqrt{1+t}} (1 + |\xi|)^{-1-|\beta|} \quad \text{for all } |\xi| \geq 1,
\]
\[
\left| \partial_x^\alpha \partial_\xi^\beta g(x, \xi, t) \right| \leq C_{\alpha, \beta} \frac{1}{1+t} (1 + |\xi|)^{-|\beta|} \quad \text{for all } |\xi| \geq 1.
\]

We introduce some notations. Let \( \vartheta(x,y) \) denote the Carnot–Carathéodory distance on \((M^3, T^1, M, \theta)\). Let \((z, t)\) be CR normal coordinates defined in a neighborhood of \( p \in M \) such that \((z(p), t(p)) = (0, 0)\). Define \( \rho^4(z, t) = |z|^4 + t^2 \). It is easy to see (cf. \text{[23, Section 3]} that for points \( x \) sufficiently close to \( p \), we have
\[
\vartheta(x, p) \approx \rho(x).
\]
Denote by $B(x,r)$ the non-isotropic ball $\{y \in M : \vartheta(x,y) < r\}$ of radius $r$ centered at $x$. Let $k \in \mathbb{N}$. We denote by $\nabla^k_b$ any differential operator of the form $L_1 \ldots L_k$, where $L_j \in C^\infty(M, T^{1,0}M \oplus T^{0,1}M)$ satisfy $\langle L_j | L_j \rangle \leq 1$ for $j = 1, \ldots, k$.

Next, we define a class of (non-isotropic) smoothing operators of order $j$. For our purposes, it suffices to restrict to the case when $0 \leq j < 4$.

Recall that a smooth function $\phi$ on $M$ is said to be a normalized bump function on $B(x,r)$ if $\operatorname{supp} \phi \subset B(x,r)$ and

\begin{equation}
\|\nabla_b^k \phi\|_{L^\infty(B(x,r))} \leq C_k r^{-k}
\end{equation}

for all $k \geq 0$; here $C_k > 0$ are absolute constants independent of $r$. If (9.15) only holds for $0 \leq k \leq N$ for some large integer $N$, we say that $\phi$ is a normalized bump function of order $N$ in $B(x,r)$.

Suppose that $A$ is a continuous linear operator $A: C^\infty(M) \to C^\infty(M)$ and its adjoint $A^*$ is also a continuous map $A^*: C^\infty(M) \to C^\infty(M)$. We say that $A$ is a smoothing operator of order $j$, $0 \leq j < 4$, if

1. there exists a kernel $A(x,y)$, defined and smooth away from the diagonal in $M \times M$, such that

\begin{equation}
Af(x) = \int_M A(x,y)f(y) dv_M(y)
\end{equation}

for any $f \in C^\infty(M)$, and every $x \notin \operatorname{supp} f$, where $dv_M = \vartheta \wedge d\vartheta$;

2. for all $x \neq y$, the kernel $A(x,y)$ satisfies

$|\langle \nabla_b \rangle_x^{\alpha_1} \langle \nabla_b \rangle_y^{\alpha_2} A(x,y) \rangle | \lesssim_{\alpha} \vartheta(x,y)^{-4+j-|\alpha|}$ for all $|\alpha| = |\alpha_1| + |\alpha_2|$;

3. the operators $A$ and $A^*$ satisfy the following cancellation conditions of order $j$: if $\phi$ is a normalized bump function in $B(x,r)$, then

\begin{align*}
\|\nabla_b^k A \phi\|_{L^\infty(B(x,r))} & \lesssim_{\alpha} r^{j-|\alpha|}, \\
\|\nabla_b^k A^* \phi\|_{L^\infty(B(x,r))} & \lesssim_{\alpha} r^{j-|\alpha|}.
\end{align*}

Since $M$ is embeddable, $\square_b$ has $L^2$ closed range. Let

\begin{equation}
N : L^2(M) \to \operatorname{Dom} \square_b
\end{equation}

be the partial inverse of $\square_b$ and let $N(x,y)$ be the distributional kernel of $N$. The following is well-known (see [23, Theorem 2.2]).

**Theorem 9.5.** The Szegő projection $S$ and the partial inverse $N$ of $\square_b$ are smoothing operators of orders 0 and 2, respectively.

We also need to study one-parameter families of smooth operators.

**Definition 9.6.** Let $A_t$ be a $t$-dependent smoothing operator of order $j$, $0 \leq j < 4$, where $t \in \mathbb{R}_+$. Let $f(t)$ be a positive continuous function of $t \in \mathbb{R}_+$. We say that $A_t$ is a smoothing operator of order $j$ with size $f(t)$ if for every $m \in \mathbb{N}_0$ and any normalized bump function $\phi$ in $B(x,r)$, there are constants $C_m, C_{m,r} > 0$, independent of $t$, such that for all $t \in \mathbb{R}_+$,

\begin{align*}
|\langle \nabla_b \rangle_x^{\alpha_1} \langle \nabla_b \rangle_y^{\alpha_2} A_t(x,y) | & \leq C_m f(t) \vartheta(x,y)^{-4+j-|\alpha|} \quad \text{for all} \quad |\alpha| = |\alpha_1| + |\alpha_2| \leq m, \\
\|\nabla_b^k A_t \phi\|_{L^\infty(B(x,r))} & \leq f(t) C_{m,r} r^{j-|\alpha|} \quad \text{for all} \quad |\alpha| \leq m, \\
\|\nabla_b^k A_t^* \phi\|_{L^\infty(B(x,r))} & \leq f(t) C_{m,r} r^{j-|\alpha|} \quad \text{for all} \quad |\alpha| \leq m.
\end{align*}

We also need the following result [23, Theorem 2.2 and Theorem 2.3].
Theorem 9.7. Let $A_t$ and $B_t$ be $t$-dependent smoothing operators of orders $j_1$ and $j_2$ with sizes $f(t)$ and $g(t)$, respectively, where $j_1, j_2 \geq 0$, $j_1 + j_2 < 4$, and $f(t), g(t)$ are positive continuous functions. Then $A_t \circ B_t$ is a smoothing operator of order $j_1 + j_2$ with size $f(t)g(t)$.

Let $P_t \in L^{-2}_{cl,2}(M, \mathbb{R}_+)$, $Q_t \in L^{-1}_{cl,2}(M, \mathbb{R}_+)$, and $R_t \in L^{0}_{cl,2}(M, \mathbb{R}_+)$. Let $p(x, \xi, t)$, $q(x, \xi, t)$ and $r(x, \xi, t)$ be symbols of $P_t$, $Q_t$ and $R_t$ respectively. It is easy to see that for every $\alpha, \beta \in \mathbb{N}_0^3$, there is a constant $C_{\alpha, \beta} > 0$ independent of $t$ such that

\begin{align}
\left| \partial_\xi^\alpha \partial_\xi^\beta p(x, \xi, t) \right| & \leq C_{\alpha, \beta} \frac{1}{1 + t} (1 + |\xi|)^{-|\beta|} \quad \text{for all } |\xi| \geq 1, \\
\left| \partial_\xi^\alpha \partial_\xi^\beta q(x, \xi, t) \right| & \leq C_{\alpha, \beta} \frac{1}{\sqrt{1 + t}} (1 + |\xi|)^{-|\beta|} \quad \text{for all } |\xi| \geq 1, \\
\left| \partial_\xi^\alpha \partial_\xi^\beta r(x, \xi, t) \right| & \leq C_{\alpha, \beta} (1 + |\xi|)^{-|\beta|} \quad \text{for all } |\xi| \geq 1.
\end{align}

(9.18)

Using the following lemma, we establish an analogue of Theorem 9.5.

