

# Bosonic Tensor Models at Large $N$ and Small $\epsilon$

Simone Giombi,<sup>1</sup> Igor R. Klebanov<sup>1,2</sup> and Grigory Tarnopolsky<sup>1</sup>

<sup>1</sup>Department of Physics, Princeton University, Princeton, NJ 08544

<sup>2</sup>Princeton Center for Theoretical Science, Princeton University, Princeton, NJ 08544

## Abstract

We study the spectrum of the large  $N$  quantum field theory of bosonic rank-3 tensors, whose quartic interactions are such that the perturbative expansion is dominated by the melonic diagrams. We use the Schwinger-Dyson equations to determine the scaling dimensions of the bilinear operators of arbitrary spin. Using the fact that the theory is renormalizable in  $d = 4$ , we compare some of these results with the  $4 - \epsilon$  expansion, finding perfect agreement. This helps elucidate why the dimension of operator  $\phi^{abc}\phi^{abc}$  is complex for  $d < 4$ : the large  $N$  fixed point in  $d = 4 - \epsilon$  has complex values of the couplings for some of the  $O(N)^3$  invariant operators. We show that a similar phenomenon holds in the  $O(N)^2$  symmetric theory of a matrix field  $\phi^{ab}$ , where the double-trace operator has a complex coupling in  $4 - \epsilon$  dimensions. We also study the spectra of bosonic theories of rank  $q - 1$  tensors with  $\phi^q$  interactions. In dimensions  $d > 1.93$  there is a critical value of  $q$ , above which we have not found any complex scaling dimensions. The critical value is a decreasing function of  $d$ , and it becomes 6 in  $d \approx 2.97$ . This raises a possibility that the large  $N$  theory of rank-5 tensors with sextic potential has an IR fixed point which is free of perturbative instabilities for  $2.97 < d < 3$ . This theory may be studied using renormalized perturbation theory in  $d = 3 - \epsilon$ .

# Contents

<b>1</b>	<b>Introduction and Summary</b>	<b>1</b>
<b>2</b>	<b>Bosonic 3-Tensor Model</b>	<b>3</b>
2.1	Spectrum of two-particle operators . . . . .	3
2.2	Spectrum of higher-spin operators . . . . .	5
<b>3</b>	<b>Complex Large <math>N</math> Fixed Point in <math>d = 4 - \epsilon</math></b>	<b>8</b>
<b>4</b>	<b>Generalization to Higher <math>q</math></b>	<b>11</b>
4.1	Higher spin operators . . . . .	13
<b>5</b>	<b>A Melonic <math>\phi^6</math> Theory in 2.99 Dimensions</b>	<b>14</b>
<b>A</b>	<b>Appendix</b>	<b>16</b>

## 1 Introduction and Summary

A remarkable feature of some theories with tensor degrees of freedom of rank 3 and higher is that they possess large  $N$  limits dominated by the so-called melonic Feynman diagrams. This was discovered and proven for a variety of theories where the different indices of a tensor are not equivalent, but rather transform under different  $O(N)$  or  $U(N)$  symmetry groups [1–8]. Recent evidence has also emerged [9,10] that, even for the symmetric traceless tensor theories which have only  $O(N)$  symmetry and are similar to the tensor models introduced in the 90s [11–13], the melonic dominance continues to apply.

One of the reasons for the renewed interest in the large  $N$  theories with tensor degrees of freedom is their connection [14,15] with the SYK-like quantum mechanical models of fermions with disordered couplings [16–23].<sup>1</sup> In the large  $N$  limit these models have a conformally invariant sector, but also have the special operators whose correlators are not fixed by the conformal invariance.

It is of obvious interest to extend the SYK and tensor models to dimensions higher than  $d = 1$ . Such extensions were considered in [15,26–29]. Some of our work in this paper will be following the observation that, in a theory of a rank-3 bosonic tensor field one may introduce quartic interactions with  $O(N)^3$  symmetry [15]. Although the action is typically unbounded

---

<sup>1</sup> Recent work on the operator spectra and the thermal phase transitions [24,25] points also to some differences between the tensor and SYK models.

from below, such a QFT is perturbatively renormalizable in  $d = 4$ , so it may be studied using the  $4 - \epsilon$  expansion [30, 31].

In this paper we further explore the  $4 - \epsilon$  expansion and compare it with the large  $N$  Schwinger-Dyson equations, finding perfect agreement. We present results on the large  $N$  scaling dimensions of two-particle operators of arbitrary spin as a function of  $d$ , found using the Schwinger-Dyson equations. A salient feature of the large  $N$  spectrum of this theory in  $d < 4$  is that the lowest scalar operator has a complex dimension of the form  $\frac{d}{2} + i\alpha(d)$ .<sup>2</sup> We confirm this using the  $4 - \epsilon$  expansion in section 3. In that calculation it is necessary to include the mixing of the basic “tetrahedron” interaction term,

$$O_t(x) = \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1} , \quad (1.1)$$

with two additional  $O(N)^3$  invariant terms: the so-called pillow and double-sum invariants (3.2). The coefficients of these additional terms turn out to be complex at the “melonic” large  $N$  IR fixed point; as a result, the scaling dimension of the leading operator  $\phi^{abc} \phi^{abc}$  is complex. A similar phenomenon for the  $O(N)^2$  symmetric theory of a matrix  $\phi^{ab}$  is discussed in the Appendix. In that case it is necessary to include the  $O(N)^2$  invariant double-trace operator  $(\phi^{ab} \phi^{ab})^2$  whose coefficient is complex at the IR fixed point; as a result, the scaling dimension of operator  $\phi^{ab} \phi^{ab}$  is complex.

We also extend our results to rank  $q - 1$  tensors with  $\phi^q$  interactions. In each dimension  $d$  it is found that the two-particle mode with complex scaling dimension disappears for  $q$  greater than some critical value  $q_{\text{crit}}$  (for example, in  $d = 2$ ,  $q_{\text{crit}} \approx 64.3$  [29]). We study the spectrum of bilinear operators in the large  $N$  bosonic theory with  $q = 6$  in  $3 - \epsilon$  dimensions and point out that it is free of the problem with the complex dimension of  $\phi^2$  for  $\epsilon < 0.03$ . Thus, this theory is a candidate for a stable large  $N$  CFT, albeit in a non-integer dimension. However, an obvious danger, which we have not investigated, is that the coupling constants for some of the  $O(N)^5$  invariant sextic operators may be complex in  $d = 3 - \epsilon$ .

A more promising direction towards finding melonic CFTs in  $d \geq 2$  is to explore the supersymmetric versions of tensor or SYK-like models [15, 29] and a successful construction of such theory in  $d = 2$  was achieved recently [29]. We hope to consider supersymmetric theories in the future.

---

<sup>2</sup>However, the scaling dimension of the lowest scalar operator is real for  $4 < d < 4.155$ .

