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Strong coupling expansion of circular Wilson loops and string theories in $\mathrm{AdS}_5 \times S^5$ and $\mathrm{AdS}_4 \times CP^3$

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Abstract: We revisit the problem of matching the strong coupling expansion of the $\frac{1}{2}$ BPS circular Wilson loops in $\mathcal{N} = 4$ SYM and ABJM gauge theories with their string theory duals in $\mathrm{AdS}_5 \times S^5$ and $\mathrm{AdS}_4 \times CP^3$, at the first subleading (one-loop) order of the expansion around the minimal surface. We observe that, including the overall factor $1/g_\mathrm{s}$ of the inverse string coupling constant, as appropriate for the open string partition function with disk topology, and a universal prefactor proportional to the square root of the string tension T, both the SYM and ABJM results precisely match the string theory prediction. We provide an explanation of the origin of the $\sqrt{T}$ prefactor based on special features of the combination of one-loop determinants appearing in the string partition function. The latter also implies a natural generalization $Z_\chi \sim (\sqrt{T}/g_\mathrm{s})^\chi$ to higher genus contributions with the Euler number χ, which is consistent with the structure of the $1/N$ corrections found on the gauge theory side.

Keywords: AdS-CFT Correspondence, Wilson, 't Hooft and Polyakov loops, Superstrings and Heterotic Strings, $1/N$ Expansion

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Contents

1 Introduction

There is a history of attempts to match gauge theory [1–3] and string theory [4–6] results for the leading terms in the strong coupling expansion of the expectation value of the $\frac{1}{2}$ BPS circular Wilson loop (WL) in $\mathcal{N}=4$ SYM theory (see [7–13]). The precise matching was recently achieved for the ratio of the $\frac{1}{2}$ and $\frac{1}{4}$ BPS WL expectation values [14, 15] (see also [16, 17] for a discussion of similar matching in the ABJM theory [18]). However, the direct computation of the string theory counterpart of the expectation value of the individual WL, that non-trivially depends on the normalization of the path integral measure, still remains a challenge.

In the SU(N) $\mathcal{N}=4$ SYM theory the Maldacena-Wilson operator defined in the fundamental representation is given by $\mathcal{W} = \mathrm{tr}\, Pe^{\int(iA+\Phi)}$ (note that we do not include the usual $1/N$ factor in the definition of $\mathcal{W}$). Then for a circular loop one finds at large N with fixed 't Hooft coupling λ [1, 2]: $\langle\mathcal{W}\rangle = N\frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda})$. Expanding at strong coupling, $\langle\mathcal{W}\rangle = N\lambda^{-3/4}\sqrt{\frac{2}{\pi}}\,e^{\sqrt{\lambda}} + \dots$. This result should be reproduced by the $\mathrm{AdS}_5\times S^5$ string perturbation theory with the string tension $T=\frac{\sqrt{\lambda}}{2\pi}=\frac{\mathrm{R}^2}{2\pi\alpha'}$, where here and below R denotes the AdS radius. It was suggested in [2] that the pre-factor $\lambda^{-3/4}\sim T^{-3/2}$ may have its origin in the normalization of three ghost 0-modes on the disk (or the Mobius volume).

This proposal, however, is problematic for several reasons. First, the effective tension T has its natural origin in the string action but should not appear in the diffeomorphism volume or the volume of residual Mobius symmetry. Furthermore, the $T^{-3/2}$ factor (which would be universal if related to the Mobius volume) would fail to explain the result for the $\frac{1}{2}$ BPS circular WL [19] in the $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ ABJM theory, where the tension is $T = \frac{1}{2}\sqrt{2\lambda}$ (with $\lambda = \frac{N}{k}$) while the gauge theory (localization) prediction [20–22] for the $\frac{1}{2}$ BPS Wilson loop in fundamental representation is $\langle \mathcal{W} \rangle = N(4\pi\lambda)^{-1} e^{\pi\sqrt{2\lambda}} + \ldots$. Note that, as above, in our definition we do not divide the Wilson loop operator by the dimension of the representation.[1]

Another indication that the explanation of the prefactor should be different is that, in general, one expects that the string counterpart of the large N term in $\langle \mathcal{W} \rangle$ should be the open-string partition function on the disk, which should contain an overall factor of the inverse power of the string coupling (corresponding to the Euler number $\chi = 1$), i.e.

$$\langle \mathcal{W} \rangle = Z_{\rm str} = \frac{1}{g_{\rm str}} \mathrm{Z}_1 + \mathcal{O}(g_{\rm str}) , \qquad \mathrm{Z}_1 = \int [dx] \ldots \, e^{-T \int d^2\sigma \, L} , \tag{1.1}$$

where $\frac{1}{g_{\rm str}}$ provides the required overall factor of N. The fact that it is natural to define the WL expectation value without the usual $1/N$ factor, and to include the $1/g_{\rm str}$ factor in its string theory counterpart, was also emphasized in [23].

In the $\mathcal{N} = 4$ SYM case we have [1, 2]

$$g_{\rm str} = \frac{g^2_{\rm YM}}{4\pi} = \frac{\lambda}{4\pi N} , \quad \lambda = g^2_{\rm YM} N , \quad T = \frac{\sqrt{\lambda}}{2\pi} , \quad \langle \mathcal{W} \rangle = \frac{N}{\lambda^{3/4}} \sqrt{\tfrac{2}{\pi}} \, e^{\sqrt{\lambda}} + \ldots , \tag{1.2}$$

while in the ABJM case [18, 22][2]

$$g_{\rm str} = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N} , \quad \lambda = \frac{N}{k} , \quad T = \frac{\sqrt{2\lambda}}{2} , \quad \langle \mathcal{W} \rangle = \frac{N}{4\pi\lambda} \, e^{\pi\sqrt{2\lambda}} + \ldots . \tag{1.3}$$

Our central observation is that both expressions for $\langle \mathcal{W} \rangle$ in (1.2) and (1.3) can be universally represented as

$$\langle \mathcal{W} \rangle = W_1 \Big[1 + \mathcal{O}(T^{-1}) \Big] + \mathcal{O}(g_{\rm str}) , \qquad W_1 = \frac{1}{g_{\rm str}} \sqrt{\frac{T}{2\pi}} \, e^{-\bar{\Gamma}_1} \, e^{2\pi T} , \tag{1.4}$$

where $\bar{\Gamma}_1$ is a numerical constant. Below we will argue that (1.4) should be the expression for the leading semiclassical result for the disk string path integral for a minimal surface in AdS_3 ending on a circle at the boundary (thus having induced AdS_2 geometry) in the $\mathrm{AdS}_n \times M^{10-n}$ string theory with tension T and coupling $g_{\rm str}$. In (1.4) the exponent $e^{2\pi T} = e^{-I_{\rm cl}}$ comes from the value of the classical string action $I_{\rm cl} = V_{\mathrm{AdS}_2} T = -2\pi T$.

[1] Our normalization of $\mathcal{W}$ in the $\frac{1}{2}$ BPS case corresponds in the localization calculation of [21, 22] to computing the matrix model expectation value $\langle \mathrm{Str} \begin{pmatrix} e^{i\mu_i} & 0 \\ 0 & e^{-i\nu_j} \end{pmatrix} \rangle$. Note that [22] defines the Wilson loop expectation value by including an extra overall factor of $g_{\rm CS} \equiv \frac{2\pi i}{k}$. Denoting by $\langle \mathrm{W} \rangle_{\rm loc}$ the expectation value given in [22], we find that the strong coupling limit of the $\frac{1}{2}$ BPS Wilson loop in the ABJM theory is $\langle \mathcal{W} \rangle = \frac{1}{g_{\rm CS}} \langle \mathrm{W} \rangle_{\rm loc} = \frac{1}{g_{\rm CS}} \frac{1}{2} e^{\pi\sqrt{2\lambda} + i\pi B} = \frac{k}{4\pi} e^{\pi\sqrt{2\lambda}} = \frac{N}{4\pi\lambda} e^{\pi\sqrt{2\lambda}}$, where we fixed the phase as $B = \frac{1}{2}$.

[2] Here the AdS_4 radius is $\mathrm{R} = (2\pi^2\lambda)^{1/4}\sqrt{\alpha'}$ with $T = \frac{\mathrm{R}^2}{2\pi\alpha'}$. The shift $\lambda \to \lambda - \frac{1}{24} + \ldots$ in the string tension [22, 24, 25] is irrelevant to the one-loop order (as discussed in [26], at the leading order we do not expect renormalization of the relation for the string tension).

The constant $\bar{\Gamma}_1$ comes from the ratio of one-loop determinants of string fluctuations near the minimal surface, and is found to be (see [6, 7, 9] and section 2 below)

$$\mathrm{AdS}_5 \times S^5 : \quad \bar{\Gamma}_1 = \frac{1}{2}\ln(2\pi)\,, \qquad\qquad \mathrm{AdS}_4 \times CP^3 : \quad \bar{\Gamma}_1 = 0\ . \tag{1.5}$$

Including also the $n = 3$ case of $\mathrm{AdS}_3 \times S^3 \times T^4$ string theory, one finds for $\mathrm{AdS}_n \times M^{10-n}$ with $n = 3, 4, 5$ that $\bar{\Gamma}_1 = \frac{1}{2}(n-4)\ln(2\pi)$ (see (2.21) below), and so in general W_1 in (1.4) is

$$W_1 = \frac{1}{(\sqrt{2\pi})^{n-3}}\,\frac{\sqrt{T}}{g_{\rm str}}\, e^{2\pi T}\ . \tag{1.6}$$

Using (1.5) one can check that the expression in (1.4) or (1.6) is in remarkable agreement with the gauge-theory expressions in (1.2) and (1.3).

As we explain below, it will also follow from our argument that at higher genera (disk with p handles with Euler number $\chi = 1 - 2p$) the $\sqrt{T}$ factor in (1.4) should be replaced by $(\sqrt{T})^{\chi}$, i.e. the corresponding term in the partition function should have a universal prefactor

$$\langle \mathcal{W} \rangle = \sum_{\chi=1,-1,\ldots} \mathrm{c}_\chi \left(\frac{\sqrt{T}}{g_s}\right)^{\chi} e^{2\pi T}\left[1 + \mathcal{O}(T^{-1})\right]\ . \tag{1.7}$$

This is indeed consistent with the structure of $1/N$ corrections found on the gauge theory side in [2] and in [22] (see section 5).

It remains to understand the origin of the simple prefactor $\sqrt{\frac{T}{2\pi}}$ in (1.4). In general, the expression for such a prefactor in the path integral is very sensitive to the definition of path integral measure which is subtle in string theory. In section 3 below we will provide an explanation for the presence of the $\sqrt{T}$ factor starting from the superstring path integral in the static gauge [6] (see also appendix A.1) but we will not be able to determine the origin of the remaining $\frac{1}{\sqrt{2\pi}}$ constant from first principles. This is already a non-trivial result: since the presence of this constant is fixed by the comparison with the SYM theory, we then have the string theory explanation for the ABJM expression in (1.3) (or vice-versa).

In section 4 we shall provide another consistency check of the universal expression for the string partition function (1.4) by considering the analog of the familiar soft dilaton insertion relation and dilaton tadpole on the disk.

In section 5 we will emphasize the fact that the universal prefactor in the disk partition function $\sim \frac{\sqrt{T}}{g_{\rm str}}$ in (1.4) has a natural generalization (1.7) to higher orders which is consistent with the structure of the $1/N$ corrections found on the gauge theory side. We will make some concluding remarks about some other WL examples in section 6.

It is interesting to note that the factor $\sqrt{\frac{T}{2\pi}}$ in (1.4) looks exactly like the one associated with just one bosonic zero mode (in the standard normalization of the path integral zero-mode measure, i.e. $\frac{1}{\sqrt{2\pi\hbar}}$, $\hbar^{-1} = T$, as was used in a similar context in [14, 15]).[3]

[3]Here we assume that the path integral measure for a scalar field is normalized so that the gaussian integral has a fixed value $\int[dx]\exp[-\frac{1}{2\hbar}(x,x)] = 1$, i.e. $[dx] = \prod_\sigma \frac{dx(\sigma)}{\sqrt{2\pi\hbar}}$. Then the factor of string tension $T = \hbar^{-1}$ appears both in the measure and in the action and cancels out in the one-loop determinant expression apart from possible 0-mode contribution.

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In appendix A.2 we will discuss a possible origin of this zero-mode factor, assuming one starts with the disk path integral in conformal gauge where there is an extra factor containing the ratio of the ghost determinant and the determinant of the two "longitudinal" string coordinates subject to "mixed" Dirichlet/Neumann boundary conditions, and thus admitting conformal Killing zero modes discussed in appendix B.

2 One-loop string correction in static gauge

consider a circular WL surface with AdS_2 induced geometry, which resides in an AdS_3 subspace of $\mathrm{AdS}_n \times M^{10-n}$, specifically:

$$\text{(i) } n=5: \quad \mathrm{AdS}_5 \times S^5; \qquad \text{(ii) } n=4: \quad \mathrm{AdS}_4 \times CP^3; \qquad \text{(iii) } n=3: \quad \mathrm{AdS}_3 \times S^3 \times T^4.$$

The string is point-like in the internal compact directions, satisfying Dirichlet boundary conditions. In general, the planar WL expectation value is given by the string path integral with a disk-like world sheet ending on a circle at the boundary of AdS space, $\langle \mathcal{W} \rangle = e^{-\Gamma}$, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \dots$. Here $\Gamma_0 = -2\pi T$ is the classical string action (proportional to the renormalized AdS_2 volume $V_{\mathrm{AdS}_2} = -2\pi$) and $\Gamma_1 = \mathcal{O}(T^0)$ is given by sum of logarithms of fluctuation determinants (in which we include possible measure-related normalization factors).