Lemma 9.8. Consider $B(x, r)$, where $x \in M$ and $r > 0$ is a small constant. Let $\chi_r \in C_0^\infty((B(x, 2r))$ be a normalized bump function on $B(x, 2r)$ with $\chi_r \equiv 1$ on $B(x, r)$. There is a constant $C > 0$ independent of $r$ such that

\begin{equation}
\|f\|_{L^\infty(B(x, r))} \leq Cr \sum_{j=0}^{3} \left\| \nabla_b^j(\chi_r f) \right\| \quad \text{for all } f \in C^\infty(M).
\end{equation}

(9.19)

Proof. Consider $\Delta_b + I : \text{Dom}(\Delta_b + I) \subset L^2(M) \rightarrow L^2(M)$, where $\text{Dom}(\Delta_b + I) = \{ u \in L^2(M) : (\Delta_b + I)u \in L^2(M) \}$. It is clear that $\Delta_b + I$ is injective, self-adjoint, has $L^2$ closed range and hence is surjective. Let $H : L^2(M) \rightarrow \text{Dom}(\Delta_b + I)$ be the inverse of $\Delta_b + I$. Put $B := H^2 : L^2(M) \rightarrow L^2(M)$. We have

\begin{equation}
B(\Delta_b + I)^2 = I \quad \text{on } C^\infty(M).
\end{equation}

(9.20)

It is known that (see [23, Appendix A])

\begin{itemize}
\item $B$ is a smoothing operator of order $4 - \varepsilon$ for every $\varepsilon > 0$,
\item $B\nabla_b$ is a smoothing operator of order $3$.
\end{itemize}

Let $f \in C^\infty(M)$. From (9.20), we have

\begin{equation}
\chi_r f = B(\Delta_b + I)^2 \chi_r f = \sum_{j=0}^{4} B\nabla_b^j \chi_r f.
\end{equation}

(9.21)

Fix $x_0 \in B(x, r)$. From (9.22), we have

\begin{equation}
f(x_0) = (\chi_r f)(x_0) = (B\nabla_b^4 \chi_r f)(x_0) + \sum_{j=0}^{3} (B\nabla_b^j \chi_r f)(x_0)
= \int (B\nabla_b)(x_0, y)\nabla_b^4 \chi_r f(y)dv_M(y) + \sum_{j=0}^{3} \int B(x_0, y)\nabla_b^j \chi_r f(y)dv_M(y),
\end{equation}

(9.22)
where \((B\nabla_b)(x, y)\) and \(B(x, y)\) denote the distribution kernels of \(B\nabla_b\) and \(B\) respectively. We then check that

\[
|f(x_0)| \leq \left( \int_{B(x_0, 2r)} |(B\nabla_b)(x_0, y)|^2 \, dv_M(y) \right)^{\frac{1}{2}} \|\nabla_b^j (\chi_r f)\| \\
+ \sum_{j=0}^{3} \left( \int_{B(x_0, 2r)} |B(x_0, y)|^2 \, dv_M(y) \right)^{\frac{1}{2}} \|\nabla_b^j (\chi_r f)\|.
\]

From (9.21), we can check that

\[
\int_{B(x_0, 2r)} |(B\nabla_b)(x_0, y)|^2 \, dv_M(y) \leq C_0 \int_{B(x_0, 2r)} \vartheta(x_0, y)^{-2} \, dy \leq C_1 r^2,
\]
\[
\int_{B(x_0, 2r)} |B(x_0, y)|^2 \, dv_M(y) \leq C_2 \int_{B(x_0, 2r)} \vartheta(x_0, y)^{-2} \, dy \leq C_3 r^2,
\]
where \(C_0, C_1, C_2, C_3\) are positive constants independent of \(r\) and the point \(x_0\). From (9.21) and (9.23), (9.19) follows. \(\square\)

**Theorem 9.9.** The operators \(SP_t, SQ_t\) and \(SR_t\) are smoothing operators of orders 0 with sizes \(1, 1, 1\) and 1, respectively. Similarly, the operators \(P_t S, Q_t S\) and \(R_t S\) are smoothing operators of orders 0 with sizes \(1, 1, 1\) and 1, respectively.

**Proof.** Let \(\phi\) be a normalized bump function in the ball \(B(x, r)\). From (9.19), we have

\[
\|SQ_t \phi\|_{L^\infty(B(x, r))} \leq C r \sum_{j=0}^{3} \left\|\nabla_b^j (\chi_r SQ_t \phi)\right\|,
\]
where \(\chi_r\) is as in Lemma 9.8 and \(C > 0\) is a constant independent of \(r, \phi, x\) and \(t\). We claim that

\[
\left\|\nabla_b^j SQ_t \phi\right\| \leq c_j \frac{1}{\sqrt{1 + t}} r^{2-j} \text{for } j = 0, 1, 2, 3, 4,
\]
where \(c_j > 0\) is a constant independent of \(r, x\) and \(t\). Fix \(j \in \{0, 1, 2, 3, 4\}\). It is known that (see [20, 31])

\[
\nabla_b^{2j} S: H^s(M) \rightarrow H^{s-j}(M) \quad \text{for all } s \in \mathbb{Z}.
\]

Moreover, from (9.18), we can check that

\[
Q_t = O\left(\frac{1}{\sqrt{1 + t}}\right): H^s(M) \rightarrow H^s(M) \quad \text{for all } s \in \mathbb{Z}.
\]

From (9.27) and (9.28), we deduce that

\[
\nabla_b^{2j} SQ_t = O\left(\frac{1}{\sqrt{1 + t}}\right): H^s(M) \rightarrow H^{s-j}(M) \quad \text{for all } s \in \mathbb{Z}.
\]
From (9.29), we have
\[
\left\| \nabla_b^j SQ_t \phi \right\|^2 = \left( \nabla_b^j SQ_t \phi \mid \nabla_b^j SQ_t \phi \right) = \left( \nabla_b^j SQ_t \phi \mid SQ_t \phi \right)
\leq \left\| \nabla_b^j SQ_t \phi \right\| \left\| SQ_t \phi \right\|
\leq \frac{1}{1+t} \left\| \phi \right\|_j \left\| \phi \right\|
\leq \frac{1}{1+t} \left\| \nabla_b^j \phi \right\| \left\| \phi \right\|
\leq \frac{1}{1+t} r^{4-2j},
\]
(9.30)
where \( \left\| \phi \right\|_j \) denotes the standard Sobolev norm of \( \phi \) of order \( j \). From (9.30), the claim (9.26) follows.

From (9.25) and (9.26) we can check that
\[
\left\| SQ_t \phi \right\|_{L^\infty(B(x,r))} \leq r \sum_{j=0}^3 \left\| \nabla_b^j (x_r SQ_t \phi) \right\|
\leq r \left\| x_r SQ_t \phi \right\| + r \sum_{j=1}^3 r^{-j+s} \left\| \nabla_b^j (SQ_t \phi) \right\|
\leq \frac{r}{\sqrt{1+t}} + r \sum_{j=1}^3 r^{-j+s} \frac{1}{\sqrt{1+t}} r^{2-s}
\leq \frac{1}{\sqrt{1+t}}
\]
We can repeat the method above with minor changes and get that for every \( j \in \mathbb{N}_0 \),
\[
\left\| \nabla_b^j SQ_t \phi \right\| \lesssim_j \frac{1}{\sqrt{1+t}}. \quad \text{Thus, } SQ_t \text{ satisfies the cancellation condition of order 0 with size } \frac{1}{\sqrt{1+t}}.
\]
Similarly, we can repeat the procedure above with minor changes to obtain that \( (SQ_t)^* \) satisfies the cancellation condition of order 0 with size \( \frac{1}{\sqrt{1+t}} \).