## 2 Bosonic 3-Tensor Model

In this section we consider the bosonic 3-tensor model with the  $O(N)^3$  symmetric action [15]

$$S = \int d^d x \left( \frac{1}{2} \partial_\mu \phi^{abc} \partial^\mu \phi^{abc} + \frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_2 b_2 c_2} \phi^{a_3 b_3 c_3} \right), \quad (2.1)$$

where each index runs from 1 to  $N$ . At the free UV fixed point the quartic interaction term has dimension  $2d - 4$ . For  $d < 4$  it is relevant and the large  $N$  theory may flow to an IR fixed point. However, the fact that the interaction term is not positive definite may cause problems with unitarity. Also, for  $d < 2$  the dimension of the interaction term lies below the unitarity bound. Nevertheless, we will see that the large  $N$  Schwinger-Dyson equations have formal solutions corresponding to the IR fixed point.

At large  $N$  in the IR limit the two-point function is a solution of the Schwinger-Dyson equation [15, 32]

$$G^{-1}(x) = -\lambda^2 G(x)^3, \quad (2.2)$$

where  $\lambda^2 = g^2 N^3$ . Using the Fourier transformation

$$\int d^d x \frac{e^{ikx}}{(x^2)^\alpha} = \frac{\pi^{d/2} \Gamma(d/2 - \alpha)}{2^{2\alpha - d} \Gamma(\alpha)} \frac{1}{(k^2)^{\frac{d}{2} - \alpha}} \quad (2.3)$$

we find the solution to the equation (2.2)

$$\begin{aligned} G(x) &= \frac{C_\phi}{\lambda^{1/2}} \frac{1}{(x^2)^{\frac{d}{4}}}, \\ C_\phi &= \left( -\frac{\Gamma(\frac{3d}{4})}{\pi^d \Gamma(-\frac{d}{4})} \right)^{1/4}. \end{aligned} \quad (2.4)$$

### 2.1 Spectrum of two-particle operators

The  $O(N)^3$  invariant two-particle operators of spin zero have the form  $\phi^{abc} (\partial_\mu \partial^\mu)^n \phi^{abc}$ , where  $n = 0, 1, 2, \dots$ . At the quantum level these operators mix with each other, although this mixing rapidly decreases as  $n$  increases, and the eigenvalues approach  $2n + \frac{d}{2}$ .

Let us denote the conformal three-point function of a general spin zero operator  $O_h$  with

two scalar fields  $\phi^{abc}$  by

$$v(x_1, x_2, x_3) = \langle O_h(x_1)\phi^{abc}(x_2)\phi^{abc}(x_3) \rangle = \frac{C_{O\phi\phi}}{(x_{12}^2 x_{13}^2)^{\frac{h}{2}} (x_{23}^2)^{\frac{1}{2}(d/2-h)}}, \quad (2.5)$$

where  $h$  and  $\Delta_\phi = d/4$  are the scaling dimensions.

In the large  $N$  limit one can write the Schwinger-Dyson equation for the three-point function [23]

$$v(x_0, x_1, x_2) = \int d^d x_3 d^d x_4 K(x_1, x_2, x_3, x_4) v(x_0, x_3, x_4), \quad (2.6)$$

where the kernel is given by the formula

$$K(x_1, x_2; x_3, x_4) = 3\lambda^2 G(x_{13})G(x_{24})G(x_{34})^2. \quad (2.7)$$

This equation determines the possible values of scaling dimension  $h$  of the operator  $O_h$ . Now using the general conformal integral [33]

$$\int d^d x_0 \frac{1}{(x_{01}^2)^{\alpha_1} (x_{02}^2)^{\alpha_2} (x_{03}^2)^{\alpha_3}} = \frac{L_d(\alpha_1, \alpha_2)}{(x_{12}^2)^{\frac{d}{2}-\alpha_3} (x_{13}^2)^{\frac{d}{2}-\alpha_2} (x_{23}^2)^{\frac{d}{2}-\alpha_1}}, \quad (2.8)$$

where  $\alpha_1 + \alpha_2 + \alpha_3 = d$  and

$$L_d(\alpha_1, \alpha_2) = \pi^{\frac{d}{2}} \frac{\Gamma(\frac{d}{2} - \alpha_1)\Gamma(\frac{d}{2} - \alpha_2)\Gamma(\frac{d}{2} - \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \quad (2.9)$$

one can find that [15]

$$\int d^d x_3 d^d x_4 K(x_1, x_2, x_3, x_4) v(x_0, x_3, x_4) = g(h) v(x_0, x_1, x_2),$$

$$g(h) = 3(C_\phi)^4 L_d\left(\frac{d}{4}, \frac{h}{2}\right) L_d\left(\frac{d-h}{2}, \frac{d}{4}\right) = -\frac{3\Gamma\left(\frac{3d}{4}\right)\Gamma\left(\frac{d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{h}{2} - \frac{d}{4}\right)}{\Gamma\left(-\frac{d}{4}\right)\Gamma\left(\frac{3d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{d}{4} + \frac{h}{2}\right)}. \quad (2.10)$$

The dimensions of the spin zero operators in large  $N$  limit are determined by  $g(h) = 1$ . In  $d = 4 - \epsilon$  this equation has solutions

$$h_0 = 2 \pm i\sqrt{6}\epsilon - \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^{3/2}), \quad h_1 = 4 + \epsilon - \frac{15\epsilon^2}{4} + \mathcal{O}(\epsilon^3),$$

$$h_n = 2(n+1) - \frac{\epsilon}{2} + \frac{3\epsilon^2}{2n^2(n^2-1)} + \mathcal{O}(\epsilon^3), \quad \text{for } n > 1. \quad (2.11)$$

We note that the first scaling dimension,  $h_0$ , is complex, which means that the critical point is unstable.<sup>3</sup> From the  $\text{AdS}_{5-\epsilon}$  side the relation between mass and scaling dimension

$$h = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2} \quad (2.12)$$

gives

$$m^2 = -4 - 4\epsilon + 11\epsilon^2 + \mathcal{O}(\epsilon^3), \quad (2.13)$$

which is slightly below the Breitenlohner-Freedman [35] bound  $m^2 > -d^2/4$ .

More generally, for  $d < 4$  the first solution of  $g(h) = 1$  has the form

$$h_0 = \frac{d}{2} \pm i\alpha(d), \quad (2.14)$$

where  $\alpha(d)$  is real. This is in agreement with (2.12) for  $m^2 < -d^2/4$ . On the other hand, for  $4 < d < 4.155$ ,  $h_0$  is real and the large  $N$  theory is free of this instability, at least formally.

## 2.2 Spectrum of higher-spin operators

Consider a higher-spin operator  $J_s(x) = z^{\mu_1} \dots z^{\mu_s} J_{\mu_1 \dots \mu_s}$ , where we introduced an auxiliary null vector  $z^\mu$ , satisfying

$$z^2 = z^\mu z^\nu \delta_{\mu\nu} = 0. \quad (2.15)$$

The three-point function  $\langle J_s \phi^{abc} \phi^{abc} \rangle$  is completely fixed by conformal invariance

$$\langle J_s(x_1) \phi^{abc}(x_2) \phi^{abc}(x_3) \rangle = C_{s00} \frac{\left( \frac{z \cdot x_{12}}{x_{12}^2} - \frac{z \cdot x_{13}}{x_{13}^2} \right)^s}{(x_{12}^2)^{\frac{\tau_s}{2}} (x_{23}^2)^{\Delta_\phi - \frac{\tau_s}{2}} (x_{31}^2)^{\frac{\tau_s}{2}}}, \quad (2.16)$$

where  $\Delta_\phi = d/4$  and  $\tau_s = \Delta_{J_s} - s$  and  $\Delta_{J_s} = 2\Delta_\phi + s + \gamma_s$ . If we set the  $J_s$  momentum to zero or equivalently, integrate over the position of  $J_s$  we get

$$v_s(x_2, x_3) = \int d^d x_1 \langle J_s(x_1) \phi^{abc}(x_2) \phi^{abc}(x_3) \rangle = \frac{(z \cdot x_{23})^s}{(x_{23}^2)^{\frac{\tau_s}{2} + s - \frac{d}{2} + \Delta_\phi}}. \quad (2.17)$$

---

<sup>3</sup>There are other indications that the melonic large  $N$  limit of bosonic tensor models is unstable [29, 34].