We shall discuss the computation of the one-loop correction $\Gamma_1 \equiv \Gamma_1^{(n)}$ in the above $\mathrm{AdS}_n \times M^{10-n}$ cases following the heat kernel method applied in the $\mathrm{AdS}_5 \times S^5$ case in [6] and [9]. In this $n=5$ case the general form of the static-gauge string one-loop correction is [6]

$$\Gamma_1^{(5)} = \frac{1}{2} \log \frac{[\det(-\nabla^2+2)]^2 \, \det(-\nabla^2 + R^{(2)} + 4) \, [\det(-\nabla^2)]^5}{[\det(-\nabla^2 + \frac{1}{4}R^{(2)} + 1)]^8} \tag{2.1}$$

$$= \frac{1}{2} \log \frac{[\det(-\nabla^2+2)]^3 \, [\det(-\nabla^2)]^5}{[\det(-\nabla^2 + \frac{1}{2})]^8} \, . \tag{2.2}$$

Here we assumed that the AdS radius R is scaled out and absorbed into the string tension $T = \frac{\mathrm{R}^2}{2\pi\alpha'}$ so that all operators are defined in the induced AdS_2 metric with radius 1 and curvature $R^{(2)} = -2$. We will come back to the radius dependence in section 3 below. In (2.1) we isolated the contribution of one special transverse AdS_5 mode that, in general, is different from the other two: this is the AdS_3 mode transverse to the minimal surface (the other two transverse modes are transverse to AdS_3), see [6, 27]. In the present case of the minimal surface being AdS_2 we have $R^{(2)} = -2$ so that its mass is actually the same as of the other two transverse AdS_5 modes.

Similar expression (2.1) is found in the conformal gauge [6], provided the contribution of the two "longitudinal" modes cancels as in flat space [28] against that of the ghost determinant and Mobius volume factor (modulo the 0-mode part of the longitudinal operator and related definition of path integral measure, see appendix A.1 for further discussion).

In the less supersymmetric cases with $\mathrm{AdS}_5 \to \mathrm{AdS}_n$ and $n = 4, 3$ there are less massive bosonic AdS directions and part of the fermions are massless, i.e. we get the following generalization of (2.1)

$$\Gamma_1^{(n)} = \frac{1}{2}\ln\frac{[\det(-\nabla^2+2)]^{n-3}\,\det(-\nabla^2+R^{(2)}+4)\,[\det(-\nabla^2)]^{10-n}}{[\det(-\nabla^2+\frac{1}{4}R^{(2)}+1)]^{2n-2}\,[\det(-\nabla^2+\frac{1}{4}R^{(2)})]^{10-2n}} \tag{2.3}$$

$$= \frac{1}{2}\ln\frac{[\det(-\nabla^2+2)]^{n-2}\,[\det(-\nabla^2)]^{10-n}}{[\det(-\nabla^2+\frac{1}{2})]^{2n-2}\,[\det(-\nabla^2-\frac{1}{2})]^{10-2n}} \tag{2.4}$$

The fermion masses are controlled by the superstring kinetic term with a projection matrix in the mass term. In the $\mathrm{AdS}_3\times S^3\times T^4$ case [6] there are 4 massless fermion modes (which are partners of T^4 bosonic modes) and 4 massive ones. In the $\mathrm{AdS}_4\times CP^3$ case one finds [29] that there are $2n-2=6$ massive and $10-2n=2$ massless fermionic modes.

Let us first discuss the divergent part of (2.3) assuming the standard heat-kernel regularization separately for each determinant contribution. The UV divergent part of $\Gamma_1=\frac{1}{2}\log\det(-\nabla^2+X)$ where $-\nabla^2+X$ is a scalar Laplacian is given by $(\Lambda\to\infty)$

$$\Gamma_{1,\infty}=-B_2\log\Lambda\,,\qquad B_2=\frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}\,b_2\,,\qquad b_2=\frac{1}{6}R^{(2)}-X\ . \tag{2.5}$$

Here we ignore boundary contributions (they contain a power of IR cutoff and are absent after renormalization of the AdS_2 volume or directly using the finite value for the Euler number of the minimal surface).

In the case of (2.2) we then find that in the total combination all $\frac{1}{6}R^{(2)}$ terms cancel out (due to balance of bosonic and fermionic d.o.f.) and the constant mass terms also cancel between bosons and fermions so that we are left only with contributions of $R^{(2)}$ terms from one special bosonic mode and the fermionic modes

$$b_{2,\mathrm{tot}}^{(5)} = -R^{(2)}-8\Big(-\frac{1}{4}R^{(2)}\Big)=R^{(2)}\,,$$
$$B_{2,\mathrm{tot}}^{(5)} = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}\,R^{(2)}=\chi=\frac{1}{4\pi}(-2\pi)(-2)=1. \tag{2.6}$$

For general n the corresponding UV divergent part of (2.3) is given by the straightforward generalization of (2.6). Again, all $\frac{1}{6}R^{(2)}$ terms in (2.5) cancel out as do the constant mass terms and we find

$$b_{2,\mathrm{tot}}^{(n)} = -(n-3)2-(R^{(2)}+4)-(2n-2)\Big(-\frac{1}{4}R^{(2)}-1\Big)-(10-2n)\Big(-\frac{1}{4}R^{(2)}\Big)$$
$$= R^{(2)}\,, \tag{2.7}$$

$$B_{2,\mathrm{tot}}^{(n)} = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}\,R^{(2)}=\chi=1\ . \tag{2.8}$$

The total result (coming again just from the $R^{(2)}$ terms in single bosonic mode and 8 fermionic modes) is thus universal, i.e. n-independent.

Moreover, the same result $B_{2\,\mathrm{tot}}=\chi$ for the coefficient of the UV divergence is found for fluctuations near any minimal surface (not even lying within AdS_3) that has disk topology (see [6, 30]): if X is a "mass matrix", the contribution of 8 transverse bosons is $b_{2b}=8\cdot\frac{1}{6}R^{(2)}-\mathrm{tr}\,X-R^{(2)}$ while of 8 fermions is $b_{2f}=8\cdot\frac{1}{12}R^{(2)}+\mathrm{tr}\,X$ so that $b_{2\,\mathrm{tot}}=R^{(2)}$.

Note that in general the Seeley coefficient is $B_2 = \zeta(0)+n_0$ where $\zeta(0)$ is the regularized number of all non-zero modes and $n_0 = n_b - \frac{1}{2}n_f$ is the effective number of all 0-modes (assuming fermions are counted as Majorana or Weyl). In the present static gauge case there are no obvious normalizable 0-modes (cf. remark below (B.10)), but we observe that the result (2.8) is formally the same as what would come just from one "uncanceled" bosonic mode.

The universality of (2.8) strongly suggests that the mechanism of cancellation of this total "topological" UV divergence should also be universal. One may absorb it into the definition of the superstring path integral measure or cancel it against other measure factors as discussed in the conformal gauge in [6].[4] An alternative is to use a special "2d supersymmetric" definition of the one-loop path integral in the static gauge (see below): the cancellation of UV divergences is, in fact, automatic if one uses a "spectral" representation for the total Γ_1 rather than heat kernel cutoff for each individual determinant.

Let us now turn to the finite part of the one-loop effective action in (2.4). We will follow [9] which completed the original computation in [6] of Γ_1 in (2.19) based on expressing the determinants in (2.2) in terms of the well known [37–41] heat kernels of the scalar and spinor Laplacians on AdS_2. $\Gamma_1^{(n)}$ in (2.3) contains the contributions of the following AdS_2 fields: (i) $n-2$ scalars with $m^2 = 2$; (ii) $10-n$ scalars with $m^2 = 0$; (iii) $2n-2$ Majorana fermions with $m^2 = 1$; (iv) $10-2n$ Majorana fermions with $m^2 = 0$. We will temporarily set the AdS_2 radius to 1 and discuss the dependence on it later. Let us first use the heat-kernel cutoff for each individual determinant in (2.3), i.e.

$$\frac{1}{2}\ln\det\Delta = -\frac{1}{2}V_{AdS_2}\int_{\Lambda^{-2}}^{\infty}\frac{dt}{t}\, K(t)\,, \qquad V_{AdS_2} = -2\pi\ . \tag{2.9}$$

The trace of heat kernel $K(t)$ for a real scalar and a Majorana 2d fermion may be written as

$$K(t) = \frac{1}{2\pi}\int_0^{\infty} dv\ \mu(v)\ e^{-t(v^2+M)}\,, \tag{2.10}$$

$$\begin{aligned}\mu_b(v) &= v\tanh(\pi v)\,, \qquad M = \frac{1}{4}+m^2\ ;\\ \mu_f(v) &= -v\coth(\pi v)\,, \qquad M = m^2\ .\end{aligned} \tag{2.11}$$

[4]To recall, the UV divergences do not cancel automatically even in the bosonic string theory in flat space. The combination of D scalar Laplacians and the conformal ghost operator $\Delta_{gh} = P^\dagger P$ gives (with all modes counted) [35] $B_2 = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}(\frac{D}{6}R^{(2)} - (\frac{2}{6}R^{(2)}+R^{(2)}) = \frac{1}{6}(D-8)\chi$. Assuming, following [36], that there are extra powers of the UV cutoff in the Mobius volume one divides over and in the integrals over moduli, the net result is that one should add to the above B_2 an extra $\delta_{top}B_2 = -3\chi = \dim\ker P^\dagger - \dim\ker P$, thus getting $B_2 = \frac{1}{6}(D-8-18)\chi = \frac{1}{6}(D-26)\chi$. A similar argument applies to the NSR string where $B_2 = \frac{1}{4}(D-10)\chi$. In the present $D=10$ GS superstring case there is an extra conformal anomaly/divergence from the Jacobian of rotation from GS fermions to 2d fermions (see [31–34]); this effectively amounts to adding 3 extra massless fermion contributions for each 2d fermion contribution (or, equivalently, multiplying the $\frac{1}{6}R^{(2)} - \frac{1}{4}R^{(2)}$ part of each fermion contribution to b_2 by 4); this gives $\delta_1 B_2 = -3\times 8\times\frac{1}{4\pi}\int d^2\sigma\sqrt{g}(\frac{1}{6}R^{(2)} - \frac{1}{4}R^{(2)}) = 2\chi$.

In the conformal gauge the divergences from the determinant of the ghost operator $(\Delta_{gh})_{ab} = -g_{ab}\nabla^2 - R_{ab}$ cancel against those of the determinant of operator Δ_{long} for 2 longitudinal scalars. As in the bosonic case, one should also add $\delta_{top}B_2 = -3\chi$ as explained above. Summing these contributions with (2.8) gives $B_{2\,tot} = B^{(n)}_{2,tot} + \delta_1 B_2 + \delta_{top}B_2 = \chi + 2\chi - 3\chi = 0$.

Here in μ_f we already accounted for the negative sign of the fermion contribution, so the total K is just the sum of the bosonic and fermionic terms. The associated ζ-function is

$$\zeta(z) = -\frac{1}{\Gamma(z)} \int_0^\infty dv\, \mu(v) \int_0^\infty dt\ t^{z-1}\ e^{-t(v^2+M)} = -\int_0^\infty dv\ \frac{\mu(v)}{(v^2+M)^z}\,. \tag{2.12}$$

For example, for AdS_2 scalars $\zeta(0) = B_2 = -\frac{1}{2}b_2 = \frac{1}{6} - \frac{1}{2}m^2$. The total value of $\zeta(0)$ is found to be 1, i.e. the same as in (2.8). In general, the one-loop correction is

$$\Gamma_1 = \sum \frac{1}{2}\log\det\Delta = -\zeta_{\rm tot}(0)\log\Lambda + \bar{\Gamma}_1\,, \quad \bar{\Gamma}_1 \equiv -\frac{1}{2}\zeta'_{\rm tot}(0)\,, \quad \zeta_{\rm tot}(0) = 1\,. \tag{2.13}$$

For the derivative of the scalar ζ-function one finds (A is the Glaisher constant)

$$\begin{aligned}
\zeta'_b(0,M) &= -\frac{1}{12}(1+\ln 2) + \ln A - \int_0^M dx\,\psi\Big(\sqrt{x}+\frac{1}{2}\Big)\,, && (2.14)\\
\zeta'_b\Big(0,\frac{9}{4}\Big) &= -\frac{25}{12} + \frac{3}{2}\ln(2\pi) - 2\ln A\,, \\
\zeta'_b\Big(0,\frac{1}{4}\Big) &= -\frac{1}{12} + \frac{1}{2}\ln(2\pi) - 2\ln A\,, && (2.15)
\end{aligned}$$

while for the massive fermion

$$\begin{aligned}
\zeta'_f(0,M) &= -\frac{1}{6} + 2\ln A + \sqrt{M} + \int_0^M dx\,\psi(\sqrt{x})\,, && (2.16)\\
\zeta'_f(0,1) &= \frac{5}{6} - \ln(2\pi) + 2\ln A\,, \qquad \zeta'_f(0,0) = -\frac{1}{6} + 2\ln A\,. && (2.17)
\end{aligned}$$

The total contribution to the finite part $\bar{\Gamma}_1^{(n)}$ in (2.13) corresponding to (2.3) then found to have a simple form

$$\begin{aligned}
\bar{\Gamma}_1^{(n)} &= -\frac{1}{2}\Big[(n-2)\zeta'_b\Big(0,\frac{9}{4}\Big) + (10-n)\zeta'_b\Big(0,\frac{1}{4}\Big) + (2n-2)\zeta'_f(0,1) + (10-2n)\zeta'_f(0,0)\Big] \\
&= \frac{1}{2}(n-4)\ln(2\pi)\,. && (2.18)\\
\bar{\Gamma}_1^{(5)} &= \frac{1}{2}\ln(2\pi)\,, \qquad \bar{\Gamma}_1^{(4)} = 0\,, \qquad \bar{\Gamma}_1^{(3)} = -\frac{1}{2}\ln(2\pi)\,. && (2.19)
\end{aligned}$$

In the $\mathrm{AdS}_5 \times S^5$ case ($n=5$) the computation of the corresponding determinants was also carried out using different methods in [7, 8] with the finite part of the resulting expression for $\bar{\Gamma}_1^{(5)}$ being as in (1.5), (2.19). Note that the finite part (2.18) happens to vanish in the $\mathrm{AdS}_4 \times CP^3$ case ($n=4$).