Now, we estimate the kernel \( SQ_t(x,y) \). Let \( x = (x_1, x_2, x_3) \) be local coordinates for \( M \) defined in an open set \( D \subset M \). From Theorem 7.2 and the complex stationary phase formula of Melin–Sjöstrand [28], it follows that
\[
(SQ_t)(x,y) = \int_0^\infty e^{i\varphi(x,y) t} b(x,y,s,t) ds + F_t(x,y) \quad \text{on } D \times D,
\]
where \( b(x,y,s,t) \in C^\infty(D \times D \times \mathbb{R}_+ \times \mathbb{R}_+) \), and for every \( \alpha, \beta \in \mathbb{N}_0^3 \), \( \gamma \in \mathbb{N}_0 \), there is a constant \( C_{\alpha,\beta,\gamma} > 0 \), independent of \( t \), such that on \( D \times D \),
\[
\left| \frac{\partial^\alpha \partial^\beta \partial^\gamma}{\partial^k x \partial^l y} b(x,y,s,t) \right| \leq C_{\alpha,\beta,\gamma} (\sqrt{s^2 + t})^{-\gamma}, \quad \text{if } \gamma \geq 1
\]
\[
\left| \frac{\partial^\alpha \partial^\beta \partial^\gamma}{\partial^k x \partial^l y} b(x,y,s,t) \right| \leq C_{\alpha,\beta,\gamma} (\sqrt{s^2 + t}), \quad \text{if } \gamma = 0,
\]
and \( F_t \) is a smoothing operator on \( D \) depending smoothly on \( t \) with the property that for all \( m \in \mathbb{N}_0 \), there is a constant \( C_m > 0 \) such that for all \( t \in \mathbb{R}_+ \),
\[
|F_t(x,y)|_{C^m(M \times M)} \leq C_m \frac{1}{\sqrt{1+t}}.
\]
From (9.32), the formula
\[ \int_0^\infty e^{i\varphi(x,y)s}b(x,y,s,t)ds = \int_0^\infty \frac{1}{(i\varphi(x,y))^2} \frac{\partial^2}{\partial s^2} (e^{i\varphi(x,y)s})b(x,y,s,t)ds, \]
and distribution theory, one can check that
\[ (9.33) \int_0^\infty e^{i\varphi(x,y)s}b(x,y,s,t)ds = \int_0^\infty \frac{1}{(i\varphi(x,y))^2} \frac{\partial^2}{\partial s^2} b(x,y,s,t)ds + \frac{1}{(i\varphi(x,y))^2} H_t(x,y), \]
where \( H_t \) is a smoothing operator on \( D \) depending smoothly on \( t \) with the property that for all \( m \in \mathbb{N}_0 \), there is a constant \( \tilde{C}_m > 0 \) such that for all \( t \in \mathbb{R}_+ \),
\[ |H_t(x,y)|_{C^m(M \times M)} \leq \tilde{C}_m \frac{1}{\sqrt{1+t}}. \]
Again, from (9.32), we have
\[ (9.34) \left| \int_0^\infty \frac{1}{(i\varphi(x,y))^2} e^{i\varphi(x,y)s} \frac{\partial^2}{\partial s^2} b(x,y,s,t)ds \right| \leq \hat{C}_1 \frac{1}{|\varphi(x,y)|^2} \int_0^\infty \frac{1}{s^2 + t} ds \leq \hat{C}_1 \frac{1}{\sqrt{1+t}} \frac{1}{|\varphi(x,y)|^2}, \]
for all \( t \geq 1 \), where \( \hat{C} > 0 \), \( \hat{C}_1 > 0 \) are constants independent of \( t \). It is known that (see [20, Theorem 1.4]) \(|\varphi(x,y)| \approx \vartheta(x,y)^2\). From this observation, (9.33) and (9.34), we conclude that
\[ |(SQ_t)(x,y)| \leq C \frac{1}{\sqrt{1+t}} \vartheta(x,y)^{-4} \]
for all \( x, y \in M \) with \( x \neq y \), where \( C > 0 \) is a constant independent of \( t \).
For every \( m \in \mathbb{N} \), we can repeat the procedure above with minor change and deduce that there is a constant \( C_m > 0 \) independent of \( t \) such that
\[ |(\nabla_b)^{\alpha_1}_{x_1}(\nabla_b)^{\alpha_2}_{y_2}(SQ_t)(x,y)| \leq C_m \frac{1}{\sqrt{1+t}} \vartheta(x,y)^{-4-|\alpha|} \]
for all \(|\alpha| = |\alpha_1| + |\alpha_2| \leq m\). Thus, \( SQ_t \) is a smoothing operator of order 0 with size \( \frac{1}{\sqrt{1+t}} \).
Arguing similarly yields that \( SP_t \) and \( SR_t \) are smoothing operators of orders 0 with sizes \( \frac{1}{\sqrt{1+t}} \) and 1, respectively. \( \square \)

We need two results about the smoothing properties of the operators \( G_t \) from Theorem 9.4.

**Lemma 9.10.** Let \( G_t \in L_{cl}^{-2}(M,\mathbb{R}_+) \) be as in Theorem 9.4. Then, \( \tau G_t \tau \) is a smoothing operator of order 2 with size \( \frac{1}{\sqrt{1+t}} \). Moreover, \( \tau G_t \tau \) is also a smoothing operator of order 0 with size \( \frac{1}{\sqrt{1+t}} \).

**Proof.** From Theorem 7.1 Lemma 8.3 and (9.7), it is straightforward to see that
\[ \tau G_t \tau = SG_t S + \overline{S} G_t \overline{S} + F_t \]
\[ = SN\overline{\square}_b G_t S + \overline{S} N \square_b G_t \overline{S} + H_t, \]
(9.35)
where \( F_t \) and \( H_t \) are smoothing operators on \( M \) depending smoothly on \( t \) with the property that for all \( m \in \mathbb{N}_0 \), there is a constant \( C_m > 0 \) such that for all \( t \in \mathbb{R}_+ \),
\[
|F_t(x, y)|_{C^m(M \times M)} \leq C_m \frac{1}{1 + t},
\]
\[
|H_t(x, y)|_{C^m(M \times M)} \leq C_m \frac{1}{1 + t}.
\]

Note that \( \square_b G_t, \square_b G_t \in \mathcal{L}_{1, 2}^0(M, \mathbb{R}_+) \). From this observation, Theorem 9.7, Theorem 9.9 and (9.35), we conclude that \( \tau G_t \tau \) is a smoothing operator of order 0 with size \( \frac{1}{1 + t} \).

From Lemma 8.2, we have
\[
\bigtriangledown \square_b G_t S = \bigtriangledown \square_b G_t S + \bigtriangledown E G_t S
\]
\[
= \bigtriangledown \square_b G_t S + \bigtriangledown E G_t S,
\]
where \( E \) is a first order partial differential operator. Note that \( [\square_b, G_t], E G_t \in \mathcal{L}_{1, 2}^0(M, \mathbb{R}_+) \). From this observation, Theorem 9.7 and Theorem 9.9 we conclude that \( \bigtriangledown \square_b, G_t S + \bigtriangledown E G_t S \) is a smoothing operator of order 2 with size \( \frac{1}{1 + t} \).

Similarly, \( \bigtriangledown \square_b G_t S \) is a smoothing operator of order 2 with size \( \frac{1}{1 + t} \). From \( (9.35) \), we conclude that \( \tau G_t \tau \) is a smoothing operator of order 2 with size \( \frac{1}{1 + t} \).

The lemma follows.

**Lemma 9.11.** Let \( E_2 \in \mathcal{L}_2^2(M) \) be as in (9.6). Then \( \tau E_2(I - \tau)G_t \tau \) is a smoothing operator of order 1 with size 1.

**Proof.** From Theorem 7.1, Lemma 8.3 and (9.7), we check that
\[
\tau E_2(I - \tau)G_t \tau = \bigtriangledown E_2(I - S)G_t S + \bigtriangledown E_2(I - S)G_t S + F_t
\]
\[
= \bigtriangledown E_2 \square_b NG_t S + \bigtriangledown E_2 \square_b (\square_b G_t S + F_t),
\]
where \( F_t \) is a smoothing operators on \( M \) depending smoothly on \( t \) with the property that for all \( m \in \mathbb{N}_0 \), there is a constant \( C_m > 0 \) such that for all \( t \in \mathbb{R}_+ \),
\[
|F_t(x, y)|_{C^m(M \times M)} \leq C_m \frac{1}{1 + t}.
\]
Again, from Theorem 7.1, Lemma 8.3 and (9.7), we check that
\[
\bigtriangledown E_2 \square_b NG_t S = \bigtriangledown [E_2, \overline{\partial}_b] \overline{\partial}_b N \bigtriangledown b G_t S + H_t,
\]
where \( H_t \) is a smoothing operator on \( M \) depending smoothly on \( t \) with the property that for all \( m \in \mathbb{N}_0 \), there is a constant \( C_m > 0 \) such that for all \( t \in \mathbb{R}_+ \),
\[
|H_t(x, y)|_{C^m(M \times M)} \leq C_m \frac{1}{1 + t}.
\]
From Lemma 8.2, we have
\[
\bigtriangledown [E_2, \overline{\partial}_b] \overline{\partial}_b N \bigtriangledown b G_t S
\]
\[
= \bigtriangledown [E_2, \overline{\partial}_b] \overline{\partial}_b N \bigtriangledown b G_t S + \bigtriangledown [E_2, \overline{\partial}_b] \overline{\partial}_b N \bigtriangledown b Z_0 S
\]
\[
= \bigtriangledown [E_2, \overline{\partial}_b] \overline{\partial}_b N \bigtriangledown b [\square_b, G_t] S + \bigtriangledown [E_2, \overline{\partial}_b] \overline{\partial}_b N \bigtriangledown b Z_1 [\square_b, G_t] S
\]
\[
+ \bigtriangledown [E_2, \overline{\partial}_b] \overline{\partial}_b N \bigtriangledown b [\square_b, Z_0] G_t S + \bigtriangledown [E_2, \overline{\partial}_b] \overline{\partial}_b N \bigtriangledown b Z_2 Z_0 G_t S,
\]
where $Z_0, Z_1, Z_2$ are first order partial differential operators. Note that
\[
[\Box_b, [\Box_b, G_t]], Z_1[\Box_b, G_t], [\Box_b, Z_0]G_t, Z_2Z_0G_t \in L^0_{cl, 2}(M, \mathbb{R}_+).
\]
From this observation and Theorem 9.11, we deduce that $[\Box_b, [\Box_b, G_t]]S, Z_1[\Box_b, G_t]S,$
$[\Box_b, Z_0]G_tS$ and $Z_2Z_0G_tS$ are smoothing operators of order 0 with sizes 1. Moreover, from the symbolic calculus of Stein–Yung [31], we check that $S[E_2, \Box_b]N\Box_bG_tS$
is a smoothing operator of order 1.