In the large  $N$  limit one can again write the Schwinger-Dyson equation for the three-point function

$$v_s(x_1, x_2) = \int d^d x_3 d^d x_4 K(x_1, x_2, x_3, x_4) v_s(x_3, x_4). \quad (2.18)$$

To perform the integral in the r.h.s of (2.18) we use the well-known integral

$$\int d^d x \frac{(z \cdot x)^s}{x^{2\alpha} (x-y)^{2\beta}} = L_{d,s}(\alpha, \beta) \frac{(z \cdot y)^s}{(y^2)^{\alpha+\beta-d/2}},$$

$$L_{d,s}(\alpha, \beta) = \pi^{d/2} \frac{\Gamma(\frac{d}{2} - \alpha + s) \Gamma(\frac{d}{2} - \beta) \Gamma(\alpha + \beta - \frac{d}{2})}{\Gamma(\alpha) \Gamma(\beta) \Gamma(d + s - \alpha - \beta)}. \quad (2.19)$$

Using (2.19) we find

$$\int d^d x_3 d^d x_4 K(x_1, x_2, x_3, x_4) v_s(x_3, x_4) = g(\tau_s, s) v_s(x_1, x_2),$$

$$g(\tau_s, s) = 3(C_\phi)^4 L_{d,s}\left(\frac{d}{4} + s + \frac{\tau_s}{2}, \frac{d}{4}\right) L_{d,s}\left(s + \frac{\tau_s}{2}, \frac{d}{4}\right) = -\frac{3\Gamma\left(\frac{3d}{4}\right) \Gamma\left(\frac{d-2\tau_s}{4}\right) \Gamma\left(\frac{4s+2\tau_s-d}{4}\right)}{\Gamma\left(-\frac{d}{4}\right) \Gamma\left(\frac{3d-2\tau_s}{4}\right) \Gamma\left(\frac{d+4s+2\tau_s}{4}\right)} \quad (2.20)$$

and to find the spectrum we have to solve the equation  $g(\tau_s, s) = 1$ . Note that for any  $d$ , there is a solution with  $s = 2$  and  $\tau_s = d - 2$ . This corresponds to the conserved stress tensor, consistently with the conformal invariance.

For general fixed spin  $s$ , the dimensions should approach, at large  $n$

$$\Delta_{J_s} = 2\Delta_\phi + s + 2n, \quad n = 0, 1, 2, \dots, \quad (2.21)$$

where  $n$  is interpreted as the number of contracted derivatives. Alternatively, one can also study the behavior for large spin  $s$ , and fixed  $n$  (say  $n = 0$ ), where the dimensions should approach  $\Delta_{J_s} \approx 2\Delta_\phi + s + c/s^{\tau_{\min}}$ , where  $\tau_{\min}$  is the lowest twist (excluding the identity) appearing in the OPE expansion of the  $\phi$  4-point function [36–38].

For  $n = 0$  we have in  $d = 4 - \epsilon$

$$\tau_s = d - 2 + \frac{(s-2)(s+3)}{2s(s+1)} \epsilon + \dots \quad (2.22)$$

Note that the correction to  $d - 2$  vanishes for  $s = 2$ , as it should since the stress tensor is conserved. The fact that this correction for  $s \neq 2$  is  $\sim \epsilon$  also makes sense, because for nearly conserved currents the anomalous dimension starts at  $\sim g^2$  on general grounds (like

$\gamma_\phi$ ). The spin dependence in the above result is the expected one for an almost conserved current near  $d = 4$ , see e.g. [39, 40].

In  $d = 2$  the equation determining the dimensions becomes elementary and reads

$$\frac{3}{(1 - \tau_s)(2s + \tau_s - 1)} = 1 \quad (2.23)$$

with solutions

$$\tau_s = 1 - s \pm \sqrt{s^2 - 3}. \quad (2.24)$$

Surprisingly, this gives only one solution with  $h > d/2$  for each spin, rather than the infinite number of solutions which are present in  $d > 2$  (already in  $d = 2 + \epsilon$  there are towers of real solutions). For  $s = 0$  in  $d = 2$  the solution (2.24) is complex

$$h \approx 1 + 1.5235i. \quad (2.25)$$

In  $d = 2 + \epsilon$  there is also a tower of real solutions:<sup>4</sup>

$$\tau_s = 2n + \frac{d}{2} + \frac{3}{3 + 4n(n + s)}\epsilon + \mathcal{O}(\epsilon^2). \quad (2.26)$$

In  $d = 1$  the primary two-particle operators have the form  $\phi^{abc}\partial_t^{2n}\phi^{abc}$ , where  $n = 0, 1, 2, \dots$ . The graphical solution of the eigenvalue equation is shown in figure 1. The equation has a symmetry under  $h \rightarrow 1 - h$ . The first real solution greater than  $1/2$  is the exact solution  $h = 2$ . It correspond to the  $n = 1$  operator, which through the use of equations of motion is proportional to the potential  $\phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$ . The first eigenvalue is complex,  $h_0 = \frac{1}{2} + 1.525i$ . Since it is of the form  $\frac{1}{2} + is$ , it is a normalizable mode which needs to be integrated over, similarly to the  $h = 2$  mode.

---

<sup>4</sup>In the  $\epsilon \rightarrow 0$  limit it appears to give additional states in  $d = 2$  which are missed by the degenerate  $d = 2$  equation (2.23).



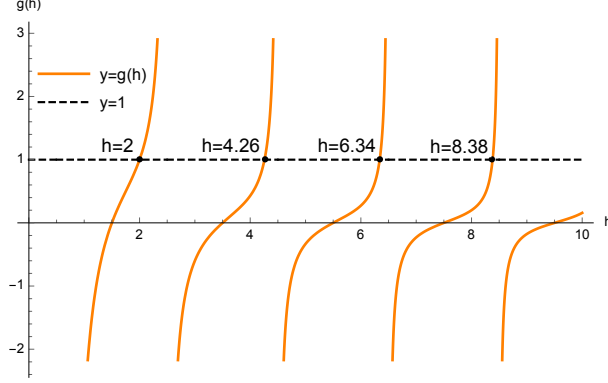


Figure 1: The graphical solution of the eigenvalue equation  $g(h) = 1$  in  $d = 1$ . This method works for finding the real solutions only; it misses the complex solution  $h_0 = \frac{1}{2} + 1.525i$ .

