Half-BPS Wilson loop and AdS$_2$/CFT$_1$

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Abstract

We study correlation functions of local operator insertions on the 1/2-BPS Wilson line in $\mathcal{N} = 4$ super Yang–Mills theory. These correlation functions are constrained by the 1d superconformal symmetry preserved by the 1/2-BPS Wilson line and define a defect CFT$_1$ living on the line. At strong coupling, a set of elementary operator insertions with protected scaling dimensions correspond to fluctuations of the dual fundamental string in AdS$_5 \times S^5$ ending on the line at the boundary and can be thought of as light fields propagating on the AdS$_2$ worldsheet. We use AdS/CFT techniques to compute the tree-level AdS$_2$ Witten diagrams describing the strong coupling limit of the four-point functions of the dual operator insertions. Using the OPE, we also extract the leading strong coupling corrections to the anomalous dimensions of the "two-particle" operators built out of elementary excitations. In the case of the circular Wilson loop, we match our results for the 4-point functions of a special type of scalar insertions to the prediction of localization to 2d Yang–Mills theory.

1. Introduction

In the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, it is natural to consider Wilson loop operators that include couplings to the six scalars Φ^I in the theory [1,2]

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$$W = \mathrm{tr}P e^{\oint dt \left(i \dot{x}^\mu A_\mu + |\dot{x}| \theta^I \Phi^I \right)}, \tag{1.1}$$

where $x^\mu(t)$ is a closed loop, and $\theta^I(t)$ is a unit 6-vector. For generic contour and scalar couplings, these operators are only locally supersymmetric, but special choices of x^μ and θ^I lead to Wilson loops preserving various fractions of the superconformal symmetry of the $\mathcal{N} = 4$ SYM theory [3,4]. The most supersymmetric operator is obtained by taking the contour to be an infinite straight line (or circle), and θ^I a constant 6-vector, corresponding to a fixed direction in the scalar space: in this case the Wilson loop is 1/2-BPS, i.e. it preserves 16 of the 32 supercharges of the superconformal group $PSU(2, 2|4)$. Making the choice $\theta^I \Phi^I = \Phi^6$, this 1/2-BPS straight line operator is given by

$$W = \mathrm{tr}P e^{\int dt \left(i A_t + \Phi^6 \right)} \tag{1.2}$$

where we have identified the Euclidean time $x^0 = t \in (-\infty, \infty)$ to be the line that defines the operator.

In this paper we will be interested in the computation of correlation functions of local operators inserted along the straight Wilson line, defined as follows. Given some local operators $O_i(t_i)$ transforming in the adjoint representation of the gauge group, one can define the gauge invariant correlator [5]

$$
\begin{aligned}
\langle\langle O_1(t_1) O_2(t_2) \cdots O_n(t_n) \rangle\rangle \\
\equiv \langle \mathrm{tr} P \big[O_1(t_1) \, e^{\int dt (i A_t + \Phi^6)} \, O_2(t_2) \, e^{\int dt (i A_t + \Phi^6)} \cdots O_n(t_n) \, e^{\int dt (i A_t + \Phi^6)} \big] \rangle \\
\equiv \langle \mathrm{tr} P \big[O_1(t_1) O_2(t_2) \cdots O_n(t_n) e^{\int dt (i A_t + \Phi^6)} \big] \rangle .
\end{aligned}
\tag{1.3}
$$

The $SU(N)$ indices are contracted with the Wilson lines joining the various points, making this a gauge invariant observable. Since the expectation value of the straight Wilson line is trivial, this definition satisfies $\langle\langle 1 \rangle\rangle = \langle W \rangle = 1$. More generally, one should normalize the correlator on the right-hand side by the expectation value of the Wilson loop without insertions (this is relevant in the case of the 1/2-BPS circular loop, which has a non-trivial expectation value [6–8]). Note that, since operator insertions are equivalent to deformations of the Wilson line [5,9], the complete knowledge of the correlators (1.3) would, at least in principle, allow to compute the expectation value of general Wilson loops which are deformations of the line or circle.

To understand the structure of the correlators (1.3), it is useful to recall the symmetries preserved by the 1/2-BPS Wilson line. First, it is clear that it preserves an $SO(5)$ subgroup of the $SO(6)_R$ R-symmetry that rotates the 5 scalars Φ^a, $a = 1, \ldots, 5$ that do not couple to the Wilson loop. In addition, it preserves an $SO(2, 1) \times SO(3)$ subgroup of the 4d conformal group $SO(2, 4)$, where the $SO(3)$ corresponds to rotations around the line, and the generators of $SO(2, 1)$ correspond to dilatations, translation and special conformal transformation along the line. This $SO(2, 1)$ is the $d = 1$ conformal group. Together with the 16 supercharges preserved by the loop, the symmetries of 1/2-BPS Wilson lines form the $d = 1$, $\mathcal{N} = 8$ superconformal group $OSp(4^*|4)$.

It follows that operator insertions along the Wilson line are classified by their representations under the $OSp(4^*|4)$ symmetry. In particular, they are labeled by their scaling dimension Δ, corresponding to a representation of $SO(2, 1)$, and by a representation of the "internal" (from the point of view of the line) symmetry group $SO(3) \times SO(5)$. The set of correlators (1.3) are then constrained by the $d = 1$ conformal symmetry in a way analogous to higher dimensional CFTs. They can be interpreted as characterizing a defect CFT$_1$ living on the Wilson line [5,10, 9]. This CFT$_1$ should then be fully determined by its spectrum of scaling dimensions and OPE

coefficients. Because the "double-bracket" correlators (1.3) satisfy all the usual properties of CFT correlation functions, we may often talk about the $O_i(t_i)$ as operators in a 1d CFT, without referring to their (non-local) origin in SYM theory.

Among the possible operator insertions, a special role is played by a set of "elementary excitations" that fall into a short representation of the $OSp(4^*|4)$ symmetry with 8 bosonic plus 8 fermionic operators, and have protected scaling dimensions. The bosonic operators are the 5 scalars Φ^a (with dimension $\Delta = 1$) that do not couple to the Wilson line, which have $\Delta = 1$, and the components of the field strength $\mathbb{F}_{ti} \equiv iF_{ti} + D_i\Phi^6$ (with dimension $\Delta = 2$) along the directions $i = 1, 2, 3$ transverse to the line. $\mathbb{F}_{ti}$ is also known as the displacement operator, which measures the change of the Wilson loop under deformations orthogonal to the contour (this can be defined for any defect in a CFT).[2]

The fact that these operators have protected scaling dimensions implies that their 2-point functions (in the sense of (1.3)) computed in planar SYM theory take the exact form

$$\langle\langle \Phi^a(t_1)\Phi^b(t_2)\rangle\rangle = \delta^{ab}\frac{C_\Phi(\lambda)}{t_{12}^2}, \qquad \langle\langle \mathbb{F}_{ti}(t_1)\mathbb{F}_{tj}(t_2)\rangle\rangle = \delta_{ij}\frac{C_{\mathbb{F}}(\lambda)}{t_{12}^4}, \qquad (1.4)$$

where the 't Hooft coupling λ dependence appears only in the normalization factors. These are proportional to the so-called Bremsstrahlung function $B(\lambda)$ defined in [14]

$$C_\Phi(\lambda) = 2B(\lambda), \qquad C_{\mathbb{F}}(\lambda) = 12B(\lambda), \qquad B(\lambda) = \frac{\sqrt{\lambda}\, I_2(\sqrt{\lambda})}{4\pi^2\, I_1(\sqrt{\lambda})}. \qquad (1.5)$$

The three-point functions of these elementary bosonic excitations vanish by the $SO(3) \times SO(5)$ symmetry. The four-point functions are expected to be non-trivial functions of the positions t_r (constrained by the 1d conformal symmetry as reviewed in Section 3 below) and of the coupling constant λ. Little is known about their structure apart from the leading perturbative term in the four-point of $\mathbb{F}_{ti}$ computed in [9].

In this paper, we will compute these four-point functions at strong coupling using the string theory in $AdS_5 \times S^5$ dual to planar $\mathcal{N} = 4$ SYM. At strong coupling, Wilson loops are related by duality to open string minimal surfaces in AdS_5 ending on the contour defining the loop operator at the boundary. In the case of the 1/2-BPS Wilson line (or circle), the relevant minimal surface is an AdS_2 embedded in AdS_5 (and sitting at a point on the S^5). The fundamental open string stretched in AdS preserves the same $OSp(4^*|4)$ as the 1/2-BPS Wilson line (see e.g. [15]). In particular, the 1d conformal group $SO(2, 1)$ is realized as the isometry of AdS_2.

As we will review in Section 2, expanding the string action in static gauge around the minimal surface solution, one finds [16] that the AdS_2 multiplet of fluctuations transverse to the string includes 5 massless scalars y^a corresponding to the S^5 directions, three massive scalars x^i with $m^2 = 2$ corresponding to AdS_5 fluctuations, and 8 fermionic modes with $m^2 = 1$. It is then natural to identify these $8 + 8$ excitations, which may be thought as fields living in AdS_2, with the elementary CFT_1 insertions described above [10,17,18]. Indeed, the standard relation $m^2 = \Delta(\Delta - d)$ between AdS_{d+1} scalar masses and the corresponding CFT_d operator dimensions in the present case implies that the massless y^a fields should be dual to $\Delta = 1$ operators in CFT_1,

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[2] The fact that the displacement operator has protected dimension $\Delta = 2$ for a line defect in a 4d CFT is a general result, and follows from a Ward identity for the breaking of translations in the directions orthogonal to the defect, see e.g. [11–13].
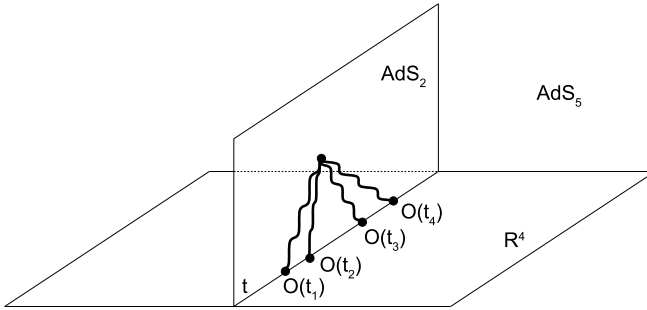
Fig. 1. Four-point function of local operators inserted on the Wilson line from a Witten diagram on the AdS_2 worldsheet.

namely the scalars Φ^a, while the three AdS_5 fluctuations x^i with $m^2 = 2$ should be dual to the field strength operators $\mathbb{F}_{ti}$ with $\Delta = 2$.[3]

In general, in AdS/CFT the closed superstring vertex operators are mapped to single-trace gauge invariant local operators in the SYM theory. Including the open-string sector (with open strings ending at the boundary) one should be able to describe the gauge-invariant operators (1.3) that correspond to insertions of general local operators along the Wilson loop. In this paper we will limit our considerations only to insertions corresponding to the operators with protected scaling dimensions, that should be dual to "light" fields on the AdS_2 string world-sheet as described above. It would be of course interesting to work out the strong coupling description of insertions that develop large anomalous dimensions at strong coupling, such as, for instance, the insertion of Φ^6 [21]. It is natural to expect that the dual of this type of "heavy" insertions have $m^2 \sim 1/\alpha' \sim \sqrt{\lambda}$, corresponding to massive states of the open string. In terms of CFT_1 scaling dimensions, this implies that at strong coupling the spectrum of operators on the Wilson line has a large gap $\Delta_{\text{gap}} \sim \lambda^{1/4}$, similarly to what happens for the closed string states.

The expansion of the Nambu action around the classical solution yields the interaction vertices between the light fields. We will use these vertices to compute the corresponding tree-level Witten diagrams in AdS_2 and extract the strong coupling prediction for the four-point functions of the protected insertions on the Wilson line. This is depicted schematically in Fig. 1. The calculation is similar to those in [22,23], however, we emphasize that the interpretation is different. In the supergravity calculations of [22,23], one computes correlation functions of single-trace local operators, dual to closed string states, and the expansion parameter is $1/N^2$. In our case, we compute the correlators (1.3) of insertions on the Wilson line, and the expansion parameter for the AdS_2 Witten diagrams is the inverse string tension, or $\frac{1}{\sqrt{\lambda}}$. Note that the 2d theory defined by the fundamental string action is expected to be UV finite, and thus the duality with the 1d CFT at the boundary should hold for any value of the coupling. In particular, the calculation of AdS_2 Witten diagrams involving loops should be well defined here.[4]

Note also that the AdS_2 worldsheet is not decoupled from the rest of the $AdS_5 \times S^5$ bulk. For instance, one can consider processes where the worldsheet interacts with closed string modes

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[3]　The spectrum of quadratic superstring fluctuations is the same as in the case of "non-relativistic limit" of $AdS_5 \times S^5$ superstring [19] and was suggested [10] to be related via AdS_2/CFT_1 to the $OSp(4^*|4)$ invariant $\mathcal{N} = 8$ superconformal quantum mechanics of [20].