From the discussion above and (9.40), we conclude that $S[E_2, \Box_b]N\Box_bG_tS$ is a smoothing operator of order 1 with size 1. Similarly, we can repeat the procedure above and conclude that $S E_2(I - \Box_b)G_tS$ is a smoothing operator of order 1 with size 1. The lemma now follows from (9.35).

Our first goal is to invert $\mathcal{F}_4 + t + \pi$. We begin by constructing a parametrix.

**Proposition 9.12.** For every $N > 0$, there are continuous operators
\[
A_{N, t} = O\left(\frac{1}{1 + t}\right); \quad H^s(M) \to H^{s + \frac{1}{2}}(M) \quad \text{for all } s \in \mathbb{Z},
\]
\[
R_{N, t} = O\left(\frac{1}{1 + t}\right); \quad H^s(M) \to H^{s + N}(M) \quad \text{for all } s \in \mathbb{Z}
\]
depending continuously on $t$ such that
\begin{enumerate}
  
  \item $A_{K, t}$ is a smoothing operator of order 3 with size $\frac{1}{\sqrt{1 + t}}$;
  
  \item $A_{K, t}$ is a smoothing operator of order 1 with size $\frac{1}{1 + t}$;
  
  \item $(\mathcal{F}_4 + t + \pi)(\tau G_t + \tau A_{K, t} \tau) = \tau + \tau R_{K, t} \tau$ on $\mathcal{P}$.
\end{enumerate}

**Proof.** From Theorem 9.11 and Theorem 9.44 we have
\[
(\mathcal{F}_4 + t + \pi)(\tau G_t + \tau A_{K, t} \tau) = \tau + \tau E_2(I - \tau)G_t \tau + \tau \tau G_t \tau
\]
\[
= \tau E_2(I - \tau)G_t \tau - \tau E_2(I - \tau)G_t + \tau \tau G_t \tau
\]
\[
= I + \tau A_t \tau \quad \text{on } \mathcal{P},
\]
where
\[
A_t = -\tau E_2(I - \tau)G_t \tau + \tau \mathcal{F}_t \tau.
\]
Here $\mathcal{F}_t$ is a smoothing operator on $M$ depending smoothly on $t$ with the property that for all $m \in \mathbb{N}_0$, there is a constant $C_m > 0$ such that for all $t \in \mathbb{R}_+$,
\[
|\mathcal{F}_t(x, y)|_{C^m(M \times M)} \leq C_m \frac{1}{1 + t}.
\]
By Lemma 9.11 we have that $A_t$ is a smoothing operator of order 1 with size 1. We claim that
\[
A_t = O(1); \quad H^s(M) \to H^{s + \frac{1}{2}}(M) \quad \text{for all } s \in \mathbb{Z}.
\]
From (9.39) and (9.40) we see that
\[
S E_2[\Box_b]N G_t S
\]
\[
= S[\Box_b]N \Box_b Z_0 [\Box_b, Z_0]G_t S + S[\Box_b]N \Box_b Z_2 G_t S + H_t,
\]
where $Z_0, Z_1, Z_2$ are first order partial differential operators and $H_t$ is a smoothing operator on $M$ depending smoothly on $t$ with the property that for all $m \in \mathbb{N}_0$, there is a constant $C_m > 0$ such that for all $t \in \mathbb{R}_+$,

$$|H_t(x, y)|_{C^m(M \times M)} \leq C_m \frac{1}{1 + t}.$$ 

It is known that (see [20, 21])

$$N, \overline{N}: H^s(M) \to H^{s+1}(M) \quad \text{for all } s \in \mathbb{Z},$$

$$\overline{\partial}_b N: H^s(M) \to H^{s+\frac{1}{2}}(M, T^{*0.1}M) \quad \text{for all } s \in \mathbb{Z}.$$ 

From this observation, (9.39) and (9.45), we deduce that

$$\overline{S}E_2 \overline{\partial}_b \overline{N} G_t S = O(1): H^s(M) \to H^{s+\frac{1}{2}}(M) \quad \text{for all } s \in \mathbb{Z}.$$ 

Similarly, we have

$$\overline{S}E_2 \overline{\partial}_b \overline{N} G_t S = O(1): H^s(M) \to H^{s+\frac{1}{2}}(M) \quad \text{for all } s \in \mathbb{Z}.$$ 

Inserting this into (9.38) yields the claim (9.44).

Now put

$$A_{K,t} = \tau G_t \tau (1 - (\tau A_t \tau)^3 - (\tau A_t \tau)^3 + \cdots + (\tau A_t \tau)^{2K+4} - \tau G_t \tau).$$

From Theorem 9.11 and Lemma 9.10 we observe that $A_{K,t}$ is a smoothing operator of order 3 with size $\frac{1}{1+\tau}$, and also $A_{K,t}$ is a smooth operator of order 1 with size $\frac{1}{1+\tau}$. Moreover, from (9.7) and (9.44) we conclude that

$$A_{K,t} = O\left(\frac{1}{1+t}\right): H^s(M) \to H^{s+\frac{1}{2}}(M)$$

for all $s \in \mathbb{Z}$. Furthermore, from (9.41), we observe that

$$(\pi G_t + t + \pi)TA_{K,t}\tau = \tau + (\tau A_t \tau)^{2K+5}.$$ 

From (9.44), we see that

$$(\tau A_t \tau)^{2K+4} = O(1): H^s(M) \to H^{s+K+2}(M)$$

for all $s \in \mathbb{Z}$. Moreover, from (9.7) and (9.42), we observe that

$$\tau A_t \tau = O\left(\frac{1}{1+t}\right): H^s(M) \to H^{s-2}(M)$$

for all $s \in \mathbb{Z}$. Thus, $(\tau A_t \tau)^{2K+5} = O\left(\frac{1}{1+t}\right): H^s(M) \to H^{s+K}(M)$ for all $s \in \mathbb{Z}$. Combining this with (9.40) yields the result. \qed

Remark 9.13. It is easy to see that $A_{K,t}$ depends continuously on $t$ in $L^2(M)$.

From now on, we identify the operator $(\overline{\mathcal{P}}_4 + t + \pi)^{-1}: \mathcal{P} \to \mathcal{P}$ with $\tau (\overline{P}_4 + t + \pi)^{-1}$. Thus $(\overline{P}_4 + t + \pi)^{-1}: L^2(M) \to L^2(M)$. We can extend and identify this operator as follows.

**Proposition 9.14.** $(\overline{P}_4 + t + \pi)^{-1}$ can be continuously extended to $(\overline{P}_4 + t + \pi)^{-1}: H^s(M) \to H^s(M)$ for every $s \in \mathbb{Z}$. Moreover, for every $K \in \mathbb{N}_0$ we have

$$(\overline{P}_4 + t + \pi)^{-1} - (\tau G_t \tau + \tau A_{2K,t} \tau) = O\left(\frac{1}{1+t}\right): H^{-K}(M) \to H^K(M),$$

where $A_{2K,t}$ is as in Proposition 9.12.
Proof. Fix $K \in \mathbb{N}_0$ and let $A_{2K,t}$ and $R_{2K,t}$ be as in Proposition 9.12. Then
\begin{equation}
\tau G_t \tau + \tau A_{2K,t} \tau = (\hat{P}_4 + t + \pi)^{-1} + (\hat{P}_4 + t + \pi)^{-1} \tau R_{2K,t} \tau.
\end{equation}
Note that $\tau R_{2K,t} \tau = O\left(\frac{1}{t+\pi}\right) : H^{-s}(M) \to L^2(M)$ for all $s \in \mathbb{Z}$ with $|s| \leq 2K$.
By [7], $\tau G_t \tau + \tau A_{2K,t} \tau = O\left(\frac{1}{t+\pi}\right) : H^s(M) \to H^s(M)$ for all $s \in \mathbb{Z}$. By [9.3], we observe that $(\hat{P}_4 + t + \pi)^{-1} = O\left(\frac{1}{t+\pi}\right) : L^2(M) \to L^2(M)$. From these observations we conclude that we can extend to $(\hat{P}_4 + t + \pi)^{-1}$ to $H^{-s}(M)$ for all $s \in \mathbb{N}_0$ with $s \leq 2K$; indeed
\begin{equation}
(\hat{P}_4 + t + \pi)^{-1} = O\left(\frac{1}{1+t}\right) : H^{-s}(M) \to H^{-s}(M)
\end{equation}
for all $s \in \mathbb{N}_0$ with $s \leq 2K$. By taking the adjoint in (9.48), we conclude that we can extend to $(\hat{P}_4 + t + \pi)^{-1}$ to
\begin{equation}
(\hat{P}_4 + t + \pi)^{-1} = O\left(\frac{1}{1+t}\right) : H^{s}(M) \to H^{s}(M)
\end{equation}
for all $s \in \mathbb{N}_0$ with $s \leq 2K$. From (9.47) and (9.49) we conclude that
\begin{equation}
(\hat{P}_4 + t + \pi)^{-1} - (\tau G_t \tau + \tau A_{2K,t} \tau) = O\left(\frac{1}{1+t}\right) : H^{-K}(M) \to H^K(M).
\end{equation}

This allows us to prove the following theorem.