### 3 Complex Large $N$ Fixed Point in $d = 4 - \epsilon$

In this section we study the renormalizable theory in  $4 - \epsilon$  dimensions with a 3-tensor degree of freedom and  $O(N)^3$  symmetric quartic interactions:

$$S = \int d^d x \left( \frac{1}{2} \partial_\mu \phi^{abc} \partial^\mu \phi^{abc} + \frac{1}{4} (g_1 O_t(x) + g_2 O_p(x) + g_3 O_{ds}(x)) \right), \quad (3.1)$$

where  $g_1, g_2, g_3$  are the bare couplings which correspond to the three possible invariant quartic interaction terms. The perturbative renormalizability of the theory requires that, in addition to the “tetrahedron” interaction term (1.1), we introduce the “pillow” and “double-sum” terms

$$\begin{aligned} O_p(x) &= \frac{1}{3} \left( \phi^{a_1 b_1 c_1} \phi^{a_1 b_1 c_2} \phi^{a_2 b_2 c_2} \phi^{a_2 b_2 c_1} + \phi^{a_1 b_1 c_1} \phi^{a_2 b_1 c_1} \phi^{a_2 b_2 c_2} \phi^{a_1 b_2 c_2} + \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_1} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_2} \right), \\ O_{ds}(x) &= \phi^{a_1 b_1 c_1} \phi^{a_1 b_1 c_1} \phi^{a_2 b_2 c_2} \phi^{a_2 b_2 c_2}. \end{aligned} \quad (3.2)$$

To find the beta functions we use a well-known result [41] for a general  $\phi^4$ -model with the interaction term  $V = \frac{1}{4} g_{ijkl} \phi^i \phi^j \phi^k \phi^l$ . In our case we can write interaction as

$$V = \frac{1}{4} g_{\kappa_1 \kappa_2 \kappa_3 \kappa_4} \phi^{\kappa_1} \phi^{\kappa_2} \phi^{\kappa_3} \phi^{\kappa_4}, \quad (3.3)$$

where  $\kappa_i = (a_i b_i c_i)$  is a set of three indices and  $g_{\kappa_1 \kappa_2 \kappa_3 \kappa_4}$  is a sum of three structures

$$g_{\kappa_1 \kappa_2 \kappa_3 \kappa_4} = g_1 T_{\kappa_1 \kappa_2 \kappa_3 \kappa_4}^t + g_2 T_{\kappa_1 \kappa_2 \kappa_3 \kappa_4}^p + g_3 T_{\kappa_1 \kappa_2 \kappa_3 \kappa_4}^{ds}. \quad (3.4)$$

Each structure is a sum of a product of Kronecker-delta terms, which after contraction with  $\phi^{\kappa_1}\phi^{\kappa_2}\phi^{\kappa_4}\phi^{\kappa_4}$  reproduce (1.1) and (3.2). For example

$$T_{\kappa_1\kappa_2\kappa_3\kappa_4}^t = \frac{1}{4!} \left( \delta_{a_1a_2}\delta_{b_1b_3}\delta_{c_1c_4}\delta_{b_2b_4}\delta_{c_2c_3}\delta_{a_3a_4} + \text{sym}(\kappa_1, \dots, \kappa_4) \right), \quad (3.5)$$

where the last term means that we have to add all terms corresponding to permutations of  $\kappa_1, \dots, \kappa_4$ . Using the explicit formulas in [41], we find the beta functions up to two loops

$$\begin{aligned} \beta_t = & -\epsilon g_1 + \frac{4}{3(4\pi)^2} \left( 3g_1g_2(N+1) + 18g_1g_3 + 2g_2^2 \right) \\ & + \frac{2}{9(4\pi)^4} \left( 9(N^3 - 15N - 10)g_1^3 - 36g_1^2((N^2 + 4N + 13)g_2 + 15Ng_3) \right. \\ & - 3g_1((N^3 + 15N^2 + 93N + 101)g_2^2 + 12(5N^2 + 17N + 17)g_2g_3 + 6(5N^3 + 82)g_3^2) \\ & \left. - 4g_2^2((2N^2 + 13N + 24)g_2 + 72g_3) \right), \end{aligned} \quad (3.6)$$

$$\begin{aligned} \beta_p = & -\epsilon g_2 + \frac{2}{3(4\pi)^2} \left( 9g_1^2(N+2) + 12g_2g_1(N+2) + g_2^2(N^2 + 5N + 12) + 36g_2g_3 \right) \\ & - \frac{2}{9(4\pi)^4} \left( 108(N^2 + N + 4)g_1^3 + 9g_1^2((N^3 + 12N^2 + 99N + 98)g_2 + 72(N+2)g_3) \right. \\ & + 36g_1g_2((4N^2 + 18N + 29)g_2 + 3(13N + 16)g_3) + g_2((5N^3 + 45N^2 + 243N + 343)g_2^2 \\ & \left. + 36(7N^2 + 15N + 29)g_2g_3 + 18(5N^3 + 82)g_3^2) \right), \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} \beta_{ds} = & -\epsilon g_3 + \frac{2}{3(4\pi)^2} \left( 3g_3^2(N^3 + 8) + 6g_3g_2(N^2 + N + 1) + g_2^2(2N + 3) + 18g_1g_3N + 6g_1g_2 \right) \\ & - \frac{2}{9(4\pi)^4} \left( 54Ng_1^3 + 9g_1^2(4(N^2 + N + 4)g_2 + 5(N^3 + 3N + 2)g_3) \right. \\ & + 36g_1(4(N+1)g_2^2 + (5N^2 + 5N + 17)g_2g_3 + 33Ng_3^2) + 14(N^2 + 3N + 5)g_2^3 \\ & \left. + 3(5N^3 + 15N^2 + 93N + 97)g_2^2g_3 + 396(N^2 + N + 1)g_2g_3^2 + 54(3N^3 + 14)g_3^3 \right). \end{aligned} \quad (3.8)$$

For the anomalous dimension we obtain

$$\begin{aligned} \gamma_\phi = & \frac{1}{6(4\pi)^4} \left( 3g_1^2(N^3 + 3N + 2) + 6g_3^2(N^3 + 2) + 12g_1(g_2(N^2 + N + 1) + 3g_3N) \right. \\ & \left. + 12g_2g_3(N^2 + N + 1) + g_2^2(N^3 + 3N^2 + 9N + 5) \right). \end{aligned} \quad (3.9)$$

Now, using the large  $N$  scaling

$$g_1 = \frac{(4\pi)^2 \tilde{g}_1}{N^{3/2}}, \quad g_2 = \frac{(4\pi)^2 \tilde{g}_2}{N^2}, \quad g_3 = \frac{(4\pi)^2 \tilde{g}_3}{N^3}, \quad (3.10)$$

where  $\tilde{g}_i$  are held fixed, we find that the anomalous dimension

$$\gamma_\phi = \frac{\tilde{g}_1^2}{2} + \mathcal{O}(1/N) \quad (3.11)$$

and the beta functions

$$\begin{aligned} \tilde{\beta}_t &= -\epsilon \tilde{g}_1 + 2\tilde{g}_1^3, \\ \tilde{\beta}_p &= -\epsilon \tilde{g}_2 + \left( 6\tilde{g}_1^2 + \frac{2}{3}\tilde{g}_2^2 \right) - 2\tilde{g}_1^2 \tilde{g}_2, \\ \tilde{\beta}_{ds} &= -\epsilon \tilde{g}_3 + \left( \frac{4}{3}\tilde{g}_2^2 + 4\tilde{g}_2 \tilde{g}_3 + 2\tilde{g}_3^2 \right) - 2\tilde{g}_1^2(4\tilde{g}_2 + 5\tilde{g}_3). \end{aligned} \quad (3.12)$$

We note that  $\tilde{\beta}_t$  depends only on the tetrahedron coupling  $\tilde{g}_1$ , while the beta functions for pillow and double-sum also contain  $\tilde{g}_1$ . This is a feature of the large  $N$  limit. Similarly, in the large  $N$  limit of the quartic matrix theory, the double-trace coupling does not affect the beta function of the single-trace coupling (see the Appendix).