It is interesting to note that there exists a special definition of Γ_1 in (2.3) that automatically gives a UV finite one-loop result. Instead of computing separately each determinant let us use (2.9) and sum up the corresponding spectral integral expressions under a common integral over v in (2.10). Interchanging the order of t- and v- integrals and first integrating over t we see that this integral is finite, i.e. the proper-time cutoff is not required. Using (2.10)–(2.11) we then get for (2.4)

$$\begin{aligned}
\bar{\Gamma}_1^{(n)} = \frac{1}{2}\frac{V_{\mathrm{AdS}_2}}{2\pi}\int_0^\infty dv\ v\Big(&\tanh(\pi v)\Big[(n-2)\ln\Big(v^2+\frac{9}{4}\Big) + (10-n)\ln\Big(v^2+\frac{1}{4}\Big)\Big] \\
&- \coth(\pi v)\big[(2n-2)\ln(v^2+1) + (10-2n)\ln(v^2)\big]\Big)\,, && (2.20)
\end{aligned}$$

where $\frac{V_{\text{AdS}_2}}{2\pi} = -1$. Remarkably, the integral over v here is convergent at both $v = 0$ and $v = \infty$ (i.e. in the UV). In general, given the structure of the eigenvalues in (2.10)–(2.11), one can see that convergence of the representation (2.20) in the UV requires the sum rule $\sum_b (m_b^2 + \frac{1}{4}) - \sum_f m_f^2 = 0$, which is satisfied for the spectra in our problem. Evaluation of (2.20) gives then a finite result equal to the one in (2.18), i.e.

$$\bar{\Gamma}_1^{(n)} = \frac{1}{2}(n-4)\ln(2\pi)\,. \tag{2.21}$$

This prescription of not using proper-time cutoff for individual log det terms, i.e. first combining the integrands and then doing the spectral integral, may be viewed as a kind of "2d supersymmetric" regularization. Indeed, the balance of the bosonic and fermionic degrees of freedom in (2.4) suggests hidden AdS_2 supersymmetry [6].[5] Then the prescription of combining the spectral integrands of the determinants together may be viewed as a result of a "superfield" computation manifestly preserving 2d supersymmetry. Note however that, even though the integral in (2.20) is finite, a dependence of Γ_1 on a normalization scale reappears on dimensional grounds if one restores the dependence on the radius R inside the logarithms, as explained in the next section. This leads to an explanation of the T-dependent prefactor in (1.4) and (1.7).

3 Dependence on AdS radius: origin of the $\sqrt{T}$ prefactor

Let us now explain the presence of the $\sqrt{T} = \frac{\text{R}}{\sqrt{2\pi\alpha'}}$ prefactor in the string one-loop partition function (1.4). As the definition of quantum string path integral (in particular, integration measure) is subtle and potentially ambiguous our aim is to identify the one that is consistent with underlying symmetries and AdS/CFT duality.

In the previous section we ignored the dependence of the one-loop correction on the AdS radius R. Let us now discuss how the string path integral may depend on it. Let us start with the classical string action in AdS_n of radius R. One possible approach is to rescale the 2d fields so that the factor of R^2 appears in front of the action[6]

$$I = \frac{1}{2}T_0 \int d^2\sigma\, \sqrt{g}\, G_{mn}(x)\, \partial^a x^m \partial_a x^n + \ldots \tag{3.1}$$

$$= \frac{1}{2}T \int d^2\sigma\, \sqrt{g}\, \bar{G}_{mn}(\bar{x})\, \partial^a \bar{x}^m \partial_a \bar{x}^n + \ldots, \qquad T = \text{R}^2 T_0, \qquad T_0 = \frac{1}{2\pi\alpha'}\,. \tag{3.2}$$

Using either (3.1) or (3.2) the expression for one-loop correction will depend also on the assumption about the path integral measure. If the measure is defined covariantly the final result should be the same.

[5] One implication is the vanishing of the corresponding vacuum energy in AdS_2 observed in [6] in the case of the strip parametrization $ds^2 = \frac{1}{\cos^2\rho}(dt^2 + d\rho^2)$, $\rho \in (-\frac{\pi}{2}, \frac{\pi}{2})$. To recall, the contributions of a scalar with mass m_b^2 and a fermion with mass m_f^2 to the AdS_2 vacuum energy are [6] $E_b(m^2) = -\frac{1}{4}(m^2 + \frac{1}{6})$ and $E_f(m^2) = \frac{1}{4}(m^2 - \frac{1}{12})$ so that for the spectrum in (2.3) we get $E_{\text{tot}} = (n-2)E_b(2) + (10-n)E_b(0) + (2n-2)E_f(1) + (10-2n)E_f(0) = (n-2)(-\frac{1}{2} - \frac{1}{24}) + (10-n)(-\frac{1}{24}) + (2n-2)(\frac{11}{48}) + (10-2n)(-\frac{1}{48}) = 0$.

[6] For example, starting with $ds^2 = dr^2 + e^{2r/\text{R}} dx_i dx_i$ we get $ds^2 = \text{R}^2(d\bar{r}^2 + e^{2\bar{r}} d\bar{x}_i d\bar{x}_i)$. Note that after the rescaling the tension T and coordinates $\bar{x}^m$ are dimensionless.

Let us consider the path integral defined by (3.1) in terms of the original unrescaled coordinates x^m of natural dimension of length, so that $G_{mn}(x)$ is dimensionless and depends on the AdS scale R. The string σ-model path integral may be defined symbolically as (cf. footnote 3)

$$Z = \int \prod_{\sigma,m} \sqrt{\frac{T_0}{2\pi}} \sqrt{G(x(\sigma))}\, [dx^m(\sigma)] \ldots \\ \times \exp\Big[-\frac{1}{2} T_0 \int d^2\sigma \sqrt{g}\, G_{mn}(x)\, \partial^a x^m \partial_a x^n + \ldots \Big] \,. \tag{3.3}$$

Expanding near the minimal surface ending on the boundary circle we will get the induced AdS_2 metric depending on the same curvature scale R as G_{mn}. Then rotating the fluctuation fields to the tangent-space components $\tilde{\mathrm{x}}^r$ and also rescaling them by $\sqrt{T_0}$ (so that they will be normalized as $|\tilde{\mathrm{x}}|^2 = \int d^2\sigma \sqrt{g}\, \tilde{\mathrm{x}}^r \tilde{\mathrm{x}}^r$) we will find that the 1-loop contribution from a single scalar is $Z_1 = (\det \Delta)^{-1/2}$ where $\Delta = -\nabla^2 + \ldots$ depends on the induced AdS_2 metric and has canonical dimension of (length)$^{-2}$ with eigenvalues scaling as R^{-2}. In the heat kernel representation $\Gamma_1 = -\log Z_1 = \frac{1}{2} \log\det\Delta = -\frac{1}{2} \int_{\Lambda^{-2}}^{\infty} \frac{dt}{t}\, \mathrm{tr}\, \exp(-t\Delta)$ the parameter t and the cutoff Λ^{-2} will now have dimension of (length)^2 and we will get instead of (2.13) (cf. (2.5), (2.8))

$$\Gamma_1 = -\zeta_{\rm tot}(0) \log(\mathrm{R}\Lambda) + \bar{\Gamma}_1 \,, \qquad\qquad \zeta_{\rm tot}(0) = \chi = 1 \tag{3.4}$$

As discussed in section 2, the UV divergence is expected to be cancelled by an extra "universal" contribution $\log(\sqrt{\alpha'}\Lambda)$ from the superstring measure (see footnote 4). We assume that this universal contribution (depending only on the Euler number of the world sheet but not on details of its metric) may only involve the string scale $\sqrt{\alpha'}$ but not the AdS radius. As a result, $\Gamma_{1\,\rm fin} = -\chi \log \frac{\mathrm{R}}{\sqrt{\alpha'}} + \bar{\Gamma}_1$. The argument of log is thus $\sim (\sqrt{T})^{\chi}$, i.e. we get

$$Z \sim e^{-\Gamma_1} \quad \to \quad \big(\sqrt{T}\big)^{\zeta_{\rm tot}(0)} = (\sqrt{T})^{\chi} = \sqrt{T} \,. \tag{3.5}$$

This explains the origin of the $\sqrt{T}$ factor in the disk partition function (1.4).

As was noted below (2.8), the coefficient of the UV divergent term in (3.4) is, in fact, the same for all minimal surfaces with disk topology and thus the dependence of the string partition function on the scale R or effective tension T through the $\sqrt{T}$ factor in (3.5) should be universal. This means, in particular, that the factors $1/g_{\rm s}$ and $\sqrt{T}$ in (1.4) will cancel in the ratio of expectation values of different Wilson loops with disk topology. Moreover, the fact that the power of T in (3.5) is controlled by the Euler number χ implies that at higher genera, for a disk with p handles, we should find that $\langle \mathcal{W} \rangle$ includes the universal prefactor $(\sqrt{T}/g_{\rm str})^{\chi}$ as in (1.7). This is in precise agreement with the large N expansion of the localization results both in $\mathcal{N} = 4$ SYM and ABJM cases, as we explain in more detail in section 5.

The result of adding the above universal counterterm $\log(\sqrt{\alpha'}\Lambda)$ is equivalent to just defining the one-loop partition function to be UV finite by first combining all the contributions using the spectral representation (2.20). There we set $\mathrm{R} = 1$ and to restore the dependence on the radius R of the AdS_2 metric we need to add the mass scale factor R^{-2} under the logs in (2.20) (cf. (2.9), (2.10)). To make the argument of the logs dimensionless we also need to introduce some normalization scale ℓ (i.e. $\log\det\Delta \to \log\det(\ell^2\Delta)$ or, equivalently, add ℓ factor in the path integral measure). Then we find that $\Gamma_1^{(n)}$ in (2.20) depends on R via the same $\zeta_{\rm tot}(0) = 1$ term as in (2.13), (3.4), i.e. via an extra contribution (to be added to (2.21))

$$\delta\Gamma_1^{(n)} = \frac{1}{2}\frac{V_{\mathrm{AdS}_2}}{2\pi}\, 8\log(\mathrm{R}^{-2}\ell^2)\int_0^\infty dv\ v\left[\tanh(\pi v) - \coth(\pi v)\right] = -\log(\mathrm{R}\,\ell^{-1})\ . \quad (3.6)$$

The dependence on ℓ illustrates the fact that as long as $\zeta_{\rm tot}(0) \neq 0$, the one-loop contribution, even if defined to be UV finite by the spectral representation (or some analytic regularization like the ζ-function one [42]), is still scheme (or measure) dependent. Choosing $\ell \sim \sqrt{\alpha'}$, which is here an obvious choice in the absence of any other available scales (and which is also suggested by the T_0 dependence in (3.3)), we again end up with the required result (3.5).

We shall discuss some other approaches to the derivation of the dependence of the one-loop correction on T in the next section and appendices A.1 and (A.2).

4 Dilaton insertion and derivative over gauge coupling

As another check of consistency and universality of the expression (1.4) for the 1-loop string partition function for a minimal surface with disk topology, let us consider a closely related object — the insertion of the dilaton operator in the expectation value or the dilaton tadpole on the disk with WL boundary conditions. Here we shall explicitly consider the SYM case but a similar discussion should apply also to the ABJM case.

Let us first recall the zero-momentum dilaton insertion relation, or the familiar "soft dilaton theorem" in flat space. The dilaton ϕ couples to the string as [43]

$$I = \int d^2\sigma\sqrt{g}\left[\frac{1}{2}T_0 G_{mn}(x)\partial^a x^m \partial_a x^n + \frac{1}{4\pi}R^{(2)}\phi(x)\right], \quad (4.1)$$

where $T_0 = \frac{1}{2\pi\alpha'}$. The string-frame metric G_{mn} expressed in terms of the Einstein-frame metric in D dimensions is $G_{mn} = e^{\frac{4}{D-2}\phi}\bar{G}_{mn}$, $\bar{G}_{mn} = \delta_{mn} + h_{mn}$ and thus the (zero-momentum) dilaton vertex operator in flat space is (cf. [45, 46])[7]

$$\begin{aligned} I &= I_0 - V_0\phi + \dots\,, \\ V_0 &= -\frac{4}{D-2}\int d^2\sigma\sqrt{g}\left[\frac{1}{2}T_0\partial^a x^m\partial_a x_m + \frac{D-2}{4}\frac{1}{4\pi}R^{(2)}\right] = -\frac{4}{D-2}I_0 - \chi\,, \end{aligned} \quad (4.2)$$

where $I_0 = \frac{1}{2}T_0\int d^2\sigma\sqrt{g}\,\partial^a x^m\partial_a x_m$ and $\chi = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R^{(2)}$. Since the expectation value of the action I_0 may be obtained by applying $-T_0\frac{\partial}{\partial T_0}$ to the string path integral (cf. (A.2)),

[7] The canonically normalized dilaton field $\bar\phi$ that appears in the generating functional for scattering amplitudes, i.e. having the same kinetic term as the graviton in the effective action, $S \sim \int d^D x\sqrt{\bar G}[-2\bar R + \frac{1}{2}(\partial\bar\phi)^2 + \dots]$, is related to ϕ as $\bar\phi = \frac{4}{\sqrt{D-2}}\phi$ so that $I = I_0 - \bar V_0\bar\phi + \dots$, $\bar V_0 = -\frac{1}{\sqrt{D-2}}(I_0 + \frac{D-2}{4}\chi)$. Note also that in (4.2) we ignored possible boundary term as its role usually is only to ensure the coupling to the correct value of the Euler number.

the insertion of the zero-momentum dilaton into the generating functional for scattering amplitudes $Z = \int [dx] e^{-I_0 + V_0\phi + V_h h + \dots}$ is then given by (here $\langle 1 \rangle$=1)

$$\frac{\partial}{\partial\phi} \log Z = \langle V_0 \rangle = -\frac{4}{D-2} \langle I_0 \rangle - \chi = \frac{4}{D-2} T_0 \frac{\partial}{\partial T_0} \log Z - \chi \ . \tag{4.3}$$

In the standard cases of a bosonic closed string or open string with Neumann boundary conditions there are D constant 0-modes, so one finds $Z \sim T_0^{D/2}$ and $\langle I_0 \rangle = -\frac{1}{2} D$ (assuming "covariant" regularization in which $\delta^{(2)}(\sigma, \sigma) = 0$, see [44]). The same relation is true also for the fermionic string as the number of bosonic translational 0-modes remains the same.