[4]　For instance, by computing loop corrections to the boundary-to-boundary propagator one can verify that the elementary excitations are protected, as well as check the strong coupling expansion of the function $B(\lambda)$.

propagating from the worldsheet to a point on the boundary away from the line (note, however, that these processes are suppressed in the large N limit). This corresponds to correlation functions such as $\langle W \operatorname{tr} Z^J \rangle$ [24–26] (and more generally one may consider mixed correlators of defect operators and operators inserted away from the defect). The picture is similar to the one discussed in [27–29] (see also, e.g., [11–13,30,31] for recent related work) where one considers an AdS_d brane inside AdS_{d+1}, and there is a defect CFT living at the boundary of AdS_d. In our case, we have an AdS_2 worldsheet inside AdS_5, and a codimension 3 defect in CFT$_4$ (the Wilson line) at the boundary of AdS_2.[5]

Using the OPE expansion, we can also extract from the tree-level four-point functions the leading strong coupling corrections to the scaling dimensions of the "two-particle" operators built of products of two of the protected insertions (with an arbitrary number of t-derivatives in between). For instance, we find that the $SO(3) \times SO(5)$ singlet operator with no derivatives built of scalar insertions has the dimension

$$\Delta_{\Phi^a \Phi^a} = 2 - \frac{5}{\sqrt{\lambda}} + \dots . \tag{1.6}$$

Let us again emphasize that these are not scaling dimensions of gauge invariant local operators in the $\mathcal{N} = 4$ SYM theory but are scaling dimensions of operator insertions on the Wilson line, defined as in (1.3), (1.4)

$$\langle\langle O(t_1) O(t_2) \rangle\rangle = \langle \operatorname{tr} P[O(t_1) O(t_2) \, e^{\int dt (iA + \Phi^6)}]\rangle = \frac{C_{OO}}{t_{12}^{2\Delta_O}} , \tag{1.7}$$

with $O(t) = \Phi^a \Phi^a(t)$ in the present case. In principle, the spectrum of dimensions of operators inserted on the Wilson line should be accessible from the TBA approach of [35,36], and it would be interesting to reproduce our results in this integrability-based framework.[6] More broadly, it would be important to see how integrability is reflected in the structure of the Witten diagrams one computes in the AdS_2 worldsheet theory, perhaps uncovering an analog of the factorization of the S-matrix in integrable theories in flat space.

While we focus on the straight line for most of the paper, our results can be also mapped to the circle by a (large) conformal transformation, as explained in Section 6. For a particular class of S^5 insertions on the circular loop that are expected to be captured by localization [37–40], we show in Section 6 that the result of the Witten diagram calculation in AdS_2 precisely matches the exact prediction derived from localization to 2d YM theory.

As was appreciated in recent discussions of $\mathrm{AdS}_2/\mathrm{CFT}_1$ in the context of dilaton-gravity models [41–45] one can think of a system in AdS_2 as having asymptotic 1d reparametrization symmetry that is spontaneously broken down to $SO(2, 1)$, which is the isometry of AdS_2 metric. In our present case the original definition of the Wilson loop (1.1) has a reparametrization invariance which is fixed by the identification $x^0 = t$ in (1.2), and the remaining conformal symmetry is the $SO(2, 1)$ subgroup of the 4d conformal group that preserves the line. It is important to

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[5] One may also consider the D3 and D5 branes dual to 1/2-BPS Wilson loops in higher-rank symmetric and antisymmetric representations [32,33,15,34]: these branes have $\mathrm{AdS}_2 \times S^2$ or $\mathrm{AdS}_2 \times S^4$ worldvolumes, and preserve the same $OSp(4^*|4)$ symmetry as the fundamental string. Computing AdS_2 Witten diagrams in this case (after KK reduction on the sphere factors) should yield the correlators (1.3) where the trace is taken to be in the symmetric or antisymmetric representations of rank $k \sim N$.

[6] It would also be interesting to use integrability to reproduce a weak coupling Feynman graph approach to dimensions of operator insertions on the Wilson line.

stress that compared to the gravitational AdS_2 models in [41–45] our bulk action (2.4) is defined in fixed AdS_2 background, i.e. does not contain gravity: before fixing the static gauge[7] the string action (2.1) is reparametrization invariant, but gravity never becomes dynamical in critical superstring theory. In line with this, the boundary theory has no analog of the pseudo-Goldstone mode [42] related to the (spontaneously broken) reparametrizations.

2. $AdS_5 \times S^5$ string action in static gauge as AdS_2 bulk theory action

The bosonic part of the superstring action in $AdS_5 \times S^5$ has the standard form

$$S_B = \frac{1}{2} T \int d^2\sigma \sqrt{h} \, h^{\mu\nu} \left[\frac{1}{z^2} \left(\partial_\mu x^r \partial_\nu x^r + \partial_\mu z \partial_\nu z \right) + \frac{\partial_\mu y^a \partial_\nu y^a}{(1 + \frac{1}{4} y^2)^2} \right], \qquad T = \frac{\sqrt{\lambda}}{2\pi}, \quad (2.1)$$

where $\sigma^\mu = (t, s)$ are Euclidean world-sheet coordinates, $r = (0, i) = (0, 1, 2, 3)$ label coordinates of the Euclidean 4-boundary and $a = 1, ..., 5$ are S^5 labels. The minimal surface corresponding to the straight Wilson line at the boundary is described by

$$z = s, \qquad x^0 = t, \qquad x^i = 0, \quad y^a = 0. \tag{2.2}$$

The corresponding induced metric is that of AdS_2, i.e. $g_{\mu\nu} d\sigma^\mu d\sigma^\nu = \frac{1}{s^2}(dt^2 + ds^2)$.

We will study correlators of small fluctuations of "transverse" string coordinates (x^i, y^a) near this minimal surface that will thus propagate in the induced AdS_2 metric. The resulting global symmetry of the bosonic action will thus be $SO(2, 1) \times [SO(3) \times SO(6)]$. To make the $SO(2, 1)$ symmetry (which will be the conformal symmetry at the corresponding 1d boundary theory) manifest it is useful to choose the AdS_2 adapted coordinates and fix the static gauge in which z and x^0 do not fluctuate. The relevant embedding of AdS_2 into AdS_5 is described by ($x^2 \equiv x^i x^i$, $i = 1, 2, 3$)

$$ds_5^2 = \frac{(1 + \frac{1}{4} x^2)^2}{(1 - \frac{1}{4} x^2)^2} ds_2^2 + \frac{dx^i dx^i}{(1 - \frac{1}{4} x^2)^2}, \qquad ds_2^2 = \frac{1}{z^2}(dx_0^2 + dz^2). \tag{2.3}$$

Starting with the Nambu action and fixing the static gauge by the conditions on x_0 and z as in (2.2) we get

$$S_B = T \int d^2\sigma \sqrt{\det\left[\frac{(1 + \frac{1}{4} x^2)^2}{(1 - \frac{1}{4} x^2)^2} g_{\mu\nu}(\sigma) + \frac{\partial_\mu x^i \partial_\nu x^i}{(1 - \frac{1}{4} x^2)^2} + \frac{\partial_\mu y^a \partial_\nu y^a}{(1 + \frac{1}{4} y^2)^2} \right]}$$

$$\equiv T \int d^2\sigma \sqrt{g} \, L_B, \tag{2.4}$$

where $g_{\mu\nu} = \frac{1}{s^2} \delta_{\mu\nu}$ is the background AdS_2 metric. This action can be interpreted as that of a straight fundamental string in $AdS_5 \times S^5$ stretched along z, i.e. from the boundary towards the center of AdS_5. It may be also viewed as a 2d field theory of $3 + 5$ scalars in AdS_2 geometry with manifest symmetry $SO(2, 1) \times [SO(3) \times SO(6)]$. Interpreted as a 2d bulk AdS_2 theory, it

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[7] Defining the Wilson loop expectation value in string theory in conformal gauge where one has two more (compared to physical static gauge) dynamical coordinates and ghosts one would end effectively with an integral over boundary reparametrizations (see [46–48]). In this case the identification between the operators on the Wilson line on the gauge theory side and the string excitations appears to become more intricate. This question deserves further study.

should thus have a CFT_1 dual living at the $z = s = 0$ boundary. As explained in the Introduction, this CFT_1 can be viewed as the defect CFT defined by operator insertions on the straight Wilson line.

Expanding this action in powers of x^i and y^a we get

$$L_B = L_2 + L_{4x} + L_{2x,2y} + L_{4y} + \dots , \tag{2.5}$$

$$L_2 = \tfrac{1}{2} g^{\mu\nu} \partial_\mu x^i \partial_\nu x^i + x^i x^i + \tfrac{1}{2} g^{\mu\nu} \partial_\mu y^a \partial_\nu y^a , \tag{2.6}$$

$$L_{4x} = \tfrac{1}{8} (g^{\mu\nu} \partial_\mu x^i \partial_\nu x^i)^2 - \tfrac{1}{4} (g^{\mu\nu} \partial_\mu x^i \partial_\nu x^i)(g^{\rho\kappa} \partial_\rho x^i \partial_\kappa x^j)$$
$$+ \tfrac{1}{4} x^i x^i (g^{\mu\nu} \partial_\mu x^j \partial_\nu x^j) + \tfrac{1}{2} x^i x^i x^j x^j , \tag{2.7}$$

$$L_{2x,2y} = \tfrac{1}{4} (g^{\mu\nu} \partial_\mu x^i \partial_\nu x^i)(g^{\rho\kappa} \partial_\rho y^a \partial_\kappa y^a) - \tfrac{1}{2} (g^{\mu\nu} \partial_\mu x^i \partial_\nu y^a)(g^{\rho\kappa} \partial_\rho x^i \partial_\kappa y^a) , \tag{2.8}$$

$$L_{4y} = -\tfrac{1}{4} (y^b y^b)(g^{\mu\nu} \partial_\mu y^a \partial_\nu y^a) + \tfrac{1}{8} (g^{\mu\nu} \partial_\mu y^a \partial_\nu y^a)^2$$
$$- \frac{1}{4} (g^{\mu\nu} \partial_\mu y^a \partial_\nu y^b)(g^{\rho\kappa} \partial_\rho y^a \partial_\kappa y^b) . \tag{2.9}$$

Thus x^i are 3 massive ($m^2 = 2$) and y^a are 5 massless scalars propagating in AdS_2.

One may also include the fermionic terms coming from the corresponding $\text{AdS}_5 \times S^5$ superstring action as in [16] (there will also be eight 2d fermions with mass 1). The resulting 2d theory should be UV finite and thus should be dual to a quantum 1d CFT at the boundary for any value of the coupling $T = \frac{\sqrt{\lambda}}{2\pi}$. The coefficients in the correlation functions computed in perturbation theory will be given by power series in $\frac{1}{\sqrt{\lambda}}$.

At strong coupling ($\lambda \gg 1$) the correlators (1.3) are expected to be reproduced by the AdS_2 amplitudes in the (super) string sigma model theory (2.4), with the operators O corresponding to particular string coordinates X, i.e.

$$\langle\!\langle O(t_1) O(t_2) \dots O(t_n) \rangle\!\rangle = \langle X(t_1) X(t_2) \dots X(t_n) \rangle_{\text{AdS}_2} , \tag{2.10}$$

where $\langle \dots \rangle_{\text{AdS}_2}$ is the expectation value in the 2d theory (2.4) corresponding to Witten diagrams with bulk-to-boundary propagators attached to the points $t_1, \dots, t_n$ at the boundary. As discussed in the Introduction, the $X \sim y^a$ in (2.4) will correspond to the scalar operators $O \sim \Phi^a$ ($a = 1, \dots, 5$) of dimension $\Delta = 1$ while $X \sim x^i$ will correspond to the generalized field strength components $O \sim \mathbb{F}_{it}$ with $\Delta = 2$.

The relation (2.10) can be understood as follows. The correlators $\langle\!\langle O(t_1) O(t_2) \dots O(t_n) \rangle\!\rangle$ in (1.3) can be found by first computing a wavy-line Wilson loop expectation value $\langle W(C) \rangle$, taking functional derivatives over the contour function $C(t)$ and then setting it to be a straight line. At weak coupling this procedure was followed in [9]. At strong coupling $\langle W(C) \rangle$ is assumed to be given by the $\text{AdS}_5 \times S^5$ open string path integral with Dirichlet boundary conditions (implying that disc-like or half-plane like world-surface ends on a contour at the boundary of $\text{AdS}_5 \times S^5$). To leading order in large $\sqrt{\lambda}$ expansion that means computing the minimal area of the corresponding surface, i.e. the value of the (Euclidean) string action on the classical solution of the Dirichlet problem. In the present case of the string action in the static gauge (2.4) interpreted as a 2d field theory in AdS_2 this is equivalent to the standard AdS/CFT procedure of computing the generating functional for the corresponding CFT_1 boundary correlators or $\langle X(t_1) X(t_2) \dots X(t_n) \rangle_{\text{AdS}_2}$. Expanding the resulting on-shell value of the string action in powers of small deviations of the boundary curve from the straight line will then give the correlators that can be equivalently found by computing the bulk Green's functions connected to the boundary points by the bulk-to-boundary propagators. One can check the relation (2.10) explicitly at the 2-point level using

the wavy-line solution of [49] (see also [50]), reproducing the string tree-level [49,51] and the 1-loop [52] corrections in the strong-coupling expansion of the $B(\lambda)$ function in (1.4), (1.5).