The large  $N$  critical point with a non-vanishing tetrahedron coupling is

$$\tilde{g}_1^* = (\epsilon/2)^{1/2}, \quad \tilde{g}_2^* = \pm 3i(\epsilon/2)^{1/2}, \quad \tilde{g}_3^* = \mp i(3 \pm \sqrt{3})(\epsilon/2)^{1/2}. \quad (3.13)$$

For the dimension of the  $O = \phi^{abc}\phi^{abc}$  operator at large  $N$  we find

$$\Delta_O = d - 2 + 2(\tilde{g}_2^* + \tilde{g}_3^*) = 2 \pm i\sqrt{6\epsilon} + \mathcal{O}(\epsilon). \quad (3.14)$$

This exactly coincides with the large  $N$  solution (2.11), providing a nice perturbative check of the fact that the scaling dimension is complex. We note that the imaginary part originates from the complex pillow and double-sum couplings.

Now if we look for the dimension of the tetrahedron operator, then using the derivative

of the beta function, we find

$$\Delta_{\text{tetra}} = d + \beta'_t(g_1^*) = 4 + \epsilon + \mathcal{O}(\epsilon^2), \quad (3.15)$$

which coincides with the scaling dimension  $h_1$  of operator  $\phi^{abc}\nabla^2\phi^{abc}$  found in (2.11).

## 4 Generalization to Higher $q$

The construction of theories for a single rank 3 tensor field with the quartic interaction (2.1) may be generalized to a single rank  $q - 1$  tensor with the  $O(N)^{q-1}$  symmetric interaction of order  $q$ . Since the indices of each  $O(N)$  group must be contracted pairwise,  $q$  has to be even. The rank  $q - 1$  tensor theories have a large  $N$  limit with  $\lambda^2 = g^2 N^{(q-1)(q-2)/2}$  held fixed, which is dominated by the melonic diagrams (this follows from the method of “forgetting” all but two colors in the graphs made out of  $q - 1$  strands by analogy with the derivation [5, 8, 14, 15] for  $q = 4$ ). For example, for  $q = 6$  the explicit form of the interaction of a real rank 5 tensor is [15]

$$V_{\text{int}} = \frac{g}{6} \phi^{a_1 b_1 c_1 d_1 e_1} \phi^{a_1 b_2 c_2 d_2 e_2} \phi^{a_2 b_2 c_3 d_3 e_1} \phi^{a_2 b_3 c_2 d_1 e_3} \phi^{a_3 b_3 c_1 d_3 e_2} \phi^{a_3 b_1 c_3 d_2 e_3}. \quad (4.1)$$

Since every pair of fields have one index in common, this interaction may be represented by a 5-simplex.

The two-point Schwinger-Dyson equation has the form

$$G^{-1}(x) = -\lambda^2 G(x)^{q-1}. \quad (4.2)$$

The general  $d$  solution to this equation is

$$G(x) = \frac{C_\phi}{\lambda^{2/q}} \frac{1}{(x^2)^{\frac{d}{q}}},$$

$$C_\phi = \left( -\frac{\pi^{-d} \Gamma(\frac{d}{q}) \Gamma(\frac{d(q-1)}{q})}{\Gamma(\frac{d(2-q)}{2q}) \Gamma(\frac{d(q-2)}{2q})} \right)^{1/q}. \quad (4.3)$$

In analogy to Section (2.1) one can find a spectrum of spin zero operators by solving Schwinger-Dyson equation for the three-point function

$$v(x_0, x_1, x_2) = \int d^d x_3 d^d x_4 K(x_1, x_2, x_3, x_4) v(x_0, x_3, x_4), \quad (4.4)$$

where the kernel is given by the formula

$$K(x_1, x_2; x_3, x_4) = (q-1)\lambda^2 G(x_{13})G(x_{24})G(x_{34})^{q-2}. \quad (4.5)$$

Using the integral (2.8) and expression (4.3) we find

$$g_q(h) = (q-1)(C_\phi)^q L_d\left(\frac{d}{q}, \frac{h}{2}\right) L_d\left(\frac{d-h}{2}, \frac{d}{q}\right) = -\frac{(q-1)\Gamma\left(\frac{d(q-2)}{2q}\right)\Gamma\left(\frac{d(q-1)}{q}\right)\Gamma\left(\frac{h}{2} - \frac{d(q-2)}{2q}\right)\Gamma\left(\frac{d}{q} - \frac{h}{2}\right)}{\Gamma\left(\frac{d(2-q)}{2q}\right)\Gamma\left(\frac{d}{q}\right)\Gamma\left(\frac{h}{2} + \frac{d(q-2)}{2q}\right)\Gamma\left(\frac{d(q-1)}{q} - \frac{h}{2}\right)}, \quad (4.6)$$

where  $C_\phi$  is given in (4.3).

By solving  $g_q(h) = 1$  we find the spectrum of dimensions of spin zero two-particle operators. As we already noticed in (2.1), for  $q = 4$  the lowest operator  $O = \phi^2$  has complex dimension, which signals an instability of the theory. However, for  $d$  greater than the critical value  $d_{\text{cr}}$ , there exists  $q_{\text{crit}}$  such that for  $q > q_{\text{crit}}$  the solutions of  $g_q(h) = 1$  are real, and the two-particle operators do not cause instabilities. The  $d_{\text{cr}}$  is determined by

$$\frac{\Gamma(-d_{\text{cr}}/4)^2\Gamma(d_{\text{cr}}/2)\Gamma(d_{\text{cr}}+1)}{\Gamma(-d_{\text{cr}}/2)\Gamma(3d_{\text{cr}}/4)^2} = -1, \quad (4.7)$$

and we find  $d_{\text{cr}} \approx 1.93427$ . Interestingly,  $q_{\text{crit}}$  diverges at  $d_{\text{cr}}$  as  $q_{\text{crit}} \approx \frac{4.092}{d-d_{\text{cr}}}$ . The plot for  $q_{\text{crit}}$  as a function of  $d$  is shown in Figure 2.