In the superstring case ($D = 10$) for the tree-level topology of a disk ($\chi = 1$) eq. (4.3) reads

$$\frac{\partial}{\partial\phi} \log Z = \langle V_0 \rangle = -\frac{1}{2} \langle I_0 \rangle - \chi = \frac{1}{2} T_0 \frac{\partial}{\partial T_0} \ln Z - 1 \ . \tag{4.4}$$

Adapting this relation to our present case of fixed contour boundary conditions with the expectation value of the action given by $\langle I \rangle = -\frac{1}{2}$ (see (A.6)) the analog of (4.4) becomes (including in $\langle I \rangle$ also the classical contribution of an AdS_2 minimal surface $\langle I \rangle_{\rm cl} = T(-2\pi) = -\sqrt{\lambda}$)

$$\frac{\partial}{\partial\phi} \log Z = \langle V_0 \rangle = -\frac{1}{2} \langle I \rangle - 1 = \frac{1}{2} \left(\sqrt{\lambda} + \frac{1}{2} - 2 \right) = \frac{1}{2} \sqrt{\lambda} - \frac{3}{4} \ . \tag{4.5}$$

Since the constant part of the dilaton is related to the string coupling which itself is related to the SYM coupling as in (1.2), i.e. $g^2_{\rm YM} = 4\pi g_{\rm str} = 4\pi e^{\phi}$, we may compare (4.5) to the derivative of the circular WL expectation value with respect to the coupling constant on the gauge theory side. The normalized gauge theory path integral is defined by $\langle \dots \rangle_{\rm SYM} \sim \int [dA \dots] e^{-S_{\rm SYM}} \dots$, $\langle 1 \rangle_{\rm SYM} = 1$ where

$$S_{\rm YM} = \int d^4x \, L_{\rm SYM} \ , \qquad\qquad L_{\rm SYM} = \frac{1}{4g^2_{\rm YM}} {\rm tr}\, (F^2_{mn} + \dots) \ . \tag{4.6}$$

We assume that the metric is Euclidean and the $SU(N)$ generators are normalized as ${\rm tr}\, (T_i T_j) = \delta_{ij}$. Since the factor in front of the action is

$$e^{-\phi} = g^{-1}_{\rm str} = \frac{4\pi}{g^2_{\rm YM}} = \frac{4\pi N}{\lambda} \ , \tag{4.7}$$

the derivative over the constant part of the dilaton ϕ corresponds on the gauge-theory side to the insertion of the SYM action into a correlator. In particular, in the case of the WL expectation value (here and below $\langle \mathcal{W} \rangle \equiv \langle \mathcal{W} \rangle_{\rm SYM}$)

$$\frac{\partial}{\partial\phi} \log \langle \mathcal{W} \rangle = \frac{\langle S_{\rm SYM} \mathcal{W} \rangle}{\langle \mathcal{W} \rangle} = \lambda \frac{\partial}{\partial\lambda} \log \langle \mathcal{W} \rangle \ . \tag{4.8}$$

Since the gauge-theory result at strong coupling is $\langle \mathcal{W} \rangle = N \lambda^{-3/4} \sqrt{\frac{2}{\pi}} \, e^{\sqrt{\lambda}} + \dots$ we conclude that

$$\frac{\partial}{\partial\phi} \log \langle \mathcal{W} \rangle = \lambda \frac{\partial}{\partial\lambda} \log \langle \mathcal{W} \rangle = \frac{1}{2} \sqrt{\lambda} - \frac{3}{4} + \dots \ , \tag{4.9}$$

which is in agreement with the string theory expression (4.5).

Note that while in the string theory relation (4.5) we used that the insertion of the string action is given by derivative over the tension, on the gauge theory side a similar relation (4.8) involves differentiation over the gauge coupling. The two are in agreement because on the string side the dependence on λ comes from both the dependence on the tension and also dependence on the string coupling (the $-\chi = -1$ term in (4.4), (4.5)). Thus, once again, one needs the independent $\frac{1}{g_{\rm str}}$ and $\sqrt{T}$ factors in the string theory disk partition function (1.4) in order to have the consistency between the dilaton derivatives, or equality of the dilaton insertions on the string and gauge theory sides.

The above discussion has a natural generalization to the string partition function on a disk with handles or $1/N$ corrections on the gauge theory side. For a surface of Euler number χ, using (A.8) we get the following analog of (4.3) generalizing (4.5)

$$\frac{\partial}{\partial\phi}\log Z = \langle V_0\rangle = -\frac{1}{2}\langle I\rangle - \chi = \frac{1}{2}T\frac{\partial}{\partial T}\log Z - \chi = \frac{1}{2}\sqrt{\lambda} - \frac{3}{4}\chi\,. \tag{4.10}$$

The subleading term $-\frac{3}{4}\chi$ is consistent with the general form of the prefactor $Z \sim (\frac{\sqrt{T}}{g_{\rm str}})^{\chi}$ in (1.7). Indeed, note that $g_{\rm str} = e^{\phi}$ and that switching to the Einstein-frame metric (cf. (4.2)) corresponds to $T \to e^{\frac{1}{2}\phi}T$ (cf. (4.2)), so that $\frac{\sqrt{T}}{g_{\rm str}} \sim e^{-\frac{3}{4}\phi}$. This is in agreement with the gauge-theory side since the dependence on the dilaton is directly correlated as in (4.7), (4.8) with the dependence on λ (which appears only as a factor in front of the SYM action), while the dependence on N may come not only from the factor (4.7) in the action but also from traces in higher order gauge-theory correlators. Indeed, according to the gauge-theory result (see (5.1)) the genus p term in $\langle\mathcal{W}\rangle$ depends on λ as $\lambda^{\frac{6p-3}{4}} = \lambda^{-\frac{3}{4}\chi}$.

One can also perform a further consistency check by considering a direct generalization of the above relations to the case of the local (i.e. "non zero-momentum") dilaton operator insertion. On the gauge theory side the derivative over a local coupling or local dilaton is essentially the Lagrangian in (4.6) and one finds [50–52][8]

$$\frac{\delta}{\delta\phi(x)}\log\langle\mathcal{W}\rangle = \frac{\langle L_{\rm SYM}(x)\,\mathcal{W}\rangle}{\langle\mathcal{W}\rangle} = -\frac{1}{8\pi^2\, d_\perp^4} f(\lambda)\,, \tag{4.11}$$

$$f(\lambda) = \lambda\frac{\partial}{\partial\lambda}\log\langle\mathcal{W}\rangle = \frac{1}{2}\sqrt{\lambda}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} = \frac{1}{2}\sqrt{\lambda} - \frac{3}{4} + \dots\,. \tag{4.12}$$

In (4.11) we assume that dependence on the local dilaton is introduced by $L_{\rm SYM} \to e^{-\phi(x)}L_{\rm SYM}$ and ϕ is set to be constant as in (4.7) after the differentiation. In (4.12) we used that $\langle\mathcal{W}\rangle = \frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda})$, i.e. $f(\lambda)$ is the same function that appeared also in (4.9). For a WL defined by a circle of unit radius on the (x_1, x_2)-plane centered at the origin, the

[8] Eq. (4.11) is a direct counterpart of the exact form of the correlation function of the $\frac{1}{2}$ BPS Wilson loop with the $\Delta = 2$ chiral primary operator which is a special case of the correlator of the Wilson loop and the $\Delta = J$ CPO first obtained in [53]. The function $f(\lambda)$ also appears in the so-called Bremsstrahlung function [54]. The dilaton operator $\mathcal{O}_4$ is a descendant of the $\Delta = 2$ chiral primary, i.e. $\mathcal{O}_4 \sim {\rm tr}\,(F^2 + \Phi D^2\Phi + \dots)$ and is different from the canonical form of the SYM Lagrangian $L_{\rm SYM}$ in (4.6) by a total derivative term (in conformal correlators one may further drop the terms proportional to the scalar and spinor equations of motion as they produce only contact terms [47–49]). Note that we use Euclidean notation (as, e.g., in [52]) and in our normalization [49] $\langle L_{\rm SYM}(x)L_{\rm SYM}(x')\rangle = \frac{3N^2}{\pi^4(x-x')^8}$.

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position dependent factor $d_\perp$ in (4.11) is given explicitly by (see, e.g., [4, 55, 56])

$$d_\perp = \frac{1}{2}\sqrt{(r^2+h^2-1)^2+4h^2}\,, \qquad r^2 = x_1^2+x_2^2\,, \qquad h^2 = x_3^2+x_4^2\,. \tag{4.13}$$

One can verify that integrating (4.11) over the position $x = (x_1, x_2, x_3, x_4)$ of the operator insertion, using the regularized expression for the integral[9]

$$\int d^4x\, \frac{1}{d_\perp^4} = (2\pi)^2 \int_0^\infty dr\, r \int_0^\infty dh\, h\, \frac{16}{\left[(r^2+h^2-1)^2+4h^2\right]^2} = -8\pi^2\,, \tag{4.14}$$

one recovers the relation (4.8).

On the string theory side, the corresponding local dilaton operator is (cf. (4.2); here $D = 10$ and $L_{\rm str} = \frac{1}{2}z^{-2}\partial^a x'^m \partial_a x'^m + \dots$ is the $\mathrm{AdS}_5\times S^5$ superstring Lagrangian)

$$V(x) = -\int d^2\sigma \sqrt{g}\left(\frac{1}{2}T L_{\rm str} + \frac{1}{4\pi}R^{(2)}\right) K(x-x';z)\,, \tag{4.15}$$

$$K(x-x';z) = c_4\, \frac{z^4}{[z^2+(x-x')^2]^4}\,, \qquad c_4 = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\Big|_{d=4,\Delta=4} = \frac{6}{\pi^2}\,. \tag{4.16}$$

K in (4.16) is the bulk-to-boundary propagator of the massless dilaton in AdS_5 ($\Delta = 4$). Integrating over the 4-dimensional boundary coordinates gives back V_0 that appeared in (4.5) (indeed, $\int d^4x\, K(x-x';z) = 2\pi^2 \frac{1}{12}c_4 = 1$). Note that the correlator in (4.11) is to be compared to the string theory dilaton insertion on the disc with the dilaton vertex operator defined relative to the Einstein-frame metric so that the 2-point functions of the graviton and dilaton (and the corresponding dual operators) are decoupled.

Note that the normalized correlator (4.11) for the case of the WL corresponding to a straight line is related to the one for the circle by a conformal transformation, and it takes the same form as (4.11), with the same function $f(\lambda)$, and $d_\perp$ being simply the distance from the straight line (i.e., for a straight line along the x_1 direction, $d_\perp = \sqrt{x_2^2+x_3^2+x_4^2}$) [4]. Using the AdS_2 surface in the straight line case ($z=\sigma$, $x^0=\tau$, $x^i=0$) we get for the contribution of the leading classical term and the $R^{(2)} = -2$ term in V in (4.15):

$$\begin{aligned}\langle V_{{\rm cl}+R^{(2)}}(x)\rangle &= -\int d^2\sigma \sqrt{g}\left(\frac{1}{2}T + \frac{1}{4\pi}R^{(2)}\right) K(x-x';z) \\ &= -c_4 \frac{1}{4\pi}(\sqrt{\lambda}-2) \int_{-\infty}^{\infty} d\tau \int_0^\infty \frac{d\sigma}{\sigma^2}\frac{\sigma^4}{(\sigma^2+\tau^2+d_\perp^2)^4} = -\frac{1}{16\pi^2\, d_\perp^4}(\sqrt{\lambda}-2)\,.\end{aligned} \tag{4.17}$$

Then the string theory expectation value $\frac{\delta}{\delta\phi(x)}\log Z = \langle V(x)\rangle$ indeed matches (4.11) if one adds in the last bracket in (4.17) an extra $+\frac{1}{2}$ coming from the 1-loop quantum fluctuations of the bosonic and fermionic string coordinates in $\langle L_{\rm str}\rangle$, in parallel to what happened in (4.5).

[9] To evaluate this integral, one may, for instance, first integrate over r, then integrate over h and finally remove the power divergence at $h = \epsilon \to 0$, i.e. $\int d^4x\, \frac{1}{d_\perp^4} = \frac{2\pi^3}{\epsilon} - 8\pi^2 + \mathcal{O}(\epsilon) \to -8\pi^2$.