**Theorem 9.15.** There is a $G \in L^{-1}_c(M)$ such that $2GE_1 - I \in L_c(M)$ for $E_1 \in L^1_c(M)$ as in Theorem 9.1 and for every $\ell \in \mathbb{N}_0$,
\begin{equation}
(\hat{P}_4)^{-\frac{1}{4}} = \tau G_t \tau + \tau A_{\ell} \tau + \tau R_{\ell} \tau
\end{equation}
on $\hat{P}$, where $A_{\ell}, R_{\ell} : C^\infty(M) \to \mathcal{D}'(M)$ are continuous operators, $R_{\ell}(x,y) \in C^\ell(M \times M)$, and $A_{\ell}$ is a smooth operator of order $3 - \varepsilon$ for every $0 < \varepsilon < 1$.

**Proof.** Fix $\ell \in \mathbb{N}_0$ and take $K \gg \ell$. Put
\begin{equation}
\Xi_{2K,t} = (\hat{P}_4 + t + \pi)^{-1} - (\tau G_t \tau + \tau A_{2K,t} \tau),
\end{equation}
where $A_{2K,t}$ is as in Proposition 9.12. By Proposition 9.14, $\Xi_{2K,t}$ is well-defined as a continuous operator $H^s(M) \to H^s(M)$ for every $s \in \mathbb{Z}$. Observe that $\Xi_{2K,t} = \tau \Xi_{2K,t} \tau$. From Lemma 9.12 we see that
\begin{equation}
(\hat{P}_4)^{-\frac{1}{4}} = c \int_0^\infty t^{-\frac{1}{4}} \tau G_t \tau dt + c \int_0^\infty t^{-\frac{1}{4}} \tau A_{2K,t} \tau dt + c \int_0^\infty t^{-\frac{1}{4}} \tau \Xi_{2K,t} \tau dt.
\end{equation}
It is known that (see [30])
\begin{equation}
c \int_0^\infty t^{-\frac{1}{4}} \tau G_t \tau dt = \tau G_t,
\end{equation}
where $G \in L^{-1}_c(M)$ with $2GE_1 - I \in L^{-1}_c(M)$.

We claim that
\begin{equation}
\Xi(x,y) := (c \int_0^\infty t^{-\frac{1}{4}} \Xi_{2K,t} dt)(x,y) \in C^\ell(M \times M)
\end{equation}
if $K$ is large enough. Fix $k \in \mathbb{N}_0$. For every $m \in \mathbb{N}$, consider
\begin{equation}
\Xi_{k,m} := c \sum_{j=1}^m \frac{1}{m} \Xi_{2K,k+j} \cdot \frac{1}{\sqrt{k + \frac{1}{m}}}.
\end{equation}
It is clear that, in $L^2(M)$,

$$
\lim_{m \to \infty} \Xi_{k,m} = c \int_k^{k+1} t^{-\frac{4}{3}} \Xi_{2K,t} dt.
$$

(9.53)

By Proposition 9.14, we see that

$$
\|\Xi_{k,m}\|_{L^2(H^{-\kappa}(M), H^\kappa(M))} \leq c_1 \sum_{j=1}^{m} \frac{1}{m} \frac{1}{1 + k + \frac{2}{m}} \frac{1}{\sqrt{k + \frac{4}{m}}} \leq c_1 \int_k^{k+1} \frac{1}{(1 + t)^{\sqrt{t}}} dt,
$$

(9.54)

where $c_1 > 0$ is a constant and $\|\Xi_{k,m}\|_{L^2(H^{-\kappa}(M), H^\kappa(M))}$ denotes the standard operator norm of $\Xi_{k,m}$ in $L^2(H^{-\kappa}(M), H^\kappa(M))$. From (9.54) and the Sobolev embedding theorem, if $K \gg \ell$, there is a subsequence $(m_s)$ such that $m_s \to \infty$ as $s \to \infty$,

$$
\lim_{s \to \infty} \Xi_{k,m_s}(x, y) = \Xi_k(x, y)
$$

in $C^\ell(M \times M)$, and

$$
\|\Xi_k(x, y)\|_{C^\ell(M \times M)} \leq \tilde{c}_1 \int_k^{k+1} \frac{1}{(1 + t)^{\sqrt{t}}} dt,
$$

(9.56)

where $\tilde{c}_1 > 0$ is a constant independent of $k$. From (9.53), (9.55) and (9.56), we conclude that

$$
\Xi_k(x, y) = \left(c \int_k^{k+1} t^{-\frac{4}{3}} \Xi_{2K,t} dt\right)(x, y) \in C^\ell(M \times M),
$$

(9.57)

$$
\left\|\int_k^{k+1} t^{-\frac{4}{3}} \Xi_{2K,t} dt\right\|_{C^\ell(M \times M)} \leq \tilde{c}_1 \int_k^{k+1} \frac{1}{(1 + t)^{\sqrt{t}}} dt.
$$

Since $\sum_{k=0}^{\infty} \int_k^{k+1} \frac{1}{(1 + t)^{\sqrt{t}}} dt = \int_0^{\infty} \frac{1}{(1 + t)^{\sqrt{t}}} dt < \infty$, we deduce that

$$
\Xi(x, y) = \left(c \int_0^{\infty} t^{-\frac{4}{3}} \Xi_{2K,t} dt\right)(x, y) = \sum_{k=0}^{\infty} \Xi_k(x, y) \in C^\ell(M \times M),
$$

as claimed.

From now on, we take $K$ large enough so that $\Xi(x, y) \in C^\ell(M \times M)$. Put

$$
A := c \int_0^{\infty} t^{-\frac{4}{3}} A_{2K,t} dt.
$$

We now study the kernel of $A$. Fix $x_0, y_0 \in M$ and set $\partial(x_0, y_0) = r$. Put

$$
B_{x_0}(\frac{r}{4}) = \left\{ z \in M; \partial(z, x_0) < \frac{r}{4} \right\}, \quad B_{y_0}(\frac{r}{4}) = \left\{ z \in M; \partial(z, y_0) < \frac{r}{4} \right\}.
$$

Take $\chi \in C_0^\infty(B_{x_0}(\frac{r}{4}))$ and $\chi_1 \in C_0^\infty(B_{y_0}(\frac{r}{4}))$ such that $\chi = 1$ near $x_0$ and $\chi_1 = 1$ near $y_0$. Consider

$$
\tilde{A} := c \int_0^{\infty} t^{-\frac{4}{3}} \chi A_{2K,t} \chi_1 dt.
$$

Then,

$$
\tilde{A} = c \int_0^{r^{-4}} t^{-\frac{4}{3}} \chi A_{2K,t} \chi_1 dt + c \int_{r^{-4}}^{\infty} t^{-\frac{4}{3}} \chi A_{2K,t} \chi_1 dt.
$$

(9.58)

For every $m \in \mathbb{N}$, consider

$$
B_m = c \sum_{j=1}^{m} \frac{r^{-4}}{m} \left(\chi A_{2K,t} \chi_1\right)\left(\frac{j}{m} r^{-4}\right)^{-\frac{4}{3}}.
$$
It is easy to see that, in \( L^2 \),

\[
\lim_{m \to +\infty} B_m = c \int_0^{r-4} t^{-\frac{3}{2}} \chi A_{2K,t} \chi_1 dt.
\]