In  $d = 2$ , the critical value of  $q$  is still large:  $q_{\text{crit}} \approx 64.3$  [29], but it drops to  $\approx 5.9$  in  $d = 3$ . For  $d < d_{\text{cr}}$  the lowest eigenvalue is complex for any  $q$ . In  $d = 1$ , in the large  $q$  limit

$$h_0 = \frac{1}{2} + i\frac{\sqrt{7}}{2} + \mathcal{O}(1/q). \quad (4.8)$$

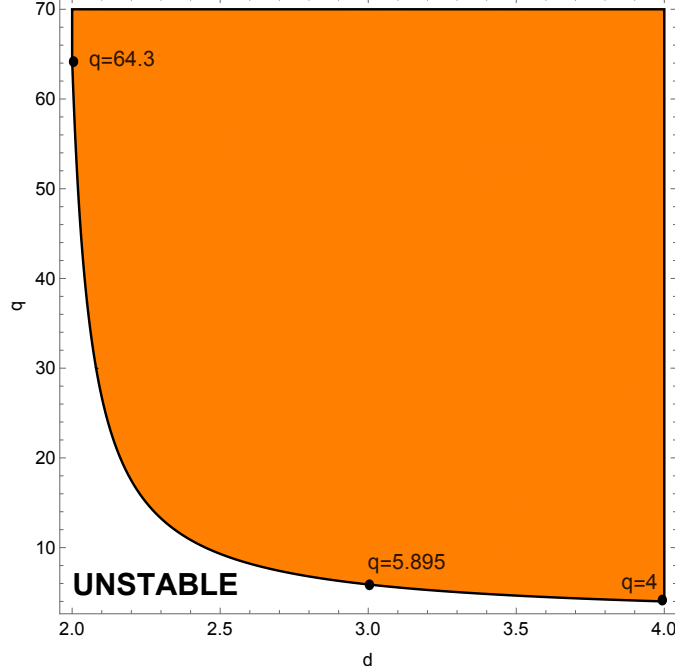


Figure 2: Plot of  $q_{\text{crit}}$  as a function of  $d$ . The orange region corresponds to  $q > q_{\text{crit}}$ , where  $\Delta_{\phi^2}$  is real and the theory is not obviously unstable. For integer dimensions we obtained  $q_{\text{crit}}(2) \approx 64.3$ ,  $q_{\text{crit}}(3) \approx 5.9$  and  $q_{\text{crit}}(4) = 4$ .

## 4.1 Higher spin operators

Similarly to the case  $q = 4$ , we may generalize the discussion of  $q > 4$  to the higher spin operators. We find that <sup>5</sup>

$$\begin{aligned}
 g_q(\tau_s, s) &= (q-1)(C_\phi)^q L_{d,s}\left(\frac{d}{2} - \frac{d}{q} + s + \frac{\tau_s}{2}, \frac{d}{q}\right) L_{d,s}\left(s + \frac{\tau_s}{2}, \frac{d}{q}\right) \\
 &= - \frac{(q-1)\Gamma\left(\frac{d(q-2)}{2q}\right)\Gamma\left(\frac{d(q-1)}{q}\right)\Gamma\left(\frac{d}{q} - \frac{\tau_s}{2}\right)\Gamma\left(s + \frac{\tau_s}{2} - \frac{d(q-2)}{2q}\right)}{\Gamma\left(\frac{d(2-q)}{2q}\right)\Gamma\left(\frac{d}{q}\right)\Gamma\left(\frac{d(q-1)}{q} - \frac{\tau_s}{2}\right)\Gamma\left(s + \frac{\tau_s}{2} + \frac{d(q-2)}{2q}\right)}. \quad (4.9)
 \end{aligned}$$

As a check of this formula, we note that the equation  $g_q(\tau_s, s) = 1$  for  $s = 2$  has a solution  $\tau_s = d - 2$  corresponding to the stress-energy tensor.

Similarly to the case  $q = 4$ , which degenerates for  $d = 2$ , we find a similar degeneration of (4.9) for  $q = 8$  and  $d = 4$ ,

$$g(\tau_s, s) = \frac{315}{(\tau_s - 5)(\tau_s - 3)(\tau_s - 1)(2s + \tau_s - 3)(2s + \tau_s - 1)(2s + \tau_s + 1)}, \quad (4.10)$$

and the equation  $g = 1$  may be solved in terms of the square and cubic roots. The physically

<sup>5</sup>For  $d = 2$ , this equation agrees with eq. (6.8) of [29] after the identifications  $h = s + \frac{\tau}{2}$ ,  $\tilde{h} = \frac{\tau}{2}$ .

relevant solution for  $\tau$  has the large  $s$  expansion

$$\tau_s = 1 + \frac{315}{64s^3} + \frac{315}{64s^5} + \dots \quad (4.11)$$

More generally, we have checked numerically that, in the large  $s$  limit,  $\tau \rightarrow 2\Delta_\phi$ , where  $\Delta_\phi = d/q$ . For example, for  $q = 6$  and  $d = 2$ , we find

$$\tau_4 = 0.456, \quad \tau_6 = 0.547, \quad \tau_{1000} \approx 0.666. \quad (4.12)$$

## 5 A Melonic $\phi^6$ Theory in 2.99 Dimensions

Using (4.6) for  $q = 6$  we find that the spin zero spectrum is free of complex solutions in a small region of dimension below 3. Working in  $d = 3 - \epsilon$ , we find that the scaling dimensions are real for  $\epsilon < 0.02819$ . Expansions of the first three solutions of the equation  $g_6(h) = 1$  are

$$\begin{aligned} h_- &= 1 + \frac{29\epsilon}{3} + \frac{400\epsilon^2}{9} + \frac{160}{27} (237 + 2\pi^2) \epsilon^3 + \mathcal{O}(\epsilon^4), \\ h_+ &= 2 - \frac{32\epsilon}{3} - \frac{400\epsilon^2}{9} - \frac{160}{27} (237 + 2\pi^2) \epsilon^3 + \mathcal{O}(\epsilon^4), \\ h_1 &= 3 + 3\epsilon - \frac{220\epsilon^2}{9} + \frac{40}{81} (503 + 3\pi^2) \epsilon^3 + \mathcal{O}(\epsilon^4), \end{aligned} \quad (5.1)$$

and the expansion coefficients grow rapidly. It appears that  $h_-$  corresponds to operator  $\phi^{abcde} \phi^{abcde}$ ,  $h_+$  to a quartic operator which mixes with it due to interactions, and  $h_1$  to  $\phi^{abcde} \partial_\mu \partial^\mu \phi^{abcde} \sim V_{\text{int}}$ .

As  $\epsilon$  increases,  $h_-$  approaches  $h_+$ , and at  $\epsilon_{\text{crit}} \approx 0.02819$  they merge and go off to complex plane (see Figure 3).

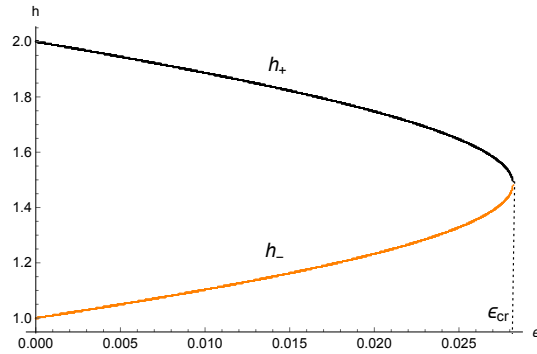


Figure 3: Plot of the two lowest operator dimensions  $h_-$  and  $h_+$  as a function of  $\epsilon$ . As  $\epsilon$  increases,  $h_-$  approaches  $h_+$ , and at  $\epsilon_{\text{crit}} \approx 0.02819$  they merge and go off to complex plane.