The Lagrangian (2.5) has no cubic terms, so the contribution to the simplest 4-point tree-level correlation functions of x^i and y^a will be given just by the contact 4-point vertices in (2.7)–(2.9). Below we will compute the corresponding Witten diagrams in AdS$_2$ connecting the 4-vertices to the boundary points by bulk-to-boundary propagators as in, e.g., [22,23]. As we will be interested only in leading large λ (tree-level) bosonic field correlators we will ignore the fermions.

Note that while we have made a particular choice of AdS$_5$ coordinates in (2.3) the result for the on-shell AdS$_2$ amplitudes (i.e. boundary operator correlation functions) should be invariant under local field redefinitions (at least in the case of separated boundary points controlled by conformal invariance).

After including fermions and fixing kappa-symmetry gauge the superstring action (generalizing (2.1), (2.4)) expanded near the 1/2 BPS straight line minimal surface should be describing a globally supersymmetric field theory [16] for the $OSp(4^*|4)$ multiplet of $8 + 8$ bosons and fermions in AdS$_2$. Same symmetry appears on the dual gauge theory side. While in this paper we will discuss only 4-point correlators of bosonic coordinates, this supersymmetry should allow also to determine the correlators involving fermionic excitations.

3. Four-point functions and conformal blocks in CFT$_1$

Before proceeding to computation of correlators of 2d fields in the AdS$_2$ theory (2.4), (2.5) let us make some general remarks about the structure of four-point functions in CFT$_1$.

Local operators in a $d = 1$ CFT defined on a line $\mathbb{R} = \{t\}$ which are covariant under the conformal group $SO(2, 1)$ are labeled just by their scaling dimension Δ (and possibly by some representation of an internal symmetry group which we suppress in this section). Let us consider the 4-point function of an operator $O_\Delta(t)$. The $SO(2, 1)$ symmetry restricts the 4-point function to take the form

$$\langle O_\Delta(t_1) O_\Delta(t_2) O_\Delta(t_3) O_\Delta(t_4) \rangle = \frac{1}{(t_{12} t_{34})^{2\Delta}} \, \mathcal{G}(\chi) \,, \tag{3.1}$$

where $\chi \in (-\infty, \infty)$ is a conformally invariant cross ratio

$$\chi = \frac{t_{12} t_{34}}{t_{13} t_{24}} \,. \tag{3.2}$$

Note that the usual cross ratios u, v are not independent in $d = 1$, i.e.

$$u \equiv \frac{t_{12}^2 t_{34}^2}{t_{13}^2 t_{24}^2} = \chi^2 \,, \qquad v \equiv \frac{t_{14}^2 t_{23}^2}{t_{13}^2 t_{24}^2} = (1 - \chi)^2 \,. \tag{3.3}$$

This is because the $SO(2, 1)$ symmetry allows one to fix three points on the line, leaving a single free real parameter as the position of the fourth point. For example, if we set $t_1 = 0$, $t_3 = 1$, $t_4 = \infty$, then χ corresponds to the position t_2 of the second operator.

3.1. OPE expansion

As in the case of higher dimensional CFT, the function $\mathcal{G}(\chi)$ in (3.1) has an OPE expansion

$$\mathcal{G}(\chi) = \sum_h c_{\Delta, \Delta; h} \, \chi^h \, {}_2F_1(h, h, 2h, \chi) \,, \tag{3.4}$$

where h is the scaling dimension of the exchanged operator, $c_{\Delta,\Delta;h} = C^2_{O_\Delta O_\Delta O_h}/(C^2_{O_\Delta O_\Delta} C_{O_h O_h})$ are normalized OPE coefficients, and $\chi^h {}_2F_1(h, h, 2h, \chi)$ is the exact conformal block in $d = 1$ [53].

We will also need the case of correlator of operators with pairwise equal dimensions

$$\langle O_{\Delta_1}(t_1) O_{\Delta_2}(t_2) O_{\Delta_1}(t_3) O_{\Delta_2}(t_4) \rangle = \frac{1}{(t_{12}t_{34})^{\Delta_1+\Delta_2}} \left| \frac{t_{24}}{t_{13}} \right|^{\Delta_{12}} \mathcal{G}(\chi), \qquad \Delta_{12} \equiv \Delta_1 - \Delta_2. \tag{3.5}$$

Here the conformal block expansion reads [53]

$$\mathcal{G}(\chi) = \sum_h c_{\Delta_1,\Delta_2;h} \, \chi^h \, {}_2F_1(h + \Delta_{12}, h - \Delta_{12}, 2h, \chi). \tag{3.6}$$

Note that in (3.5) we have written the result by choosing the $12 \to 34$ channel (corresponding to $\chi \to 0$), which will be more convenient below. Of course, one may also write the 4-point function in the form

$$\langle O_{\Delta_1}(t_1) O_{\Delta_1}(t_2) O_{\Delta_2}(t_3) O_{\Delta_2}(t_4) \rangle = \frac{1}{t_{12}^{2\Delta_1} t_{34}^{2\Delta_2}} \tilde{\mathcal{G}}(\chi), \tag{3.7}$$

where $\tilde{\mathcal{G}}(\chi)$ is related to $\mathcal{G}(\chi)$ in (3.5) by $\quad \tilde{\mathcal{G}}(\chi) = \chi^3 \mathcal{G}(\chi^{-1})$.

3.2. Generalized free field OPE coefficients

It will be useful for what follows to collect some results for the OPE coefficients of generalized free fields (see, e.g., [54–57,12]). In the case of the 4-point function of identical operators of dimension Δ, the generalized free field 4-point function has $\mathcal{G}(u, v) = 1 + u^\Delta + (u/v)^\Delta$, i.e. in $d = 1$ (3.3) is given by

$$\langle O_\Delta(t_1) O_\Delta(t_2) O_\Delta(t_3) O_\Delta(t_4) \rangle = \frac{1}{(t_{12}t_{34})^{2\Delta}} \left[1 + \chi^{2\Delta} + \frac{\chi^{2\Delta}}{(1-\chi)^{2\Delta}} \right], \tag{3.8}$$

where we assumed unit normalization of the 2-point function. The operators exchanged in the OPE are just the identity and the tower of "two-particle" operators

$$\left[O_\Delta O_\Delta \right]_{2n} \sim O_\Delta \partial_t^{2n} O_\Delta \tag{3.9}$$

of dimension $2\Delta + 2n$, $n = 0, 1, \ldots$. The corresponding OPE coefficients are given explicitly by

$$c_{\Delta,\Delta;2\Delta+2n} = \frac{2 \left[\Gamma(2n + 2\Delta) \right]^2 \Gamma(2n + 4\Delta - 1)}{\left[\Gamma(2\Delta) \right]^2 \Gamma(2n+1) \Gamma(4n + 4\Delta - 1)}, \tag{3.10}$$

as one can verify from the identity

$$\sum_{n=0}^\infty c_{\Delta,\Delta;2\Delta+2n} \, \chi^{2\Delta+2n} \, {}_2F_1(2\Delta + 2n, 2\Delta + 2n, 4\Delta + 4n, \chi) = \chi^{2\Delta} + \frac{\chi^{2\Delta}}{(1-\chi)^{2\Delta}}. \tag{3.11}$$

While operators with odd number of derivatives do not appear in the OPE of identical O_Δ's, it will be useful for the case of operators carrying a flavor index (where odd n can appear in the antisymmetric channel) to note the following result for the sum over odd n:

$$\sum_{n=0}^{\infty} c_{\Delta,\Delta;2\Delta+2n+1}\, \chi^{2\Delta+2n+1}\, {}_2F_1(2\Delta+2n+1, 2\Delta+2n+1, 4\Delta+4n+2, \chi)$$

$$= -\chi^{2\Delta} + \frac{\chi^{2\Delta}}{(1-\chi)^{2\Delta}}\,. \tag{3.12}$$

In the case of pairwise identical operators, we have (cf. (3.5))

$$\langle O_{\Delta_1}(t_1) O_{\Delta_2}(t_2) O_{\Delta_1}(t_3) O_{\Delta_2}(t_4)\rangle = \frac{1}{t_{13}^{2\Delta_1} t_{24}^{2\Delta_2}} = \frac{1}{(t_{12}t_{34})^{\Delta_1+\Delta_2}} \left|\frac{t_{24}}{t_{13}}\right|^{\Delta_{12}} \chi^{\Delta_1+\Delta_2}\,. \tag{3.13}$$

Here the operators exchanged in the small χ expansion are $[O_{\Delta_1} O_{\Delta_2}]_n \sim O_{\Delta_1} \partial_t^n O_{\Delta_2}$ for all integer n (both even and odd). The corresponding OPE coefficients are found to be

$$c_{\Delta_1,\Delta_2;\Delta_1+\Delta_2+n} = \frac{(-1)^n \Gamma(n+2\Delta_1)\, \Gamma(n+2\Delta_2)\, \Gamma(n+2\Delta_1+2\Delta_2-1)}{\Gamma(2\Delta_1)\, \Gamma(2\Delta_2)\, \Gamma(n+1)\, \Gamma(2n+2\Delta_1+2\Delta_2-1)}\,. \tag{3.14}$$

Indeed, one may verify that this agrees with the OPE expansion in (3.6) by checking that

$$\sum_{n=0}^{\infty} c_{\Delta_1,\Delta_2;\Delta_1+\Delta_2+n}\, \chi^{\Delta_1+\Delta_2+n}\, {}_2F_1(\Delta_1+n, \Delta_2+n, 2\Delta_1+2\Delta_2+2n, \chi) = \chi^{\Delta_1+\Delta_2}\,. \tag{3.15}$$

4. Four-point function of S^5 fluctuations

In this section we compute the tree-level 4-point Witten diagram of the S^5 fluctuations y^a in the AdS$_2$ action in (2.5). As reviewed above, these are dual to the 5 SYM scalars Φ^a, $a = 1, \ldots, 5$ (that do not appear in the exponent of the half-BPS Wilson line operator) inserted along the line. The strong-coupling limit of the SYM correlator 1.3 should be given by the tree-level string coordinate amplitude as in (2.10).

By conformal symmetry, the 4-point function should take the form (3.1), i.e.

$$\langle y^{a_1}(t_1) y^{a_2}(t_2) y^{a_3}(t_3) y^{a_4}(t_4)\rangle_{\text{AdS}_2} = \langle\langle \Phi^{a_1}(t_1) \Phi^{a_2}(t_2) \Phi^{a_3}(t_3) \Phi^{a_4}(t_4)\rangle\rangle$$

$$= \frac{[C_\Phi(\lambda)]^2}{t_{12}^2 t_{34}^2}\, G^{a_1 a_2 a_3 a_4}(\chi)\,, \tag{4.1}$$

where χ is the conformally invariant cross ratio (3.2). In writing (4.1), we used the fact that the operators Φ^a have protected dimension $\Delta = 1$, i.e. that their exact two-point function is[8]

$$\langle y^{a_1}(t_1) y^{a_2}(t_2)\rangle_{\text{AdS}_2} = \langle\langle \Phi^{a_1}(t_1) \Phi^{a_2}(t_2)\rangle\rangle = \delta^{a_1 a_2} \frac{C_\Phi(\lambda)}{t_{12}^2}\,. \tag{4.2}$$

In (4.1) we factored out $[C_\Phi(\lambda)]^2$ so that in the OPE limit $\chi \to 0$ we have $G^{a_1 a_2 a_3 a_4}(\chi) = \delta^{a_1 a_2} \delta^{a_3 a_4} + O(\chi)$. The two-point normalization factor $C_\Phi(\lambda)$ is related to the Bremsstrahlung function defined in [14,36]. We can always absorb this factor in the normalization of the operators, and we will do so in the following by choosing a canonical form of the bulk-to-boundary propagators.

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[8] Once again, when referring to operators of 1d CFT we understand them as insertions on the Wilson line.
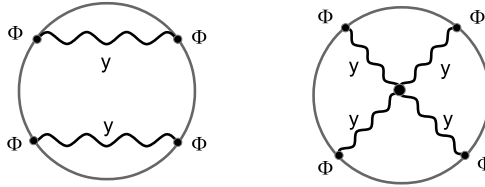
Fig. 2. The disconnected and connected contributions to the 4-point function.