Recall from Proposition 9.12 that \( A_{2K,t} \) is a smoothing operator of order 3 with size \( \frac{1}{\sqrt{1+t}} \). From this and (9.59), we have that, for any \( x \in B_{x_0}(\frac{r}{4}) \), \( y \in B_{y_0}(\frac{r}{4}) \),

\[
|B_m(x,y)| \leq c \sum_{j=1}^{m} \left( \chi A_{2K,\frac{r}{m}} \chi_1 \right)(x,y) \left( \frac{r-4}{m} \right)^{\frac{3}{2}}
\]

\[
\leq c \sum_{j=1}^{m} \frac{r-4}{m} \frac{1}{\sqrt{1 + \frac{1}{m}}} \vartheta(x,y)^{-1} \left( \frac{r-4}{m} \right)^{\frac{3}{2}}
\]

\[
\leq c_2 \vartheta(x,y)^{-1} \int_0^{r-4} \frac{1}{\sqrt{1 + t}} t^{-\frac{3}{2}} dt
\]

\[
\leq c_3 \vartheta(x,y)^{-1} |\log \vartheta(x,y)|,
\]

where \( c_2 > 0, c_3 > 0 \) are constants independent of \( m, r, \chi, \chi_1, x_0, y_0 \). Similarly, for every \( \alpha_1, \alpha_2 \in \mathbb{N}_0 \) and \( \varepsilon > 0 \), there is a constant \( C_{\alpha_1,\alpha_2,\varepsilon} \), independent of \( m, r, \chi, \chi_1, x_0, y_0 \), such that

\[
|\left( \nabla_b \right)_{x}^{\alpha_1} \left( \nabla_b \right)_{y}^{\alpha_2} B_m(x,y)| \leq C_{\alpha_1,\alpha_2,\varepsilon} \vartheta(x,y)^{-1 - |\alpha_1| - |\alpha_2| - \varepsilon}.
\]

From \( 9.60 \), we deduce that there is a subsequence \( (m_s) \) such that \( m_s \to \infty \) as \( s \to \infty \) for which \( B_{m_s}(x,y) \) converges to some \( B(x,y) \) in the \( C^\infty(M \times M) \) topology with the property that for every \( \alpha_1, \alpha_2 \in \mathbb{N}_0 \) and every \( \varepsilon > 0 \), there is a constant \( C_{\alpha_1,\alpha_2,\varepsilon} \), independent of \( m, r, \chi, \chi_1, x_0, y_0 \), such that

\[
|\left( \nabla_b \right)_{x}^{\alpha_1} \left( \nabla_b \right)_{y}^{\alpha_2} B(x,y)| \leq C_{\alpha_1,\alpha_2,\varepsilon} \vartheta(x,y)^{-1 - |\alpha_1| - |\alpha_2| - \varepsilon}.
\]

In particular, from (9.59) we have that

\[
(c \int_0^{r-4} t^{-\frac{3}{2}} \chi A_{2K,t} \chi_1 dt)(x,y) = B(x,y).
\]

Fix \( k \in \mathbb{N}_0 \). For every \( m \in \mathbb{N} \), consider

\[
D_{k,m} := c \sum_{j=1}^{m} \frac{r-4}{m} \chi A_{2K,\frac{r}{m}} \chi_1 \frac{1}{\sqrt{r-4k + \frac{1}{m} r^{-4}}}
\]

It is clear that

\[
\lim_{m \to \infty} D_{k,m} = c \int_{r-4k}^{r-4(k+1)} t^{-\frac{3}{2}} \chi A_{2K,t} \chi_1 dt
\]

in \( L^2(M) \). Recall from Proposition 9.12 that \( A_{2K,t} \) is a smoothing operator of order 1 with size \( \frac{1}{\sqrt{1+t}} \). From this observation, we find that for every \( x \in B_{x_0}(\frac{r}{4}) \),
y \in B_{\rho_0}(\frac{\epsilon}{4}), \text{ we have that }

\begin{align*}
|D_{k,m}(x,y)| \leq \bar{c}_1 \sum_{j=1}^{m} \frac{r^{-4}}{m} \left| \left( \chi A_{2K, r^{-4}k + \frac{1}{m} r^{-4} \chi_1} \right)(x,y) \right| (r^{-4}k + \frac{1}{m} r^{-4})^{-\frac{1}{2}} \\
\leq \bar{c}_2 \int_{r^{-4}k}^{r^{-4}(k+1)} (x,y)^{-3} \frac{1}{1 + r^{-4}k + \frac{1}{m} r^{-4}} \left( r^{-4}k + \frac{1}{m} r^{-4} \right)^{-\frac{1}{2}} \frac{1}{1 + t \sqrt{t}} dt,
\end{align*}

(9.64)

where \( \bar{c}_1 > 0, \bar{c}_2 > 0 \) are constants independent of \( k, m, r, \chi, \chi_1, x_0, y_0 \). Similarly, for every \( \alpha_1, \alpha_2 \in \mathbb{N}_0 \), there is a constant \( \bar{C}_{\alpha_1, \alpha_2} \), independent of \( k, m, r, \chi, \chi_1, x_0, y_0 \), such that

\begin{align*}
\left| (\nabla_b)^{\alpha_1}_x (\nabla_b)^{\alpha_2}_y D_{k,m}(x,y) \right| \leq \bar{C}_{\alpha_1, \alpha_2} \int_{r^{-4}k}^{r^{-4}(k+1)} (x,y)^{-3-\alpha_1-\alpha_2} \frac{1}{1 + t \sqrt{t}} dt.
\end{align*}

(9.65)

Therefore there is a subsequence \( (m_s) \) such that \( m_s \to s \) as \( s \to \infty \) for which \( D_{k,m_s}(x,y) \) converges to some \( D_k(x,y) \) in the \( C^\infty(M \times M) \) topology with the property that for every \( \alpha_1, \alpha_2 \in \mathbb{N}_0 \) and every \( \epsilon > 0 \), there is a constant \( \tilde{C}_{\alpha_1, \alpha_2, \epsilon} \) such that

\begin{align*}
\left| (\nabla_b)^{\alpha_1}_x (\nabla_b)^{\alpha_2}_y D_k(x,y) \right| \leq \tilde{C}_{\alpha_1, \alpha_2, \epsilon} \int_{r^{-4}k}^{r^{-4}(k+1)} (x,y)^{-3-\alpha_1-\alpha_2-\epsilon} \frac{1}{1 + t \sqrt{t}} dt.
\end{align*}

(9.66)

In particular, from (9.63) we find that

\begin{align*}
\left( c \int_{r^{-4}k}^{r^{-4}(k+1)} t^{-\frac{1}{2} \chi A_{2K, \epsilon} \chi_1 dt} \right)(x,y) = D_k(x,y).
\end{align*}

(9.66)

Note that for \( x \in B_{\rho_0}(\frac{\epsilon}{4}) \) and \( y \in B_{\rho_0}(\frac{\epsilon}{4}) \),

\begin{align*}
\sum_{k=1}^{\infty} \int_{r^{-4}k}^{r^{-4}(k+1)} (x,y)^{-3-\alpha_1-\alpha_2-\epsilon} \frac{1}{1 + t \sqrt{t}} \leq \tilde{c}_0 \vartheta(x,y)^{-1-\alpha_1-\alpha_2-\epsilon},
\end{align*}

(9.67)

where \( \tilde{c}_0 > 0 \) is a constant. From (9.65), (9.66), and (9.67) we deduce that

\begin{align*}
\left( c \int_{r^{-4}}^{\infty} t^{-\frac{1}{2} \chi A_{2K, \epsilon} \chi_1 dt} \right)(x,y) \in C^\infty(M \times M) \text{ and for every } \alpha_1, \alpha_2 \in \mathbb{N}_0 \text{ and } \epsilon > 0,
\end{align*}

there is a constant \( \tilde{C}_{\alpha_1, \alpha_2, \epsilon} \), independent of \( r, \chi, \chi_1, x_0, y_0 \), such that

\begin{align*}
\left| (\nabla_b)^{\alpha_1}_x (\nabla_b)^{\alpha_2}_y \left( c \int_{r^{-4}}^{\infty} t^{-\frac{1}{2} \chi A_{2K, \epsilon} \chi_1 dt} \right)(x,y) \right| \leq \tilde{C}_{\alpha_1, \alpha_2, \epsilon} \vartheta(x,y)^{-1-\alpha_1-\alpha_2-\epsilon}.
\end{align*}

(9.68)

From (9.65), (9.61), (9.62) and (9.68), we deduce that \( A(x,y) \) satisfies the following differential inequalities when \( x \neq y \): For every \( \epsilon > 0 \) and every \( \alpha_1, \alpha_2 \in \mathbb{N}_0 \), there is a constant \( C_{\alpha_1, \alpha_2, \epsilon} > 0 \) independent of \( x \) and \( y \) such that

\begin{align*}
\left| (\nabla_b)^{\alpha_1}_x (\nabla_b)^{\alpha_2}_y A(x,y) \right| \leq C_{\alpha_1, \alpha_2, \epsilon} \vartheta(x,y)^{-1-\epsilon-\alpha} \text{ for all } |\alpha| = |\alpha_1| + |\alpha_2|.
\end{align*}