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5 Universal form of higher genus corrections

An important feature of the $\frac{\sqrt{T}}{g_{\rm str}}$ prefactor in (1.4) is that it has a natural generalization $(\frac{\sqrt{T}}{g_{\rm str}})^\chi$ to contributions from higher genera (1.7) (cf. also (4.10), (A.8)). Let us recall that in the case of the SU(N) $\mathcal{N}=4$ SYM theory the exact expression for the expectation value of the $\frac{1}{2}$ BPS circular WL $\mathcal{W} = {\rm tr}\, P e^{\int (iA+\Phi)}$ expanded at large N and then at large λ is [2, 3] (L^1_{N-1} is the Laguerre polynomial)[10]

$$\langle \mathcal{W} \rangle = \ e^{\frac{\lambda}{8N}}\, L^1_{N-1}(-\frac{\lambda}{4N}) = N \sum_{p=0}^{\infty} \frac{\sqrt{2}}{96^p\, \sqrt{\pi}\, p!}\, \frac{\lambda^{\frac{6p-3}{4}}}{N^{2p}}\, e^{\sqrt{\lambda}} \left[1 + \mathcal{O}\Big(\frac{1}{\sqrt{\lambda}}\Big)\right] . \tag{5.1}$$

It was suggested in [2] that the sum over p may be interpreted as a genus expansion on the string side. Remarkably, we observe that once the overall factor of N is included, i.e. one considers the expectation value of ${\rm tr}\,(\dots)$ rather that $\frac{1}{N}{\rm tr}\,(\dots)$, the full dependence on N and λ in the prefactor of $e^{\sqrt{\lambda}}$ in (5.1) combines just into $(N\lambda^{-3/4})^{1-2p}$. Rewriting (5.1) in terms of the string tension $T = \frac{\sqrt{\lambda}}{2\pi}$ and string coupling $g_{\rm str} = \frac{\lambda}{4\pi N}$ as defined in (1.2) we then get

$$\langle \mathcal{W} \rangle = \sum_{p=0}^{\infty} {\rm c}_p \left(\frac{\sqrt{T}}{g_{\rm str}}\right)^{1-2p} e^{2\pi T} \left[1 + \mathcal{O}(T^{-1})\right], \qquad {\rm c}_p = \frac{1}{2\pi\, p!} \left(\frac{\pi}{12}\right)^p , \tag{5.2}$$

which is the same as (1.7) where $\chi = 1 - 2p$ is the Euler number of a disk with p handles.

Furthermore, the sum that represents the coefficient of the leading large T term in (5.2) has a simple closed expression: since $\sum_{p=0}^{\infty} {\rm c}_p\, z^p = \frac{1}{2\pi} \exp(\frac{\pi}{12} z)$ we find as in ref. [2]

$$\langle \mathcal{W} \rangle = e^H\, W_1 \left[1 + \mathcal{O}(T^{-1})\right], \qquad W_1 = \frac{\sqrt{T}}{2\pi g_{\rm str}}\, e^{2\pi T} , \quad H \equiv \frac{\pi}{12}\, \frac{g_{\rm str}^2}{T} . \tag{5.3}$$

Here W_1 is the leading large N or disk contribution in the SYM theory given by (1.4) (with $e^{-\bar{\Gamma}_1} = \sqrt{2\pi}$ according to (1.5)). H may be interpreted as representing a handle insertion operator, i.e. higher order string loop corrections here simply exponentiate. Such exponentiation is expected in the "dilute handle gas" approximation of thin far-separated handles which should be relevant to the leading order in the large tension expansion considered in (5.1), (5.2) (cf. [57–63]).

It has another interesting interpretation suggested in [64]. If one considers a circular Wilson loop in the totally symmetric rank k representation of SU(N) then for large k, N and λ with $\kappa = \frac{k\sqrt{\lambda}}{4N}$ =fixed its expectation value should be given by the exponent $\exp(-S_{\rm D3})$ of the action of the classical D3-brane solution. In the limit of $1 \ll k \ll N$ this description should apply also to the case of the WL in the k-fundamental representation described by minimal surface ending on multiply wrapped circle and here one finds [64]

[10] Let us note that this expression applies to the SYM theory with the U(N) gauge group; the result in the SU(N) case is obtained by multiplying (5.1) by $\exp(-\frac{\lambda}{8N^2})$ [2]. This factor expressed in terms of $g_{\rm str}$ and T in (1.2) is $\exp(-\frac{g_{\rm str}^2}{2T^2})$ and thus is subleading compared to H in (5.3) at large T; we therefore ignore it here.

that $S_{\rm D3} = Nf(\kappa) = -k\sqrt{\lambda} - \frac{k^3\lambda^{3/2}}{96N^2} + \mathcal{O}(\frac{k^5\lambda^{5/2}}{N^4})$. If one *formally* extrapolates this expression to $k=1$, i.e. a single circle case discussed above, then one finds that it becomes $S_{\rm D3} = -2\pi T + \frac{\pi}{12}\frac{g_{\rm str}^2}{T} + \mathcal{O}(\frac{g_{\rm str}^4}{T^3})$, i.e. $\exp(-S_{\rm D3})$ reproduces precisely the exponential factor $e^{2\pi T+H}$ in (5.3).

A similar structure (5.2) of the topological expansion should appear in the case of the $\frac{1}{2}$ BPS circular WL in the ABJM theory which was computed from localization in [22]. According to (1.3), in that case we have $\frac{\sqrt{T}}{g_{\rm str}} = \frac{N}{\sqrt{8\pi}\,\lambda} = \frac{k}{\sqrt{8\pi}}$ where k is the CS level so that $\langle\mathcal{W}\rangle$ should be a series in $(\frac{\sqrt{T}}{g_{\rm str}})^\chi \sim k^\chi \sim (\frac{1}{g_{\rm CS}})^\chi$ (cf. footnote 1). Translating the leading and the first subleading $1/N$ corrections to the WL expectation value found explicitly in [22] into our notation we get[11]

$$\langle\mathcal{W}\rangle = \Big(\frac{N}{4\pi\lambda} + \frac{\pi\lambda}{6N} + \ldots\Big)\, e^{\pi\sqrt{2\lambda}} = \Big(1 + \frac{\pi}{12}\,\frac{g_{\rm str}^2}{T} + \ldots\Big)\, W_1\,, \qquad W_1 = \frac{\sqrt{T}}{\sqrt{2\pi}\, g_{\rm str}}\, e^{2\pi T}\,. \tag{5.4}$$

Here W_1 is the leading disk term in $\langle\mathcal{W}\rangle$ in the ABJM theory given by (1.4) (with $\bar{\Gamma}_1 = 0$ according to (1.5)). Thus, to this order, the genus expansion in the ABJM case has the same universal structure as (5.2), (5.3) in the SYM case. It would be interesting to check if the prefactor in (5.4) exponentiates as in (5.3) (e.g. using the results of [65]) and also if there is a D2-brane description of this similar to the one in the SYM case discussed above (cf. [66–68]).

6 Concluding remarks

As was noted below (2.8), the coefficient of the UV divergent term in (3.4) is, in fact, the same for all minimal surfaces with disk topology, and thus the dependence of the string partition function on the scale R or effective tension T through the $\sqrt{T}$ factor in (3.5) should be universal.

A check of the universality of the prefactor in (1.4) is that it applies also to the circular WL in the k-fundamental representation dual to a minimal surface ending on k-wrapped circle at the boundary of AdS_5. In this case the classical action is $I_{\rm cl} = -k\sqrt{\lambda}$ but the Euler number of the minimal surface is still equal to 1 [12] so that the coefficient $\zeta_{\rm tot}(0)$ in (3.4), (3.5) is also 1 and thus the disk partition function is $\langle\mathcal{W}\rangle \sim \frac{\sqrt{T}}{g_{\rm str}}\, e^{2\pi kT}$. This is consistent with the SYM (localization) result in the k-fundamental case [3, 64, 69] given by the $k=1$ expression with $\sqrt{\lambda} \to k\sqrt{\lambda}$. The overall k-dependent constant that should come from $\bar{\Gamma}_1$ in (1.4) still remains to be explained, despite several earlier attempts in [7, 9, 12, 70].

The universality of (1.4) implies, in particular, that the prefactor $\frac{\sqrt{T}}{g_s}$ should cancel in the ratio of expectation values of similar Wilson loops. In particular, this applies to the case of $\frac{1}{4}$ BPS latitude WL parametrized by an angle θ_0. Matching with the gauge theory

[11] Note that we use the notation $g_{\rm CS} \equiv \frac{2\pi i}{k}$ for what was called g_s in [22] in order not to confuse it with the type IIA string theory coupling $g_{\rm str}$ in (1.3). The leading correction in eq. (8.19) in [22] is to be multiplied by $g_{\rm CS}^2$ according to the definition of the topological expansion in (8.1) there. Also, as already mentioned in footnote (1), with our definition of the WL expectation value $\langle\mathcal{W}\rangle = \langle{\rm tr}\,(\ldots)\rangle = \frac{1}{g_{\rm CS}}\langle{\rm W}\rangle_{\rm loc}$, where $\langle{\rm W}\rangle_{\rm loc}$ is the gauge theory localization expression of [22].

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prediction for the ratio of the latitude WL and simple circular WL was checked in the SYM case in [13–15] and in the ABJM case in [16, 17].

Let us note that (1.4) actually requires a generalization in special cases when there are 0-modes in the internal (non-AdS) directions of $\mathrm{AdS}_n \times M^{10-n}$ space, each producing extra factor of $\sqrt{T}$ (cf. (A.10)). This is what happens in the case of the $\frac{1}{4}$ supersymmetric ($\theta_0 = \frac{\pi}{2}$) latitude WL discussed in [14, 15, 71] where we then get for the disk partition function

$$\langle \mathcal{W}_{\frac{1}{4}} \rangle \sim \frac{\sqrt{T}}{g_{\rm str}} (\sqrt{T})^3 \sim N \ . \tag{6.1}$$

Here all λ-dependence cancels out and the finite proportionality constant should be equal to 1, i.e. $N^{-1}\langle \mathcal{W}_{\frac{1}{4}} \rangle = 1$, in agreement with [71].

A similar remark applies to the case of the $\frac{1}{6}$ (bosonic) BPS WL [66, 67] in the ABJM theory. According to [22] here we get instead of $\langle \mathcal{W} \rangle$ for the $\frac{1}{2}$ BPS WL in (1.3) (cf. footnotes 1, 11 and eq. (5.4))

$$\langle \mathcal{W}_{\frac{1}{6}} \rangle = \frac{1}{g_{\rm CS}} \langle \mathrm{W}_{\frac{1}{6}} \rangle_{\rm loc} = i e^{i\pi\lambda} \frac{N}{4\pi\lambda} \frac{1}{2} \sqrt{2\lambda}\, e^{\pi\sqrt{2\lambda}} + \dots \ . \tag{6.2}$$

As was argued in [66, 67] (see also [72, 73]), here the minimal surface solution is smeared over $S^2 = CP^1$ in CP^3 so there are two scalar 0-modes. This explains the extra factor $\frac{1}{2}\sqrt{2\lambda} = (\sqrt{T})^2$ in (6.2) compared to $\langle \mathcal{W} \rangle$ in (1.3) [22]. More generally, for contributions from each genus p one finds [21, 22, 65] that the ratio of the $\frac{1}{6}$ and $\frac{1}{2}$ BPS WL's is given by this universal $(\sqrt{T})^2$ term (ignoring phase factors)

$$\frac{\langle \mathcal{W}_{\frac{1}{6}} \rangle_p}{\langle \mathcal{W} \rangle_p} = (\sqrt{T})^2 + \mathcal{O}(T^{-1}) \ . \tag{6.3}$$

It would be interesting to match the precise numerical coefficient in the ratio between the $\frac{1}{6}$ BPS and $\frac{1}{2}$ BPS Wilson loops by carefully fixing the normalization of the two zero modes on the string side.

Finally, let us note that while in this paper we focused on the case of 4d and 3d gauge theories, as explained in section 2 our results also apply to string theory in $\mathrm{AdS}_3 \times S^3 \times T^4$ with RR flux. This case corresponds to $n = 3$ in (1.6) (cf. (1.4), (2.21)), i.e. $\langle \mathcal{W} \rangle = Z_{\rm str} = \frac{1}{g_{\rm str}} \sqrt{T}\, e^{2\pi T} + \dots$. It would be interesting to see if this string-theory prediction can be matched to localization calculations for Wilson loops in 2d supersymmetric gauge theory (cf. [74, 75]).

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A Comments on tension dependence of the string partition function

In section 3 we discussed how to explain the prefactor $\sqrt{T}$ in the one-loop string partition function (1.4) starting with the string action (3.1) and using the static gauge expression (2.3). We emphasized that the result is sensitive to the choice of the path integral measure, i.e. the definition of the quantum theory (which, in general, is not unique, unless completely fixed by symmetry requirements or extra consistency conditions). In the appendices below we shall discuss some other approaches to derive this prefactor, which again involve certain assumptions about the measure or regularization procedure.

A.1 T-derivative of the partition function in static gauge

Suppose we start with the string action (3.2) in terms of the rescaled (dimensionless) coordinates so that there is an explicit factor of the effective string tension T in front of the action with the induced AdS_2 metric having radius 1. Then we would get the same result as in (3.5) if we assume that the norm or the measure is defined so that the resulting one-loop correction from a single scalar has the form $\Gamma_1 = \frac{1}{2}\log\det\hat{\Delta}$ where $\hat{\Delta} = T^{-1}\Delta$.[12] Indeed, using the ζ-function regularization with $\zeta(0)$ being the regularized total number of eigenvalues we get $\frac{1}{2}\log\det(T^{-1}\Delta) = \frac{1}{2}\zeta(0)\log(T^{-1}) + \ldots = -\zeta(0)\log\sqrt{T} + \ldots$. This leads again to (3.5) once we use that the total value of $\zeta(0)$ corresponding to the static-gauge partition function (2.4) is $\zeta_{\rm tot}(0) = 1$ (see (2.8), (2.13)).

Another way to obtain the same result (which will be again based on a particular choice of a regularization prescription) is to find the dependence of the string partition function on the tension by first computing its derivative over T. This is closely related to the argument appearing in the context of the "soft dilaton theorem" [44], see section 4.