The function $G^{a_1a_2a_3a_4}(\chi)$ (which is also a non-trivial function of the coupling λ) can be decomposed into its $SO(5)$ singlet, symmetric traceless and antisymmetric parts,

$$G^{a_1a_2a_3a_4}(\chi) = G_S(\chi)\delta^{a_1a_2}\delta^{a_3a_4} + G_T(\chi)\left(\delta^{a_1a_3}\delta^{a_2a_4} + \delta^{a_2a_3}\delta^{a_1a_4} - \tfrac{2}{5}\delta^{a_1a_2}\delta^{a_3a_4}\right)$$
$$+ G_A(\chi)\left(\delta^{a_1a_3}\delta^{a_2a_4} - \delta^{a_2a_3}\delta^{a_1a_4}\right). \tag{4.3}$$

At strong coupling, these functions are expected to have the expansion (working in perturbation theory)

$$G_{S,T,A}(\chi) = G^{(0)}_{S,T,A}(\chi) + \frac{1}{\sqrt{\lambda}}G^{(1)}_{S,T,A}(\chi) + \dots . \tag{4.4}$$

The leading terms here correspond to the disconnected contribution to the 4-point function, namely diagrams with two "boundary-to-boundary" propagators, (see Fig. 2), and are given by the generalized free field expression (cf. (3.8))[9]

$$\langle\langle \Phi^{a_1}(t_1)\Phi^{a_2}(t_2)\Phi^{a_3}(t_3)\Phi^{a_4}(t_4)\rangle\rangle_{\text{disconn.}}$$
$$= \frac{[C_\Phi(\lambda)]^2}{t_{12}^2 t_{34}^2}\left[\delta^{a_1a_2}\delta^{a_3a_4} + \chi^2\delta^{a_1a_3}\delta^{a_2a_4} + \frac{\chi^2}{(1-\chi)^2}\delta^{a_1a_4}\delta^{a_2a_3}\right] \tag{4.5}$$

which yields

$$G^{(0)}_S(\chi) = 1 + \tfrac{2}{5}G^{(0)}_T(\chi), \qquad G^{(0)}_T(\chi) = \tfrac{1}{2}\left[\chi^2 + \frac{\chi^2}{(1-\chi)^2}\right],$$
$$G^{(0)}_A(\chi) = \tfrac{1}{2}\left[\chi^2 - \frac{\chi^2}{(1-\chi)^2}\right]. \tag{4.6}$$

The functions appearing at order $\frac{1}{\sqrt{\lambda}}$ in (4.4) correspond to the leading contribution to the connected 4-point function at strong coupling, which comes from tree-level connected Witten diagrams. These are given by the 4-vertices in (2.9) with four bulk-to-boundary propagators attached.

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[9] Note that the separation between a connected and a disconnected contribution defined in (4.5) is natural from the point of view of the AdS$_2$ worldsheet perturbation theory, to all orders in $1/\sqrt{\lambda}$: in general, the disconnected contribution is given by a pair of loop-corrected boundary-to-boundary propagators. In the weak coupling limit, on the other hand, it is straightforward to see that the leading contribution in the planar limit is $\langle\langle \Phi^{a_1}(t_1)\Phi^{a_2}(t_2)\Phi^{a_3}(t_3)\Phi^{a_4}(t_4)\rangle\rangle \sim \frac{\lambda^2}{t_{12}^2 t_{34}^2}\left(\delta^{a_1a_2}\delta^{a_3a_4} + \frac{\chi^2}{(1-\chi)^2}\delta^{a_1a_4}\delta^{a_2a_3}\right)$, which is not exactly of the form (4.5), indicating that the connected contribution, defined from the point of view of the AdS$_2$ perturbation theory, should, in fact, contribute at leading order at small λ.

We will adopt the following normalization of the bulk-to-boundary propagator (in general dimension d)

$$K_\Delta(z, x; x') = C_\Delta \left[\frac{z}{z^2 + (x - x')^2} \right]^\Delta \equiv C_\Delta \, \tilde{K}_\Delta(z, x; x') \, ,$$

$$C_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}} \Gamma\left(\Delta + 1 - \frac{d}{2}\right)} \, . \tag{4.7}$$

In this normalization [58,59], the tree level two-point function of the dual boundary operator is $\langle O_\Delta(x_1) O_\Delta(x_2) \rangle = \frac{C_\Delta}{x_{12}^{2\Delta}}$.[10] In the present case of $d = 1$ and $\Delta = 1$, we then have ($t \equiv x^0$)

$$K_{\Delta=1}(z, t; t') = \frac{1}{\pi} \frac{z}{z^2 + (t - t')^2} \, , \qquad\qquad C_{\Delta=1} = \frac{1}{\pi} \, . \tag{4.8}$$

When one has only quartic contact diagrams (as in our present case), all tree-level 4-point functions can be written in terms of the D-functions [22,61,62] defined in the general case of AdS_{d+1} as

$$D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4)$$

$$= \int \frac{dz d^d x}{z^{d+1}} \tilde{K}_{\Delta_1}(z, x; x_1) \tilde{K}_{\Delta_2}(z, x; x_2) \, \tilde{K}_{\Delta_3}(z, x; x_3) \tilde{K}_{\Delta_4}(z, x; x_4) \, . \tag{4.9}$$

Note that derivatives in the vertices can be dealt with by using the identity (here $\partial_\mu = (\partial_z, \partial_r)$, $r = 0, 1, 2, ..., d - 1$ and $g^{\mu\nu} = z^2 \delta^{\mu\nu}$)

$$g^{\mu\nu} \partial_\mu \tilde{K}_{\Delta_1}(z, x; x_1) \, \partial_\nu \tilde{K}_{\Delta_2}(z, x; x_2)$$

$$= \Delta_1 \Delta_2 \left[\tilde{K}_{\Delta_1}(z, x; x_1) \tilde{K}_{\Delta_2}(z, x; x_2) - 2x_{12}^2 \tilde{K}_{\Delta_1+1}(z, x; x_1) \tilde{K}_{\Delta_2+1}(z, x; x_2) \right] . \tag{4.10}$$

4.1. Connected part of the four-point function

Returning to our case of $d = 1$, let us write the tree-level connected 4-point function (4.1) of y^a coordinates in (2.5) as

$$\langle\langle \Phi^{a_1}(t_1) \Phi^{a_2}(t_2) \Phi^{a_3}(t_3) \Phi^{a_4}(t_4) \rangle\rangle_{\text{conn}} = \frac{2\pi}{\sqrt{\lambda}} (C_{\Delta=1})^4 \, Q_{4y}^{a_1 a_2 a_3 a_4} \, , \tag{4.11}$$

where Q_{4y} is obtained from the vertex L_{4y} in (2.9). Explicitly, we find

---

[10] Note that this differs from the normalization adopted in [60], where $C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)}$ was used. In that normalization, the two-point function of the dual operator is $\langle O_\Delta(x_1) O_\Delta(x_2) \rangle = \frac{(2\Delta - d) C_\Delta}{x_{12}^{2\Delta}}$.

$$Q_{4y}^{a_1a_2a_3a_4} = \Big[3D_{1111} - 2t_{13}^2 D_{2121} - 2t_{14}^2 D_{2112} - 2t_{23}^2 D_{1221} - 2t_{24}^2 D_{1212}$$
$$+ 4(t_{13}^2 t_{24}^2 + t_{14}^2 t_{23}^2 - t_{12}^2 t_{34}^2) D_{2222}\Big]\delta^{a_1a_2}\delta^{a_3a_4}$$
$$+ \Big[3D_{1111} - 2t_{12}^2 D_{2211} - 2t_{14}^2 D_{2112} - 2t_{23}^2 D_{1221} - 2t_{34}^2 D_{1122}$$
$$+ 4(t_{12}^2 t_{34}^2 + t_{14}^2 t_{23}^2 - t_{13}^2 t_{24}^2) D_{2222}\Big]\delta^{a_1a_3}\delta^{a_2a_4}$$
$$+ \Big[3D_{1111} - 2t_{12}^2 D_{2211} - 2t_{13}^2 D_{2121} - 2t_{24}^2 D_{1212} - 2t_{34}^2 D_{1122}$$
$$+ 4(t_{12}^2 t_{34}^2 + t_{13}^2 t_{24}^2 - t_{14}^2 t_{23}^2) D_{2222}\Big]\delta^{a_1a_4}\delta^{a_2a_3} . \tag{4.12}$$

To write the result in a manifestly conformally invariant form, it is convenient to introduce the "reduced" $\bar{D}$-functions that are functions of cross-ratios only. In general d, they are defined in terms of (4.9) as [62] ($\Sigma \equiv \frac{1}{2}\sum_i \Delta_i$)

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}$$
$$= \frac{\pi^{\frac{d}{2}}\Gamma\left(\Sigma - \frac{d}{2}\right)}{2\,\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)\Gamma(\Delta_4)} \frac{x_{14}^{2(\Sigma-\Delta_1-\Delta_4)}x_{34}^{2(\Sigma-\Delta_3-\Delta_4)}}{x_{13}^{2(\Sigma-\Delta_4)}x_{24}^{2\Delta_2}}\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) . \tag{4.13}$$

$\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}$ can be written explicitly as the following Feynman parameter integral

$$\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)$$
$$= \int d\alpha d\beta d\gamma\, \delta(\alpha + \beta + \gamma - 1)\, \alpha^{\Delta_1-1}\beta^{\Delta_2-1}\gamma^{\Delta_3-1}\frac{\Gamma(\Sigma - \Delta_4)\Gamma(\Delta_4)}{\left(\alpha\gamma + \alpha\beta u + \beta\gamma v\right)^{\Sigma-\Delta_4}} . \tag{4.14}$$

In $d = 1$ where $u = \chi^2$, $v = (1 - \chi)^2$ we get $\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}$ as a function of a single variable χ. When the indices Δ_i are integers, the integral (4.14) can be evaluated explicitly. The basic example appearing in our calculations is

$$\bar{D}_{1111}(\chi) = \frac{1}{\chi - 1}\log\left(\chi^2\right) - \frac{1}{\chi}\log\left[(1 - \chi)^2\right]. \tag{4.15}$$

One can check that this agrees with the $d = 1$ limit of the well-known result in general d

$$\bar{D}_{1111}(u,v) = \frac{1}{z - \bar{z}}\Big[\log(z\bar{z})\log(\frac{1-z}{1-\bar{z}}) + 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z})\Big],$$
$$u = z\bar{z}, \quad v = (1 - z)(1 - \bar{z}) \tag{4.16}$$

after we set $z = \bar{z} = \chi$ (cf. (3.3)). The $\bar{D}$-functions with higher integer indices can be either evaluated directly using (4.14), or expressed in terms of derivatives of $\bar{D}_{1111}$ using the identities listed in [62].

Evaluating all the relevant integrals, the final result for (4.11) takes the form

$$\langle\langle\Phi^{a_1}(t_1)\Phi^{a_2}(t_2)\Phi^{a_3}(t_3)\Phi^{a_4}(t_4)\rangle\rangle_{\mathrm{conn}} = \frac{(\mathcal{C}_{\Delta=1})^2}{t_{12}^2 t_{34}^2}G_{(1)}^{a_1a_2a_3a_4}(\chi) , \tag{4.17}$$

where we factored out $(\mathcal{C}_{\Delta=1})^2$ so that $G_{(1)}^{a_1a_2a_3a_4}(\chi)$ corresponds to a canonical unit normalization. Separating out the singlet (S), symmetric traceless (T) and antisymmetric (A) channels as in (4.3), (4.4)

$$G_{(1)}^{a_1a_2a_3a_4}(\chi)$$
$$= \frac{1}{\sqrt{\lambda}}\Big[G_S^{(1)}(\chi)\,\delta^{a_1a_2}\delta^{a_3a_4} + G_T^{(1)}(\chi)\left(\delta^{a_1a_3}\delta^{a_2a_4} + \delta^{a_2a_3}\delta^{a_1a_4} - \tfrac{2}{5}\delta^{a_1a_2}\delta^{a_3a_4}\right) \quad (4.18)$$
$$+ G_A^{(1)}(\chi)\left(\delta^{a_1a_3}\delta^{a_2a_4} - \delta^{a_2a_3}\delta^{a_1a_4}\right)\Big],$$

we find

$$G_S^{(1)}(\chi) = -\frac{2\left(\chi^4 - 4\chi^3 + 9\chi^2 - 10\chi + 5\right)}{5(\chi-1)^2}$$
$$+ \frac{\chi^2\left(2\chi^4 - 11\chi^3 + 21\chi^2 - 20\chi + 10\right)}{5(\chi-1)^3}\log|\chi|$$
$$- \frac{2\chi^4 - 5\chi^3 - 5\chi + 10}{5\chi}\log|1-\chi|,$$

$$G_T^{(1)}(\chi) = -\frac{\chi^2\left(2\chi^2 - 3\chi + 3\right)}{2(\chi-1)^2} + \frac{\chi^4\left(\chi^2 - 3\chi + 3\right)}{(\chi-1)^3}\log|\chi| - \chi^3\log|1-\chi|, \quad (4.19)$$

$$G_A^{(1)}(\chi) = \frac{\chi\left(-2\chi^3 + 5\chi^2 - 3\chi + 2\right)}{2(\chi-1)^2} + \frac{\chi^3\left(\chi^3 - 4\chi^2 + 6\chi - 4\right)}{(\chi-1)^3}\log|\chi|$$
$$- (\chi^3 - \chi^2 - 1)\log|1-\chi|$$

Here and in what follows $\log|\chi| \equiv \frac{1}{2}\log(\chi^2)$ and $\log|1-\chi| \equiv \frac{1}{2}\log\left[(1-\chi)^2\right]$ where $\chi \in (-\infty,\infty)$. Alternatively, we may assume that $\chi \in (0,1)$ (which, in particular, is sufficient for considerations of the OPE below) and thus omit the absolute values, $\log|\chi| \to \log\chi$ and $\log|1-\chi| \to \log(1-\chi)$. This is sufficient for obtaining the expressions on the entire real line using analytic continuation in χ (and crossing symmetry).