The scaling dimension of operators  $\phi^{abcde}(\partial_\mu\partial^\mu)^n\phi^{abcde}$  with  $n > 1$  are found to be

$$h_n = 2n + 1 - \frac{\epsilon}{3} + \frac{20}{3(n-1)n(4n^2-1)}\epsilon^2 + \frac{80\left(H_{2n-3} - \frac{92n^4 - 128n^3 + 13n^2 + 23n - 45}{12n(n-1)(4n^2-1)}\right)}{9n(n-1)(4n^2-1)}\epsilon^3 + \mathcal{O}(\epsilon^4), \quad (5.2)$$

where  $H_n$  is the Harmonic number. For large  $n$  we get

$$h_n = 2n + 1 - \frac{\epsilon}{3} + \frac{5\epsilon^2}{3n^4} + \frac{5\epsilon^3(12\log(2ne^\gamma) - 23)}{27n^4} + \mathcal{O}(\epsilon^4). \quad (5.3)$$

This agrees with the fact that the dimension of operators  $\phi^{abcde}(\partial_\mu\partial^\mu)^n\phi^{abcde}$  should approach  $2n + \frac{d}{3}$  for large  $n$ .

For operators of  $s > 0$ , we may use (4.9) to obtain for  $n = 0$

$$h(s) = d - 2 + s + \frac{8(s^2 - 4)}{3(4s^2 - 1)}\epsilon - \frac{20}{3(4s^2 - 1)}\left(\psi\left(s - \frac{1}{2}\right) - \psi\left(\frac{3}{2}\right) - \frac{2(s-2)(20s^3 + 4s^2 + 43s + 5)}{3(4s^2 - 1)^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (5.4)$$

The first term is the dimension of the operator in free field theory, while the additional terms appear due to the  $\phi^6$  interactions.

It would be interesting to reproduce the  $3 - \epsilon$  expansions found in this section using perturbative calculations in the  $O(N)^5$  invariant renormalizable  $\phi^6$  theory. This is technically more complicated than the similar calculation we carried out in  $4 - \epsilon$  dimensions, because there are several invariant  $\phi^6$  terms. An obvious danger is that the coupling constants for some of the sextic operators will be complex in  $d = 3 - \epsilon$ . We hope to return to these issues in the future.

## Acknowledgments

We thank S. Chester, V. Kirilin, F. Popov, D. Stanford and E. Witten for very useful discussions. The work of SG was supported in part by the US NSF under Grant No. PHY-1620542. The work of IRK and GT was supported in part by the US NSF under Grant No. PHY-1620059. GT also acknowledges the support of a Myhrvold-Havranek Innovative Thinking Fellowship.



# A Appendix

In this appendix we consider renormalizable theory in  $4 - \epsilon$  dimensions with a matrix degree of freedom and  $O(N)^2$  symmetric quartic interactions:

$$S = \int d^d x \left( \frac{1}{2} \partial_\mu \phi^{ab} \partial^\mu \phi^{ab} + \frac{1}{4} g_1 O_{st}(x) + \frac{1}{4} g_2 O_{dt}(x) \right), \quad (\text{A.1})$$

where  $g_1, g_2$  are the bare couplings which correspond to the two possible invariant quartic interaction terms. The perturbative renormalizability of the theory requires that, in addition to the single-trace term

$$O_{st}(x) = \phi^{ab} \phi^{cb} \phi^{cd} \phi^{ad} = \text{Tr} \phi \phi^T \phi \phi^T, \quad (\text{A.2})$$

we introduce the double-trace term

$$O_{dt}(x) = \phi^{ab} \phi^{ab} \phi^{cd} \phi^{cd} = \text{Tr} \phi \phi^T \text{Tr} \phi \phi^T. \quad (\text{A.3})$$

In analogy with the section 3 we find the beta functions using a well-known result [41] for a general  $\phi^4$ -model with the interaction vertex  $V = \frac{1}{4} g_{ijkl} \phi^i \phi^j \phi^k \phi^l$ . The beta functions up to two loops are

$$\begin{aligned} \beta_{st} &= -\epsilon g_1 + \frac{g_1(g_1(N+2) + 6g_2)}{4\pi^2} \\ &\quad - \frac{g_1(3g_1^2(N(N+6) + 17) + 4g_1g_2(22N+29) + 2g_2^2(5N^2+82))}{128\pi^4}, \\ \beta_{dt} &= -\epsilon g_2 + \frac{3g_1^2 + 2g_1g_2(2N+1) + g_2^2(N^2+8)}{8\pi^2} \\ &\quad - \frac{6g_1^3(2N+3) + g_1^2g_2(5N(N+2) + 87) + 44g_1g_2^2(2N+1) + 6g_2^3(3N^2+14)}{128\pi^4}. \end{aligned} \quad (\text{A.4})$$

Now, using the large  $N$  scaling

$$g_1 = \frac{(4\pi)^2 \tilde{g}_1}{N}, \quad g_2 = \frac{(4\pi)^2 \tilde{g}_2}{N^2}, \quad (\text{A.5})$$

where  $\tilde{g}_i$  are held fixed, we find the beta functions

$$\begin{aligned}\tilde{\beta}_{st} &= -\epsilon\tilde{g}_1 + 4\tilde{g}_1^2 - 6\tilde{g}_1^3, \\ \tilde{\beta}_{dt} &= -\epsilon\tilde{g}_2 + (6\tilde{g}_1^2 + 2\tilde{g}_2^2 + 8\tilde{g}_1\tilde{g}_2) - 2\tilde{g}_1^2(12\tilde{g}_1 + 5\tilde{g}_2).\end{aligned}\tag{A.6}$$

We note that  $\tilde{\beta}_{st}$  depends only on the single-trace coupling  $\tilde{g}_1$ , while the double-trace beta function depends on both couplings. This is a familiar phenomenon for beta functions in large  $N$  matrix theories [42]. Comparing with the beta functions (3.6–3.8) of the large  $N$  3-tensor theory, we observe that the tetrahedron coupling in the tensor model is analogous to the single-trace coupling in the matrix model, while the pillow and double-sum couplings in the tensor model are analogous to the double-trace coupling in the matrix model.

The large  $N$  critical point with a non-vanishing single-trace coupling is

$$\tilde{g}_1^* = \frac{\epsilon}{4} + \frac{3\epsilon^2}{32}, \quad \tilde{g}_2^* = -\frac{1}{4} \left(1 \pm i\sqrt{2}\right) \epsilon - \frac{1}{32} \left(1 \mp 2i\sqrt{2}\right) \epsilon^2.\tag{A.7}$$

For the dimension of the  $O = \phi^{ab}\phi^{ab}$  operator at large  $N$  we find

$$\Delta_O = d - 2 + 4\tilde{g}_1^* + 2\tilde{g}_2^* = 2 - \frac{1}{2} \left(1 \pm i\sqrt{2}\right) \epsilon + \mathcal{O}(\epsilon^2).\tag{A.8}$$

The imaginary part originates from the double-trace coupling. So, in spite of the positivity of the interaction term  $O_{st}$ , this large  $N$  critical point is unstable due to an operator dimension being complex. The form of the dimension,  $\frac{d}{2} + i\alpha$ , corresponds to a field violating the Breitenlohner-Freedman bound in the dual AdS space.