(9.69)
Now, we prove that $A$ satisfies the cancellation condition of order $3 - \varepsilon$ for every $0 < \varepsilon < 1$. Let $\phi$ be a normalized bump function in $B(x, r)$. Then
\[
\left\| \nabla^2_0 A \phi \right\|_{L^\infty(B(x, r))} 
\leq \int_0^r t^{-\frac{5}{2}} \left\| \nabla^0_0 A_{2K,t} \phi \right\|_{L^\infty(B(x, r))} + c \int_0^r t^{-\frac{5}{2}} \left\| \nabla^0_0 A_{2K,t} \phi \right\|_{L^\infty(B(x, r))},
\]
(9.70)
Since $A_{2K,t}$ is a smoothing operator of order 3 with size $\frac{1}{\sqrt{1+t}}$, we have that
\[
\int_0^r t^{-\frac{5}{2}} \left\| \nabla^0_0 A_{2K,t} \phi \right\|_{L^\infty(B(x, r))} \leq c_2 t^{3-|\alpha|} |\log r|,
\]
(9.71)
where $c_1 > 0$ and $c_2 > 0$ are constants independent of $r$. Since $A_{2K,t}$ is also a smoothing operator of order 1 with size $\frac{1}{1+t}$, we have
\[
\int_0^r t^{-\frac{5}{2}} \left\| \nabla^0_0 A_{2K,t} \phi \right\|_{L^\infty(B(x, r))} \leq \hat{c}_1 t^{1-|\alpha|} \int_0^r t^{-\frac{5}{2}} \frac{1}{1+t} dt \leq \hat{c}_2 t^{3-|\alpha|},
\]
(9.72)
where $\hat{c}_1 > 0$ and $\hat{c}_2 > 0$ are constants independent of $r$. From (9.70), (9.71) and (9.72), we deduce that $A$ satisfies the cancellation condition of order $3 - \varepsilon$ for every $0 < \varepsilon < 1$. Similarly, $A^*$ satisfies the cancellation condition of order $3 - \varepsilon$ for every $0 < \varepsilon < 1$. The conclusion follows from (9.69). $\Box$

Now let us consider $2\Delta_b + \frac{1}{2}R$ extended to $L^2(M)$ in the standard way. Since $2\Delta_b + \frac{1}{2}R$ is hypoelliptic with loss of one derivative,
\[
2\Delta_b + \frac{1}{2}R: \text{Dom}(2\Delta_b + \frac{1}{2}R) \to L^2(M)
\]
has closed range, is self-adjoint, and $\text{Ker}(2\Delta_b + \frac{1}{2}R)$ is a finite-dimensional subspace of $C^\infty(M)$. Let $\tilde{N}: L^2(M) \to \text{Dom}(2\Delta_b + \frac{1}{2}R)$ be the partial inverse and let $p: L^2(M) \to \text{Ker}(2\Delta_b + \frac{1}{2}R)$ be the orthogonal projection. Then $p$ is a smoothing operator on $M$ and we have
\[
\tilde{N}(2\Delta_b + \frac{1}{2}R) + p = I \quad \text{on } \text{Dom}(2\Delta_b + \frac{1}{2}R),
\]
\[
(2\Delta_b + \frac{1}{2}R)\tilde{N} + p = I \quad \text{on } L^2(M).
\]
Note that $\tilde{N}: H^s(M) \to H^{s+1}(M)$ for all $s \in \mathbb{Z}$. Moreover, $\tilde{N}$ is a smoothing operator of order 2 (see [23, Section 10] and [31]).

**Proposition 9.16.** With the notations above, $\tau G r - \tau \tilde{N}$ is a smoothing operator of order 3, where $G \in L^2_{cl}(M)$ is as in Theorem 9.15.

**Proof.** From Theorem 8.1 and Theorem 9.15 we have that
\[
(\tau G r)(\tau(2\Delta_b + \frac{R}{2})r) = \tau + \tau P_{-1} \tau - \tau G(I - \tau)(2E_1 + \frac{R}{2})r,
\]
(9.73)
where $P_{-1} \in L^2_{cl}(M)$.

We claim that
\[
\tau P_{-1} \tau \text{ is a smoothing operator of order 2},
\]
(9.74)
\[
\tau G(I - \tau)(2E_1 + \frac{R}{2})\tau \text{ is a smoothing operator of order 1}.
\]
(9.75)
From Theorem 7.1 and Lemma 8.3, we have that
\( (9.76) \quad \tau P_{-1} \tau \equiv SP_{-1}S + \overline{SP_{-1}S} \).

From Lemma 8.2 and Lemma 8.3, we have that
\( (9.77) \quad SP_{-1}S \equiv S\nabla \Box \tau P_{-1}S = S\nabla \Box \tau P_{-1}S + S\nabla L_{1}P_{-1}S \)
\[ = S\nabla |\Box \tau P_{-1}|S + S\nabla L_{1}P_{-1}S, \]
where \( L_{1} \) is a first order partial differential operator. From the symbolic calculus of Stein–Yung [31], we check that \([\Box \tau P_{-1}]S \) and \( L_{1}P_{-1}S \) are smoothing operators of order 0. From this observation, \((9.78)\) and Theorem 9.7, we conclude that \( SP_{-1}S \) is a smoothing operator of order 2. Similarly, \( \overline{SP_{-1}S} \) is a smoothing operator of order 2. From \((9.76)\), we obtain \((9.74)\).

Again, from Theorem 7.1, Lemma 8.3, and Lemma 8.2, we have that
\( \tau G(I - \tau)(2E_{1} + \frac{R}{2}) \tau \equiv SG(I - S)(2E_{1} + \frac{R}{2})S + \overline{SG(I - S)}(2E_{1} + \frac{R}{2})S \)
\[ \equiv SG\nabla_{b}N\nabla_{b}(2E_{1} + \frac{R}{2})S + \overline{SG\nabla_{b}N\nabla_{b}}(2E_{1} + \frac{R}{2})S \]
\[ = S[G, \Box_{b}]\nabla_{b}\nabla_{b}[\partial_{b}, 2E_{1} + \frac{R}{2}]S + SGL_{1}N\nabla_{b}[\partial_{b}, 2E_{1} + \frac{R}{2}]S \]
\[ + \overline{S[G, \Box_{b}]\nabla_{b}\nabla_{b}[\partial_{b}, 2E_{1} + \frac{R}{2}]S + SGL_{1}N\nabla_{b}[\partial_{b}, 2E_{1} + \frac{R}{2}]S}, \]
where \( L_{1} \) is a first order partial differential operator. From the symbolic calculus of Stein–Yung [31], we check that \([\nabla_{b}, \Box_{b}]S \) and \( SGL_{1}, \overline{S[G, \Box_{b}]} \) are smoothing operators of order 1 and \( S[G, \Box_{b}], SGL_{1}, \overline{S[G, \Box_{b}]} \) are smoothing operators of order 0. From this observation, \((9.78)\) and Theorem 9.7, we obtain \((9.75)\).

Now, from Theorem 7.1, Lemma 8.3, Lemma 8.2, and recall that \( \Delta_{b} = \Box_{b} + \Box_{b} \), we have that
\( (9.79) \quad (\tau(2\Delta_{b} + \frac{R}{2}) \tau)\hat{N} \equiv \tau - (2\Delta_{b} + \frac{R}{2})(I - \tau)\hat{N} \)
\[ \equiv \tau - S(2\Delta_{b} + \frac{R}{2})(I - S)\hat{N} - \overline{S(2\Delta_{b} + \frac{R}{2})(I - S)\hat{N}} \]
\[ = \tau - S(\Box_{b} + \frac{R}{2})\Box_{b}N\hat{N} - \overline{S(\Box_{b} + \frac{R}{2})\Box_{b}N\hat{N}} \]
\[ \equiv \tau - S[L_{1} + \frac{R}{2}\Box_{b}]\Box_{b}N\hat{N} - \overline{S[L_{1} + \frac{R}{2}\Box_{b}]\Box_{b}N\hat{N}}, \]
where \( L_{1} \) is a first order partial differential operator. From the symbolic calculus of Stein–Yung [31], we check that \([L_{1} + \frac{R}{2}\Box_{b}]\Box_{b}N\hat{N} \) and \([\overline{L_{1} + \frac{R}{2}\Box_{b}}]\Box_{b}N\hat{N} \) are smoothing operators of order 1. From this observation, \((9.79)\) and Theorem 9.7, we obtain that
\( (9.80) \quad (\tau(2\Delta_{b} + \frac{R}{2}) \tau)\hat{N} = \tau + H, \)
where \( H \) is a smoothing operator of order 1. From \((9.73)\) and \((9.81)\), we find that
\( \tau G\tau + (\tau G\tau)H = \tau \hat{N} + (\tau P_{-1} \tau)\hat{N} - \tau G(I - \tau)(2E_{1} + \frac{R}{2})\tau \hat{N}. \)
We can repeat the proof of (9.74) and deduce that $\tau G\tau$ is a smoothing operator of order 2 and hence
\begin{equation}
(\tau G\tau)H \text{ is a smoothing operator of order 3.}
\end{equation}

From (9.74), (9.75), (9.81) and (9.82) we deduce that $\tau G\tau - \hat{N}$ is a smoothing operator of order 3. $\square$

Fix a point $\zeta \in X$. The Green’s function of $(\overline{P}_4)^{\frac{1}{2}}$ at $\zeta$ is given by
\begin{equation}
G_{\zeta} := (\overline{P}_4)^{-\frac{1}{2}} \tau \delta \zeta \tau \in \mathcal{D}'(M).
\end{equation}

It is easy to see that
\begin{equation}
(\overline{P}_4)^{\frac{1}{2}} G_{\zeta} = \delta_{\zeta} - \pi(x, \zeta) \text{ on } \mathcal{P}.
\end{equation}

Note that $\pi(x, \zeta) \in C^\infty(M) \cap \text{Ker}(\overline{P}_4)^{-\frac{1}{2}}$.