We can expand the above functions in the OPE limit $\chi \to 0$ as

$$G_S^{(1)}(\chi) = \frac{1}{30}\chi^2\left(-60\log|\chi| - 43\right) + \frac{1}{30}\chi^3\left(-60\log|\chi| - 73\right)$$
$$+ \frac{1}{60}\chi^4\left(-252\log|\chi| - 117\right) + \dots, \quad (4.20)$$

$$G_T^{(1)}(\chi) = -\frac{3}{2}\chi^2 - \frac{3}{2}\chi^3 + \frac{1}{12}\chi^4\left(-36\log|\chi| - 18\right) + \dots, \quad (4.21)$$

$$G_A^{(1)}(\chi) = \frac{1}{6}\chi^3\left(24\log|\chi| + 7\right) + \frac{3}{4}\chi^4\left(8\log|\chi| + 5\right) + \dots. \quad (4.22)$$

Since the term of order $\chi^2\log|\chi|$ is absent from $G_T^{(1)}(\chi)$, this result implies that the symmetric traceless "two-particle" operators $\Phi^{(a}\Phi^{b)}$ do not have an anomalous dimension. This is as expected since these operators, such as Z^J with $Z = \Phi^1 + i\Phi^2$ (inserted into the Wilson line) are BPS and hence protected [5,36].

On the other hand, the singlet $\Phi^a\Phi^a$ acquires an anomalous dimension due to the presence of the $\chi^2\log|\chi|$ term in (4.20). The same is true for other two-particle operators encoded in the higher powers $\chi^{2+n}\log|\chi|$. We will extract their scaling dimensions systematically in the next subsection. Note that in the antisymmetric channel there is no operator of dimension $2 + O(\frac{1}{\sqrt{\lambda}})$.

While one can consider the insertion of the operator $[\Phi^a, \Phi^b]$ on the Wilson line, this operator sits in the same supersymmetry multiplet as Φ^6 [9] and thus is expected to acquire a large anomalous

dimension at strong coupling (presumably of order $\lambda^{1/4}$ [21] at large λ). Thus, it is expected to decouple from our perturbative calculations.

4.2. Scaling dimensions of two-particle operators from OPE

Each of the functions $G_S(\chi)$, $G_T(\chi)$ and $G_A(\chi)$ in (4.18) is expected to have an OPE expansion of the form (3.4). To the leading order, where the 4-point function is given by the generalized free field expressions, the exchanged operators entering this expansion are the "two-particle" operators of the form $\sim \Phi^a \partial_t^n \Phi^b$ (as always understood as attached to the Wilson line), as reviewed in Section 3.2. These can be decomposed in the irreducible representations of $SO(5)$

$$[\Phi\Phi]_{2n}^S \sim \Phi^a \partial_t^{2n} \Phi^a \,, \qquad [\Phi\Phi]_{2n}^T \sim \Phi^{(a} \partial_t^{2n} \Phi^{b)} \,, \qquad [\Phi\Phi]_{2n+1}^A \sim \Phi^{[a} \partial_t^{2n+1} \Phi^{b]} \,. \tag{4.23}$$

The connected 4-point functions computed in the previous subsection encode the $\frac{1}{\sqrt{\lambda}}$ corrections to the scaling dimension of these operators, as well as the correction to the corresponding OPE coefficient. However, a difficulty arises in directly extracting this CFT data from the 4-point functions because of operator mixing.[11] Due to degeneracies in the leading order two-particle spectrum, at the interacting level some of the operators in (4.23) can mix with two-particle operators with the appropriate quantum numbers built out of generalized gauge field strength $\mathbb{F}_{ti}$ or fermions (recall that the 8 fermions transform in the $(2, 4)$ representation of $SU(2) \times Sp(4) \simeq SO(3) \times SO(5)$). The singlet operators $[\Phi\Phi]_{2n}^S$ with $n > 0$ can mix with $\mathbb{FF}$ and two-fermion states, while the antisymmetric $[\Phi\Phi]_{2n+1}^A$ can mix with two-fermion states in the $(1, 10)$ of $SU(2) \times Sp(4)$.

Let us start our analysis with the symmetric traceless channel. In this case, we expect that the corresponding operators $[\Phi\Phi]_{2n}^T$ should not be affected by mixing because there are no other two-particle operators with the same quantum numbers. We can write

$$G_T(\chi) = \sum_h c_h \, \chi^h \, F_h(\chi) = G_T^{(0)}(\chi) + \frac{1}{\sqrt{\lambda}} G_T^{(1)}(\chi) + \dots \,, \tag{4.24}$$

$$F_h(\chi) \equiv {}_2F_1(h, h, 2h, \chi) \,, \tag{4.25}$$

where the sum is over the primaries $[\Phi\Phi]_{2n}^T$ appearing in the OPE. At large λ, we can write their dimension and the OPE coefficient as

$$h = 2 + 2n + \frac{1}{\sqrt{\lambda}} \gamma^{(1)}_{[\Phi\Phi]_{2n}^T} + \dots \,, \qquad c_h = c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n}^T} + \frac{1}{\sqrt{\lambda}} c^{(1)}_{\Phi\Phi[\Phi\Phi]_{2n}^T} + \dots \,. \tag{4.26}$$

Plugging the expansion (4.26) into (4.24), we get (see (4.6))

$$\sum_{n=0}^{\infty} c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n}^T} \chi^{2+2n} F_{2+2n}(\chi) = G_T^{(0)}(\chi) = \frac{1}{2} \Big[\chi^2 + \frac{\chi^2}{(1-\chi)^2} \Big] \,. \tag{4.27}$$

Comparing this with the generalized free field result in (3.10), the leading OPE coefficients are found to be

$$c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n}^T} = \frac{\big[\Gamma(2n+2)\big]^2 \, \Gamma(2n+3)}{\Gamma(2n+1) \, \Gamma(4n+3)} \,. \tag{4.28}$$

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[11] We thank Marco Meineri and Carlo Meneghelli for useful discussions on these issues.

From the terms of order $\frac{1}{\sqrt{\lambda}}$ in the expansions (4.26), (4.24) we find the anomalous dimensions and corrections to the OPE coefficients. Expanding

$$\chi^h = \chi^{2+2n+\frac{1}{\sqrt{\lambda}}\gamma^{(1)}+\cdots} = \chi^{2+2n}\left(1 + \frac{1}{\sqrt{\lambda}}\gamma^{(1)}\log|\chi| + \dots\right), \tag{4.29}$$

we see that the anomalous dimensions are determined by the $\log|\chi|$ terms in $G_T^{(1)}(\chi)$, via

$$\sum_{n=0}^{\infty} c_{\Phi\Phi[\Phi\Phi]_{2n}^T}^{(0)} \gamma_{[\Phi\Phi]_{2n}^T}^{(1)} \chi^{2+2n} F_{2+2n}(\chi) = \left[G_T^{(1)}(\chi)\right]_{\log|\chi|}, \tag{4.30}$$

where $\left[G_T^{(1)}(\chi)\right]_{\log|\chi|}$ is the function multiplying $\log|\chi|$ in (4.19). This equation can be solved for any n with the help of the orthogonality relation [54][12]

$$\oint \frac{dz}{2\pi i} \frac{1}{z^2} z^{\Delta+n} F_{\Delta+n}(z) z^{1-\Delta-n'} F_{1-\Delta-n'}(z) = \delta_{n,n'}, \tag{4.31}$$

where the integral is over a contour around the origin in the complex plane. This result is valid for any Δ, and can be verified, for instance, by using the series expansion of the hypergeometric function in (4.25). Using (4.31), we get from (4.30)

$$\gamma_{[\Phi\Phi]_{2n}^T}^{(1)} = \frac{1}{c_{\Phi\Phi[\Phi\Phi]_{2n}^T}^{(0)}} \oint \frac{d\chi}{2\pi i} \chi^{-3-2n} F_{-1-2n}(\chi) \left[G_T^{(1)}(\chi)\right]_{\log|\chi|}. \tag{4.32}$$

Evaluating the residue, we find that the result takes the remarkably simple form

$$\gamma_{[\Phi\Phi]_{2n}^T}^{(1)} = -2n^2 - 3n. \tag{4.33}$$

Thus the strong-coupling expansion of the scaling dimension of the operator $O(t) = [\Phi\Phi]_{2n}^T \sim \Phi^{(a}\partial_t^{2n}\Phi^{b)}$ inserted in the Wilson line as in (1.3) is given by

$$\Delta_{[\Phi\Phi]_{2n}^T} = 2 + 2n - \frac{2n^2 + 3n}{\sqrt{\lambda}} + O(\frac{1}{\lambda}). \tag{4.34}$$

The vanishing of the anomalous dimension for $n = 0$ reflects the fact that the operator $\Phi^{(a}\Phi^{b)}$ is protected. The operators with $n > 0$ are unprotected and belong to a long superconformal multiplet. Note that the anomalous dimension is negative for all $n > 0$, indicating an effective attractive interaction between single-particle states.

Plugging the expansion (4.26) into (4.24) results also in the following equation which determines the leading strong-coupling correction to the OPE coefficients

$$\sum_{n=0}^{\infty} \chi^{2+2n}\left[c_{\Phi\Phi[\Phi\Phi]_{2n}^T}^{(1)} F_{2+2n}(\chi) + \frac{1}{2} c_{\Phi\Phi[\Phi\Phi]_{2n}^T}^{(0)} \gamma_{[\Phi\Phi]_{2n}^T}^{(1)} \partial_n F_{2+2n}(\chi)\right] = \left[G_T^{(1)}(\chi)\right]_{\text{no}-\log|\chi|}, \tag{4.35}$$

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where $\left[G_T^{(1)}(\chi)\right]_{\text{no}-\log|\chi|}$ denotes the part of (4.19) which does not involve the $\log|\chi|$ term. It is straightforward to use this to extract $c^{(1)}_{\Phi\Phi[\Phi\Phi]_{2n}^T}$ for any given n. For example, the results for $n = 0, 1, 2$ including both order zero (4.28) and $\frac{1}{\sqrt{\lambda}}$ correction are

$$c_{\Phi\Phi[\Phi\Phi]_0^T} = 1 - \frac{3}{2\sqrt{\lambda}} + \dots , \quad c_{\Phi\Phi[\Phi\Phi]_2^T} = \frac{3}{5} - \frac{3}{20\sqrt{\lambda}} + \dots ,$$
$$c_{\Phi\Phi[\Phi\Phi]_4^T} = \frac{5}{42} + \frac{335}{378\sqrt{\lambda}} + \dots \tag{4.36}$$

For general n, we observe that the $O(\frac{1}{\sqrt{\lambda}})$ correction to the OPE coefficients in (4.26) is given by the simple formula

$$c^{(1)}_{\Phi\Phi[\Phi\Phi]_{2n}^T} = \frac{1}{2}\frac{\partial}{\partial n}\left(c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n}^T}\gamma^{(1)}_{[\Phi\Phi]_{2n}^T}\right). \tag{4.37}$$

A relation of this type was found empirically in [54] and proved in [55] (see also [63]). Explicitly, we get

$$c^{(1)}_{\Phi\Phi[\Phi\Phi]_{2n}^T} = \frac{[\Gamma(2n+2)]^2}{\Gamma(4n+3)}\Big[-(3+34n+56n^2+24n^3)$$
$$+4n(n+1)(2n+1)(2n+3)(H_{4n+3}-H_{2n})\Big], \tag{4.38}$$

where $H_n = \sum_{k=1}^n \frac{1}{k}$ is the harmonic number.