Let us assume that the tension dependence of the string partition function may come only from the factor of T in the string action (3.2) in the static gauge (i.e. in the action for the "physical" fluctuations whose determinants are present in (2.3)), i.e. the measure is defined so that it does not depend on T. For example, for a single scalar field

$$Z = \int [dx]\exp(-I), \qquad I = \frac{1}{2}T\int d^2\sigma\sqrt{g}\,x\Delta_{(m^2)}x\,, \qquad \Delta_{(m^2)} = -\nabla^2 + m^2\,, \tag{A.1}$$

$$T\frac{\partial}{\partial T}\log Z = -\langle I\rangle\,, \qquad \langle I\rangle = \int d^2\sigma\sqrt{g}\,[\Delta \mathrm{G}_{(m^2)}(\sigma,\sigma')]_{\sigma=\sigma'} = \int d^2\sigma\sqrt{g}\,\delta_{(m^2)}(\sigma,\sigma), \tag{A.2}$$

where $\langle I\rangle = Z^{-1}\int[dx]\,I\,\exp(-I)$, $\mathrm{G}_{(m^2)}(\sigma,\sigma') = \langle\sigma|\Delta^{-1}_{(m^2)}|\sigma'\rangle$ is the Green's function and $\delta_{(m^2)}(\sigma,\sigma)$ is a regularized value of the bosonic delta-function at the coinciding points. Let

[12] Explicitly, $(x,\hat{\Delta}x) = T\int d^2\sigma\sqrt{g}\,x\hat{\Delta}x = \int d^2\sigma\sqrt{g}\,x\Delta x$, where x are rescaled fluctuations. In general, one can of course move T-dependence from the action to the measure by a field redefinition (taking into account the resulting regularized Jacobian of the transformation). If the path integral measure is $\prod_\sigma \frac{\mu}{\sqrt{2\pi}}dx(\sigma)$ and the action is simply $\frac{1}{2}\int d^2\sigma\sqrt{g}\,x\Delta x$ then $Z = (\prod_n \frac{\lambda_n}{\mu^2})^{-1/2} \sim \mu^{\zeta(0)}$ where λ_n are the eigenvalues of Δ [42]. If $\zeta(0)$ is non-zero the result is thus sensitive to the definition of the measure.

us use the heat-kernel cutoff, i.e. assume that

$$\delta_{(m^2)}(\sigma,\sigma) = \langle\sigma|e^{-\epsilon\,\Delta^{-1}_{(m^2)}}|\sigma\rangle = \frac{1}{4\pi}\Big[\Lambda^2 + \frac{1}{6}R^{(2)} - m^2\Big]\,, \qquad \epsilon \equiv \Lambda^{-2} \to 0\ . \tag{A.3}$$

The expectation value of the action corresponding to the full static-gauge expression (2.3) is then

$$\begin{aligned}\langle I\rangle = \frac{1}{2}\int d^2\sigma\sqrt{g}\,\Big\{&(n-2)[\Delta_{(2)}\mathrm{G}_{(2)}(\sigma,\sigma')]_{\sigma=\sigma'} + (10-n)[\Delta_{(0)}\mathrm{G}_{(0)}(\sigma,\sigma')]_{\sigma=\sigma'}\\ &- (2n-2)[\mathrm{D}^f_{(1)}\mathrm{G}^f_{(1)}(\sigma,\sigma')]_{\sigma=\sigma'} - (10-2n)[\mathrm{D}^f_{(0)}\mathrm{G}^f_{(0)}(\sigma,\sigma')]_{\sigma=\sigma'}\Big\}\,,\\ = \frac{1}{2}\int d^2\sigma\sqrt{g}\,\Big\{&(n-2)\delta_{(2)}(\sigma,\sigma) + (10-n)\delta_{(0)}(\sigma,\sigma)\\ &- (2n-2)\delta^f_{(1)}(\sigma,\sigma) - (10-2n)\delta^f_{(0)}(\sigma,\sigma)\Big\}\,,\end{aligned} \tag{A.4}$$

where D^f is the fermionic 1st order operator and G^f and δ^f stand for the corresponding Green's function and δ-function.

A key next step is to assume a special "2d supersymmetric" regularization in which the bosonic and fermionic Green's functions and thus the corresponding regularized delta-functions are related to each other as[13]

$$\delta^f_{(m)}(\sigma,\sigma) = \frac{1}{2}\Big[\delta_{(m^2-m)}(\sigma,\sigma) + \delta_{(m^2+m)}(\sigma,\sigma)\Big]\ . \tag{A.5}$$

Then (A.4) reduces simply to

$$\langle I\rangle = \frac{1}{2}\int d^2\sigma\sqrt{g}\,\Big[\delta_{(0)}(\sigma,\sigma) - \delta_{(2)}(\sigma,\sigma)\Big] = \frac{1}{2}\times\frac{1}{2\pi}\times V_{\mathrm{AdS}_2} = -\frac{1}{2}\,, \tag{A.6}$$

where we used (A.3).[14]

The relation $T\frac{\partial}{\partial T}\log Z = -\langle I\rangle$ in (A.2) implies once again that

$$Z \;\sim\; \sqrt{T}\ . \tag{A.7}$$

Let us note that the result for the expectation value of the action (A.6) should be more universal than a particular prescription used above. The integrand in (A.6) should be in

[13] This is an effective consequence of the fact that the 2d supersymmetric Ward identity (cf. [76, 77]) relates a fermion of mass $m>0$ to a boson of mass m^2-m (e.g. in a special regularization the trace of the Green's function for a single 2d fermion $\mathrm{G}^f_{(m)}$ is related to $2m\,\mathrm{G}_{(m^2-m)}$ [77]). In the present case we have half of the massive fermions with mass $m=1$ and the other half — with mass $m=-1$. Alternatively, one may use a particular representation for $\mathrm{G}^f_{(m)}$ (for $m>0$) as $\mathrm{G}^f_{(m)}(\sigma,\sigma') = [(i\gamma^a\partial_a + m)\mathrm{G}_{(m^2-m)}]S(\sigma,\sigma')$ implying that one has $\mathrm{D}^f_{(m)}\mathrm{G}^f_{(1)}(\sigma,\sigma') = \delta_{(m^2-m)}(\sigma,\sigma')\,S(\sigma,\sigma')$.

[14] Let us note that the use of (A.5) may be interpreted as a specific regularization prescription for the fermions which is different from the heat-kernel or ζ-function one applied to the squared fermionic operator $\Delta^f_{(m^2)} = (\mathrm{D}^f_{(m)})^2 = -\nabla^2 + \frac{1}{4}R^{(2)} + m^2$ in (2.3), (2.7). Indeed, if we assume that $\delta^f_{(m)}(\sigma,\sigma)$ in (A.5) is defined as in (A.3), i.e. $\delta^f_{(m)}(\sigma,\sigma) = \langle\sigma|e^{-\epsilon\Delta_f}|\sigma\rangle = \frac{1}{4\pi}\big(\Lambda^2 + \frac{1}{6}R^{(2)} - \frac{1}{4}R^{(2)} - m^2\big)$ then we find that $\langle I\rangle = +\frac{1}{2}$ which is consistent with the $\zeta_{\mathrm{tot}}(0)=1$ value in (2.8), (2.13), i.e. $Z \sim \prod[\det(T\Delta)]^{-1/2} \sim T^{-1/2}$. In this regularization the l.h.s. of (A.5) is $2(\Lambda^2 + \frac{1}{6}R^{(2)}) - \frac{1}{2}R^{(2)} - 2m^2$ while the r.h.s. is $2(\Lambda^2 + \frac{1}{6}R^{(2)}) - \frac{1}{2}R^{(2)} - 2m^2$ so the difference $-\frac{1}{2}R^{(2)}$ may be attributed to the presence of $-\frac{1}{4}R^{(2)}$ term in the squared fermionic operator which is thus effectively omitted in the prescription (A.5).

general $\delta_{(0)}(\sigma,\sigma)-\delta_{(2)}(\sigma,\sigma)\to-\frac{1}{4\pi}R^{(2)}$. In the case of a more general topology of a disk with p handles with the Euler number $\chi=1-2p$ we should then find that

$$\langle I\rangle=-\frac{1}{2}\chi\,,\qquad\qquad Z\sim(\sqrt{T})^{\chi}\,, \tag{A.8}$$

which is in agreement with (1.7), (5.2).

A.2 T-dependence from zero modes in conformal gauge

The conformal gauge expression for the string partition function contains, in addition to the ratio of determinants in $Z=e^{-\Gamma_1}$ in (2.3), also an extra factor [6, 28]

$$Z_{\rm c}=\Omega^{-1}\left[\frac{\det'\Delta_{\rm gh}}{\det\Delta_{\rm long}}\right]^{1/2}\,. \tag{A.9}$$

Here Ω is the SL$(2,R)$ Mobius group volume. The 2-derivative ghost operator $\Delta_{\rm gh\,ab}$ and the operator on the two "longitudinal" fluctuations $\Delta_{\rm long\,ab}=-(\nabla^2)_{ab}-\frac{1}{2}R^{(2)}g_{ab}$ have the same structure (and the same "mixed" boundary conditions) so their non-zero-mode contributions should effectively cancel each other. The integral over the collective coordinates of the three 0-modes of $\Delta_{\rm long}$ (or conformal Killing vectors) which is implicit in (A.9) should cancel against the Mobius volume factor. As a result, one may assume that $Z_{\rm c}$ in (A.9) is effectively equal to 1, thus getting back to the static gauge partition function expression (2.3).

However, this depends on the definition of path integral measure. An alternative possibility compared to the one in the static gauge discussed in the main text and section A.1 is to assume that in the conformal gauge the measure is defined so that the path integral over all non-zero modes does not produce any T-dependent factor, while the presence of the $\sqrt{T}$ factor is (1.4) is due to the normalization of the 0-modes, i.e. of the collective coordinate integral implicit in (A.9).[15] Each bosonic 0-mode absent in the fluctuation action then contributes a measure factor $\sim(\frac{T}{2\pi})^{1/2}$, leading to

$$Z\quad\sim\quad(\sqrt{T})^{n_0}\,, \tag{A.10}$$

where n_0 is the total number of the 0-modes.

Equivalently, this result will follow assuming that one uses a regularization (e.g., dimensional one) in which the delta-functions at coinciding points vanish, $\delta^{(2)}(\sigma,\sigma)=0$ and thus the factors of T in the measure and in front of the action do not contribute, cf. (A.2), apart from the 0-mode contribution.

It is useful to recall that in the familiar case of the open strings with free ends where the bosonic coordinates x^m $(m=1,\ldots,D)$ are subject to the Neumann boundary conditions one finds D constant zero modes and thus an overall factor of $T^{D/2}$ in the disk path integral. The same result can be found also using $T\frac{\partial}{\partial T}$ argument by using that the delta-function

[15] As was already mentioned above, this corresponds to a specific choice of the measure factors implying that the normalization of the gaussian path integral is 1, i.e. $\int[dx]\exp[-\frac{1}{2\hbar}(x,x)]=1$, with $[dx]=\prod_\sigma\frac{dx(\sigma)}{\sqrt{2\pi\hbar}}$. Then the factor of string tension $T=\hbar^{-1}$ should appear not only in the action but also in the measure so that it cancels out in the integrals for all modes with non-zero eigenvalues.

appearing in (A.2) is the "projected" one, i.e. $\delta^{(2)}(\sigma,\sigma)$ (set to 0) minus the trace of the projector to the 0-mode subspace (see, e.g., [44, 78]).

In the present WL case of path integral with the Dirichlet-type (or fixed-contour) boundary conditions one could expect to have no 0-modes. However, as the two "longitudinal" string coordinates are subject to "mixed" Dirichlet/Neumann b.c. [28, 36, 79–81] (motivated by the requirement of preservation of the reparametrization invariance of the boundary contour) there is, in particular, a special 0-mode corresponding to a constant shift of a point on the boundary circle. There are, in fact, two more 0-modes of the longitudinal operator (see appendix B). As already mentioned above, these three bosonic 0-modes are direct counterparts of the conformal Killing vectors associated to the $\mathrm{SL}(2,R)$ Mobius symmetry on the disk surviving in the conformal gauge.

Thus if the path integral measure is normalized so that the integral over non-zero modes does not produce any T-dependence we then get a factor $Z \sim (\sqrt{T})^3$ associated to the $n_0 = 3$ "longitudinal" 0-modes on the disk. To reduce the effective number of 0-modes to $n_0 = 1$ (required to match the $\sqrt{T}$ factor in (1.4)) one may contemplate the following possibilities:

(i) assume that 2 longitudinal 0-modes are lifted due to some boundary contributions to the string action leaving only one translational mode (corresponding to a constant shift on the boundary circle);[16]

(ii) assume that the GS fermion contribution effectively conspires to mimic the NSR contribution on the disk with $n_f = 2$ fermionic super-Mobius 0-modes,[17] producing the effective number $n_0 = n_b - n_f = 3 - 2 = 1$. A relation to the NSR formulation with manifest 2d supersymmetry may of course be expected and was mentioned already in the discussion of the static gauge approach above. Note also that the super-Mobius volume is finite [86] so it is not necessary to cancel it explicitly.

B Conformal Killing vectors as longitudinal zero modes

Here we shall record the expressions for the conformal gauge ghost zero-modes or conformal Killing vectors (CKV) on a flat disk D^2 and on a euclidean hyperbolic space $H^2 = \mathrm{AdS}_2$

[16] This, at first, may look unnatural as then we would not have a cancellation between the integral over the corresponding collective coordinates and the Mobius volume factor in the path integral. Yet, that may not be a problem as the Mobius volume on the disk may be regularized to a finite value [82, 83] (similarly to how this is done for the AdS_2 volume).