**Proof of Theorem 1.3.** Fix $\zeta \in M$ and let $(z, t)$ be CR normal coordinates defined in a neighborhood of $\zeta$ such that $(z(\zeta), t(\zeta)) = (0, 0)$. For $m \in \mathbb{R}$, let $\mathcal{E}(\rho^m)$ be as in the discussion before Theorem 1.3. Let $\ell_0 \in \mathbb{N}_0$ and fix $\ell \gg \ell_0$. From Theorem 9.15 and Proposition 9.16, we have
\begin{equation}
G_{\zeta} = \tau G\tau \delta \zeta \tau + \tau A_\ell \tau \delta \zeta \tau + \tau R_\ell \tau \delta \zeta \tau,
\end{equation}
where $K$ is a smoothing operator of order 3. Since $R_\ell(x, y) \in C^\ell(M \times M)$, we can take $\ell$ large enough so that
\begin{equation}
R_\ell \tau \delta \zeta \tau \in \mathcal{E}^\ell_0(M).
\end{equation}

Since $K$ is a smoothing operator of order 3,
\begin{equation}
K \delta \zeta \tau \in \mathcal{E}(\rho^{-1}).
\end{equation}

From Theorem 9.7 and Theorem 9.15 we see that $A_\ell \tau$ is a smoothing operator of order $3 - \varepsilon$, for every $0 < \varepsilon < 1$. Hence,
\begin{equation}
A_\ell \tau \delta \zeta \tau \in \mathcal{E}(\rho^{-1-\varepsilon})
\end{equation}
for all $\varepsilon > 0$.

Finally, we consider $\hat{N} \delta \zeta$. It is clear that $\hat{N} \delta \zeta$ is the Green’s function of $2\Delta_k + \frac{1}{2} R$.

It was shown in [10, Section 5] that, near $\zeta$, $\hat{N} \delta \zeta$ has the form
\begin{equation}
\hat{N} \delta \zeta(z, t) = \rho(z, t)^{-2} + \omega_0
\end{equation}
for some $\omega_0 \in C^1(M)$. Moreover, repeating the method in [23, Section 10], we conclude that
\begin{equation}
\omega_0 \in \mathcal{E}(\rho^{-\varepsilon})
\end{equation}
for all $\varepsilon > 0$. The conclusion follows from (9.85), (9.86), (9.87), (9.88), (9.89) and (9.90). $\square$

In the proof of Theorem 4.2, we need the following result.

**Theorem 9.17.** For every $\ell \in \mathbb{N}_0$, we have
\begin{equation}
\tau B_\ell \tau \overline{P}_4 = \tau + \tau C_\ell \tau \text{ on } \overline{P},
\end{equation}
where $B_\ell, C_\ell : C^\infty(M) \rightarrow \mathcal{D}'(M)$ are continuous operators, $B_\ell$ is a smoothing operator of order $4 - \varepsilon$ for all $0 < \varepsilon < 1$, and $(\tau C_\ell \tau)(x, y) \in C^\ell(M \times M)$. 

Proof. In view of (9.92), we see that $\overline{\tau F_4} = \tau(4\Delta_b^2 + L_2)\tau$, where $L_2 = \nabla_b^2 + \nabla_b + r$, $r \in C^\infty(X)$. Let $H$ be a parametrix of $4\Delta_b^2 + L_2$. Then $H^*: (M) \to H^{s+2}(M)$ for every $s \in \mathbb{Z}$ and $H$ is a smoothing operator of order 4. From Theorem 9.18 and Lemma 8.3, we have that

$$\tau H \overline{\tau F_4} = (\tau H \tau)(\tau(4\Delta_b^2 + L_2)\tau)$$

$$= \tau - \tau H(I - \tau)(4\Delta_b^2 + L_2)\tau - F_0$$

$$= \tau - S\overline{H}(I - S)(4\Delta_b^2 + L_2)\tau - \overline{S\overline{H}(I - S)(4\Delta_b^2 + L_2)\tau - F_1},$$

where $F_0$ and $F_1$ are smoothing operators on $M$. Put

$$\tau = \overline{S\overline{H}(I - S)(4\Delta_b^2 + L_2)\tau + F_1}.$$ 

Note that $\tau = \tau \tau^2$. Repeating the procedure in (9.89), we conclude that

$$S\overline{H}(I - S)(4\Delta_b^2 + L_2)\tau = S\overline{H}N\overline{\tau}Q_2\tau,$$

where $Q_2, \overline{Q}_2 \in L_2^2(M)$. From (9.92) and (9.93), we conclude that $\tau: H^s(M) \to H^{s+\frac{1}{2}}(M)$ for all $s \in \mathbb{Z}$ and $\tau$ is a smoothing operator of order 1. Fix $K \in \mathbb{N}$. Put

$$B_K := (\tau H \tau)(\tau + \tau + \tau^2 + \cdots + \tau^K).$$

Then, $B_K$ is a smoothing operator of order $4 - \varepsilon$ for all $0 < \varepsilon < 1$. From (9.91), we have that

$$B_K \overline{\tau F_4} = \tau - \tau^{K+1}.$$

Since $\tau^{K+1}: H^s(M) \to H^{s+\frac{K+1}{2}}(M)$ for every $s \in \mathbb{Z}$, given $\ell \in \mathbb{N}_0$, we can take $K$ large enough so that $\tau^{K+1}(x, y) \in C^\infty(M \times M)$. The theorem follows. □

In the proof of Theorem 9.2, we also need the following result.

**Theorem 9.18.** Let $w \in L^2(M)$. If $\Delta_b w \in L^2(M)$, then there is a constant $c > 0$ such that $e^{i|w|^2} \in L^1(M)$.

To prove Theorem 9.18, we need the following Adams-type theorem of Fontana and Morpurgo [12].

**Theorem 9.19.** Let $A: L^2(M) \to L^2(M)$ be a continuous operator with distribution kernel $A(x, y) \in C^\infty(M \times M \setminus \text{diag}(M \times M))$. Suppose that the kernel $A(x, y)$ satisfies

$$\sup_{x \in M} \{|y \in M: |A(x, y)| > s|\} \leq Ks^{-2},$$

$$\sup_{y \in M} \{|x \in M: |A(x, y)| > s|\} \leq Ks^{-2}$$

as $s \to \infty$, where $K > 0$ is a constant and

$$|\{y \in M: |A(x, y)| > s\}|, \quad |\{x \in M: |A(x, y)| > s\}|$$

denote the volumes of the sets $\{y \in M: |A(x, y)| > s\}$ and $\{x \in M: |A(x, y)| > s\}$, respectively with respect to the given volume form on $M$. Then, for any $f \in L^2(M)$ with $Tf \in L^2(M)$, there is a constant $c > 0$ such that $e^{i|f|^2} \in L^1(M)$.
Proof of Theorem 9.18. Put $g := (\Delta_b + I)w \in L^2(M)$. Let $Q$ be the inverse of $\Delta_b + I$. Then, $w = Qg$. It is known that (see [10, Section 2] and [23, Section 10])

\begin{equation}
|Q(x, y)| \lesssim \vartheta(x, y)^{-2}.
\end{equation}

From (9.95), one readily checks that

\begin{align}
\sup_{x \in M} \{ y \in M; |Q(x, y)| > s \} & \lesssim s^{-2}, \\
\sup_{y \in M} \{ x \in M; |Q(x, y)| > s \} & \lesssim s^{-2},
\end{align}

as $s \to \infty$. The conclusion follows from (9.95) and Theorem 9.19. □

References


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