In the singlet and antisymmetric channels, the leading order OPE coefficients are determined by

$$1 + \sum_{n=0}^{\infty} c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n}^S}\chi^{2+2n}F_{2+2n}(\chi) = G_S^{(0)}(\chi) = 1 + \frac{1}{5}\left[\chi^2 + \frac{\chi^2}{(1-\chi)^2}\right],$$
$$\sum_{n=0}^{\infty} c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n+1}^A}\chi^{3+2n}F_{3+2n}(\chi) = G_A^{(0)}(\chi) = \frac{1}{2}\left[\chi^2 - \frac{\chi^2}{(1-\chi)^2}\right], \tag{4.39}$$

and using the results in Section 3.2 are found to be

$$c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n}^S} = \frac{2[\Gamma(2n+2)]^2\,\Gamma(2n+3)}{5\,\Gamma(2n+1)\,\Gamma(4n+3)},$$
$$c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n+1}^A} = -\frac{[\Gamma(2n+3)]^2\,\Gamma(2n+4)}{\Gamma(2n+2)\,\Gamma(4n+2+3)}. \tag{4.40}$$

In view of the orthogonality relation (4.31), we can extract the anomalous dimensions as

$$\gamma^{(1)}_{[\Phi\Phi]_{2n}^S} = \frac{1}{c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n}^S}}\oint\frac{d\chi}{2\pi i}\chi^{-3-2n}F_{-1-2n}(\chi)\left[G_S^{(1)}(\chi)\right]_{\log|\chi|},$$
$$\gamma^{(1)}_{[\Phi\Phi]_{2n+1}^A} = \frac{1}{c^{(0)}_{\Phi\Phi[\Phi\Phi]_{2n+1}^A}}\oint\frac{d\chi}{2\pi i}\chi^{-4-2n}F_{-2-2n}(\chi)\left[G_A^{(1)}(\chi)\right]_{\log|\chi|}. \tag{4.41}$$

Evaluating the residues we find, as in (4.33), simple quadratic polynomials in n

$$\gamma^{(1)}_{[\Phi\Phi]^S_{2n}} = -2n^2 - 3n - 5\,, \qquad\qquad \gamma^{(1)}_{[\Phi\Phi]^A_{2n+1}} = -2n^2 - 5n - 4\,. \tag{4.42}$$

However, due to the mixing issues described above, these expressions should be viewed as "averages" of the anomalous dimensions over the operators appearing in the mixing (weighted by the corresponding OPE coefficients).[13] The exception is the singlet operator $\Phi^a\Phi^a$ with $n = 0$, which cannot mix with any other operator. In this case, (4.42) yields

$$\Delta_{\Phi^a\Phi^a} = 2 - \frac{5}{\sqrt{\lambda}} + O(\frac{1}{\lambda})\,. \tag{4.43}$$

Following similar approach as used above in the $[\Phi\Phi]^T_{2n}$ case, we can also extract the corresponding OPE coefficient

$$c_{\Phi\Phi[\Phi\Phi]^S_0} = \frac{2}{5} - \frac{43}{30\sqrt{\lambda}} + \dots\,. \tag{4.44}$$

It is interesting to notice that in all of the above expressions (4.34), (4.42) the large n limit of the scaling dimensions has the same asymptotic form

$$\Delta_{n\gg1} = 2n - \frac{2n^2}{\sqrt{\lambda}} + \dots\,. \tag{4.45}$$

We will find below that (4.45) is true also for the scaling dimensions extracted from the mixed x^2y^2-correlators and x^4-correlators. Note that this implies that the perturbative result should not be trusted when n becomes of order $\sqrt{\lambda}$, because then the leading term is comparable to the first perturbative correction (also, in such regime, contributions of massive string states should be already important, presumably corresponding to non-perturbative corrections to the 4-point function). Nevertheless, the form (4.45) is suggestive of a semiclassical limit with $n, \sqrt{\lambda} \gg 1$ and $\nu \equiv \frac{n}{\sqrt{\lambda}}$ fixed. Our results then suggest that in this limit the dimensions of such "two-particle" operators have a universal strong-coupling form

$$\Delta_n = \sqrt{\lambda}\, f(\nu)\,, \qquad\qquad f(\nu) = 2\nu - 2\nu^2 + O(\nu^3)\,. \tag{4.46}$$

This behavior may be captured by a semiclassical string calculation, analogous to the one in [5,65,66] where the insertions carried large R-charge, while here we just need large $SO(2, 1)$ quantum number and no R-charge.

5. Four-point functions with AdS$_5$ fluctuations

Starting with the AdS$_2$ Lagrangian (2.5)–(2.9) we may also compute other four-point correlators involving AdS$_5$ coordinates x^i which are dual to the dimension $\Delta = 2$ operator $\mathbb{F}_{it}$ inserted on the Wilson line. Explicitly, below we will compute (cf. (2.10))

$$\langle x^{i_1}(t_1)x^{i_2}(t_2)y^{a_1}(t_3)y^{a_2}(t_4)\rangle_{\text{AdS}_2}$$
$$= \langle\!\langle \mathbb{F}^{i_1}_t(t_1)\,\mathbb{F}^{i_2}_t(t_2)\,\Phi^{a_1}(t_3)\,\Phi^{a_2}(t_4)\,\rangle\!\rangle = \delta^{i_1i_2}\delta^{a_1a_2}\frac{G(\chi)}{t^4_{12}t^2_{34}}\,, \tag{5.1}$$

$$\langle x^{i_1}(t_1) x^{i_2}(t_2) x^{i_3}(t_3) x^{i_4}(t_4) \rangle_{\mathrm{AdS}_2}$$
$$= \langle\langle \mathbb{F}_t^{i_1}(t_1) \mathbb{F}_t^{i_2}(t_2) \mathbb{F}_t^{i_3}(t_3) \mathbb{F}_t^{i_4}(t_4) \rangle\rangle = \frac{G^{i_1 i_2 i_3 i_4}(\chi)}{t_{12}^4 t_{34}^4}. \tag{5.2}$$

Since the Wilson line is 1/2-BPS these two correlators should be related to the correlation function of four S^5 fluctuations by supersymmetry transformations.

5.1. Two AdS_5 and two S^5 fluctuations

The leading-order contribution to the connected part of the correlator (5.1) may be written as (cf. (4.7), (4.8))

$$\frac{G_{\mathrm{conn}}(\chi)}{t_{12}^4 t_{34}^2} = \frac{2\pi}{\sqrt{\lambda}}(\mathcal{C}_{\Delta=1}\mathcal{C}_{\Delta=2})^2 Q_{xy}, \qquad G_{\mathrm{conn}}(\chi) \equiv \mathcal{C}_{\Delta=1}\mathcal{C}_{\Delta=2}\, G_{(1)}(\chi), \tag{5.3}$$

where

$$\begin{aligned}
Q_{xy} = &-\int \frac{dt\,ds}{s^2}\Big[g^{\mu\nu}\partial_\mu \tilde{K}_2(t_1)\partial_\nu \tilde{K}_2(t_2) g^{\rho\sigma}\partial_\rho \tilde{K}_1(t_3)\partial_\sigma \tilde{K}_1(t_4) \\
&- g^{\mu\nu}\partial_\mu \tilde{K}_2(t_1)\partial_\nu \tilde{K}_1(t_3) g^{\rho\sigma}\partial_\rho \tilde{K}_2(t_2)\partial_\sigma \tilde{K}_1(t_4) \\
&- g^{\mu\nu}\partial_\mu \tilde{K}_2(t_1)\partial_\nu \tilde{K}_1(t_4) g^{\rho\sigma}\partial_\rho \tilde{K}_2(t_2)\partial_\sigma \tilde{K}_1(t_3) \Big] \\
= &\,4\Big(D_{2211} + 2t_{12}^2 D_{3311} - 2t_{13}^2 D_{3221} - 2t_{23}^2 D_{2321} - 2t_{14}^2 D_{3212} - 2t_{24}^2 D_{2312} \\
&+ 2t_{34}^2 D_{2222} + 4t_{14}^2 t_{23}^2 D_{3322} + 4t_{13}^2 t_{24}^2 D_{3322} - 4t_{12}^2 t_{34}^2 D_{3322} \Big).
\end{aligned} \tag{5.4}$$

As a result, the function $G_{(1)}(\chi)$ in (5.3)

$$G_{(1)}(\chi) = -\frac{4}{\sqrt{\lambda}}\left[1 - (\tfrac{1}{2} - \chi^{-1}) \ln|1-\chi| \right]. \tag{5.5}$$

Similarly to the discussion in Section 4.2, we may also extract the scaling dimensions of two-particle operators appearing in the OPE. In this case the relevant operators are

$$[\Phi^a \mathbb{F}_{it}]_n \sim \Phi^a \partial_t^n \mathbb{F}_{it} \tag{5.6}$$

that have dimension $3 + n + O(\frac{1}{\sqrt{\lambda}})$ and correspond to mixed xy two-particle states. Let us first rewrite the 4-point function (5.1), (5.3) by relabeling $t_2 \leftrightarrow t_3$

$$\langle\langle \mathbb{F}_{it}(t_1)\Phi^a(t_2)\mathbb{F}_{jt}(t_3)\Phi^b(t_4) \rangle\rangle_{\mathrm{conn}} = \delta^{ab}\delta_{ij} \frac{\mathcal{C}_{\Delta=1}\mathcal{C}_{\Delta=2}}{(t_{12}^2 t_{34}^2)^{3/2}}\left(\frac{t_{24}^2}{t_{13}^2}\right)^{1/2} G_{xy}^{(1)}(\chi), \tag{5.7}$$

where from (5.5) we get

$$G_{xy}^{(1)}(\chi) = \chi^3 G_{(1)}(\chi^{-1}) = -\frac{4\chi^3}{\sqrt{\lambda}}\left[1 + (\tfrac{1}{2} - \chi)\log\frac{|\chi|}{|1-\chi|} \right]. \tag{5.8}$$

The corresponding disconnected contribution appearing at leading order is (see (3.13))

$$G^{(0)}(\chi) = \frac{1}{(t_{12}^2 t_{34}^2)^{3/2}}\left(\frac{t_{24}^2}{t_{13}^2}\right)^{1/2} \chi^3. \tag{5.9}$$

Using (3.14), this determines the leading order OPE coefficients appearing in the expansion (3.6)

$$c_{\Phi\mathbb{F}[\Phi\mathbb{F}]_n}^{(0)} = \frac{\Gamma(n+2)\,\Gamma(n+4)\,\Gamma(n+5)}{6\,\Gamma(n+1)\,\Gamma(2n+5)}\,. \tag{5.10}$$

To extract the anomalous dimensions, we can use the following generalization of the orthogonality relation (4.31)

$$\oint \frac{dz}{2\pi i}\frac{1}{z^2} z^{\Delta+n} F_{\Delta+n,a}(z) z^{1-\Delta-n'} F_{1-\Delta-n',a}(z) = \delta_{n,n'}\,, \tag{5.11}$$

$$F_{h,a}(z) \equiv {}_2F_1(h+a, h-a, 2h, z)\,.$$

In our case, we need $\Delta = 3$ and $a = 1$, see (3.6). Then the anomalous dimensions are given by

$$\gamma_{[\Phi\mathbb{F}]_n}^{(1)} = \frac{1}{c_{\Phi\mathbb{F}[\Phi\mathbb{F}]_n}^{(0)}} \oint \frac{d\chi}{2\pi i}\,\chi^{-n-4} F_{-n-2,1}(\chi)\,2\chi^3(2\chi - 1)\,, \tag{5.12}$$

where we have used that $\left[G_{xy}^{(1)}(\chi)\right]_{\log|\chi|} = 2\chi^3(2\chi - 1)$. Evaluating the residue, we find

$$\gamma_{[\Phi\mathbb{F}]_n}^{(1)} = -\frac{n^2}{2} - \frac{5n}{2} - 2\,. \tag{5.13}$$

Let us separate the cases of even and odd n. For even n we expect that the operators $[\Phi\mathbb{F}]_{2n}$ can mix with two-fermion states in the same representation.[14] For odd n, on the other hand, we do not expect mixing with two fermion states, and from (5.13) we get

$$\Delta_{[\Phi\mathbb{F}]_{2n+1}} = 4 + 2n - \frac{2n^2 + 7n + 5}{\sqrt{\lambda}} + O(\frac{1}{\lambda})\,. \tag{5.14}$$

For large n, we recover the universal form (4.45), (4.46) found from the analysis of the y-correlators. Note that the dimension $\Delta_{[\Phi\mathbb{F}]_{2n+1}}$ in (5.14) is the same as $\Delta_{[\Phi\Phi]_{2n'}^T}$ in (4.34) for $n' = n + 1$. This is consistent with the fact that these operators should belong to the same long supermultiplet.