[17] To compare, in the case of the one-loop instanton partition function in super YM theory (see [84, 85]) the contributions of all non-zero modes cancel (i.e. $\zeta_{\rm tot}(0) = 0$) and as a result the UV cutoff dependence (and thus one-loop beta function) is controlled just by the 0-modes — the total Seeley coefficient is $B_4 = \zeta_{\rm tot}(0) + n_{\rm tot} = n_b - \frac{1}{2} n_f$. At the same time, the dependence on the inverse gauge coupling $1/g^2_{\rm YM}$ (which is the analog of string tension T in our case) is controlled by the coefficient $n_b - n_f$. Note, however, that the prescription for $g_{\rm YM}$ dependence becomes unambiguous only in physical correlation functions with external fermionic legs saturating the fermionic 0-mode integral [85].

with the metric[18]

$$ds^2 = e^{2\rho}(dr^2 + r^2 d\phi^2)\,, \qquad (e^{2\rho})_{D^2} = 1\,, \qquad (e^{2\rho})_{H^2} = \frac{4}{(1-r^2)^2}\,. \tag{B.1}$$

The CKV are also the zero-modes of the longitudinal Laplacian in (A.9) which is equivalent to the 2nd derivative conformal ghost operator [28]. The defining relation $\nabla_a \xi_b + \nabla_b \xi_a - g_{ab}\nabla^c \xi_c = 0$ does not depend on the conformal factor ρ when written in terms of the contravariant components ξ^a: $\partial_a \xi^b + \partial_b \xi^b - \delta_{ab}\partial_c \xi^c = 0$. The expressions for the three Killing vectors ξ^a corresponding to the SL$(2, R)$ transformations on the plane are (here $\mathrm{a}, \mathrm{b}_1, \mathrm{b}_2$ are real parameters)

$$\begin{aligned} z' &= e^{i\mathrm{a}}\frac{z+\mathrm{b}}{1+\mathrm{b}^* z}, && \mathrm{b} = \mathrm{b}_1 + i\mathrm{b}_2, \\ \delta z &= \xi^1 + i\xi^2 = \mathrm{b} + i\mathrm{a}\, z - \mathrm{b}^* z^2, && z = re^{i\phi}\,, \end{aligned} \tag{B.2}$$

$$\begin{aligned} \xi^1 &= \mathrm{b}_1 - \mathrm{a}\, r\sin\phi - r^2(\mathrm{b}_1\cos 2\phi + \mathrm{b}_2 \sin 2\phi)\,, \\ \xi^2 &= \mathrm{b}_2 + \mathrm{a}\, r\cos\phi - r^2(-\mathrm{b}_2\cos 2\phi + \mathrm{b}_1 \sin 2\phi) \\ \xi^r &= \cos\phi\ \xi^1 + \sin\phi\ \xi^2, \\ \xi^\phi &= r^{-1}(-\sin\phi\ \xi^1 + \cos\phi\ \xi^2) \\ \xi^r &= (1-r^2)(\mathrm{b}_1\cos\phi + \mathrm{b}_2\sin\phi)\,, \\ \xi^\phi &= \mathrm{a} + (r + r^{-1})(\mathrm{b}_2\cos\phi - \mathrm{b}_1\sin\phi)\ . \end{aligned} \tag{B.3}$$

Then the standard conformal Killing vectors on the disk satisfy mixed boundary conditions: $\xi^r = 0$ (normal component) and $\partial_r \xi^\phi = 0$ (normal derivative of tangential component) vanish at the $r = 1$ boundary. Once we consider a metric with a non-trivial conformal factor these conditions are modified to:

$$g_{ab} = n_a n_b + t_a t_b\,, \qquad \xi_n\big|_\partial = 0, \qquad (\partial_n - K)\xi_t\big|_\partial = 0\,, \qquad K = \nabla_a n^a\,. \tag{B.4}$$

The mixed boundary conditions were discussed in [79, 80] and [28]. The condition $(\partial_n - K)\xi_t\big|_\partial = 0$ was used in [81] (and implicitly in [36]).

The r, ϕ components of n_a and t_a are: $n_a = e^\rho\{1, 0\}$, $t_a = e^\rho\{0, r\}$ so that

$$\xi_n = n_a\xi^a = e^\rho \xi^r\,, \qquad \xi_t = t_a \xi^a = re^\rho\xi^\phi\,, \tag{B.5}$$

$$\partial_n = e^{-\rho}\partial_r\,, \qquad K = e^{-2\rho}r^{-1}\partial_r(re^\rho)\,, \qquad (\partial_n - K)\xi_t = r\partial_r\xi^\phi\ . \tag{B.6}$$

Note that for a flat disk $K = r^{-1}$ and $\chi = \frac{1}{4\pi}(\int R + 2\int_\partial K) = 1$.

Thus ξ^ϕ in (B.3) satisfies $(\partial_n - K)\xi_t\big|_\partial = 0$ at $r = 1$ but there is an issue with $\xi_n\big|_\partial = 0$: $e^\rho\xi^r = \frac{2}{1-r^2} \times (1-r^2)(\mathrm{b}_1\cos\phi + \mathrm{b}_2\sin\phi)$ so ξ_n is a non-zero function at the boundary. This suggests that either we should set $\mathrm{b}_1, \mathrm{b}_2 = 0$ or the boundary condition $\xi_n\big|_\partial = 0$ is to be modified. One option is to define it with the flat metric as in (2.15)

[18]Alternatively, for the AdS$_2$ metric we have $ds^2 = (\sinh^2 s)^{-1}(ds^2 + d\phi^2)$, $r = e^{-s}$. Another form of the AdS$_2$ metric that follows from AdS$_3$ metric $ds^2 = z^{-2}(d\mathrm{r}^2 + \mathrm{r}^2 d\phi^2 + dz^2)$ with $z = \sqrt{1-\mathrm{r}^2}$ is $ds^2 = \frac{d\mathrm{r}^2}{(1-\mathrm{r}^2)^2} + \frac{\mathrm{r}^2 d\phi^2}{1-\mathrm{r}^2}$ is related to the above one via $\mathrm{r} = \frac{1}{\cosh s} = \frac{2r}{1+r^2}$.

in [28]: $\tilde{n}_a \xi^a|_\partial = 0$ where $\tilde{n}_a$ is the normal in flat metric. This condition just says that the boundary condition $x^m|_\partial = c^m(\phi)$ should be preserved under diffeomorphisms up to a boundary reparametrization, so $\delta x^m = \xi^a \partial_a x^m$ should vanish at the boundary for ξ^a along the normal direction (the definition of normal formally depends on the metric, but here all we need is $\xi^r\big|_\partial = 0$).

The norms of CKV depend on the conformal factor:[19]

$$|\xi|^2 = \int d^2z \sqrt{g}\, g_{ab}\, \xi^a \xi^b = \int_0^{2\pi} d\phi \int_0^1 dr\, r\, e^{4\rho}\, \xi^a \xi^a \ . \tag{B.7}$$

For the three CKV proportional to a, $\mathrm{b}_1, \mathrm{b}_2$ in (B.3) we have ($\xi = \{\xi^r, \xi^\phi\}$): $\xi_{(\mathrm{a})} = \mathrm{a}\{0,1\}$, $\xi_{(\mathrm{b}_1)} = \mathrm{b}_1\{(1-r^2)\cos\phi,\ -(r+r^{-1})\sin\phi\}$, $\xi_{(\mathrm{b}_2)} = \mathrm{b}_2\{(1-r^2)\sin\phi,\ (r+r^{-1})\cos\phi\}$. Thus for $\xi^a\xi^a$ in (B.7) we get: $\xi_{(\mathrm{a})}\cdot\xi_{(\mathrm{a})} = \mathrm{a}^2 r^2$, $\xi_{(\mathrm{b}_1)}\cdot\xi_{(\mathrm{b}_1)} = \mathrm{b}_1^2[2(1+r^4) - 2r^2\cos 2\phi]$, $\xi_{(\mathrm{b}_2)}\cdot\xi_{(\mathrm{b}_2)} = \mathrm{b}_1^2[2(1+r^4) + 2r^2\cos 2\phi]$, so that these vectors have a finite norm for a flat disk ($e^{2\rho} = 1$) or half-sphere ($e^{2\rho} = 4(1+r^2)^{-2}$) but their norm formally diverges for H^2 ($e^{2\rho} = 4(1-r^2)^{-2}$). One option then is to regularize the norms in the same way as we do for the H^2 volume — introduce a cutoff and drop power divergences. We find (with a cutoff at $r = e^{-\epsilon}$) that for the H^2 volume $\int_0^{e^{-\epsilon}} dr \frac{4r}{(1-r^2)^2} = \frac{1}{\epsilon} - 1 + \ldots$ while for the norms $\int_0^{e^{-\epsilon}} dr \frac{4^2 r^3}{(1-r^2)^4} = \frac{4}{3\epsilon^3} - \frac{8}{3\epsilon} + \frac{8}{3} + \ldots$, $\int_0^{e^{-\epsilon}} dr \frac{4^2 2r(1+r^4)}{(1-r^2)^4} = \frac{16}{3\epsilon^3} + \frac{64}{3\epsilon} - \frac{64}{3} + \ldots$.

As a side remark, let us comment on the possibility of having zero modes for the transverse $m^2 = 2$ fluctuation operator in the AdS directions in (2.2), (2.4). If we focus on just a single transverse fluctuation within AdS_3, then one can formally find 3 zero modes related to the fact that the string solution breaks the SO(3,1) isometries of AdS_3 down to SO(2,1). Explicitly, taking the Poincaré coordinates on AdS_3 with metric $ds^2 = \frac{1}{z^2}(dz^2 + dr^2 + r^2 d\phi^2)$, the general AdS_2 string solution ending on a circle (of radius α and center at (β_1, β_2)) at the boundary can be written as

$$z^2 + (r\cos\phi - \beta_1)^2 + (r\cos\phi - \beta_2)^2 = \alpha^2 \ . \tag{B.8}$$

The parameters β_1, β_2 and α correspond to broken translations and dilatation. The zero modes of the transverse fluctuation operator can be obtained as usual by taking derivatives of the classical solution with respect to these parameters. Expressing the result in the coordinates where the induced worldsheet metric is $ds^2 = \frac{1}{\sinh^2\sigma}(d\sigma^2 + d\tau^2)$ ($0 < \tau < 2\pi$, $\sigma > 0$), the 3 zero modes are found to be

$$\psi_{(\alpha)} = \coth\sigma\,, \qquad \psi_{(\beta_1)} = \frac{\cos\tau}{\sinh\sigma}\,, \qquad \psi_{(\beta_2)} = \frac{\sin\tau}{\sinh\sigma}\,. \tag{B.9}$$

One can verify that these indeed satisfy

$$\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2} - \frac{2}{\sinh^2\sigma}\right)\psi_{(\alpha,\beta_1,\beta_2)} = 0\,. \tag{B.10}$$

[19] The definition of the norm for the diffeomorphism vectors via $|\xi|^2 = \int d^2z\sqrt{g}\, g_{ab}\, \xi^a\xi^b$ is a natural one; while it involves the conformal factor it is conformal factor dependence in the corresponding determinants that should cancel in the critical dimension. This definition is different from the one used in [36] but agrees with the one of [81, 87].

However, these zero modes are not normalizable. Moreover, they do not satisfy the Dirichlet boundary conditions at $\sigma = 0$, as required for the transverse fluctuations, so they should not be relevant for our problem. Note also that, when considering all of the $n-2$ transverse directions in AdS_n, there would be, in fact, $3(n-2)$ such zero modes (i.e., 9 in the AdS_5 case).

References

[1] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **582** (2000) 155 [hep-th/0003055] [INSPIRE].

[2] N. Drukker and D.J. Gross, *An exact prediction of $N = 4$ SUSYM theory for string theory*, J. Math. Phys. **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[3] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[4] D.E. Berenstein, R. Corrado, W. Fischler and J.M. Maldacena, *The operator product expansion for Wilson loops and surfaces in the large N limit*, Phys. Rev. D **59** (1999) 105023 [hep-th/9809188] [INSPIRE].

[5] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, Phys. Rev. D **60** (1999) 125006 [hep-th/9904191] [INSPIRE].

[6] N. Drukker, D.J. Gross and A.A. Tseytlin, *Green-Schwarz string in $AdS_5 \times S^5$: semiclassical partition function*, JHEP **04** (2000) 021 [hep-th/0001204] [INSPIRE].

[7] M. Kruczenski and A. Tirziu, *Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling*, JHEP **05** (2008) 064 [arXiv:0803.0315] [INSPIRE].

[8] C. Kristjansen and Y. Makeenko, *More about one-loop effective action of open superstring in $AdS_5 \times S^5$*, JHEP **09** (2012) 053 [arXiv:1206.5660] [INSPIRE].

[9] E.I. Buchbinder and A.A. Tseytlin, *$1/N$ correction in the D3-brane description of a circular Wilson loop at strong coupling*, Phys. Rev. D **89** (2014) 126008 [arXiv:1404.4952] [INSPIRE].

[10] V. Forini, V. Giangreco M. Puletti, L. Griguolo, D. Seminara and E. Vescovi, *Precision calculation of $1/4$-BPS Wilson loops in $AdS_5 \times S^5$*, JHEP **02** (2016) 105 [arXiv:1512.00841] [INSPIRE].

[11] A. Faraggi, L.A. Pando Zayas, G.A. Silva and D. Trancanelli, *Toward precision holography with supersymmetric Wilson loops*, JHEP **04** (2016) 053 [arXiv:1601.04708] [INSPIRE].

[12] V. Forini, A.A. Tseytlin and E. Vescovi, *Perturbative computation of string one-loop corrections to Wilson loop minimal surfaces in $AdS_5 \times S^5$*, JHEP **03** (2017) 003 [arXiv:1702.02164] [INSPIRE].

[13] A. Cagnazzo, D. Medina-Rincon and K. Zarembo, *String corrections to circular Wilson loop and anomalies*, JHEP **02** (2018) 120 [arXiv:1712.07730] [INSPIRE].

[14] D. Medina-Rincon, A.A. Tseytlin and K. Zarembo, *Precision matching of circular Wilson loops and strings in $AdS_5 \times S^5$*, JHEP **05** (2018) 199 [arXiv:1804.08925] [INSPIRE].

JHEP10(2020)130

[15] L. Botao and D. Medina-Rincon, *On precision holography for the circular Wilson loop in $AdS_5 \times S^5$*, *Phys. Lett. B* **810** (2020) 135789 [arXiv:2007.15760] [INSPIRE].

[16] D. Medina-Rincon, *Matching quantum string corrections and circular Wilson loops in $AdS_4 \times CP^3$*, *JHEP* **08** (2019) 158 [arXiv:1907.02984] [INSPIRE].