## References

- [1] R. Gurau, “Colored Group Field Theory,” *Commun. Math. Phys.* **304** (2011) 69–93, 0907.2582.
- [2] R. Gurau and J. P. Ryan, “Colored Tensor Models - a review,” *SIGMA* **8** (2012) 020, 1109.4812.
- [3] R. Gurau and V. Rivasseau, “The 1/ $N$  expansion of colored tensor models in arbitrary dimension,” *Europhys. Lett.* **95** (2011) 50004, 1101.4182.
- [4] R. Gurau, “The complete 1/ $N$  expansion of colored tensor models in arbitrary dimension,” *Annales Henri Poincaré* **13** (2012) 399–423, 1102.5759.

- [5] V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau, “Critical behavior of colored tensor models in the large  $N$  limit,” *Nucl. Phys.* **B853** (2011) 174–195, 1105.3122.
- [6] A. Tanasa, “Multi-orientable Group Field Theory,” *J. Phys.* **A45** (2012) 165401, 1109.0694.
- [7] V. Bonzom, R. Gurau, and V. Rivasseau, “Random tensor models in the large  $N$  limit: Uncoloring the colored tensor models,” *Phys. Rev.* **D85** (2012) 084037, 1202.3637.
- [8] S. Carrozza and A. Tanasa, “ $O(N)$  Random Tensor Models,” *Lett. Math. Phys.* **106** (2016), no. 11 1531–1559, 1512.06718.
- [9] I. R. Klebanov and G. Tarnopolsky, “On Large  $N$  Limit of Symmetric Traceless Tensor Models,” 1706.00839.
- [10] R. Gurau, “The  $1/N$  expansion of tensor models with two symmetric tensors,” 1706.05328.
- [11] J. Ambjorn, B. Durhuus, and T. Jonsson, “Three-dimensional simplicial quantum gravity and generalized matrix models,” *Mod. Phys. Lett.* **A6** (1991) 1133–1146.
- [12] N. Sasakura, “Tensor model for gravity and orientability of manifold,” *Mod. Phys. Lett.* **A6** (1991) 2613–2624.
- [13] M. Gross, “Tensor models and simplicial quantum gravity in  $> 2$ -D,” *Nucl. Phys. Proc. Suppl.* **25A** (1992) 144–149.
- [14] E. Witten, “An SYK-Like Model Without Disorder,” 1610.09758.
- [15] I. R. Klebanov and G. Tarnopolsky, “Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models,” *Phys. Rev.* **D95** (2017), no. 4 046004, 1611.08915.
- [16] S. Sachdev and J. Ye, “Gapless spin fluid ground state in a random, quantum Heisenberg magnet,” *Phys. Rev. Lett.* **70** (1993) 3339, cond-mat/9212030.
- [17] O. Parcollet and A. Georges, “Non-Fermi-liquid regime of a doped Mott insulator,” *Physical Review B* **59** (Feb., 1999) 5341–5360, cond-mat/9806119.
- [18] A. Georges, O. Parcollet, and S. Sachdev, “Mean Field Theory of a Quantum Heisenberg Spin Glass,” *Physical Review Letters* **85** (July, 2000) 840–843, cond-mat/9909239.

- [19] A. Kitaev, “A simple model of quantum holography,”  
<http://online.kitp.ucsb.edu/online/entangled15/kitaev/>,  
<http://online.kitp.ucsb.edu/online/entangled15/kitaev2/>. Talks at KITP, April 7,  
 2015 and May 27, 2015.
- [20] J. Polchinski and V. Rosenhaus, “The Spectrum in the Sachdev-Ye-Kitaev Model,”  
*JHEP* **04** (2016) 001, 1601.06768.
- [21] J. Maldacena and D. Stanford, “Comments on the Sachdev-Ye-Kitaev model,” *Phys. Rev. D* **94** (2016), no. 10 106002, 1604.07818.
- [22] A. Jevicki, K. Suzuki, and J. Yoon, “Bi-Local Holography in the SYK Model,” *JHEP* **07** (2016) 007, 1603.06246.
- [23] D. J. Gross and V. Rosenhaus, “A Generalization of Sachdev-Ye-Kitaev,” *JHEP* **02** (2017) 093, 1610.01569.
- [24] K. Bulycheva, I. R. Klebanov, A. Milekhin, and G. Tarnopolsky, “Spectra of Operators in Large  $N$  Tensor Models,” 1707.09347.
- [25] S. Choudhury, A. Dey, I. Halder, L. Janagal, S. Minwalla, and R. Poojary, “Notes on Melonic  $O(N)^{q-1}$  Tensor Models,” 1707.09352.
- [26] Y. Gu, X.-L. Qi, and D. Stanford, “Local criticality, diffusion and chaos in generalized Sachdev-Ye-Kitaev models,” *JHEP* **05** (2017) 125, 1609.07832.
- [27] G. Turiaci and H. Verlinde, “Towards a 2d QFT Analog of the SYK Model,” 1701.00528.
- [28] P. Narayan and J. Yoon, “SYK-like Tensor Models on the Lattice,” 1705.01554.
- [29] J. Murugan, D. Stanford, and E. Witten, “More on Supersymmetric and 2d Analogs of the SYK Model,” 1706.05362.
- [30] K. G. Wilson and M. E. Fisher, “Critical exponents in 3.99 dimensions,” *Phys. Rev. Lett.* **28** (1972) 240–243.
- [31] K. G. Wilson, “Quantum field theory models in less than four-dimensions,” *Phys. Rev. D* **7** (1973) 2911–2926.

- [32] A. Z. Patashinskii and V. L. Pokrovskii, “Second Order Phase Transitions in a Bose Fluid,” *JETP* **19** (1964) 677.
- [33] K. Symanzik, “On Calculations in conformal invariant field theories,” *Lett. Nuovo Cim.* **3** (1972) 734–738.
- [34] T. Azeyanagi, F. Ferrari, and F. I. Schaposnik Massolo, “Phase Diagram of Planar Matrix Quantum Mechanics, Tensor and SYK Models,” 1707.03431.
- [35] P. Breitenlohner and D. Z. Freedman, “Stability in Gauged Extended Supergravity,” *Annals Phys.* **144** (1982) 249.
- [36] L. F. Alday and J. M. Maldacena, “Comments on operators with large spin,” *JHEP* **11** (2007) 019, 0708.0672.
- [37] A. L. Fitzpatrick, J. Kaplan, D. Poland, and D. Simmons-Duffin, “The Analytic Bootstrap and AdS Superhorizon Locality,” *JHEP* **1312** (2013) 004, 1212.3616.
- [38] Z. Komargodski and A. Zhiboedov, “Convexity and Liberation at Large Spin,” *JHEP* **11** (2013) 140, 1212.4103.
- [39] E. D. Skvortsov, “On (Un)Broken Higher-Spin Symmetry in Vector Models,” 1512.05994.
- [40] S. Giombi and V. Kirilin, “Anomalous Dimensions in CFT with Weakly Broken Higher Spin Symmetry,” 1601.01310.
- [41] I. Jack and H. Osborn, “Analogues for the  $c$  Theorem for Four-dimensional Renormalizable Field Theories,” *Nucl. Phys.* **B343** (1990) 647–688.
- [42] A. Dymarsky, I. R. Klebanov, and R. Roiban, “Perturbative search for fixed lines in large  $N$  gauge theories,” *JHEP* **08** (2005) 011, hep-th/0505099.