5.2. Four AdS₅ fluctuations

Finally, let us compute the four-point function of the three AdS fluctuations x^i (5.2) using similar normalization for the connected part as in (5.3)

$$G_{\text{conn}}^{i_1i_2i_3i_4}(\chi) = (\mathcal{C}_{\Delta=2})^2\,G_{(1)}^{i_1i_2i_3i_4}(\chi)\,, \tag{5.15}$$

with (cf. (4.18))

$$G_{(1)}^{i_1i_2i_3i_4}(\chi) = \delta^{i_1i_2}\delta^{i_3i_4}G_S^{(1)}(\chi) + G_A^{(1)}(\delta^{i_1i_3}\delta^{i_2i_4} - \delta^{i_1i_4}\delta^{i_2i_3})$$
$$+ G_T^{(1)}(\delta^{i_1i_3}\delta^{i_2i_4} + \delta^{i_1i_4}\delta^{i_2i_3} - \tfrac{2}{3}\delta^{i_1i_2}\delta^{i_3i_4})\,. \tag{5.16}$$

The irreducible $SO(3)$ singlet, symmetric traceless and antisymmetric parts are found to be

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[14] The product of two $(2,4)$ representations of $SU(2) \times Sp(4)$ contains the $(3,5)$ of $SO(3) \times SO(5)$. The vector of $SO(5)$ corresponds to the antisymmetric symplectic-traceless representation of $Sp(4)$.

$$G_S^{(1)}(\chi) = -\frac{\left(24\chi^8 - 90\chi^7 + 125\chi^6 - 76\chi^5 + 125\chi^4 - 306\chi^3 + 438\chi^2 - 288\chi + 72\right)}{9(\chi - 1)^4}$$

$$-\frac{2\left(4\chi^6 - \chi^5 - 6\chi + 12\right)}{3\chi}\log|1 - \chi|$$

$$+\frac{2\chi^4\left(4\chi^6 - 21\chi^5 + 45\chi^4 - 50\chi^3 + 30\chi^2 - 6\chi + 2\right)}{3(\chi - 1)^5}\log|\chi|, \qquad (5.17)$$

$$G_T^{(1)}(\chi) = -\frac{\left(48\chi^4 - 198\chi^3 + 313\chi^2 - 230\chi + 115\right)\chi^4}{12(\chi - 1)^4} - \frac{1}{2}(8\chi - 5)\chi^4\log|1 - \chi|$$

$$+\frac{4\left(8\chi^6 - 45\chi^5 + 105\chi^4 - 130\chi^3 + 90\chi^2 - 30\chi + 10\right)\chi^4}{9(\chi - 1)^5}\log|\chi|, \quad (5.18)$$

$$G_A^{(1)}(\chi) = -\frac{(\chi - 2)\left(48\chi^6 - 90\chi^5 + 91\chi^4 + 4\chi^3 - 17\chi^2 + 18\chi - 6\right)\chi}{12(\chi - 1)^4}$$

$$-\frac{1}{4}\left(4\chi^5 - 3\chi^4 + 2\right)\log|1 - \chi|$$

$$+\frac{(\chi - 2)\left(8\chi^4 - 27\chi^3 + 41\chi^2 - 28\chi + 14\right)\chi^5}{2(\chi - 1)^5}\log|\chi|. \qquad (5.19)$$

The two-particle states encoded in the OPE of the 4-point function of x fluctuations are

$$[\mathbb{FF}]_{2n}^S \sim \mathbb{F}_{ti}\partial_t^{2n}\mathbb{F}_{it}, \qquad [\mathbb{FF}]_{2n}^T \sim \mathbb{F}_{t(i}\partial_t^{2n}\mathbb{F}_{j)t}, \qquad [\mathbb{FF}]_{2n+1}^A \sim \mathbb{F}_{t[i}\partial_t^{2n+1}\mathbb{F}_{j]t}. \quad (5.20)$$

The calculation of their anomalous dimensions follows the same steps as outlined in the previous sections. The disconnected contributions to the 4-point function are (cf. (4.6))

$$G_S^{(0)}(\chi) = 1 + \frac{2}{3}G_T^{(0)}(\chi), \qquad G_T^{(0)}(\chi) = \frac{1}{2}\left[\chi^4 + \frac{\chi^4}{(1 - \chi)^4}\right],$$

$$G_A^{(0)}(\chi) = \frac{1}{2}\left[\chi^4 - \frac{\chi^4}{(1 - \chi)^4}\right], \qquad (5.21)$$

from which, using (3.10), we find the leading OPE coefficients

$$c_{\mathbb{FF}[\mathbb{FF}]_{2n}^S}^{(0)} = \frac{\left[\Gamma(2n + 4)\right]^2 \Gamma(2n + 7)}{54\,\Gamma(2n + 1)\,\Gamma(4n + 7)},$$

$$c_{\mathbb{FF}[\mathbb{FF}]_{2n}^T}^{(0)} = \frac{\left[\Gamma(2n + 4)\right]^2 \Gamma(2n + 7)}{36\,\Gamma(2n + 1)\,\Gamma(4n + 7)}, \qquad c_{\mathbb{FF}[\mathbb{FF}]_{2n+1}^A}^{(0)} = -\frac{\left[\Gamma(2n + 5)\right]^2 \Gamma(2n + 8)}{36\,\Gamma(2n + 2)\,\Gamma(4n + 9)}. \qquad (5.22)$$

Then starting with the OPE (3.4), expanding in powers of $\frac{1}{\sqrt{\lambda}}$ and using the orthogonality relation (4.31), we find

$$\gamma^{(1)}_{[\mathbb{F}\mathbb{F}]^S_{2n}} = \frac{1}{c^{(0)}_{\mathbb{F}\mathbb{F}[\mathbb{F}\mathbb{F}]^S_{2n}}} \oint \frac{d\chi}{2\pi i} \chi^{-5-2n} F_{-3-2n}(\chi) \left[G^{(1)}_S(\chi)|_{\log|\chi|} = -2n^2 - 7n - 2 \right.$$

$$\gamma^{(1)}_{[\mathbb{F}\mathbb{F}]^T_{2n}} = \frac{1}{c^{(0)}_{\mathbb{F}\mathbb{F}[\mathbb{F}\mathbb{F}]^T_{2n}}} \oint \frac{d\chi}{2\pi i} \chi^{-5-2n} F_{-3-2n}(\chi) \left[G^{(1)}_T(\chi) \right]_{\log|\chi|} = -2n^2 - 7n - 5$$

$$\gamma^{(1)}_{[\mathbb{F}\mathbb{F}]^A_{2n+1}} = \frac{1}{c^{(0)}_{\mathbb{F}\mathbb{F}[\mathbb{F}\mathbb{F}]^A_{2n}}} \oint \frac{d\chi}{2\pi i} \chi^{-6-2n} F_{-4-2n}(\chi) \left[G^{(1)}_A(\chi) \right]_{\log|\chi|} = -2n^2 - 9n - 7 .$$

$$(5.23)$$

As explained in Section 4.2, the singlet operators $[\mathbb{F}\mathbb{F}]^S_{2n}$ can mix with $\Phi\Phi$ and two-fermion operators, and also $[\mathbb{F}\mathbb{F}]^A_{2n+1}$ can mix with two-fermion states in the same representation. Therefore, the corresponding anomalous dimensions above should be viewed as averages and more work would be needed to disentangle the mixing. The symmetric traceless operators $[\mathbb{F}\mathbb{F}]^T_{2n}$ are not expected to mix, and from (5.23) we hence get their dimensions to be

$$\Delta_{[\mathbb{F}\mathbb{F}]^T_{2n}} = 4 + 2n - \frac{2n^2 + 7n + 5}{\sqrt{\lambda}} + O(\frac{1}{\lambda}) . \tag{5.24}$$

Note that $\Delta_{[\mathbb{F}\mathbb{F}]^T_{2n}}$ is the same as $\Delta_{[\Phi\mathbb{F}]_{2n+1}}$ in (5.14) and also $\Delta_{[\Phi\Phi]^T_{2n+2}}$ in (4.34), indicating that these operators belong to the same supermultiplet.

6. Circular Wilson loop: comparison to localization

In the above calculations we assumed the straight Wilson line at the boundary. However, one can map the straight line to the circle by a conformal transformation, which allows then to translate correlators of operator insertions on the line to those on the circle. Explicitly, we can perform the transformation $t \to \tan(\tau/2)$, where $-\pi < \tau < \pi$ is the coordinate along the circle. Under this transformation, the two-point function of an operator O_Δ inserted in the Wilson loop changes as

$$\langle\langle O_\Delta(t_1) O_\Delta(t_2) \rangle\rangle_{\text{line}} = \frac{C_O}{t_{12}^{2\Delta}} \quad \to \quad \langle\langle O_\Delta(\tau_1) O_\Delta(\tau_2) \rangle\rangle_{\text{circle}} = \frac{C_O}{\left(2 \sin \frac{\tau_{12}}{2}\right)^{2\Delta}} . \tag{6.1}$$

Note that the expectation value of the circular half-BPS Wilson loop is not trivial and given at large N by the well-known expression [6–8]

$$\langle W_{\text{circle}} \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) , \tag{6.2}$$

and hence the double-bracket correlator in (6.1) requires a normalization factor given by this expectation value.

On the string theory side, the transformation from boundary line to circle simply amounts to changing coordinates on the Euclidean AdS$_2$ worldsheet from the Poincare metric we have been assuming above to the hyperbolic disk metric

$$ds_2^2 = d\rho^2 + \sinh^2 \rho \, d\tau^2 . \tag{6.3}$$

All of our results for the four-point functions of insertions on the line can be then translated to the circle by simply replacing the coordinate-dependent prefactors as

$$\frac{1}{t_{12}^{2\Delta_1} t_{34}^{2\Delta_2}} \quad \rightarrow \quad \frac{1}{(2 \sin \frac{\tau_1 - \tau_2}{2})^{2\Delta_1} (2 \sin \frac{\tau_3 - \tau_4}{2})^{2\Delta_2}} \,, \tag{6.4}$$

and the conformally invariant cross ratio χ in (3.2) is mapped to

$$\chi = \frac{\sin \frac{\tau_1 - \tau_2}{2} \, \sin \frac{\tau_3 - \tau_4}{2}}{\sin \frac{\tau_1 - \tau_3}{2} \, \sin \frac{\tau_2 - \tau_4}{2}} \,. \tag{6.5}$$

In the case of the four-point function of S^5 fluctuations, it appears to be possible to compare our results to some localization prediction. In a series of papers [4,37–39,67] it was proposed that correlation functions in a subsector of supersymmetric Wilson loops and local operators in $\mathcal{N} = 4$ SYM can be computed via localization in terms of 2d YM theory. The relevant Wilson loops, first introduced in [68], are defined on generic contours on an S^2 subspace of R^4 (or S^4), and couple to three of the scalar fields in the SYM theory, say Φ_1, Φ_2, Φ_3, in a way prescribed by supersymmetry:

$$W(C) = \text{tr} P e^{\oint_C \left(i A_j + \epsilon_{klj} x^k \Phi^l \right) dx^j} \,, \tag{6.6}$$

where x_i parametrize a unit two-sphere $x_1^2 + x_2^2 + x_3^2 = 1$. With such couplings to scalars Φ^i, the Wilson loops (6.6) are 1/8-BPS for generic contour C. These operators are mapped under localization[15] to usual Wilson loops in 2d YM on S^2. The 1/2-BPS circular Wilson loop is a special case obtained by choosing the contour to be a great circle on S^2. For instance, taking the equator $x_1 = \cos(\tau)$, $x_2 = \sin(\tau)$ gives the 1/2-BPS operator which couples to Φ^3 only. In this section, we will use this convention for the scalar that couples to the 1/2-BPS Wilson loop, to adhere with the definition (6.6) used in the original papers.

The relevant local operators appearing in the localization setup are chiral primaries with specific position-dependent combination of scalars, which were first studied in [69]. Recall that a convenient way to write a chiral primary is in terms of an auxiliary null 6-vector ϵ

$$(\epsilon \cdot \Phi)^J \,, \qquad \epsilon^2 = 0 \,. \tag{6.7}$$

The local operators that are captured by localization are inserted on the S^2 and have the form

$$(x_1 \Phi_1 + x_2 \Phi_2 + x_3 \Phi_3 + i \Phi_4)^J \,, \qquad x_1^2 + x_2^2 + x_3^2 = 1 \,, \tag{6.8}$$

where x_1, x_2, x_3 is the point on S^2 where the operator is inserted. This means that the null 6-vector is position-dependent and given by $\epsilon(x) = (x_1, x_2, x_3, i, 0, 0)$. These operators are mapped by localization [39,38] to powers of the Hodge dual $(i * F)^J$ of the 2d YM field strength, and one can then compute general mixed correlation functions of Wilson loops and local operators using the 2d YM theory.

A crucial property of the operators (6.8) is that their correlation functions are position independent, at any coupling [69]. From the point of view of localization to 2d YM theory, this can be understood as the fact that correlation functions of the field strength dual $*F$ are position independent.[16] In addition to considering correlation functions of Wilson loops with local operators inserted away from the loop, as in [25], one can also insert the local operators (6.8) along the Wilson loop, which is our main interest here. A calculation of this type was carried out in [70],

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[15] To be precise, this has not yet been proven completely rigorously, as the calculation of the determinant for the fluctuations around the localization locus was not computed in [39].

[16] This is because the 2d YM equation of motion is $d * F = 0$, and $*F$ is a scalar.

where the calculation in the 2d theory (to the leading order considered there) was found to be in agreement with the integrability-based results of [71].