[17] M. David, R. De León Ardón, A. Faraggi, L.A. Pando Zayas and G.A. Silva, *One-loop holography with strings in $AdS_4 \times \mathbb{CP}^3$*, *JHEP* **10** (2019) 070 [arXiv:1907.08590] [INSPIRE].

[18] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$N = 6$ superconformal Chern-Simons-matter theories, $M2$-branes and their gravity duals*, *JHEP* **10** (2008) 091 [arXiv:0806.1218] [INSPIRE].

[19] N. Drukker and D. Trancanelli, *A supermatrix model for $N = 6$ super Chern-Simons-matter theory*, *JHEP* **02** (2010) 058 [arXiv:0912.3006] [INSPIRE].

[20] A. Kapustin, B. Willett and I. Yaakov, *Exact results for Wilson loops in superconformal Chern-Simons theories with matter*, *JHEP* **03** (2010) 089 [arXiv:0909.4559] [INSPIRE].

[21] M. Mariño and P. Putrov, *Exact results in ABJM theory from topological strings*, *JHEP* **06** (2010) 011 [arXiv:0912.3074] [INSPIRE].

[22] N. Drukker, M. Mariño and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511 [arXiv:1007.3837] [INSPIRE].

[23] A. Lewkowycz and J. Maldacena, *Exact results for the entanglement entropy and the energy radiated by a quark*, *JHEP* **05** (2014) 025 [arXiv:1312.5682] [INSPIRE].

[24] O. Bergman and S. Hirano, *Anomalous radius shift in AdS_4/CFT_3*, *JHEP* **07** (2009) 016 [arXiv:0902.1743] [INSPIRE].

[25] N. Gromov and G. Sizov, *Exact slope and interpolating functions in $N = 6$ supersymmetric Chern-Simons theory*, *Phys. Rev. Lett.* **113** (2014) 121601 [arXiv:1403.1894] [INSPIRE].

[26] T. McLoughlin, R. Roiban and A.A. Tseytlin, *Quantum spinning strings in $AdS_4 \times CP^3$: testing the Bethe Ansatz proposal*, *JHEP* **11** (2008) 069 [arXiv:0809.4038] [INSPIRE].

[27] S. Förste, D. Ghoshal and S. Theisen, *Stringy corrections to the Wilson loop in $N = 4$ super-Yang-Mills theory*, *JHEP* **08** (1999) 013 [hep-th/9903042] [INSPIRE].

[28] E.S. Fradkin and A.A. Tseytlin, *On quantized string models*, *Annals Phys.* **143** (1982) 413 [INSPIRE].

[29] H. Kim, N. Kim and J. Hun Lee, *One-loop corrections to holographic Wilson loop in AdS4xCP3*, *J. Korean Phys. Soc.* **61** (2012) 713 [arXiv:1203.6343] [INSPIRE].

[30] V. Forini, V.G.M. Puletti, L. Griguolo, D. Seminara and E. Vescovi, *Remarks on the geometrical properties of semiclassically quantized strings*, *J. Phys. A* **48** (2015) 475401 [arXiv:1507.01883] [INSPIRE].

[31] R. Kallosh and A.Y. Morozov, *Green-Schwarz action and loop calculations for superstring*, *Int. J. Mod. Phys. A* **3** (1988) 1943 [INSPIRE].

[32] A.H. Diaz and F. Toppan, *Towards the quantization of the Green-Schwarz heterotic string*, *Phys. Lett. B* **211** (1988) 285 [INSPIRE].

[33] P.B. Wiegmann, *Extrinsic geometry of superstrings*, *Nucl. Phys. B* **323** (1989) 330 [INSPIRE].

[34] F. Bastianelli, P. van Nieuwenhuizen and A. Van Proeyen, *Superstring anomalies in the semilight cone gauge*, *Phys. Lett. B* **253** (1991) 67 [INSPIRE].

JHEP10(2020)130

[35] E.S. Fradkin and A.A. Tseytlin, *Quantization of two-dimensional supergravity and critical dimensions for string models*, *Phys. Lett. B* **106** (1981) 63.

[36] O. Alvarez, *Theory of strings with boundaries: fluctuations, topology, and quantum geometry*, *Nucl. Phys. B* **216** (1983) 125 [INSPIRE].

[37] R. Camporesi, *Harmonic analysis and propagators on homogeneous spaces*, *Phys. Rept.* **196** (1990) 1 [INSPIRE].

[38] R. Camporesi, *zeta function regularization of one loop effective potentials in Anti-de Sitter space-time*, *Phys. Rev. D* **43** (1991) 3958 [INSPIRE].

[39] R. Camporesi, *The spinor heat kernel in maximally symmetric spaces*, *Commun. Math. Phys.* **148** (1992) 283 [INSPIRE].

[40] R. Camporesi and A. Higuchi, *Arbitrary spin effective potentials in anti-de Sitter space-time*, *Phys. Rev. D* **47** (1993) 3339 [INSPIRE].

[41] R. Camporesi and A. Higuchi, *Spectral functions and zeta functions in hyperbolic spaces*, *J. Math. Phys.* **35** (1994) 4217 [INSPIRE].

[42] S.W. Hawking, *Zeta function regularization of path integrals in curved space-time*, *Commun. Math. Phys.* **55** (1977) 133 [INSPIRE].

[43] E.S. Fradkin and A.A. Tseytlin, *Effective field theory from quantized strings*, *Phys. Lett. B* **158** (1985) 316.

[44] A.A. Tseytlin, *Renormalization of string loop corrections on the disk and the annulus*, *Phys. Lett. B* **208** (1988) 228 [INSPIRE].

[45] A.A. Tseytlin, *Vector field effective action in the open superstring theory*, *Nucl. Phys. B* **276** (1986) 391 [*Erratum ibid.* **291** (1987) 876] [INSPIRE].

[46] S.P. de Alwis, *The dilaton vertex in the path integral formulation of strings*, *Phys. Lett. B* **168** (1986) 59.

[47] I.R. Klebanov, *World volume approach to absorption by nondilatonic branes*, *Nucl. Phys. B* **496** (1997) 231 [hep-th/9702076] [INSPIRE].

[48] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *String theory and classical absorption by three-branes*, *Nucl. Phys. B* **499** (1997) 217 [hep-th/9703040] [INSPIRE].

[49] H. Liu and A.A. Tseytlin, *Dilaton - fixed scalar correlators and $AdS_5 \times S^5$ - SYM correspondence*, *JHEP* **10** (1999) 003 [hep-th/9906151] [INSPIRE].

[50] U.H. Danielsson, E. Keski-Vakkuri and M. Kruczenski, *Vacua, propagators, and holographic probes in AdS/CFT*, *JHEP* **01** (1999) 002 [hep-th/9812007] [INSPIRE].

[51] B. Fiol, B. Garolera and A. Lewkowycz, *Exact results for static and radiative fields of a quark in $N = 4$ super Yang-Mills*, *JHEP* **05** (2012) 093 [arXiv:1202.5292] [INSPIRE].

[52] J.L. Hovdebo, M. Kruczenski, D. Mateos, R.C. Myers and D.J. Winters, *Holographic mesons: adding flavor to the AdS/CFT duality*, *Int. J. Mod. Phys. A* **20** (2005) 3428 [INSPIRE].

[53] G.W. Semenoff and K. Zarembo, *More exact predictions of SUSYM for string theory*, *Nucl. Phys. B* **616** (2001) 34 [hep-th/0106015] [INSPIRE].

[54] D. Correa, J. Henn, J. Maldacena and A. Sever, *An exact formula for the radiation of a moving quark in $N = 4$ super Yang-Mills*, *JHEP* **06** (2012) 048 [arXiv:1202.4455] [INSPIRE].

[55] L.F. Alday and A.A. Tseytlin, *On strong-coupling correlation functions of circular Wilson loops and local operators*, J. Phys. A **44** (2011) 395401 [arXiv:1105.1537] [INSPIRE].

[56] S. Giombi and S. Komatsu, *More exact results in the Wilson loop defect CFT: bulk-defect OPE, nonplanar corrections and quantum spectral curve*, J. Phys. A **52** (2019) 125401 [arXiv:1811.02369] [INSPIRE].

[57] H. Ooguri and N. Sakai, *String loop corrections from fusion of handles and vertex operators*, Phys. Lett. B **197** (1987) 109 [INSPIRE].

[58] H. Ooguri and N. Sakai, *String multiloop corrections to equations of motion*, Nucl. Phys. B **312** (1989) 435 [INSPIRE].

[59] U. Ellwanger and M.G. Schmidt, *Loop corrected string field equations from world sheet Weyl invariance*, Z. Phys. C **43** (1989) 485 [INSPIRE].

[60] U. Ellwanger, *Multiloop string equations of motions*, Nucl. Phys. B **326** (1989) 254 [INSPIRE].

[61] A.A. Tseytlin, *Renormalization group and string loops*, Int. J. Mod. Phys. A **5** (1990) 589 [INSPIRE].

[62] A.A. Tseytlin, *On 'macroscopic string' approximation in string theory*, Phys. Lett. B **251** (1990) 530 [INSPIRE].

[63] D. Lüst and D. Skliros, *Handle operators in string theory*, arXiv:1912.01055 [INSPIRE].

[64] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, JHEP **02** (2005) 010 [hep-th/0501109] [INSPIRE].

[65] A. Klemm, M. Mariño, M. Schiereck and M. Soroush, *Aharony-Bergman-Jafferis-Maldacena Wilson loops in the Fermi gas approach*, Z. Naturforsch. A **68** (2013) 178 [arXiv:1207.0611] [INSPIRE].

[66] N. Drukker, J. Plefka and D. Young, *Wilson loops in* 3*-dimensional* $N = 6$ *supersymmetric Chern-Simons Theory and their string theory duals*, JHEP **11** (2008) 019 [arXiv:0809.2787] [INSPIRE].

[67] B. Chen and J.-B. Wu, *Supersymmetric Wilson loops in* $N = 6$ *super Chern-Simons-Matter theory*, Nucl. Phys. B **825** (2010) 38 [arXiv:0809.2863] [INSPIRE].

[68] J. Cookmeyer, J.T. Liu and L.A. Pando Zayas, *Higher rank ABJM Wilson loops from matrix models*, JHEP **11** (2016) 121 [arXiv:1609.08165] [INSPIRE].

[69] N. Drukker and B. Fiol, *On the integrability of Wilson loops in* $AdS_5 \times S^5$*: some periodic ansatze*, JHEP **01** (2006) 056 [hep-th/0506058] [INSPIRE].

[70] R. Bergamin and A.A. Tseytlin, *Heat kernels on cone of* AdS_2 *and* k*-wound circular Wilson loop in* $AdS_5 \times S^5$ *superstring*, J. Phys. A **49** (2016) 14LT01 [arXiv:1510.06894] [INSPIRE].

[71] K. Zarembo, *Supersymmetric Wilson loops*, Nucl. Phys. B **643** (2002) 157 [hep-th/0205160] [INSPIRE].

[72] S.-J. Rey, T. Suyama and S. Yamaguchi, *Wilson Loops in Superconformal Chern-Simons Theory and Fundamental Strings in Anti-de Sitter Supergravity Dual*, JHEP **03** (2009) 127 [arXiv:0809.3786] [INSPIRE].

[73] J. Kluson and K.L. Panigrahi, *Wilson loops in* 3*d QFT from D-branes in* $AdS_4 \times CP^3$, Eur. Phys. J. C **61** (2009) 339 [arXiv:0809.3355].

JHEP10(2020)130

[74] O. Ohlsson Sax, A. Sfondrini and B. Stefanski, *Integrability and the Conformal Field Theory of the Higgs branch*, JHEP **06** (2015) 103 [arXiv:1411.3676] [INSPIRE].

[75] N. Doroud, J. Gomis, B. Le Floch and S. Lee, *Exact results in $D = 2$ supersymmetric gauge theories*, JHEP **05** (2013) 093 [arXiv:1206.2606] [INSPIRE].

[76] W.A. Bardeen and D.Z. Freedman, *On the energy crisis in Anti-de Sitter supersymmetry*, Nucl. Phys. B **253** (1985) 635 [INSPIRE].

[77] T. Inami and H. Ooguri, *Dynamical breakdown of supersymmetry in two-dimensional Anti-de Sitter space*, Nucl. Phys. B **273** (1986) 487 [INSPIRE].

[78] S. Randjbar-Daemi, A. Salam and J.A. Strathdee, *σ models and string theories*, Int. J. Mod. Phys. A **2** (1987) 667 [INSPIRE].

[79] B. Durhuus, P. Olesen and J.L. Petersen, *Polyakov's quantized string with boundary terms*, Nucl. Phys. B **198** (1982) 157.

[80] B. Durhuus, P. Olesen and J.L. Petersen, *On the static potential in Polyakov's theory of the quantized string*, Nucl. Phys. B **232** (1984) 291.

[81] H. Lückock, *Quantum geometry of strings with boundaries*, Annals Phys. **194** (1989) 113 [INSPIRE].

[82] A.A. Tseytlin, *Renormalization of Mobius infinities and partition function representation for string theory effective action*, Phys. Lett. B **202** (1988) 81 [INSPIRE].

[83] J. Liu and J. Polchinski, *Renormalization of the Mobius Volume*, Phys. Lett. B **203** (1988) 39 [INSPIRE].

[84] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Exact Gell-Mann-Low function of supersymmetric Yang-Mills theories from instanton calculus*, Nucl. Phys. B **229** (1983) 381.

[85] M. Shifman, *Advanced topics in quantum field theory: a lecture course*, Cambridge University Press, Camrbidge U.K. (2012).

[86] O.D. Andreev and A.A. Tseytlin, *Partition function representation for the open superstring effective action: cancellation of Mobius infinities and derivative corrections to Born-Infeld Lagrangian*, Nucl. Phys. B **311** (1988) 205 [INSPIRE].

[87] G.W. Moore and P.C. Nelson, *Absence of nonlocal anomalies in the Polyakov string*, Nucl. Phys. B **266** (1986) 58 [INSPIRE].