To make contact with the calculation in Section 4, we should consider the 4-point function of the operators (6.8) inserted along the circular loop

$$\langle\!\langle \epsilon(\tau_1)\cdot\Phi(\tau_1)\,\epsilon(\tau_2)\cdot\Phi(\tau_2)\,\epsilon(\tau_3)\cdot\Phi(\tau_3)\,\epsilon(\tau_4)\cdot\Phi(\tau_4)\rangle\!\rangle_{\text{circle}}\,, \tag{6.9}$$

where, since the operators are inserted on the great circle in the (12)-plane, the null 6-vectors are given by[17]

$$\epsilon(\tau_k) = \big(\cos\tau_k,\ \sin\tau_k,\ 0,\ i,\ 0,\ 0\big) \equiv \epsilon_k\,, \qquad k = 1,\dots,4\,. \tag{6.10}$$

Let us first check that the two-point function of such operators along the circle is indeed position independent. We have

$$\langle\!\langle \epsilon(\tau_1)\cdot\Phi(\tau_1)\,\epsilon(\tau_2)\cdot\Phi(\tau_2)\rangle\!\rangle_{\text{circle}} = C_\Phi(\lambda)\frac{\epsilon(\tau_1)\cdot\epsilon(\tau_2)}{\big(2\sin\frac{\tau_{12}}{2}\big)^2} = -\frac{1}{2}C_\Phi(\lambda)\,. \tag{6.11}$$

As expected, the factor in the numerator coming from the τ-dependent null vector cancels the position dependence of the denominator.

To find (6.9), we just have to contract the $SO(5)$ index structures in the result in Section 4 with the vectors ϵ_k. Note that

$$\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4 = \big(2\sin\tfrac{\tau_1-\tau_2}{2}\,2\sin\tfrac{\tau_3-\tau_4}{2}\big)^2\,, \qquad \frac{\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4}{\epsilon_1\cdot\epsilon_3\,\epsilon_2\cdot\epsilon_4} = \chi^2\,,$$
$$\frac{\epsilon_1\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_4}{\epsilon_1\cdot\epsilon_4\,\epsilon_2\cdot\epsilon_3} = \frac{\chi^2}{(1-\chi)^2}\,. \tag{6.12}$$

Using these relations and the decomposition in (4.18), we find for the unit-normalized connected part of the 4-point function

$$\frac{\langle\!\langle\epsilon_1\cdot\Phi(\tau_1)\,\epsilon_2\cdot\Phi(\tau_2)\,\epsilon_3\cdot\Phi(\tau_3)\,\epsilon_4\cdot\Phi(\tau_4)\rangle\!\rangle_{\text{circle}}^{\text{conn}}}{\langle\!\langle\epsilon\cdot\Phi\epsilon\cdot\Phi\rangle\!\rangle_{\text{circle}}^2}$$
$$= \frac{1}{\sqrt{\lambda}}\Big[G_S^{(1)}(\chi) - \frac{2}{5}G_T^{(1)}(\chi) + \frac{1}{\chi^2}\big(G_T^{(1)}(\chi)+G_A^{(1)}(\chi)\big)$$
$$+ \frac{(1-\chi)^2}{\chi^2}\big(G_T^{(1)}(\chi)-G_A^{(1)}(\chi)\big)\Big]\,. \tag{6.13}$$

Plugging in the explicit functions of cross-ratio from (4.19), one can verify that the position dependence completely cancels out and we end up with

$$\frac{\langle\!\langle\epsilon_1\cdot\Phi(\tau_1)\,\epsilon_2\cdot\Phi(\tau_2)\,\epsilon_3\cdot\Phi(\tau_3)\,\epsilon_4\cdot\Phi(\tau_4)\rangle\!\rangle_{\text{circle}}^{\text{conn}}}{\langle\!\langle\epsilon\cdot\Phi\,\epsilon\cdot\Phi\rangle\!\rangle_{\text{circle}}^2} = -\frac{3}{\sqrt{\lambda}} + O(\frac{1}{\lambda})\,. \tag{6.14}$$

Let us now compare this result with the prediction of localization. One should compute the 4-point function $\langle\!\langle \tilde{F}(\tau_1)\tilde{F}(\tau_2)\tilde{F}(\tau_3)\tilde{F}(\tau_4)\rangle\!\rangle_{\text{YM}_2}$ in 2d YM, where we introduced for convenience the shorthand $\tilde{F}\equiv i*F$ for the dual of the field strength, which is inserted four times along the circular Wilson loop. A shortcut to this calculation may be obtained by starting from a more

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[17] Recall that in this section we are assuming that Φ^3 is the scalar that couples to the Wilson loop, so the S^5 fluctuations y^a are dual to $\Phi_1, \Phi_2, \Phi_4, \Phi_5, \Phi_6$.

general contour C and noticing that insertions of $i * F$ are equivalent to taking derivatives of the Wilson loop expectation value with respect to the area.[18] For a general contour C singling out areas A_1, A_2 on S^2, with $A_1 + A_2 = 4\pi$ (we take unit radius), the invariance under area preserving diffeomorphisms of 2d YM implies that the expectation value is given by the same expression (6.2) up to an area-dependent rescaling of the coupling

$$\langle W_{A_1} \rangle = \frac{2}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'}), \qquad \lambda' \equiv \frac{A_1 A_2}{4\pi^2} \lambda = \frac{A_1(4\pi - A_1)}{4\pi^2} \lambda. \tag{6.15}$$

This is the expectation value of the general 1/8-BPS operator (6.6). The 1/2-BPS circle corresponds to the special case $A_1 = 2\pi$ when $\lambda' = \lambda$. Then, taking derivatives of $\log\langle W_{A_1} \rangle$ with respect to A_1 and setting $A_1 = 2\pi$ after that yields the connected correlators of $i * F$ inserted along the circle.

For instance, the two-point function is given by

$$\langle\langle \tilde{F}(\tau_1) \tilde{F}(\tau_2) \rangle\rangle_{\mathrm{YM}_2} = \frac{\partial^2}{\partial A_1^2} \log\langle W_{A_1} \rangle \Big|_{A_1 = 2\pi} = -\frac{\sqrt{\lambda} I_2(\sqrt{\lambda})}{4\pi^2 I_1(\sqrt{\lambda})}. \tag{6.16}$$

This implies that in (6.11) $C_\Phi(\lambda) = \frac{\sqrt{\lambda} I_2(\sqrt{\lambda})}{2\pi^2 I_1(\sqrt{\lambda})}$, in agreement with the Bremsstrahlung function of [14].

For the connected 4-point function, we get

$$\frac{\langle\langle \tilde{F}(\tau_1) \tilde{F}(\tau_2) \tilde{F}(\tau_3) \tilde{F}(\tau_4) \rangle\rangle_{\mathrm{YM}_2}^{\mathrm{conn}}}{\langle\langle \tilde{F} \tilde{F} \rangle\rangle_{\mathrm{YM}_2}^2} = \frac{\frac{\partial^4}{\partial A_1^4} \log\langle W_{A_1} \rangle \big|_{A_1 = 2\pi}}{\left(\frac{\partial^2}{\partial A_1^2} \log\langle W_{A_1} \rangle \big|_{A_1 = 2\pi} \right)^2}$$

$$= \frac{3(\lambda + 4)\left[I_1(\sqrt{\lambda}) \right]^2 - 3\lambda \left[I_0(\sqrt{\lambda}) \right]^2}{\lambda \left[I_2(\sqrt{\lambda}) \right]^2}. \tag{6.17}$$

Expanding at large λ, this gives

$$\frac{3(\lambda + 4)\left[I_1(\sqrt{\lambda}) \right]^2 - 3\lambda \left[I_0(\sqrt{\lambda}) \right]^2}{\lambda \left[I_2(\sqrt{\lambda}) \right]^2} = -\frac{3}{\sqrt{\lambda}} + \frac{45}{8\lambda^{3/2}} + \dots, \tag{6.18}$$

and we see that the leading term agrees with our result (6.14) coming from tree-level connected diagrams in AdS$_2$.

Acknowledgements

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[18] In general, under a small deformation of the contour, the Wilson loop expectation value varies as $\langle W(C + \delta C) \rangle = \langle \mathrm{tr} P(1 + \int d\tau \delta x^\mu \dot{x}^\nu i F_{\mu\nu} + \dots) e^{\int iA} \rangle$. In 2d, we can write $\int d\tau \delta x^\mu \dot{x}^\nu i F_{\mu\nu} = \int d\tau \delta x^\mu \dot{x}^\nu \sqrt{g} \epsilon_{\mu\nu} i * F$. The factor $\int d\tau \delta x^\mu \dot{x}^\nu \sqrt{g} \epsilon_{\mu\nu}$ measures the change in area of the Wilson loop.

Appendix A. Toy model: scalar in AdS$_2$ with φ^4 interaction

As a simple toy model, let us consider a scalar in AdS$_2$ with a simple quartic self-interaction

$$S = \int d^2x \sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{g}{4!}\varphi^4\right), \tag{A.1}$$

where we assume the Poincare metric $ds^2 = \frac{1}{z^2}(dz^2 + dt^2)$. Tree level Witten diagrams obtained from this model yield conformally invariant correlation functions of an operator $O(t)$ at the boundary with scaling dimension given by $\Delta(\Delta - 1) = m^2$. The tree-level 4-point function is straightforward to compute

$$\langle O(t_1)O(t_2)O(t_2)O(t_3)\rangle = -g\mathcal{C}_\Delta^4 D_{\Delta\Delta\Delta\Delta}(t_1, t_2, t_3, t_4)$$
$$= -g\frac{\mathcal{C}_\Delta^4\sqrt{\pi}\Gamma(2\Delta - \frac{1}{2})}{2[\Gamma(\Delta)]^4}\frac{1}{t_{12}^{2\Delta}t_{34}^{2\Delta}}\chi^{2\Delta}\bar{D}_{\Delta\Delta\Delta\Delta}(\chi) \tag{A.2}$$

Specializing to the case of a massless scalar, so that $\Delta = 1$, this may be written as (cf. (4.8))

$$\langle O(t_1)O(t_2)O(t_2)O(t_3)\rangle = -\frac{g}{4\pi}\frac{(\mathcal{C}_{\Delta=1})^2}{t_{12}^2 t_{34}^2}\chi^2\bar{D}_{1111}(\chi), \tag{A.3}$$

with $\bar{D}_{1111}(\chi)$ given in (4.15). From this result we can extract the anomalous dimension of the $[OO]_{2n} \sim O\partial_t^{2n}O$ operators as explained in the main text. The leading order OPE coefficients are given by (3.10) with $\Delta = 1$

$$c_{OO[OO]_{2n}}^{(0)} = \frac{2[\Gamma(2n+2)]^2\Gamma(2n+3)}{\Gamma(2n+1)\Gamma(4n+3)}, \tag{A.4}$$

and extracting the coefficient of $\log(\chi)$ in $\bar{D}_{1111}(\chi)$, the anomalous dimensions are given by

$$\gamma_{[OO]_{2n}}^{(1)} = \frac{1}{c_{OO[OO]_{2n}}^{(0)}}\oint\frac{d\chi}{2\pi i}\chi^{-3-2n}F_{-1-2n}(\chi)\frac{g\chi^2}{2\pi(1-\chi)}, \tag{A.5}$$

which yields

$$\Delta_{[OO]_{2n}} = 2 + 2n + \frac{g}{4\pi}\frac{1}{(2n+1)(n+1)} + O(g^2). \tag{A.6}$$

Note that unlike the results we obtained above from the worldsheet model (2.4), the anomalous dimensions in (A.6) are positive and also they go to zero at large n.

One can similarly consider the case of a $m^2 = 2$ scalar, i.e. $\Delta = 2$. Then we get

$$\langle O(t_1)O(t_2)O(t_2)O(t_3)\rangle = -\frac{5g}{12\pi}\frac{(\mathcal{C}_{\Delta=2})^2}{t_{12}^4 t_{34}^4}\chi^4\bar{D}_{2222}(\chi),$$
$$\bar{D}_{2222}(\chi) = \frac{\chi(2\chi - 5) + 5}{30(\chi - 1)^3}\log(\chi^2) - \frac{2\chi^2 + \chi + 2}{30\chi^3}\log\left((1-\chi)^2\right) - \frac{2((\chi - 1)\chi + 1)}{15(\chi - 1)^2\chi^2}. \tag{A.7}$$

The leading OPE coefficients are obtained from (3.10) with $\Delta = 2$

$$c^{(0)}_{OO[OO]_{2n}} = \frac{\left[\Gamma(2n+4)\right]^2 \Gamma(2n+7)}{18\,\Gamma(2n+1)\,\Gamma(4n+7)}\,, \tag{A.8}$$

and the anomalous dimensions are then given by

$$\gamma^{(1)}_{[OO]_{2n}} = \frac{1}{c^{(0)}_{OO[OO]_{2n}}} \oint \frac{d\chi}{2\pi i} \chi^{-5-2n} F_{-3-2n}(\chi) \frac{g\chi^4((5-2\chi)\chi-5)}{36\pi(\chi-1)^3}\,. \tag{A.9}$$

This yields the result

$$\Delta_{[OO]_{2n}} = 4 + 2n + \frac{g}{4\pi}\frac{(n+1)(2n+5)}{(n+2)(n+3)(2n+1)(2n+3)} + O(g^2)\,. \tag{A.10}$$

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